# Galois theory, functional Lindemann-Weierstrass, and Manin maps

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February 14, revised July 7, 2015

**Abstract**

We prove several new results of Ax-Lindemann type for semi-abelian varieties over the algebraic closure $K$ of $\mathbb{C}(t)$, making heavy use of the Galois theory of logarithmic differential equations. Using related techniques, we also give a generalization of the theorem of the kernel for abelian varieties over $K$. This paper is a continuation of [7] as well as an elaboration on the methods of Galois descent introduced in [4] and [5] in the case of abelian varieties.

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*Thanks to the ANR-13-BS01-0006 for support, as well as MSRI, Berkeley
†Thanks to EPSRC grant EP/I002294/1, MSRI, Berkeley, as well as NSF grant DMS-1360702

# 1 Introduction

The paper has three related themes, the common feature being differential Galois theory and its applications.

Firstly, given a semiabelian variety $B$ over the algebraic closure $K$ of $\mathbb{C}(t)$, a $K$-rational point $a$ of the Lie algebra $LG$ of its universal vectorial extension $G = \tilde{B}$, and a solution $y \in G(K^{diff})$ of the logarithmic differential equation

$$\partial \ell n_G(y) = a, \;\; a \in LG(K),$$

we want to describe $tr.deg(K_G^\sharp(y)/K_G^\sharp)$ in terms of "gauge transformations" *over* $K$. Here $K_G^\sharp$ is the differential field generated over $K$ by solutions in $K^{diff}$ of $\partial \ell n_G(-) = 0$. Introducing this field as base presents both advantages and difficulties. On the one hand, it allows us to use the differential Galois theory developed by the second author in [16], [17], [19], thereby replacing the study of transcendence degrees by the computation of a Galois group. On the other hand, we have only a partial knowledge of the extension $K_G^\sharp/K$. However, it was observed by the first author in [4], [5] that in the case of an abelian variety, what we do know essentially suffices to perform a Galois descent from $K_G^\sharp$ to the field $K$ of the searched-for gauge transform. In §2.2 and §3 of the present paper, we extend this principle to semi-abelian varieties $B$ whose toric part is $\mathbb{G}_m$, and give a definitive description of $tr.deg(K_G^\sharp(y)/K_G^\sharp)$ when $B$ is an abelian variety.

The main application we have in mind of these Galois theoretic results forms the second theme of our paper, and concerns Lindemann-Weierstrass statements for the semiabelian variety $B$ over $K$, by which we mean the description of the transcendence degree of $exp_B(x)$ where $x$ is a $K$-rational point of the Lie algebra $LB$ of $B$. The problem is covered in the above setting by choosing as data

$$a := \partial_{LG}(\tilde{x}) \in \partial_{LG}(LG(K)),$$

where $\tilde{x}$ is an arbitrary $K$-rational lift of $x$ to $G = \tilde{B}$. This study was initiated in our joint paper [7], where the Galois approach was mentioned, but only under the hypothesis that $K_G^\sharp = K$, described as $K$*-largeness* of $G$. There are natural conjectures in analogy with the well-known "constant" case (where $B$ is over $\mathbb{C}$), although as pointed out in [7], there are also counterexamples provided by nonconstant extensions of a constant elliptic curve by the multiplicative group. In §2.3 and §4 of the paper, we extend the main result of [7] to the base $K_G^\sharp$, but assuming the toric part of $B$ is at most 1-dimensional. Furthermore, we give in this case a full solution of the Lindemann-Weierstrass statement when the abelian quotient of $B$ too is 1-dimensional. This uses results from [6] which deal with the "logarithmic" case. In this direction, we will also formulate an "Ax-Schanuel" type conjecture for abelian varieties over $K$.

The third theme of the paper concerns the "theorem of the kernel", which we generalize in §2.4 and §5 by proving that linear independence with respect to $End(A)$ of points $y_1, \ldots, y_n$ in $A(K)$ implies linear independence of $\mu_A(y_1)$,..., $\mu_A(y_n)$ with respect to $\mathbb{C}$ (this answers a question posed to us by Hrushovski). Here $A$ is an abelian variety over $K = \mathbb{C}(t)^{alg}$ with $\mathbb{C}$-trace 0 and $\mu_A$ is the differential-algebraic Manin map. However, we will give an example showing that its $\mathbb{C}$-linear extension $\mu_A \otimes 1$ on $A(K) \otimes_{\mathbb{Z}} \mathbb{C}$ is not always injective. In contrast, we observe that the $\mathbb{C}$-linear extension $M_{K,A} \otimes 1$ of the classical (differential-arithmetic) Manin map $M_{K,A}$ is always injective. Differential Galois theory and the logarithmic case of nonconstant Ax-Schanuel are involved in the proofs.

# 2 Statements of results

## 2.1 Preliminaries on logarithmic equations

We will here give a quick background to the basic notions and objects so as to be able to state our main results in the next subsections. The remaining parts 3, 4, 5 of the paper are devoted to the proofs. We refer the reader to [7] for more details including differential algebraic preliminaries.

We fix a differential field $(K, \partial)$ of characteristic 0 whose field of constants $C_K$ is algebraically closed (the reader will loose nothing by taking $C_K = \mathbb{C}$). We usually assume that $K$ is algebraically closed, and denote by $K^{diff}$ the differential closure of $K$. We let $\mathcal{U}$ denote a universal differential field containing $K$, with constant field $\mathcal{C}$. If $X$ is an algebraic variety over $K$ we will identify $X$ with its set $X(\mathcal{U})$ of $\mathcal{U}$ points, unless we say otherwise.

We start with algebraic $\partial$-groups, which provide the habitat of the (generalised) differential Galois theory of [16], [17], [19] discussed later on. A (connected) *algebraic $\partial$-group* over $K$ is a (connected) algebraic group $G$ over $K$ together with a lifting $D$ of the derivation $\partial$ of $K$ to a derivation of the structure sheaf $\mathcal{O}_G$ which respects the group structure. The derivation $D$ may be identified with a section $s$, in the category of algebraic groups, of the projection map $T_\partial(G) \to G$, where $T_\partial(G)$ denotes the twisted tangent bundle of $G$. This $T_\partial(G)$ is a (connected) algebraic group over $K$, which is a torsor under the tangent bundle $TG$, and is locally defined by equations $\sum_{i=1,..n} \partial P/\partial x_i(\bar{x})u_i + P^\partial(\bar{x})$, for polynomials $P$ in the ideal of $G$, where $P^\partial$ is obtained by applying the derivation $\partial$ of $K$ to the coefficients of $P$. Notice for later use that for any differential extension $L/K$, there is a group homomorphism $G(L) \to T_\partial G(L)$, which is given in coordinates by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \partial x_1, \ldots, \partial x_n)$ and will be denoted by $\partial$.

We write the algebraic $\partial$-group as $(G, D)$ or $(G, s)$. Not every algebraic group over $K$ has a $\partial$-structure. But when $G$ is defined over the constants $C_K$ of $K$, there is a privileged $\partial$-structure $s_0$ on $G$ which is precisely the 0-section of $TG = T_\partial G$. Given an algebraic $\partial$-group $(G, s)$ over $K$ we obtain an associated "logarithmic derivative" $\partial\ell n_{G,s}(-)$ from $G$ to the Lie algebra $LG$ of $G$: $\partial\ell n_{G,s}(y) = \partial(y)s(y)^{-1}$, where the product is computed in the algebraic group $T_\partial(G)$. This is a differential rational crossed homomorphism from $G$ onto $LG$ (at the level of $\mathcal{U}$-points or points in a differentially closed field) defined over $K$. Its kernel $\ker(\partial\ell n_{G,s})$ is a differential algebraic subgroup of

$G$ which we denote $(G,s)^\partial$, or simply $G^\partial$ when the context is clear. Now $s$ equips the Lie algebra $LG$ of $G$ with its own structure of a $\partial$-group (in this case a $\partial$-module) which we call $\partial_{LG}$ (depending on $(G,s)$) and again the kernel is denoted $(LG)^\partial$.

In the case where $G$ is defined over $C_K$ and $s = s_0$, $\partial\ell n_{G,s}$ is precisely Kolchin's logarithmic derivative, taking $y \in G$ to $\partial(y)y^{-1}$. In general, as soon as $s$ is understood, we will abbreviate $\partial\ell n_{G,s}$ by $\partial\ell n_G$.

By a *logarithmic differential equation* over $K$ on the algebraic $\partial$-group $(G,s)$, we mean a differential equation $\partial\ell n_{G,s}(y) = a$ for some $a \in LG(K)$. When $G = GL_n$ and $s = s_0$ this is the equation for a fundamental system of solutions of a linear differential equation $Y' = aY$ in vector form. And more generally for $G$ an algebraic group over $C_K$ and $s = s_0$ this is a logarithmic differential equation on $G$ over $K$ in the sense of Kolchin. There is a well-known Galois theory here. In the given differential closure $K^{diff}$ of $K$, any two solutions $y_1, y_2$ of $\partial\ell n_G(-) = a$, in $G(K^{diff})$, differ by an element in the kernel $G^\partial$ of $\partial\ell n_G(-)$. But $G^\partial(K^{diff})$ is precisely $G(C_K)$. Hence $K(y_1) = K(y_2)$. In particular $tr.deg(K(y)/K)$ is the same for all solutions $y$ in $K^{diff}$. Moreover $Aut(K(y)/K)$ has the structure of an algebraic subgroup of $G(C_K)$: for any $\sigma \in Aut(K(y)/K)$, let $\rho_\sigma \in G(C_K)$ be such that $\sigma(y) = y\rho_\sigma$. Then the map taking $\sigma$ to $\rho_\sigma$ is an isomorphism between $Aut(K(y)/K)$ and an algebraic subgroup $H(C_K)$ of $G(C_K)$, which we call the differential Galois group of $K(y)/K$. This depends on the choice of solution $y$, but another choice yields a conjugate of $H$. Of course when $G$ is commutative, $H$ is independent of the choice of $y$. In any case $tr.deg(K(y)/K) = dim(H)$, so computing the differential Galois group gives us a transcendence estimate.

Continuing with this Kolchin situation, we have the following well-known fact, whose proof we present in the setting of the more general situation considered in Fact 2.2.(i).

**Fact 2.1** (for $G/C_K$). *Suppose $K$ algebraically closed. Then, $tr.deg(K(y)/K)$ is the dimension of a minimal connected algebraic subgroup $H$ of $G$, defined over $C_K$, such that for some $g \in G(K)$, $gag^{-1} + \partial\ell n_G(g) \in LH(K)$. Moreover $H(C_K)$ is the differential Galois group of $K(y)/K$.*

*Proof.* Let $H$ be a connected algebraic subgroup of $G$, defined over $C_K$ such that $H^\partial(K^{diff}) = H(C_K)$ is the differential Galois group of $K(y)$ over $K$. Now the $H^\partial(K^{diff})$-orbit of $y$ is defined over $K$ in the differential algebraic sense, so the $H$-orbit of $y$ is defined over $K$ in the differential algebraic sense.

A result of Kolchin on constrained cohomology (see Proposition 3.2 of [16], or Theorem 2.2 of [5]) implies that this orbit has a $K$-rational point $g^{-1}$. So, there exists $z^{-1} \in H$ such that $g^{-1} = yz^{-1}$, and $z = gy$, which satisfies $K(y) = K(z)$, is a solution of $\partial \ell n_G(-) = a'$ where $a' = gag^{-1} + \partial \ell n_G(g)$. □

(Such a map $LG(K) \to LG(K)$ taking $a \in LG(K)$ to $gag^{-1} + \partial \ell n_G(g)$ for some $g \in G(K)$ is called a gauge transformation.)

Now in the case of an arbitrary algebraic $\partial$-group $(G, s)$ over $K$, and logarithmic differential equation $\partial \ell n_{G,s}(-) = a$ over $K$, two solutions $y_1, y_2$ in $G(K^{diff})$ differ by an element of $(G, s)^\partial(K^{diff})$ which in general may not be contained in $G(K)$. (For instance, if $(G = \mathbb{G}_a, s)$ is the $\partial$-module attached to $\partial y - y = 0$, and $a = 1-t$, then, $y_1 = t$ is rational over $K = \mathbb{C}(t)$, while $y_2 = t + e^t$ is transcendental over $K$.) So both to obtain a transcendence statement independent of the choice of solution, as well as a Galois theory, we should work over $K^\sharp_{G,s}$ which is the (automatically differential) field generated by $K$ and $(G, s)^\partial(K^{diff})$. This field may be viewed as a field of "new constants", and its algebraic closure in $K^{diff}$ will be denoted by $K^{\sharp\, alg}_{G,s}$. As with $\partial \ell n_G$ and $G^\partial$, we will abbreviate $K^\sharp_{G,s}$ as $K^\sharp_G$ , or even $K^\sharp$, when the context is clear, and similarly for its algebraic closure. Fixing a solution $y \in G(K^{diff})$ of $\partial \ell n_G(-) = a$, for $\sigma \in Aut(K^\sharp(y)/K^\sharp)$, $\sigma(y) = y\rho_\sigma$ for unique $\rho_\sigma \in G^\partial(K^{diff}) = G^\partial(K^\sharp) \subseteq G(K^\sharp)$, and again the map $\sigma \to \rho_\sigma$ defines an isomorphism between $Aut(K^\sharp(y)/K^\sharp)$ and $(H, s)^\partial(K^{diff})$ for an algebraic $\partial$ subgroup $H$ of $(G, s)$, ostensibly defined over $K^\sharp$. The $\partial$-group $H$ (or more properly $H^\partial$ , or $H^\partial(K^\sharp)$) is called the *(differential) Galois group* of $K^\sharp(y)$ over $K^\sharp$, and when $G$ is commutative does not depend on the choice of $y$, just on the data $a \in LG(K)$ of the logarithmic equation, and in fact only on the image of $a$ in the cokernel $LG(K)/\partial \ell n_G G(K)$ of $\partial \ell n_G$. Again $tr.deg(K^\sharp(y)/K^\sharp) = dim(H)$. In any case, Fact 1.1 extends to this context with essentially the same proof. This can also be extracted from Proposition 3.4 of [16] and the set-up of [19]. For the commutative case (part (ii) below) see [5], Theorem 3.2. Note that in the present paper, it is this Fact 2.2.(ii) which we will use. Going to the algebraic closure of $K^\sharp$ as in Fact 2.2.(i) would force us to consider profinite groups, for which our descent arguments may not work.

**Fact 2.2** (for $G/K$)**.** *Let $y$ be a solution of $\partial \ell n_{G,s}(-) = a$ in $G(K^{diff})$, and let $K^\sharp = K(G^\partial)$, with algebraic closure $K^{\sharp\, alg}$. Then* *(i) tr.deg$(K^\sharp(y)/K^\sharp)$ is the dimension of a minimal connected algebraic $\partial$-subgroup $H$ of $G$, defined over $K^{\sharp\, alg}$ such that $gag^{-1}+\partial\ell n_{G,s}(g) \in LH(K^{\sharp\, alg})$ for some $g \in G(K^{\sharp\, alg})$. And $H^\partial(K^{\sharp\, alg})$ is the differential Galois group of $K^{\sharp\, alg}(y)/K^{\sharp\, alg}$.*

*(ii) Suppose that $G$ is commutative. Then, the identity component of the differential Galois group of $K^\sharp(y)/K^\sharp$ is $H^\partial(K^\sharp)$, where $H$ is the smallest algebraic $\partial$-subgroup of $G$ defined over $K^\sharp$ such that $a \in LH+\mathbb{Q}.\partial\ell n_{G,s}G(K^\sharp)$.*

**Remark**. - We point out that when $G$ is commutative, then in Facts 2.1 and 2.2, the Galois group, say $\tilde{H}$, of $K^\sharp(y)/K^\sharp$ is a unique subgroup of $G$, so its identity component $H$ must indeed be the smallest algebraic subgroup of $G$ with the required properties (see also [5], §3.1). Of course, $\tilde{H}$ is automatically connected in 2.2.(i), where the base $K^{\sharp\, alg}$ is algebraically closed, but as just announced, our proofs in §3 will be based on 2.2.(ii). Now, in this commutative case, the map $\sigma \to \rho_\sigma$ described above depends $\mathbb{Z}$-linearly on $a$. So, if $N = [\tilde{H} : H]$ denotes the number of connected components of $\tilde{H}$, then replacing $a$ by $Na$ turns the Galois group into a connected algebraic group, without modifying $K^\sharp$ nor *tr.deg*$(K^\sharp(y)/K^\sharp) =$ *tr.deg*$(K^\sharp(Ny)/K^\sharp)$. Therefore, in the computations of Galois groups later on, we will tacitly replace $y$ by $Ny$ and determine the connected component $H$ of $\tilde{H}$. But it turns out that in our main Conjecture 2.3 and in all its cases under study here, we can then assume that $y$ itself lies in $H$. Indeed, $y$ appears only via its class modulo $G(K)$, and in particular, modulo its torsion subgroup (recall that $K$ is algebraically closed). So, once we have proven that $Ny$ lies in $H$, then a translate $y'$ of $y$ by a $N$-torsion point will lie in $H$. Replacing $y$ by $y'$ does not modify the Galois group $\tilde{H}$ of $K^\sharp(y)$ over $K^\sharp$, so we may assume that $y$ lies in $H$, in which case $\tilde{H}$ coincides with $H$, and will in the end always be connected[1].

## 2.2 Galois theoretic results

The question which we deal with in this paper is when and whether in Fact 2.2, it suffices to consider $H$ defined over $K$ and $g \in G(K)$. In fact it is not hard to see that the Galois group is defined over $K$, but the second point is

[1] We take opportunity of this remark to mention two errata in [5] : in the proof of its Theorem 3.2, replace "of finite index" by "with quotient of finite exponent"; in the proof of Theorem 4.4, use the reduction process described above to justify that the Galois group is indeed connected.

problematic. The case where $(G,s)$ is a $\partial$-module, namely $G$ is a vector space $V$, and the logarithmic derivative $\partial\ell n_{G,s}(y)$ has the form $\nabla_V(y) = \partial y - By$ for some $n\times n$ matrix $B$ over $K$, was considered in [2], and shown to provide counterexamples, unless the $\partial$-module $(V,\nabla_V)$ is semisimple. The rough idea is that the Galois group $Gal(K_V^\sharp/K)$ of $\nabla_V$ is then reductive, allowing an argument of Galois descent from $K_V^\sharp$ to $K$ to construct a $K$-rational gauge transformation $g$. The argument was extended in [4], [5] to $\partial$-groups $(G,s)$ attached to abelian varieties, which by Poincaré reducibility, are in a sense again semi-simple.

We will here focus on the *almost semiabelian* case: namely certain $\partial$-groups attached to semiabelian varieties, which provide the main source of non semi-simple situations. If $B$ is a semiabelian variety over $K$, then $\tilde{B}$, the universal vectorial extension of $B$, is a (commutative) algebraic group over $K$ which has a *unique* algebraic $\partial$-group structure. Let $U$ be any unipotent algebraic $\partial$-subgroup of $\tilde{B}$. Then $\tilde{B}/U$, which by [7], Lemma 3.4 also has a unique $\partial$-group structure, is what we mean by an almost semiabelian $\partial$-group over $K$. When $B$ is an abelian variety $A$ we call $\tilde{A}/U$ an almost abelian algebraic $\partial$-group over $K$. If $G$ is an almost semiabelian algebraic $\partial$-group over $K$, then because the $\partial$-group structure $s$ on $G$ is unique, the abbreviation $K_G^\sharp$ for $K_{G,s}^\sharp$ is now unambiguous. Under these conditions, we make the following conjecture.

**Conjecture 2.3.** *Let $G$ be an almost semiabelian $\partial$-group over $K = \mathbb{C}(t)^{alg}$. Let $a \in LG(K)$, and $y \in G(K^{diff})$ be such that $\partial\ell n_G(y) = a$. Then $tr.deg(K_G^\sharp(y)/K_G^\sharp)$ is the dimension of the smallest algebraic $\partial$-subgroup $H$ of $G$ defined over $K$ such that $a \in LH+\partial\ell n_G(G(K))$, i.e. $a+\partial\ell n_G(g) \in LH(K)$ for some $g \in G(K)$ . Equivalently the smallest algebraic $\partial$-subgroup $H$ of $G$, defined over $K$, such that $y \in H + G(K) + G^\partial(K^{diff})$. Moreover $H^\partial(K^{diff})$ is the Galois group of $K_G^\sharp(y)$ over $K_G^\sharp$.*

The conjecture can be restated as: there is a smallest algebraic $\partial$-subgroup $H$ of $(G,s)$ defined over $K$ such that $a \in LH + \partial\ell n_G(G(K))$ and it coincides with the Galois group of $K_G^\sharp(y)$ over $K_G^\sharp$. In comparison with Fact 2.2.(ii), notice that since $K$ is algebraically closed, $\partial\ell n_G(G(K))$ is already a $\mathbb{Q}$-vector space, so we do not need to tensor with $\mathbb{Q}$ in the condition on $a$.

A corollary of Conjecture 2.3 is the following special *generic case*, where an additional assumption on non-degeneracy is made on $a$:

**Conjecture 2.4.** *Let $G$ be an almost semiabelian $\partial$-group over $K = \mathbb{C}(t)^{alg}$. Let $a \in LG(K)$, and $y \in G(K^{diff})$ be such that $\partial\ell n_G(y) = a$. Assume that*

*$a \notin LH + \partial\ell n_G G(K)$ for any proper algebraic $\partial$-subgroup $H$ of $G$, defined over $K$ (equivalently $y \notin H + G(K) + G^{\partial}(K^{diff})$ for any proper algebraic $\partial$-subgroup of $G$ defined over $K$). Then $tr.deg(K_G^{\sharp}(y)/K_G^{\sharp}) = dim(G)$.*

We will prove the following results in the direction of Conjectures 2.3 and (the weaker) 2.4.

**Proposition 2.5.** *Conjecture 2.3 holds when $G$ is "almost abelian".*

The truth of the weaker Conjecture 2.4 in the almost abelian case is already established in [4], Section 8.1(i). This reference does not address Conjecture 2.3 itself, even if in this case, the ingredients for its proof are there (see also [5]). So we take the liberty to give a reasonably self-contained proof of Proposition 2.5 in Section 3.

As announced above, one of the main points of the Galois-theoretic part of this paper is to try to extend Proposition 2.5 to the almost semiabelian case. Due to technical complications, which will be discussed later, we restrict our attention to the simplest possible extension of the almost abelian case, namely where the toric part of the semiabelian variety is 1-dimensional, and also we sometimes just consider the generic case. For simplicity we will state and prove our results for an almost semiabelian $G$ of the form $\tilde{B}$ for $B$ semiabelian. So, the next proposition gives Conjecture 2.4 for an extension by $\mathbb{G}_m$ of the universal vectorial extension of an abelian variety.

**Theorem 2.6.** *Suppose that $B$ is a semiabelian variety over $K = \mathbb{C}(t)^{alg}$ with toric part of dimension $\leq 1$. Let $G = \tilde{B}$, $a \in LG(K)$ and $y \in G(K^{diff})$ a solution of $\partial\ell n_G(-) = a$. Suppose that for no proper algebraic $\partial$-subgroup $H$ of $G$ defined over $K$ is $y \in H + G(K)$. Then $tr.deg(K_G^{\sharp}(y)/K_G^{\sharp}) = dim(G)$ and $G^{\partial}(K^{diff})$ is the differential Galois group.*

Note that the hypothesis above " $y \notin H + G(K)$ for any proper algebraic $\partial$-subgroup of $G$ over $K$ " is formally weaker than " $y \notin H + G(K) + G^{\partial}(K^{diff})$ for any proper algebraic $\partial$-subgroup of $G$ over $K$ " but nevertheless suffices, as shown by the proof of 2.6 in Section 3.2. More specifically, assume that $G = \tilde{A}$ for a simple abelian variety $A/K$, that $A$ is traceless (i.e. that there is no non-zero morphism from an abelian variety defined over $\mathbb{C}$ to $A$), that the maximal unipotent $\partial$-subgroup $U_A$ of $\tilde{A}$ vanishes, and that $a = 0 \in L\tilde{A}(K)$. Theorem 2.6 then *implies* that any $y \in \tilde{A}^{\partial}(K^{diff})$ is actually defined over $K$, so $K_{\tilde{A}}^{\sharp} = K$. As in [4], [5], this property of $K$-largeness of $\tilde{A}$ (when $U_A = 0$)

is in fact one of the main *ingredients* in the proof of 2.6. As explained in [14] it is based on the strong minimality of $\tilde{A}^{\partial}$ (see [11]) in the context above. But it has recently been noted in [1] that this $K$-largeness property can be seen rather more directly, using only the simplicity of $A$.

Our last Galois-theoretic result requires the *semiconstant* notions introduced in [7], although our notation will be a slight modification of that in [7]. First a connected algebraic group $G$ over $K$ is said to be constant if $G$ is isomorphic (as an algebraic group) to an algebraic group defined over $\mathbb{C}$ (equivalent $G$ arises via base change from an algebraic group $G_{\mathbb{C}}$ over $\mathbb{C}$). For $G$ an algebraic group over $K$, $G_0$ will denote the largest (connected) constant algebraic subgroup of $G$. We will concentrate on the case $G = \tilde{B}$ for a semiabelian variety $B$ over $K$, with $0 \to T \to B \to A \to 0$ the canonical exact sequence, where $T$ is the maximal linear algebraic subgroup of $B$ (which is an algebraic torus) and $A$ is an abelian variety. So now $A_0$, $B_0$ denote the constant parts of $A, B$ respectively. The inverse image of $A_0$ in $B$ will be called the semiconstant part of $B$ and will now be denoted by $B_{sc}$. We will call $B$ semiconstant if $B = B_{sc}$ which is equivalent to requiring that $A = A_0$, and moreover allows the possibility that $B = B_0$ is constant. (Of course, when $B$ is constant, $\tilde{B}$, which is also constant, obviously satisfies Conjecture 2.3, in view of Fact 2.1.)

**Theorem 2.7.** *Suppose that $K = \mathbb{C}(t)^{alg}$ and that $B = B_{sc}$ is a semiconstant semiabelian variety over $K$ with toric part of dimension $\leq 1$. Then Conjecture 2.3 holds for $G = \tilde{B}$.*

## 2.3 Lindemann-Weierstrass via Galois theory

We are now ready to describe the impact of the previous Galois theoretic results on Ax-Lindemann problems, where $a = \partial_{LG}(\tilde{x}) \in \partial_{LG}(LG(K))$.

Firstly, from Theorem 2.6 we will deduce directly the main result of [7] (Theorem 1.4), when $B$ is semiabelian with toric part at most $\mathbb{G}_m$, but now with transcendence degree computed over $K^{\sharp}_{\tilde{B}}$:

**Corollary 2.8.** *Let $B$ be a semiabelian variety over $K = \mathbb{C}(t)^{alg}$ such that the toric part of $B$ is of dimension $\leq 1$ and $B_{sc} = B_0$ (i.e. the semiconstant part $B_{sc}$ of $B$ is constant). Let $x \in LB(K)$, and lift $x$ to $\tilde{x} \in L\tilde{B}(K)$. Assume that*
(*) *for no proper algebraic subgroup $H$ of $\tilde{B}$ defined over $K$ is $\tilde{x} \in LH(K) +$*

$(L\tilde{B})^\partial(K)$,
*which under the current assumptions is equivalent to demanding that for no proper semiabelian subvariety $H$ of $B$, is $x \in LH(K) + LB_0(\mathbb{C})$. Then,*

*(i) any solution $\tilde{y} \in B(\mathcal{U})$ of $\partial\ell n_{\tilde{B}}(-) = \partial_{L\tilde{B}}(\tilde{x})$ satisfies*

$$tr.deg(K^\sharp_{\tilde{B}})(\tilde{y})/K^\sharp_{\tilde{B}}) = dim(\tilde{B}) \quad ;$$

*(ii) in particular, $y := exp_B(x)$ satisfies $tr.deg(K^\sharp_{\tilde{B}}(y)/K^\sharp_{\tilde{B}}) = dim(B)$, i.e. is a generic point over $K^\sharp_{\tilde{B}}$ of $B$.*

See [7] for the analytic description of $exp_B(x)$ in (ii) above. In particular $exp_B(x)$ can be viewed as a a point of $B(\mathcal{U})$. We recall briefly the argument. Consider $B$ as the generic fibre of a family $\mathbf{B} \to S$ of complex semiabelian varieties over a complex curve $S$, and $x$ as a rational section $x : S \to L\mathbf{B}$ of the corresponding family of Lie algebras. Fix a small disc $U$ in $S$, such that $x : U \to L\mathbf{B}$ is holomorphic, and let $exp(x) = y : U \to \mathbf{B}$ be the holomorphic section obtained by composing with the exponential map in the fibres. So $y$ lives in the differential field of meromorphic functions on $U$, which contains $K$, and can thus be embedded over $K$ in the universal differentially closed field $\mathcal{U}$. So talking about $tr.deg(K^\sharp_{\tilde{B}}(y)/K^\sharp_{\tilde{B}})$ makes sense.

Let us comment on the methods. In [7] an essential use was made of the so-called "socle theorem" (see §4.1 of [7] for a discussion of this expression) in order to prove Theorem 1.4 there. As recalled in the introduction, a differential Galois theoretic approach was also mentioned ([7], §6), but could be worked out only when $\tilde{B}$ is $K$-large. In the current paper, we dispose of this hypothesis, and obtain a stronger result, namely over $K^\sharp_{\tilde{B}}$ , but for the time being at the expense of restricting the toric part of $B$.

When $B = A$ is an abelian variety one obtains a stronger statement than Corollary 2.8. This is Theorem 4.4 of [5], which for the sake of completeness we restate, and will deduce from Proposition 2.5 in Section 4.1.

**Corollary 2.9.** *Let $A$ be an abelian variety over $K = \mathbb{C}(t)^{alg}$. Let $x \in LA(K)$, and let $B$ be the smallest abelian subvariety of $A$ such that $x \in LB(K) + LA_0(\mathbb{C})$. Let $\tilde{x} \in L\tilde{A}(K)$ be a lift of $x$ and let $\tilde{y} \in \tilde{A}(\mathcal{U})$ be such that $\partial\ell n_{\tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x})$. Then, $\tilde{B}^\partial$ is the Galois group of $K^\sharp_{\tilde{A}}(\tilde{y})$ over $K^\sharp_{\tilde{A}}$, so*

*(i) $tr.deg(K^\sharp_{\tilde{A}}(\tilde{y})/K^\sharp_{\tilde{A}}) = dim(\tilde{B}) = 2dim(B)$, and in particular:*

*(ii) $y := exp_A(x)$ satisfies $tr.deg(K^\sharp_{\tilde{A}}(y)/K^\sharp_{\tilde{A}}) = dim(B)$.*

We now return to the semiabelian context. Corollary 2.8 is not true without the assumption that the semiconstant part of $B$ is constant. The simplest possible counterexample is given in section 5.3 of [7]: $B$ is a nonconstant extension of a constant elliptic curve $E_0$ by $\mathbb{G}_m$, with judicious choices of $x$ and $\tilde{x}$. Moreover $\tilde{x}$ will satisfy assumption (*) in Corollary 2.8, but $tr.deg(K(\tilde{y})/K) \leq 1$, which is strictly smaller than $dim(\tilde{B}) = 3$. We will use 2.6 and 2.7 as well as material from [6] to give a full account of this situation (now over $K^{\sharp}_{\tilde{B}}$, of course), and more generally, for all semiabelian surfaces $B/K$, as follows:

**Corollary 2.10.** *Let $B$ be an extension over $K = \mathbb{C}(t)^{alg}$ of an elliptic curve $E/K$ by $\mathbb{G}_m$. Let $x \in LB(K)$ satisfy*
*(*) for any proper algebraic subgroup $H$ of $B$, $x \notin LH + LB_0(\mathbb{C})$.*
*Let $\tilde{x} \in L\tilde{B}(K)$ be a lift of $x$, let $\overline{x}$ be its projection to $LE(K)$, and let $\tilde{y} \in \tilde{B}(\mathcal{U})$ be such that $\partial\ell n_{\tilde{B}}(\tilde{y}) = \tilde{x}$. Then, $tr.deg(K^{\sharp}_{\tilde{B}}(\tilde{y})/K^{\sharp}_{\tilde{B}}) = 3$, unless $\bar{x} \in LE_0(\mathbb{C})$ in which case $tr.deg(K^{\sharp}_{\tilde{B}}(\tilde{y})/K^{\sharp}_{\tilde{B}})$ is precisely 1.*

Here, $E_0$ is the constant part of $E$. Notice that in view of Hypothesis (*), $E$ must descend to $\mathbb{C}$ and $B$ must be non-constant (hence not isotrivial) if $x$ projects to $LE_0(\mathbb{C})$.

## 2.4 Manin maps

We finally discuss the results on the Manin maps attached to abelian varieties. The expression "Manin map" covers at least two maps. The original one was introduced by Manin in [13], see also [10], and is discussed at the end of this section. We are here mainly concerned with the model-theoretic or *differential algebraic Manin map* (see [8], §2.5, and [17]). We identify our algebraic, differential algebraic, groups with their sets of points in a universal differential field $\mathcal{U}$ (or alternatively, points in a differential closure of whatever differential field of definition we work over). So for now let $K$ be a differential field, and $A$ an abelian variety over $K$. $A$ has a smallest Zariski-dense differential algebraic (definable in $\mathcal{U}$) subgroup $A^{\sharp}$, which can also be described as the smallest definable subgroup of $A$ containing the torsion. The definable group $A/A^{\sharp}$ embeds definably in a commutative unipotent algebraic group (i.e. a vector group) by Buium, and results of Cassidy on differential algebraic vector groups yield a (non canonical) differential algebraic isomorphism between $A/A^{\sharp}$ and $\mathbb{G}_a^n$ where $n = dim(A)$. This differential algebraic isomorphism is defined over $K$, and we call it the Manin homomorphism.

There is a somewhat more intrinsic account of this isomorphism. Let $\tilde{A}$ be the universal vectorial extension of $A$ as discussed above, equipped with its unique algebraic $\partial$-group structure, and let $W_A$ be the unipotent part of $\tilde{A}$. We have the surjective differential algebraic homomorphism $\partial\ell n_{\tilde{A}} : \tilde{A} \to L\tilde{A}$. Note that if $\tilde{y} \in \tilde{A}$ lifts $y \in A$, then the image of $\tilde{y}$ under $\partial\ell n_{\tilde{A}}$, modulo the subgroup $\partial\ell n_{\tilde{A}}(W_A)$ depends only on $y$. This gives a surjective differential algebraic homomorphism from $A$ to $L\tilde{A}/\partial\ell n(W_A)$, which is defined over $K$, and which we call $\mu_A$.

**Remark 2.11.** *Any abelian variety $A/K$ satisfies:* $\ker(\mu_A) = A^\sharp$.

*Proof.* Let $U_A$ be the maximal algebraic subgroup of $W_A$ which is a $\partial$-subgroup of $\tilde{A}$. Then $\tilde{A}/U_A$ has the structure of an algebraic $\partial$-group, and as explained in [7], the canonical map $\pi : \tilde{A} \to A$ induces an isomomorphism between $(\tilde{A}/U_A)^\partial$ and $A^\sharp$. As (by functoriality) $(\tilde{A})^\partial$ maps onto $(\tilde{A}/U_A)^\partial$, $\pi : \tilde{A} \to A$ induces a surjective map $(\tilde{A})^\partial \to A^\sharp$. Now as the image of $\mu_A$ is torsion-free, $ker(\mu_A)$ contains $A^\sharp$. On the other hand, if $y \in ker(\mu_A)$ and $\tilde{y} \in \tilde{A}$ lifts $y$, then there is $z \in W_A$ such that $\partial\ell n_{\tilde{A}}(\tilde{y}) = \partial\ell n_{\tilde{A}}(z)$. So $\partial\ell n_{\tilde{A}}(\tilde{y} - z) = 0$ and $\pi(\tilde{y} - z) = y$, hence $y \in A^\sharp$. □

Hence we call $\mu_A$ the (differential algebraic) Manin map. The target space embeds in an algebraic vector group hence has the structure of a $\mathcal{C}$-vector space which is unique (any definable isomorphism between two commutative unipotent differential algebraic groups is an isomorphism of $\mathcal{C}$-vector spaces).

Now assume that $K = \mathbb{C}(t)^{alg}$ and that $A$ is an abelian variety over $K$ with $\mathbb{C}$-trace $A_0 = 0$. Then the "model-theoretic/differential algebraic theorem of the kernel" is (see Corollary K3 of [7]):

**Fact 2.12** ($K = \mathbb{C}(t)^{alg}, A/K$ traceless)**.** $\ker(\mu_A) \cap A(K)$ *is precisely the subgroup $Tor(A)$ of torsion points of $A$.*

In section 5 we generalize Fact 2.12 by proving:

**Theorem 2.13** ($K = \mathbb{C}(t)^{alg}, A/K$ traceless)**.** *Let $y_1, \ldots, y_n \in A(K)$. Suppose that $a_1, \ldots, a_n \in \mathbb{C}$ are not all $0$, and that $a_1\mu_A(y_1) + \ldots + a_n\mu_A(y_n) = 0$ in $L\tilde{A}(K)/\partial\ell n_{\tilde{A}}(W_A)$. Then $y_1, \ldots, y_n$ are linearly dependent over $End(A)$.*

Note that on reducing to a simple abelian variety, Fact 2.12 is the special case when $n = 1$. Hrushovski asked whether the conclusion of Theorem 2.13 can be strengthened to the linear dependence of $y_1, .., y_n$ over $\mathbb{Z}$. Namely is the extension $\mu_A \otimes 1$ of $\mu_A$ to $A(K) \otimes_{\mathbb{Z}} \mathbb{C}$ injective ? We found that an example of André (see [7], p. 504, as well as [12], IX.6) of a traceless abelian variety $A$ with $U_A \neq W_A$ yields a counterexample. Namely:

**Proposition 2.14.** *There exist*

*- a simple traceless 4-dimensional abelian variety $A$ over $K = \mathbb{C}(t)^{alg}$, such that $End(A)$ is an order in a CM number-field $F$ of degree 4 over $\mathbb{Q}$,*

*- four points $y_1, \dots, y_4$ in $A(K)$ which are linearly dependent over $End(A)$, but linearly independent over $\mathbb{Z}$,*

*- and four complex numbers $a_1, \dots, a_4$, not all zero, such that $a_1\mu_A(y_1) + \dots + a_4\mu_A(y_4) = 0$.*

In fact, for $i = 1, \dots, 4$, we will construct lifts $\tilde{y}_i \in \tilde{A}(K)$ of the points $y_i$, and solutions $\tilde{x}_i \in L\tilde{A}(K^{diff})$ to the equations $\nabla(\tilde{x}_i) = \partial\ell n_{\tilde{A}}\tilde{y}_i$ (where we have set $\nabla := \nabla_{L\tilde{A}} = \partial_{L\tilde{A}}$, with $\nabla_{|LW_A} = \partial\ell n_{\tilde{A}|W_A}$ in the identification $W_A = LW_A$), and will find a non-trivial relation

$$a_1\tilde{x}_1 + \dots + a_4\tilde{x}_4 := u \in U_A(K^{diff}) \qquad (\mathfrak{R}).$$

Since $U_A$ is a $\nabla$-submodule of $L\tilde{A}$, this implies that $a_1\partial\ell n_{\tilde{A}}\tilde{y}_1 + \dots + a_4\partial\ell n_{\tilde{A}}\tilde{y}_4$ lies in $U_A$. And since $U_A \subseteq W_A$, this in turn shows that

$$a_1\mu_A(y_1) + \dots + a_4\mu_A(y_4) = 0 \text{ in } L\tilde{A}/\partial\ell n_{\tilde{A}}(W_A),$$

contradicting the injectivity of $\mu_A \otimes 1$.

We conclude with a remark on the more classical *differential arithmetic* Manin map $M_{K,A}$ , where the stronger version *is* true. Again $A$ is an abelian variety over $K = \mathbb{C}(t)^{alg}$ with $\mathbb{C}$-trace 0. As above, we let $\nabla$ denote $\partial_{L\tilde{A}} : L\tilde{A} \to L\tilde{A}$. The map $M_{K,A}$ is then the homomorphism from $A(K)$ to $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$ which attaches to a point $y \in A(K)$ the class $M_{K,A}(y)$ of $\partial\ell n_{\tilde{A}}(\tilde{y})$ in $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$, for any lift $\tilde{y}$ of $y$ to $\tilde{A}(K)$. This class is independent of the lift, since $\partial\ell n_{\tilde{A}}$ and $\partial_{L\tilde{A}}$ coincide on $W_A = LW_A$. Again $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$ is a $\mathbb{C}$-vector space. The initial theorem of Manin (see [10]) says that $\ker(M_{K,A}) = Tor(A) + A_0(\mathbb{C})$, so in the traceless case is precisely $Tor(A)$.

**Proposition 2.15** ($K = \mathbb{C}(t)^{alg}, A/K$ traceless)**.** *The $\mathbb{C}$-linear extension $M_{K,A} \otimes 1 : A(K) \otimes_Z \mathbb{C} \to L\tilde{A}(K)/\nabla(L\tilde{A}(K))$ is injective.*

# 3 Computation of Galois groups

Here we prove the Galois theoretic statements 2.5, 2.6 and 2.7 announced in §2.2. We assume throughout that $K = \mathbb{C}(t)^{alg}$.

## 3.1 The abelian case

Let us first set up the notations. Let $A$ be an abelian variety over $K$, and let $A_0$ be its $\mathbb{C}$-trace, which we view as a subgroup of $A$ defined over $\mathbb{C}$. Let $\tilde{A}$ be the universal vectorial extension of $A$. We have the short exact sequence $0 \to W_A \to \tilde{A} \to A \to 0$. Let $U_A$ denote the (unique) maximal $\partial$-subgroup of $\tilde{A}$ contained in $W_A$. By Remark 7.2 of [4], we have:

**Fact 3.1.** $\tilde{A}^{\partial}(K^{diff}) = \tilde{A}_0(\mathbb{C}) + Tor(\tilde{A}) + U_A^{\partial}(K^{diff})$.

Let us briefly remark that the ingredients behind Fact 3.1 include Chai's theorem (see [9] and §K of [7]), as well as the strong minimality of $A^{\sharp}$ when $A$ is simple and traceless from [11]. As already pointed out in connection with $K$-largeness, the reference to [11] can be replaced by the easier arguments from [1]. Let $K^{\sharp}_{\tilde{A}}$ be the (automatically differential) field generated over $K$ by $\tilde{A}^{\partial}(K^{diff})$, and likewise with $K^{\sharp}_{U_A}$ for $(U_A)^{\partial}(K^{diff})$. So by Fact 3.1 $K^{\sharp}_{\tilde{A}} = K^{\sharp}_{U_A}$. Also, as recalled at the beginning of Section 8 of [4], we have:

**Remark 3.2.** *$K^{\sharp}_{U_A}$ is a Picard-Vessiot extension of $K$ whose Galois group (a linear algebraic group over $\mathbb{C}$) is semisimple.*

### Proof of Proposition 2.5

Here, $G$ is an *almost abelian* $\partial$-group over $K$. We first treat the case where $G = \tilde{A}$.

Let $y \in G(K^{diff})$ be such that $a = \partial \ell n_G(y)$ lies in $LG(K)$. Note that in the set-up of Conjecture 2.3, $y$ could be very well be an element of $U_A$, for instance when $a \in LU_A \simeq U_A$, so in a sense we move outside the almost abelian context. In any case, let $H$ be a minimal $\partial$-subgroup of $G$ defined over $K$ such that $y \in H + G(K) + G^{\partial}(K^{diff})$. Since $G(K)$ contains all the torsion points, $H$ is automatically connected. We will prove that $H^{\partial}(K^{diff})$ is the differential Galois group of $K^{\sharp}(y)$ over $K^{\sharp}$ where $K^{\sharp} = K^{\sharp}_G$. We recall from the Remark after Fact 2.2 on the commutative case that we can and do assume that this Galois group is connected. Also, these statements imply that $H$ is actually the smallest $\partial$-subgroup of $G$ over $K$ such that $y \in H + G(K) + G^{\partial}(K^{diff})$, as required.

Let $H_1^{\partial}$ be the Galois group of $K^{\sharp}(y)$ over $K^{\sharp}$ with $H_1$ a $\partial$-subgroup of $G$ which on the face of it is defined over $K^{\sharp}$. So, $H_1$ is a connected $\partial$-subgroup of $H$, and we aim to show that $H = H_1$.

*Claim.* $H_1$ is defined over $K$ as an algebraic group.
*Proof.* It is enough to show that $H_1^{\partial}$ is defined over $K$ as a differential algebraic group. This is a very basic model-theoretic argument, but may be a bit surprizing at the algebraic-geometric level, as $K^{\sharp}(y)$ need not be a "differential Galois extension" of $K$ in any of the usual meanings. We use the fact that any definable (with parameters) set in the differentially closed field $K^{diff}$ which is $Aut(K^{diff}/K)$-invariant, is definable over $K$. This follows from model-theoretic homogeneity of $K^{diff}$ over $K$ as well as elimination of imaginaries in $DCF_0$. Now $H_1^{\partial}(K^{diff})$ is the set of $g \in G^{\partial}(K^{diff})$ such that $y_1 g$ and $y_1$ have the same type over $K^{\sharp}$ for some/any $y_1 \in G(K^{diff})$ such that $\partial\ell n_G(y_1) = a$. As $a \in LG(K)$ and $K^{\sharp}$ is setwise invariant under $Aut(K^{diff}/K)$ it follows that $H_1^{\partial}(K^{diff})$ is also $Aut(K^{diff}/K)$-invariant, so defined over $K$. This proves the claim.

Note that since one of its translate by $G(K)$ lies in $H$, we may assume that $y \in H$, whereby $\partial\ell n_G(y) = a \in LH(K)$.

Let $B$ be the image of $H$ in $A$, and $B_1$ the image of $H_1$ in $A$. So $B_1 \leq B$ are abelian subvarieties of $A$. Let $V$ be the maximal unipotent $\partial$-subgroup of $H$, and $V_1$ the maximal unipotent subgroup of $H_1$. So $V_1 \leq V$, and using the assumptions and the claim, everything is defined over $K$. Note also that the surjective homomorphism $H \to B$ induces an isomorphism between $H/V$ and $\tilde{B}/U_B$ (where as above $U_B$ denotes the maximal unipotent $\partial$-subgroup of $\tilde{B}$). Likewise for $H_1/V_1$ and the quotient of $\tilde{B}_1$ by its maximal unipotent $\partial$-subgroup.

*Case (I).* $B = B_1$.
Then by the previous paragraph, we have a canonical isomorphism $\iota$ (of $\partial$-groups) between $H/H_1$ and $V/V_1$, defined over $K$, so there is no harm in identifying them, although we need to remember where they came from. Let us denote $V/V_1$ by $\overline{V}$, a unipotent $\partial$-group. This isomorphism respects the logarithmic derivatives in the obvious sense. Let $\bar{y}$ denote the image of $y$ in $H/H_1$. So $\partial\ell n_{H/H_1}(\bar{y}) = \bar{a}$ where $\bar{a}$ is the image of $a$ in $L(H/H_1)(K)$. Via $\iota$ we identify $\bar{y}$ with a point in $\bar{V}(K^{\sharp})$ and $\bar{a}$ with $\partial\ell n_{\bar{V}}(\bar{y}) \in L(\bar{V})(K)$.

By 3.2 we identify $Aut(K^\sharp/K)$ with a group $J(\mathbb{C})$ where $J$ is a semisimple algebraic group. We have a natural action of $J(\mathbb{C})$ on $\bar{V}^\partial(K^{diff}) = \bar{V}^\partial(K^\sharp)$. Now the latter is a $\mathbb{C}$-vector space, and this action can be checked to be a (rational) representation of $J(\mathbb{C})$. On the other hand, for $\sigma \in J(\mathbb{C})$, $\sigma(\bar{y})$ (which is well-defined since $\bar{y}$ is $K^\sharp$-rational) is also a solution of $\partial\ell n_{\bar{V}}(-) = \bar{a}$, hence $\sigma(\bar{y}) - \bar{y} \in \bar{V}^\partial(K^{diff})$. The map taking $\sigma$ to $\sigma(\bar{y}) - \bar{y}$ is then a cocycle $c$ from $J(\mathbb{C})$ to $V^\partial(K^{diff})$ which is a morphism of algebraic varieties. Now the corresponding $H^1(J(\mathbb{C}), \bar{V}^\partial(K^{diff}))$ is trivial as it equals $Ext_{J(\mathbb{C})}(1, \bar{V}^\partial(K^{diff}))$, the group of isomorphism classes of extensions of the trivial representation of $J(\mathbb{C})$ by $\bar{V}^\partial(K^{diff})$. But $J(\mathbb{C})$ is semisimple, so reductive, whereby every rational representation is completely reducible (see p.26 and 27 of [15], and [2] for Picard-Vessiot applications, which actually cover the case when $a$ lies in $LU_A$). Putting everything together the original cocycle is trival. So there is $\bar{z} \in \bar{V}^\partial(K^\sharp)$ such that $\sigma(\bar{y}) - \bar{y} = \sigma(z) - z$ for all $\sigma \in J(\mathbb{C})$. So $\sigma(\bar{y} - \bar{z}) = \bar{y} - \bar{z}$ for all $\sigma$. Hence $\bar{y} - \bar{z} \in (H/H_1)(K)$. Lift $\bar{z}$ to a point $z \in H^\partial(K^{diff})$. So $\overline{y - z} \in \bar{V}(K)$. As $K$ is algebraically closed, there is $d \in H(K)$ such that $y - z + d \in H_1$. This contradicts the minimal choice of $H$, unless $H = H_1$. So the proof is complete in Case (I).

*Case (II).* $B_1$ is a proper subgroup of $B$.
Consider the group $H_1.V$ a $\partial$-subgroup of $H$, defined over $K$, which also projects onto $B_1$. It is now easy to extend $H_1.V$ to a $\partial$-subgroup $H_2$ of $H$ over $K$ such that $H/H_2$ is canonically isomorphic to $\overline{B}_2$, where $B_2$ is a simple abelian variety, and $\overline{B_2}$ denotes the quotient of $\tilde{B}_2$ by its maximal unipotent subgroup. Now let $\bar{y}$ denote $y/H_2 \in H/H_2$. Hence $\partial\ell n_{\overline{B}_2}(\bar{y}) = \bar{a} \in L(\overline{B_2})(K)$. As $H_1 \subseteq H_2$, $\bar{y} \in \overline{B}_2(K^\sharp)$. Now we have two cases. If $B_2$ descends to $\mathbb{C}$, then $\bar{y}$ generates a strongly normal extension of $K$ with Galois group a connected algebraic subgroup of $B_2(\mathbb{C})$. As this Galois group will be a homomorphic image of the linear (in fact semisimple) complex algebraic group $Aut(K^\sharp/K)$ we have a contradiction, unless $\bar{y}$ is $K$-rational. On the other hand, if $B_2$ does not descend to $\mathbb{C}$, then by Fact 2.2.(ii) $\bar{y}$ generates over $K$ a (generalized) differential Galois extension of $K$ with Galois group contained in $\overline{B}_2{}^\partial(K^{diff})$, which again will be a homomorphic image of a complex semisimple linear algebraic group (cf. [4], 8.2.i). We get a contradiction by various possible means (for example as in Remarque 8.2 of [4]) unless $\bar{y}$ is $K$-rational. So either way we are forced into $\bar{y} \in (H/H_2)(K)$. But then, as $K$ is algebraically closed, $y - d \in H_2$ for some $d \in H(K)$, again a contradiction. So Case (II) is impossible. This concludes the proof of 2.5 when $G = \tilde{A}$.

Finally, consider a general almost abelian $\partial$-group $G$, given as a quotient of $\tilde{A}$ by a unipotent $\partial$-subgroup $U \subset U_A$ defined over $K$. Taking the quotient by $U^\partial(K^{diff})$ of the decomposition of $\tilde{A}^\partial(K^{diff})$ given by Fact 3.1, we obtain a similar decomposition for $G^\partial(K^{diff})$. Therefore $K^\sharp_G = K((U_A/U)^\partial)$ too is a Picard-Vessiot extension of $K$, and we deduce from Remark 3.2 that its Galois group is again semi-simple. The various cases of the previous proof therefore also apply to the quotient $G = \tilde{A}/U$, and Proposition 2.5 holds for any almost abelian $\partial$-group.

## 3.2 The semiabelian case

We now aim towards proofs of Theorems 2.6 and 2.7. Here, $G = \tilde{B}$ for $B$ a semiabelian variety over $K$, equipped with its unique algebraic $\partial$-group structure.

We have:
$0 \to T \to B \to A \to 0$, where $T$ is an algebraic torus and $A$ an abelian variety, all over $K$,
$G = \tilde{B} = B \times_A \tilde{A}$, where $\tilde{A}$ is the universal vectorial extension of $A$, and
$0 \to T \to G \to \tilde{A} \to 0$.
We use the same notation for $A$ as at the beginning of this section, namely $0 \to W_A \to \tilde{A} \to A \to 0$. We denote by $A_0$ the $\mathbb{C}$-trace of $A$ (so up to isogeny we can write $A$ as a product $A_0 \times A_1$, all defined over $K$, where $A_1$ has $\mathbb{C}$-trace 0), and by $U_A$ the maximal $\partial$-subgroup of $\tilde{A}$ contained in $W_A$. So $U_A$ is a unipotent subgroup of $G$, though not necessarily one of its $\partial$-subgroups. Finally, we have the exact sequence:

$$0 \longrightarrow T^\partial \longrightarrow G^\partial \xrightarrow{\pi} \tilde{A}^\partial \longrightarrow 0 \ .$$

Note that $T^\partial = T(\mathbb{C})$. Let $K^\sharp_G$ be the (differential) field generated over $K$ by $G^\partial(K^{diff})$. We have already noted above that $K^\sharp_{\tilde{A}}$ equals $K^\sharp_{U_A}$. So $K^\sharp_{U_A} < K^\sharp_G$, and we deduce from the last exact sequence above:

**Remark 3.3.** *$G^\partial(K^{diff})$ is the union of the $\pi^{-1}(b)$ for $b \in \tilde{A}^\partial$, each $\pi^{-1}(b)$ being a coset of $T(\mathbb{C})$ defined over $K^\sharp_{U_A}$. Hence $K^\sharp_G$ is (generated by) a union of Picard-Vessiot extensions over $K^\sharp_{U_A}$ each with Galois group contained in $T(\mathbb{C})$.*

**Proof of Theorem 2.6**

Bearing in mind Proposition 2.5 we may assume that $T = \mathbb{G}_m$. We have $a \in LG(K)$ and $y \in G(K^{diff})$ such that $\partial\ell n_G(y) = a$, and that $y \notin H + G(K)$ for any proper $\partial$-subgroup $H$ of $G$. The latter is a little weaker than the condition that $a \notin LH(K) + \partial\ell n_G(G(K))$ for any proper $H$, but (thanks to Fact 3.1) will suffice for the special case we are dealing with.

Fix a solution $y$ of $\partial\ell n_G(-) = a$ in $G(K^{diff})$ and let $H^\partial(K^{diff})$ be the differential Galois group of $K_G^\sharp(y)$ over $K_G^\sharp$. As said after Fact 2.2, there is no harm in assuming that $H$ is connected. So $H$ is a connected $\partial$-subgroup of $G$, defined over $K_G^\sharp$.

As in the proof of the claim in the proof of Proposition 2.5 we have:

*Claim 1.* $H$ (equivalently $H^\partial$) is defined over $K$.

We assume for a contradiction that $H \neq G$.
*Case (I).* $H$ maps onto a proper ($\partial$-)subgroup of $\tilde{A}$.
This is similar to the Case (II) in the proof of Proposition 2.5 above. Some additional complications come from the structure of $K_G^\sharp$. We will make use of Remark 3.3 all the time.
As $\tilde{A}$ is an essential extension of $A$ by $W_A$, it follows that we can find a connected $\partial$-subgroup $H_1$ of $G$ containing $H$ and defined over $K$ such that the surjection $G \to \tilde{A}$ induces an isomorphism between $G/H_1$ and $\overline{A_2}$, where $A_2$ is a simple abelian subvariety of $A$ (over $K$ of course) and $\overline{A_2}$ is the quotient of $\tilde{A}_2$ by its maximal unipotent $\partial$-subgroup. The quotient map taking $G$ to $\overline{A_2}$ takes $y$ to $\eta$ say and also induces a surjection $LG \to L(\overline{A_2})$ which takes $a$ to $\alpha \in L(\overline{A_2})$ say.

As $\eta = y/H_1$ and $H \subseteq H_1$, we see that $\eta$ is fixed by $Aut(K_G^\sharp(y)/K_G^\sharp)$, hence

*Claim 2.* $\eta \in \overline{A_2}(K_G^\sharp)$.

On the other hand $\eta$ is a solution of the logarithmic differential equation $\partial\ell n_{\overline{A_2}}(-) = \alpha$ over $K$. By the $K$-largeness of $\overline{A_2}$, $K_{\overline{A_2}}^\sharp = K$, hence $K(\eta)$ is a differential Galois extension of $K$ whose Galois group is either trivial (in which case $\eta \in \overline{A_2}(K)$), or equal to $\overline{A_2}^\partial(K^{diff})$, in view of the strong minimality of $\overline{A_2}^\partial$.

*Claim 3.* $\eta \in \overline{A_2}(K)$.
*Proof.* Suppose not. We first claim that $\eta$ is independent from $K_{U_A}^\sharp$ over $K$ (in the sense of differential fields). Indeed, the Galois theory would other-

wise give us some proper definable subgroup in the product of $\overline{A_2}^{\partial}(K^{diff})$ by the Galois group of $K^{\sharp}_{U_A}$ over $K$ (or equivalently, these two groups would share a non-trivial definable quotient). As the latter is a complex semisimple algebraic group (Remark 3.2), we get a contradiction. Alternatively we can proceed as in Remarque 8.2 of [4].
So the Galois group of $K^{\sharp}_{U_A}(\eta)$ over $K^{\sharp}_{U_A}$ is $\overline{A_2}^{\partial}(K^{diff})$. As there are no nontrivial definable subgroups of $\overline{A_2}(K^{diff}) \times \mathbb{G}_m(\mathbb{C})^n$, we see that $\eta$ is independent of $K^{\sharp}_G$ over $K^{\sharp}_{U_A}$ contradicting Claim 2.

By Claim 3, the coset of $y$ modulo $H_1$ is defined over $K$ (differential algebraically), so as in the proof of Fact 2.1, as $K$ is algebraically closed there is $y_1 \in G(K)$ in the same coset of $H_1$ as $y$. So $y \in H_1 + G(K)$, contradicting the assumptions. So Case (I) is complete.

*Case (II). H* projects on to $\tilde{A}$.
Our assumption that $H$ is a proper subgroup of $G$ and that the toric part is $\mathbb{G}_m$ implies that (up to isogeny) $G$ splits as $T \times H = T \times \tilde{A}$. The case is essentially dealt with in [4]. But nevertheless we continue with the proof. We identify $G/H$ with $T$. So $y/H = d \in T$ and the image $a_0$ of $a$ under the projection $G \to T$ is in $LT(K)$. As $H^{\partial}(K^{diff})$ is the Galois group of $K^{\sharp}_G(y)$ over $K^{\sharp}_G$, we see that $y \in T(K^{\sharp}_G)$. Now $K(d)$ is a Picard-Vessiot extension of $K$ with Galois group a subgroup of $\mathbb{G}_m(\mathbb{C})$. Moreover as $G$ splits as $T \times \tilde{A}$, $G^{\partial} = T^{\partial} \times \tilde{A}^{\partial}$. Hence by Fact 3.1, $K^{\sharp}_G = K^{\sharp}_{\tilde{A}}$ and by Remark 3.2, it is a Picard-Vessiot extension of $K$ with Galois group a semisimple algebraic group in the constants. We deduce from the Galois theory that $d$ is independent from $K^{\sharp}_G$ over $K$, hence $d \in T(K)$. So the coset of $y$ modulo $H$ has a representative $y_1 \in G(K)$ and $y \in H + G(K)$, contradicting our assumption. This concludes Case (II) and the proof of Theorem 2.6.

**Proof of Theorem 2.7**

So $G = \tilde{B}$ for $B = B_{sc}$ a semiconstant semiabelian variety over $K$ and we may assume it has toric part $\mathbb{G}_m$. So although the toric part is still $\mathbb{G}_m$ both the hypothesis and conclusion of 2.7 are stronger than in 2.6.

We have $0 \to \mathbb{G}_m \to B \to A$ where $A = A_0$ is over $\mathbb{C}$, hence also $\tilde{A}$ is over $\mathbb{C}$ and we have $0 \to \mathbb{G}_m \to \tilde{B} \to \tilde{A} \to 0$, and $G = \tilde{B}$. As $\tilde{A}^{\partial} = \tilde{A}(\mathbb{C}) \subseteq \tilde{A}(K)$, we see that

**Fact 3.4.** *$G^{\partial}(K^{diff})$ is a union of cosets of $\mathbb{G}_m(\mathbb{C})$, each defined over $K$.*

We are given a logarithmic differential equation $\partial\ell n_G(-) = a \in LG(K)$ and solution $y \in G(K^{diff})$. We let $H$ be a minimal connected $\partial$-subgroup of $G$, defined over $K$, such that $a \in LH + \partial\ell n_G(G(K))$, equivalently $y \in H + G(K) + G^{\partial}(K^{diff})$. We want to prove that $H^{\partial}(K^{diff})$ is the Galois group of $K^{\sharp}_G(y)$ over $K^{\sharp}_G$.
By Theorem 2.6 we may assume that $H \neq G$. Note that after translating $y$ by an element of $G(K)$ plus an element of $G^{\partial}(K^{diff})$ we can assume that $y \in H$. If $H$ is trivial then everything is clear.

We go through the cases.
*Case (I).* $H = \mathbb{G}_m$.
Then by Fact 2.1, $K(y)$ is a Picard-Vessiot extension of $K$, with Galois group $\mathbb{G}_m(\mathbb{C})$, and all that remains to be proved is that $y$ is algebraically independent from $K^{\sharp}$ over $K$. Let $z_1, \ldots, z_n \in G^{\partial}(K^{diff})$, and we want to show that $y$ is independent from $z_1, \ldots, z_n$ over $K$ (in the sense of $DCF_0$). By Fact 3.4, $K(z_1, \ldots, z_n)$ is a Picard-Vessiot extension of $K$ and we can assume the Galois group is $\mathbb{G}_m^n(\mathbb{C})$. Suppose towards a contradiction that $tr.deg(K(y, z_1, \ldots, z_n)/K) < n+1$ so has to equal $n$. Hence the differential Galois group of $K(y, z_1, \ldots, z_n)/K)$ is of the form $L(\mathbb{C})$ where $L$ is the algebraic subgroup of $\mathbb{G}_m^{n+1}$ defined by $x^k x_1^{k_1} \ldots x_n^{k_n} = 1$ for $k, k_i$ integers, $k \neq 0$, not all $k_i$ zero. It easily follows that in additive notations, $ky + k_1 z_1 + \ldots + k_n z_n \in G(K)$. So $ky$ is of the form $z + g$ for $z \in G^{\partial}(K^{diff})$ and $g \in G(K)$. Let $z' \in G^{\partial}$ and $g' \in G(K)$ be such that $kz' = z$ and $kg' = g$. Then $k(y - (z' + g')) = 0$, so $y - (z' + g)$ is a torsion point of $G$ hence also in $G^{\partial}$. We conclude that $y \in G^{\partial}(K^{diff}) + G(K)$, contradicting our assumptions on $y$. This concludes the proof in Case (I).

*Case (II).* $H$ projects onto $\tilde{A}$.
So our assumption that $G \neq H$ implies that up to isogeny $G$ is $T \times \tilde{A}$ so defined over $\mathbb{C}$, and everything follows from Fact 2.1.

*Case (III).* Otherwise.
This is more or less a combination of the previous cases.
To begin, suppose $H$ is disjoint from $T$ (up to finite). So $H \leq \tilde{A}$ is a constant group, and by Fact 2.1, $H^{\partial}(K^{diff}) = H(\mathbb{C})$ is the Galois group of $K(y)$ over $K$. By Fact 3.4 the Galois theory tells us that $y$ is independent from $K^{\sharp}_G$ over $K$, so $H(\mathbb{C})$ is the Galois group of $K^{\sharp}(y)$ over $K^{\sharp}$ as required.
So we may assume that $T \leq H$. Let $H_1 \leq H$ be the differential Galois group

of $K^\sharp_G(y)$ over $K^\sharp_G$, and we suppose for a contradiction that $H_1 \neq H$. As in the proof of 2.5, $H_1$ is defined over $K$. By the remark after Fact 2.2, we can assume that $H_1$ is connected.

*Case (III)(a).* $H_1$ is a complement of $T$ in $H$ (in the usual sense that $H_1 \times T \to H$ is an isogeny).
So $y/H_1 \in T(K^\sharp_G)$. Let $y_1 = y/H_1$. If $y_1 \notin T(K)$, $K(y_1)$ is a Picard-Vessiot extension of $K$ with Galois group $\mathbb{G}_m(\mathbb{C})$. The proof in Case (I) above shows that $y_1 \in G^\partial(K^{diff}) + G(K)$ whereby $y \in H_1 + G^\partial(K^{diff}) + G(K)$, contradicting the minimality assumptions on $H$.

*Case (III)(b).* $H_1 + T$ is a proper subgroup of $H$.
Note that as we are assuming $H_1 \neq H$, then the negation of Case (III)(a) forces Case (III)(b) to hold. Let $H_2 = H_1 + T$, so $H/H_2$ is a constant group $H_3$ say which is a vectorial extension of an abelian variety. Then $y_2 = y/H_2 \in H_3(K^\sharp_G)$, and $K(y_2)$ is a Picard-Vessiot extension of $K$ with Galois group a subgroup of $H_3(\mathbb{C})$. Fact 3.4 and the Galois theory implies that $y_2 \in H_3(K)$. Hence $y \in H_2 + G(K)$, contradicting the minimality of $H$ again.
This completes the proof of Theorem 2.7.

## 3.3 Discussion on non generic cases

We complete this section with a discussion of some complications arising when one would like to drop either the genericity assumption in Theorem 2.6, or the restriction on the toric part in both Theorems 2.6 and 2.7.

Let us first give an example which will have to be considered if we drop the genericity assumption in 2.6, and give some positive information as well as identifying some technical complications. Let $A$ be a simple abelian variety over $K$ which has $\mathbb{C}$-trace 0 and such that $U_A \neq 0$. (Note that such an example appears below in Section 5.2 connected with Manin map issues.) Let $B$ be a nonsplit extension of $A$ by $\mathbb{G}_m$, and let $G = \tilde{B}$. We have $\pi : G \to \tilde{A}$ with kernel $\mathbb{G}_m$, and let $H$ be $\pi^{-1}(U_A)$, a $\partial$-subgroup of $G$. Let $a \in LH(K)$ and $y \in H(K^{diff})$ with $d\ell n_H(y) = a$. We have to compute $tr.deg(K^\sharp_G(y)/K^\sharp_G)$. Conjecture 2.3 predicts that it is the dimension of the smallest algebraic $\partial$-subgroup $H_1$ of $H$ such that $y \in H_1 + G(K) + G^\partial(K^{diff})$.

**Lemma 3.5.** *With the above notation: Suppose that $y \notin H_1 + G(K) + G^\partial(K^{diff})$ for any proper algebraic $\partial$-subgroup $H_1$ of $H$ over $K$. Then*

*$tr.deg(K^\sharp_G(y)/K^\sharp_G) = dim(H)$ (and $H$ is the Galois group).*

*Proof.* Let $z$ and $\alpha$ the images of $y$, $a$ respectively under the maps $H \to U_A$ and $LH \to L(U_A) = U_A$ induced by $\pi : G \to \tilde{A}$. So $\partial\ell n_{\tilde{A}}(z) = \alpha$ with $\alpha \in L\tilde{A}(K)$.

*Claim.* $z \notin U + \tilde{A}(K) + \tilde{A}^\partial(K^{diff})$ for any proper algebraic $\partial$-subgroup $U$ of $U_A$, over $K$.
*Proof of claim.* Suppose otherwise. Then lifting suitable $z_2 \in \tilde{A}(K)$, $z_3 \in \tilde{A}(K^{diff})$, to $y_2 \in G(K)$, $y_3 \in G^\partial(K^{diff})$ respectively, we see that $y - (y_2 + y_3) \in \pi^{-1}(U)$, a proper algebraic $\partial$-subgroup of $H$, a contradiction.

As in Case (I) in the proof of 2.5, we conclude that $tr.deg(K^\sharp_{\tilde{A}}(z)/K^\sharp_{\tilde{A}}) = dim(U_A)$, and $U_A$ is the Galois group. Now $K^\sharp_G$ is a union of Picard-Vessiot extensions of $K^\sharp_{\tilde{A}} = K^\sharp_{U_A}$, each with Galois group $\mathbb{G}_m$ (by 3.3) so the Galois theory tells us that $z$ is independent from $K^\sharp_G$ over $K^\sharp_{\tilde{A}}$. Hence the differential Galois group of $K^\sharp_G(z)$ over $K^\sharp_G$ is $U_A^\partial$. But then the Galois group of $K^\sharp_G(y)$ over $K^\sharp_G$ will be the group of $\partial$-points of a $\partial$-subgroup of $H$ which projects onto $U_A$. The only possibility is $H$ itself, because otherwise $H$ splits as $\mathbb{G}_m \times U_A$ as a $\partial$-group, which contradicts (v) of Section 2 of [4]. This completes the proof. □

Essentially the same argument applies if we replace $H$ by the preimage under $\pi$ of some nontrivial $\partial$-subgroup of $U_A$. So this shows that the scenario described right before Lemma 3.5, reduces to the case where $a \in LT$ where $T$ is the toric part $\mathbb{G}_m$ (of both $G$ and $H$), and we may assume $y \in T(K^{diff})$. We would like to show (in analogy with 3.5) that if $y \notin G(K) + G^\partial(K^{diff})$ then $tr.deg(K^\sharp_G(y)/K^\sharp_G) = 1$. Of course already $K(y)$ is a Picard-Vessiot extension of $K$ with Galois group $T(\mathbb{C})$, and we have to prove that $y$ is independent from $K^\sharp_G$ over $K$. One deduces from the Galois theory that $y$ is independent from $K^\sharp_{U_A}$ over $K$. It remains to show that for any $z_1, .. z_n \in G^\partial(K^{diff})$, $y$ is independent from $z_1, .., z_n$ over $K^\sharp_{U_A}$. If not, the discussion in Case (I) of the proof of Theorem 2.7, gives that $y = z + g$ for some $z \in G^\partial(K^{diff})$ and $g \in G(K^\sharp_{U_A})$, but an additional argument seems necessary to yield a contradiction.

Similar and other issues arise when we want to drop the restriction on the toric part. For example in Case (ii) in the proof of Theorem 2.6, we can no longer deduce the splitting of $G$ as $T \times \tilde{A}$. And in the proof of Theorem

2.7, both the analogues of Case (I) $H = T$ and Case (II) $H$ projects on to $\tilde{A}$, present technical difficulties.

# 4 Lindemann-Weierstrass

We here prove Corollaries 2.8, 2.9, and 2.10.

## 4.1 General results

### Proof of Corollary 2.8

We first prove (i). Write $G$ for $\tilde{B}$. Let $\tilde{x} \in LG(K)$ be a lift of $x$ and $\tilde{y} \in G(\mathcal{U})$ a solution of $\partial \ell n_G(-) = \tilde{x}$. We refer to Section 1.2 and Lemma 4.2 of [7] for a discussion of the equivalence of the hypotheses " $x \notin LH(K) + LB_0(\mathbb{C})$ for any proper semiabelian subvariety $H$ of $B$ " and " (*) $\tilde{x} \notin LH(K)+(LG)^{\partial}(K)$ for any proper algebraic subgroup $H$ of $G$ over $K$ ".

Let $a = \partial_{LG}(\tilde{x})$. So $\tilde{y}$ is a solution of the logarithmic differential equation (over $K$) $\partial \ell n_G(-) = a$. We want to show that $tr.deg(K_G^{\sharp}(\tilde{y})/K_G^{\sharp}) = \dim(G)$. If not, we may assume that $\tilde{y} \in G(K^{diff})$, and so by Theorem 2.6, $\tilde{y} \in H + G(K)$ for some proper connected algebraic $\partial$-subgroup $H$ of $G$ defined over $K$. Extend $H$ to a maximal proper connected $\partial$-subgroup $H_1$ of $G$, defined over $K$. Then $G/H_1$ is either (i) $\mathbb{G}_m$ or (ii) a simple abelian variety $A_0$ over $\mathbb{C}$, or (iii) the quotient of $\tilde{A}_1$ by a maximal unipotent $\partial$-subgroup, where $A_1$ is a simple abelian variety over $K$ with $\mathbb{C}$-trace 0. Let $x'$, $y'$ be the images of $\tilde{x}, \tilde{y}$ under the map $G \to G/H_1$ and induced $LG \to L(G/H_1)$. So both $x', y'$ are $K$-rational. Moreover the hypothesis (*) is preserved in $G/H_1$ (by our assumptions on $G$ and Lemma 4.2(ii) of [7]). As $\partial \ell n_{G/H_1}(y') = \partial_{L(G/H_1)}(x')$, we have a contradiction in each of the cases (i), (ii), (iii) listed above, by virtue of the truth of Ax-Lindemann in the constant case, as well as Manin-Chai (Proposition 4.4 in [7]).

(ii) Immediate as in [7]: Choosing $\tilde{y} = exp_G(\tilde{x})$, then $exp_B(y)$ is the projection of $\tilde{y}$ on $B$.

### Proof of Corollary 2.9

This is like the proof of Corollary 2.8. So $x \in LA(K)$. Let $\tilde{x} \in L\tilde{A}(K)$ lift $x$ and let $\tilde{y} \in \tilde{A}(K^{diff})$ be such that $\partial \ell n_{\tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x}) = a$, say. Let $B$ be

a minimal abelian subvariety of $A$ such that $x \in LB(K) + LA_0(\mathbb{C})$, and we want to prove that $tr.deg(K^\sharp_{\tilde{A}}(\tilde{y})/K^\sharp_{\tilde{A}}) = dim(\tilde{B})$.
*Claim.* We may assume that $x \in LB(K)$, $\tilde{x} \in L\tilde{B}(K)$ and $\tilde{y} \in \tilde{B}(K^{diff})$.
*Proof of claim.* Let $x = x_1 + c$ for $x_1 \in LB$ and $c \in LA_0(\mathbb{C})$. Let $\tilde{x}_1 \in L\tilde{B}(K)$ be a lift of $x_1$ and $\tilde{c} \in L\tilde{A}_0(\mathbb{C})$ be a lift of $c$. Finally let $\tilde{y}_1 \in \tilde{B}(K^{diff})$ be such that $\partial\ell n_{\tilde{A}}(\tilde{y}_1) = \partial_{L\tilde{A}}(\tilde{x}_1) = a_1$, say. As $\tilde{x}_1 + \tilde{c}$ projects on to $x$, it differs from $\tilde{x}$ by an element $z \in LW(K)$. Now $\partial_{L\tilde{A}}(z) = \partial\ell n_{\tilde{A}}(z)$. So $a = \partial_{L\tilde{A}}(\tilde{x}) = \partial_{L\tilde{A}}(\tilde{x}_1 + \tilde{c} + z) = \partial_{L\tilde{A}}(\tilde{x}_1) + \partial\ell n_{\tilde{A}}(z) = a_1 + \partial\ell n_{\tilde{A}}(z)$. Hence $\partial\ell n(\tilde{y}_1 + z) = a$, and so $\tilde{y}_1 + z$ differs from $\tilde{y}$, by an element of $\tilde{A}^\partial$. Hence $tr.deg(K^\sharp_{\tilde{A}}(\tilde{y}_1)/K^\sharp_{\tilde{A}}) = tr.deg(K^\sharp_{\tilde{A}}(\tilde{y}_1)/K^\sharp_{\tilde{A}})$. Moreover the same hypothesis remains true of $x_1$ (namely $B$ is minimal such that $x_1 \in LB + LA_0(\mathbb{C})$). So we can replace $x, \tilde{x}, \tilde{y}$ by $x_1, \tilde{x}_1, \tilde{y}_1$.

As recalled in the proof of Corollary 2.8 (see Corollary H.5 of [7]), the condition that $x \notin B_1(K) + LA_0(\mathbb{C})$ for any proper abelian subvariety $B_1$ of $B$ is equivalent to (*) $\tilde{x} \notin LH(K) + (L\tilde{A})^\partial(K)$ for any proper algebraic subgroup $H$ of $\tilde{B}$ defined over $K$. Now we can use the Galois-theoretic result Proposition 2.5, namely the truth of Corollary 2.3 for $\tilde{A}$, as above. That is, if by way of contradiction $tr.deg(K^\sharp_{\tilde{A}}(\tilde{y})/K^\sharp_{\tilde{A}}) < dim(\tilde{B})$, then $\tilde{y} \in H + \tilde{A}(K) + (\tilde{A})^\partial(K^{diff})$ for some proper connected algebraic $\partial$-subgroup of $\tilde{B}$, defined over $K$, and moreover $H^\partial$ is the differential Galois group of $K^\sharp_{\tilde{A}}(\tilde{y})/K^\sharp_{\tilde{A}}$. As at the end of the proof of Corollary 2.8 above we get a contradiction by choosing $H_1$ to be a maximal proper connected algebraic $\partial$-subgroup of $\tilde{A}$, containing $H$ and defined over $K$. This concludes the proof of 2.9.

## 4.2 Semiabelian surfaces

We first recall the counterexample from Section 5.3 of [7]. This example shows that in Corollary 2.8, we cannot drop the assumption that the semi-constant part is constant. We go through it again briefly. Let $B$ over $K$ be a nonconstant extension of a constant elliptic curve $E = E_0$ by $\mathbb{G}_m$, and let $G = \tilde{B}$. Let $\tilde{x} \in LG(K)$ map onto a point $\check{x}$ in $L\tilde{E}(\mathbb{C})$ which itself maps onto a nonzero point $\bar{x}$ of $LE(\mathbb{C})$. As pointed out in [7] $(LG)^\partial(K) = (L\mathbb{G}_m)(\mathbb{C})$, whereby $\tilde{x}$ satisfies the hypothesis (*) from 2.8: $\tilde{x} \notin LH(K) + (LG)^\partial(K)$ for any proper algebraic subgroup $H$ of $G$. Let $a = \partial_{LG}(\tilde{x}) \in LG(K)$, and $\tilde{y} \in G(K^{diff})$ such that $\partial\ell n_G(\tilde{y}) = a$. Then as the image of $a$ in $L\tilde{E}$ is 0, $\tilde{y}$ projects onto a point of $\tilde{E}(\mathbb{C})$, and hence $\tilde{y}$ is in a coset of $\mathbb{G}_m$ defined over $K$

whereby $tr.deg(K(\tilde{y})/K) \leq 1$, so a fortiori the same with $K_G^\sharp$ in place of $K$. A consequence of Corollary 2.10, in fact the main part of its proof, is that with the above choice of $\tilde{x}$, we have $tr.deg(K_G^\sharp(\tilde{y})/K_G^\sharp) = 1$ (as announced in [6], Footnote 5).

**Proof of Corollary 2.10**

Let us fix notations: $B$ is a semiabelian variety over $K$ with toric part $\mathbb{G}_m$ and abelian quotient a non-necessarily constant elliptic curve $E/K$, with constant part $E_0$; $G$ denotes the universal vectorial extension $\tilde{B}$ of $B$ and $\tilde{E}$ the universal vectorial extension of $E$. For $x \in LB(K)$, $\tilde{x}$ denotes a lift of $x$ to a point of $LG(K)$, $\bar{x}$ denotes the projection of $x$ to $LE(K)$, and $\check{x}$ denotes the projection of $\tilde{x}$ to $L\tilde{E}(K)$.

Recall the hypothesis (*) in 2.10: $x \notin LH + LB_0(\mathbb{C})$ for any proper algebraic subgroup $H$ of $B$. As pointed out after the statement of Corollary 2.10, under this hypothesis, the condition $\bar{x} \in LE_0(\mathbb{C})$ can occur only if $B$ is semiconstant and not constant. Indeed, if $B$ were not semiconstant then $E_0 = 0$ so $x \in L\mathbb{G}_m$ contradicting the hypothesis on $x$. And if $B$ is constant then $B = B_0$ and $\bar{x}$ has a lift in $LB_0(\mathbb{C})$ whereby $x \in L\mathbb{G}_m + LB_0(\mathbb{C})$, contradicting the hypothesis.

Now if the semiconstant part of $B$ is constant, then we can simply quote Corollary 2.8, bearing in mind the paragraph above which rules out the possibility that $\bar{x} \in LE_0(\mathbb{C})$. So we will assume that $B_{sc} \neq B_0$, namely $E = E_0$ and $B_0 = \mathbb{G}_m$.

*Case (I).* $\bar{x} \in LE(\mathbb{C})$ ($= LE_0(\mathbb{C})$ as $E = E_0$).
This is where the bulk of the work goes. We first check that we are essentially in the situation of the "counterexample" mentioned above. The argument is a bit like in the proof of the claim in Corollary 2.9. Note that $\bar{x} \neq 0$ by hypothesis (*). Let $\check{x}'$ be a lift of $\bar{x}$ to a point in $L\tilde{E}(\mathbb{C})$ (noting that $\tilde{E}$ is also over $\mathbb{C}$). Then $\check{x}' = \check{x} - \beta$ for some $\beta \in L\mathbb{G}_a(K)$. Let $\tilde{x}' = \tilde{x} - \beta$. Let $a' = \partial_{LG}(\tilde{x}')$. Then (as $\partial_{LG}(\beta) = \partial\ell n_G(\beta)$, under the usual identifications) $a' = a + \partial\ell n_G(\beta)$, and if $\tilde{y}' \in G$ is such that $\partial\ell n_G(\tilde{y}') = a'$ then $\partial\ell n_G(\tilde{y}' - \beta) = a$. As $\beta \in G(K)$, $tr.deg(K_G^\sharp(\tilde{y}')/K_G^\sharp) = tr.deg(K_G^\sharp(\tilde{y})/K_G^\sharp)$.
The end result is that we can assume that $\tilde{x} \in LG(K)$ maps onto $\check{x}' \in L\tilde{E}(\mathbb{C})$ which in turn maps on to our nonzero $\bar{x} \in LE(\mathbb{C})$, precisely the situation in the example above from Section 5.1 of [7]. So to deal with Case (I), we have to prove:

*Claim 1.* $tr.deg(K_G^\sharp(\tilde{y})/K_G^\sharp) = 1$.
*Proof of claim 1.* Remember that $a$ denotes $\partial_{LG}(\tilde{x})$. Now by Theorem 2.7, it suffices to prove that $a \notin \partial\ell n_G(G(K))$.

We assume for a contradiction that there is $\tilde{s} \in G(K)$ such that

$$a = \partial_{LG}(\tilde{x}) = \partial\ell n_G(\tilde{s}). \qquad (\dagger)$$

This is the semi-abelian analogue of a Manin kernel statement, which can probably be studied directly, but we will deduce the contradiction from [6]. Let $\tilde{x}_1 = log_G(\tilde{s})$ be a solution given by complex analysis to the linear inhomogeneous equation $\partial_{LG}(-) = \partial\ell n_G(\tilde{s})$. Namely, with notations as in the appendix to [7] (generalizing those given after Corollary 2.8 above), a local analytic section of $L\mathbf{G}^{an}/S^{an}$ such that $exp_{\mathbf{G}}(\tilde{x}_1) = \tilde{s}$. Let $\xi \in (LG)^\partial$ be $\tilde{x} - \tilde{x}_1$. Then $\xi$ lives in a differential field (of meromorphic functions on some disc in $S$) which extends $K$ and has the same constants as $K$, namely $\mathbb{C}$. As $\xi$ is the solution of a linear homogeneous differential equation over $K$, $\xi$ lives in $(LG)^\partial(K^{diff})$. Hence as $\tilde{x} \in LG(K)$ this implies that $\tilde{x}_1 \in LG(K_{LG}^\sharp)$ where $K_{LG}^\sharp$ is the differential field generated over $K$ by $(LG)^\partial(K^{diff})$.

Now from Section 5.1 of [6], $K_{LG}^\sharp$ coincides with the "field of periods" $F_q$ attached to the point $q \in \hat{E}(K)$ which parametrizes the extension $B$ of $E$ by $\mathbb{G}_m$. Hence from ($\dagger$) we conclude that $F_q(log_G(\tilde{s})) = F_q$.

Let $s \in B(K)$ be the projection of $\tilde{s}$, and let $p \in E(K)$ be the projection of $s$. By [6], discussion in Section 5.1, we have that $F_{pq}(log_B(s)) = F_q(log_G(\tilde{s}))$. Therefore, $F_q = F_{pq} = F_{pq}(log_B(s))$.

Now as $\tilde{x} \in LG(K)$ maps onto the constant point $\check{x} \in L\tilde{E}(\mathbb{C})$, so also $\tilde{s}$ maps onto a constant point $\check{p} \in \tilde{E}(\mathbb{C})$ and hence $p \in E(\mathbb{C})$. So we are in Case (SC2) of the proof of the Main Lemma of [6], §6, namely $p$ constant while $q$ nonconstant. The conclusion of (SC2) is that $log_B(s)$ is transcendental over $F_{pq}$ if $p$ is nontorsion. So the previous equality forces $p \in E(\mathbb{C})$ to be torsion.

Let $\tilde{s}_{tor} \in G(K)$ be a torsion point lifting $p$, hence $\tilde{s} - \tilde{s}_{tor}$ is a $K$-point of the kernel of the surjection $G \to E$. Hence $\tilde{s} = \tilde{s}_{tor} + \delta + \beta$ where $\beta \in \mathbb{G}_a(K)$ and $\delta \in \mathbb{G}_m(K)$. Taking logs, putting again $\xi = \tilde{x} - \tilde{x}_1$, and using that $log_G(-)$ restricted to $\mathbb{G}_a(K)$ is the identity, we see that $\tilde{x} = \xi + log_G(\tilde{s}_{tor}) + log_G(\delta) + \beta = \xi' + log_{\mathbb{G}_m}(\delta) + \beta$ where $\xi' \in (LG)^\partial$. It follows that $\ell = log_{\mathbb{G}_m}(\delta) \in K_G^\sharp = F_q$. But by Lemma 1 of [7] (proof of Main Lemma in isotrivial case, but reversing roles of $p$ and $q$), such $\ell$ is transcendental over $F_q$ unless $\delta$ is constant.

Hence $\delta \in \mathbb{G}_m(\mathbb{C})$, whereby $log_{\mathbb{G}_m}(\delta) \in L\mathbb{G}_m(\mathbb{C})$ so is in $(LG)^\partial(K^{diff})$, and we conclude that $\tilde{x} - \beta \in (LG)^\partial(K^{diff})$. As also $\tilde{x} - \beta \in LG(K)$, from

Claim III in Section 5.3 of [7] (alternatively, using the fact that $K^{\sharp}_{LG} = F_q$ has transcendence degree 2 over $K$), we conclude that $\tilde{x} - \beta \in L\mathbb{G}_m(\mathbb{C})$ whereby $\tilde{x} \in L\mathbb{G}_a(K) + L\mathbb{G}_m(\mathbb{C})$, contradicting that $x$ projects onto a nonzero element $LE$. This contradiction completes the proof of Claim 1 and hence of Case (I) of Corollary 2.10.

*Case (II).* $\bar{x} \in LE(K) \setminus LE(\mathbb{C})$ is a nonconstant point of $LE(K) = LE_0(K)$. Let $\tilde{y} \in G(K^{diff})$ be such that $\partial\ell n_G(\tilde{y}) = a = \partial_{LG}(\tilde{x})$. Let $\check{y}$ be the projection of $\tilde{y}$ to $\tilde{E}$. Hence $\partial\ell n_{\tilde{E}}(\check{y}) = \partial_{L\tilde{A}}(\check{x})$ (where remember $\check{x}$ is the projection of $\tilde{x}$ to $L\tilde{E}$). Noting that $\check{x}$ lifts $\bar{x} \in LE(K)$, and using our case hypothesis, we can apply Corollary 2.9 to $E$ to conclude that $tr.deg(K(\check{y})/K) = 2$ with Galois group $\tilde{E}^{\partial}(K^{diff}) = \tilde{E}(\mathbb{C})$. (In fact as $E$ is constant this is already part of the Ax-Kolchin framework and appears in [3].)

*Claim 2.* $tr.deg(K^{\sharp}_G(\check{y})/K^{\sharp}_G) = 2$.
*Proof of Claim 2.* Fact 3.4 applies to the current situation, showing that $K^{\sharp}_G$ is a directed union of Picard-Vessiot extensions of $K$ each with Galois group some product of $\mathbb{G}^n_m(\mathbb{C})$'s. As there are no proper algebraic subgroups of $\tilde{E}(\mathbb{C}) \times \mathbb{G}^n_m(\mathbb{C})$ projecting onto each factor, it follows from the Galois theory, that $\check{y}$ is independent from $K^{\sharp}_G$ over $K$, yielding Claim 2.

Now $K^{\sharp}_G(\tilde{y})/K^{\sharp}_G$ is a differential Galois extension with Galois group of the form $H^{\partial}(K^{diff})$ where $H$ is connected algebraic $\partial$-subgroup of $G$. So $H^{\partial}$ projects onto the (differential) Galois group of $K^{\sharp}_G(\check{y})$ over $K^{\sharp}_G$, which by Claim 2 is $\tilde{E}^{\partial}(K^{diff})$. In particular $H$ projects onto $\tilde{E}$. If $H$ is a proper subgroup of $G$, then projecting $H$, $\tilde{E}$ to $B$, $E$, respectively, shows that $B$ splits (up to isogeny), so $B = B_0$ is constant, contradicting the current assumptions. Hence the (differential) Galois group of $K^{\sharp}_G(\tilde{y})$ over $K^{\sharp}_G$ is $G^{\partial}(K^{diff})$, whereby $tr.deg(K^{\sharp}_G(\tilde{y}/K^{\sharp}_G)$ is 3. This concludes the proof of Corollary 2.10.

### 4.3 An Ax-Schanuel conjecture

As a conclusion to the first two themes of the paper, we may say that both at the Galois theoretic level and for Lindemann-Weierstrass, we have obtained rather definitive results for families of abelian varieties, and working over a suitable base $K^{\sharp}$. There remain open questions for families of semiabelian varieties, such as Conjecture 2.3, as well as dropping the restriction on the toric part in 2.6, 2.7, 2.8, and 2.10. It also remains to formulate a qualitative description of $tr.deg(K^{\sharp}(exp_B(x)/K^{\sharp})$ where $B$ is a semiabelian variety over

$K$ of dimension $> 2$, and $x \in LB(K)$, under the nondegeneracy hypothesis that $x \notin LH + LB_0(\mathbb{C})$ for any proper semiabelian subvariety $H$ of $B$.

Before turning to our third theme, it seems fitting to propose a more general *Ax-Schanuel* conjecture for families of abelian varieties, as follows.

**Conjecture 4.1.** *Let $A$ be an abelian variety over $K = \mathbb{C}(S)$ for a curve $S/\mathbb{C}$, and let $F$ be the field of meromorphic functions on some disc in $S$. Let $K^\sharp$ now denote $K^\sharp_{L\tilde{A}}$ (which contains $K^\sharp_{\tilde{A}}$). Let $\tilde{x}$, $\tilde{y}$ be $F$-rational points of $L\tilde{A}, \tilde{A}$ respectively such that $exp_{\tilde{A}}(\tilde{x}) = \tilde{y}$, and let $y$ be the projection of $\tilde{y}$ on $A$. Assume that $y \notin H + A_0(\mathbb{C})$ for any proper algebraic subgroup $H$ of $A$. Then $tr.deg(K^\sharp(\tilde{x}, \tilde{y})/K^\sharp) \geq dim(\tilde{A})$.*

We point out that the assumption concerns $y$, and not the projection $x$ of $\tilde{x}$ to $LA$. Indeed, the conclusion would in general not hold true under the weaker hypothesis that $x \notin LH + LA_0(\mathbb{C})$ for any proper abelian subvariety $H$ of $A$. As a counterexample, take for $A$ a simple non constant abelian variety over $K$, and for $\tilde{x}$ a non-zero period of $L\tilde{A}$. Then, $x \neq 0$ satisfies the hypothesis above and $\tilde{x}$ is defined over $K^\sharp = K^\sharp_{L\tilde{A}}$, but $\tilde{y} = exp_{\tilde{A}}(\tilde{x}) = 0$, so $tr.deg(K^\sharp(\tilde{x}, \tilde{y})/K^\sharp) = 0$.

Finally, here is a concrete corollary of the conjecture. Let $\mathbf{E} : y^2 = x(x-1)(x-t)$ be the universal Legendre elliptic curve over $S = \mathbb{C} \setminus \{0, 1\}$, and let $\omega_1(t), \omega_2(t)$ be a basis of the group of periods of $\mathbf{E}$ over some disk, so $K^\sharp = K^\sharp_{L\tilde{E}}$ is the field generated over $K = \mathbb{C}(t)$ by $\omega_1, \omega_2$ and their first derivatives. Let $\wp = \wp_t(z), \zeta = \zeta_t(z)$ be the standard Weierstrass functions attached to $\{\omega_1(t), \omega_2(t)\}$. For $g \geq 1$, consider $2g$ algebraic functions $\alpha_1^{(i)}(t), \alpha_2^{(i)}(t) \in K^{alg}, i = 1, \ldots, g$, and assume that the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_1^{(g)} \\ \alpha_2^{(g)} \end{pmatrix}$ are linearly independent over $\mathbb{Z}$. Then, the $2g$ functions

$$\wp(\alpha_1^{(i)}\omega_1 + \alpha_2^{(i)}\omega_2)\ ,\ \zeta(\alpha_1^{(i)}\omega_1 + \alpha_2^{(i)}\omega_2)\ , i = 1, \ldots, g,$$

of the variable $t$ are algebraically independent over $K^\sharp$. In the language of [6], §3.3, this says in particular that a $g$-tuple of $\mathbb{Z}$-linearly independent local analytic sections of $\mathbf{E}/S$ with algebraic *Betti* coordinates forms a generic point of $\mathbf{E}^g/S$. Such a statement is not covered by our Lindemann-Weierstrass results, which concern analytic sections with algebraic logarithms.

# 5 Manin maps

## 5.1 Injectivity

We here prove Theorem 2.13. and Proposition 2.15. Both statements will follow fairly quickly from Fact 5.1 below, which is Theorem 4.3 of [5] and relies on the strongest version of "Manin-Chai", namely formula ($2^*$) from Section 4.1 of [5]. We should mention that a more direct proof of Proposition 2.15 can be extracted from the proof of Proposition J.2 (Manin-Coleman) in [7]. But we will stick with the current proof below, as it provides a good introduction to the counterexample in Section 5.2.

We set up notations : $K$ is $\mathbb{C}(t)^{alg}$ as usual, $A$ is an an abelian variety over $K$ and $A_0$ is the $\mathbb{C}$-trace of $A$. For $y \in \tilde{A}(K)$, we let $\overline{y}$ be its image in $A(K)$. Let $b = \partial\ell n_{\tilde{A}}(y)$. We consider the differential system in unknown $x$:

$$\nabla_{L\tilde{A}}(x) = b \ ,$$

where we write $\nabla_{L\tilde{A}}$ for $\partial_{L\tilde{A}}$. Let $K^\sharp_{L\tilde{A}}$ be the differential field generated, over $K$, by $(L\tilde{A})^\partial(K^{diff})$. So for $x$ a solution in $L\tilde{A}(K^{diff})$, the differential Galois group of $K^\sharp_{L\tilde{A}}(x)$ over $K^\sharp_{L\tilde{A}}$ pertains to Picard-Vessiot theory, and is well-defined as a $\mathbb{C}$-subpace of the $\mathbb{C}$-vector space $(L\tilde{A})^\partial(K^{diff})$.

**Fact 5.1** ($A =$ any abelian variety over $K = \mathbb{C}(t)^{alg}$). *Let $y \in \tilde{A}(K)$. Let $B$ be the smallest abelian subvariety of $A$ such that a multiple of $\overline{y}$ by a nonzero integer is in $B + A_0(\mathbb{C})$. Let $x$ be a solution of $\nabla_{L\tilde{A}}(-) = b$ in $L\tilde{A}(K^{diff})$. Then the differential Galois group of $K^\sharp_{L\tilde{A}}(x)$ over $K^\sharp_{L\tilde{A}}$ is $(L\tilde{B})^\partial(K^{diff})$. In particular $tr.deg(K^\sharp_{L\tilde{A}}(x)/K^\sharp_{L\tilde{A}}) = dim\tilde{B} = 2dimB$.*

### Proof of Theorem 2.13

Here, $A$ has $\mathbb{C}$-trace 0. By assumption we have $y_1, .., y_n \in A(K)$ and $a_1, .., a_n \in \mathbb{C}$, not all 0 such that $a_1\mu_A(y_1) + \ldots + a_n\mu_A(y_n) = 0$ in $L\tilde{A}(K)/\partial\ell n_{\tilde{A}}(W_A)$. Lifting $y_i$ to $\tilde{y}_i \in \tilde{A}(K)$, we derive that

$$a_1\partial\ell n_{\tilde{A}}(\tilde{y}_1) + \ldots + a_n\partial\ell n_{\tilde{A}}(\tilde{y}_n) = \partial\ell n_{\tilde{A}}(z)$$

for some $z \in W_A$. Via our identification of $W_A$ with $LW_A$ we write the right hand side as $\nabla_{L\tilde{A}}z$ with $z \in LW_A \subset L\tilde{A}$. Let $\tilde{x}_i \in L\tilde{A}$ be such that

$\nabla_{L\tilde{A}}(\tilde{x}_i) = \partial\ell n_{\tilde{A}}(\tilde{y}_i)$. Hence $a_1\tilde{x}_1 + \ldots + a_n\tilde{x}_n - z \in (L\tilde{A})^\partial$, and there exists $d \in (L\tilde{A})^\partial$ such that

$$a_1\tilde{x}_1 + \ldots + a_n\tilde{x}_n - d = z \in LW_A.$$

Suppose for a contradiction that $y_1, .., y_n$ are linearly independent with respect to $End(A)$. Then no multiple of $y = (y_1, .., y_n)$ by a nonzero integer lies in any proper abelian subvariety $B$ of the traceless abelian variety $A^n = A \times .. \times A$. By Fact 5.1, $tr.deg(K^\sharp(\tilde{x}_1, .., \tilde{x}_n)/K^\sharp) = dim(\tilde{A}^n)$, where we have set $K^\sharp := K^\sharp_{L\tilde{A}^n} = K^\sharp_{L\tilde{A}}$. So $\tilde{x}_1, .., \tilde{x}_n$ are generic independent, over $K^\sharp$, points of $L\tilde{A}$. Hence, as $a_1, .., a_n$ are in $\mathbb{C}$ so in $K^\sharp$, $a_1\tilde{x}_1 + ... + a_n\tilde{x}_n$ is a generic point of $L\tilde{A}$ over $K^\sharp$. And as $d$ is a $K^\sharp$-rational point of $(L\tilde{A})^\partial$, $a_1\tilde{x}_1 + .. + a_n\tilde{x}_n - d = z$ too is a generic point of $L\tilde{A}$ over $K^\sharp$, so cannot lie in its strict subspace $LW_A$. This contradiction concludes the proof of 2.13.

**Proof of Proposition 2.15.**

We use the same notation as at the end of Section 2.4, and recall that $A$ is traceless. Furthermore, the functoriality of $M_{K,A}$ in $A$ allows us to assume that $A$ is a simple abelian variety.

*Step (I).* We show as in the proof of 2.13 above that if $M_{K,A}(y_1), .., M_{K,A}(y_n)$ are $\mathbb{C}$-linearly dependent, then $y_1, .., y_n$ are $End(A)$-linearly dependent. Indeed, assume that $a_i \in \mathbb{C}$ are not all 0 and that $a_1 M_{K,A}(y_1) + .. + a_n M_{K,A}(y_n) = 0$ in the target space $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$. Lifting $y_i$ to $\tilde{y}_i \in \tilde{A}(K)$, we derive that

$$a_1\partial\ell n_{\tilde{A}}(\tilde{y}_1) + \ldots . + a_n\partial\ell n_{\tilde{A}}(\tilde{y}_n) \in \nabla(L\tilde{A}(K))$$

Letting $\tilde{x}_i \in L\tilde{A}(K^{diff})$ be such that $\nabla\tilde{x}_i = \partial\ell n_{\tilde{A}}(\tilde{y}_i)$, we obtain a $K$-rational point $z \in L\tilde{A}(K)$ such that

$$a_1\tilde{x}_1 + \ldots + a_n\tilde{x}_n - z := d \in (L\tilde{A})^\partial(K^{diff}).$$

Taking $K^\sharp := K^\sharp_{L\tilde{A}}$ as in the proof of 2.13, we get $tr.deg(K^\sharp(\tilde{x}_1, .., \tilde{x}_n)/K^\sharp) < dim(\tilde{A}^n)$. Hence by Fact 5.1, some integral multiple of $(y_1, .., y_n)$ lies in a proper abelian subvariety of $A^n$, whereby $y_1, .., y_n$ are $End(A)$-linearly dependent.

*Step (II).* Assuming that $y_1, .., y_n$ are $End(A)$-linearly dependent, given by Step (I), as well as the relation on the point $d$ above with not all $a_i = 0$, we will show that the points $y_i$ are $\mathbb{Z}$-linearly dependent. Equivalently we will show that if a similar relation holds with the $a_i$ linearly independent over $\mathbb{Z}$, then $y = (y_1, .., y_n)$ is a torsion point of $A^n$. Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$. Let $B$ be the connected component of the Zariski closure of the group $\mathbb{Z} \cdot y$ of multiples of $y$ in $A^n$. By Fact 5.1, the differential Galois group of $K^\sharp(\tilde{x})$ over $K^\sharp := K^\sharp_{L\tilde{A}}$ is $(L\tilde{B})^\partial$. More precisely, the set of $\sigma(\tilde{x}) - \tilde{x}$ as $\sigma$ varies in $Aut_\partial(K^\sharp(\tilde{x})/K^\sharp)$ is precisely $(L\tilde{B})^\partial \subseteq (L\tilde{A^n})^\partial$. Since $z$ and $d$ are defined over $K^\sharp$, the relation on $d$ implies that

$$\forall(\tilde{c}_1, .., \tilde{c}_n) \in (L\tilde{B})^\partial, a_1\tilde{c}_1 + \ldots + a_n\tilde{c}_n = 0.$$

Let now

$$\mathcal{B} = \{\alpha = (\alpha_1, .., \alpha_n) \in (End(A))^n = Hom(A, A^n) : \alpha(A) \subseteq B \subset A^n\}.$$

*Claim.* Assume that $a_1, \ldots, a_n$ are linearly independent over $\mathbb{Z}$. Then, any $\alpha \in \mathcal{B}$ is identically 0.

It follows from the claim that $B = 0$ and hence some multiple of $y$ by a nonzero integer vanishes, namely $y$ is a torsion point of $A^n$. This completes the proof of Step (II), hence of Proposition 2.15, and we are now reduced to proving the claim.

*Proof of claim.* Since $A$ is simple, $End(A)$ is an order in a simple algebra $D$ over $\mathbb{Q}$. For $i = 1, .., n$, denote by $\rho(\alpha_i)$ the $\mathbb{C}$-linear map induced on $(L\tilde{A})^\partial$ by the endomorphism $\alpha_i$ of $A$. So we view $(L\tilde{A})^\partial$ as a complex representation, of degree $2dimA$, of the $\mathbb{Z}$-algebra $End(A)$, or more generally, of $D$. Let $f^2$ be the dimension of $D$ over its centre $F$, let $e$ be the degree of $F$ over $\mathbb{Q}$ and let $R$ be a reduced representation of $D$, viewed as a complex representation of degree $ef$. As the representation $\rho$ is defined over $\mathbb{Q}$ (since it preserves the Betti homology), $\rho$ is equivalent to the direct sum $R^{\oplus r}$ of $r = 2dimA/ef$ copies of $R$ (cf. [20], §5.1). Furthermore, $R : D \to Mat_f(F \otimes \mathbb{C}) \simeq (Mat_f(\mathbb{C}))^e \subset Mat_{ef}(\mathbb{C})$ extends by $\mathbb{C}$-linearity to an injection $R \otimes 1 : D \otimes \mathbb{C} \simeq (Mat_f(\mathbb{C}))^e \subset Mat_{ef}(\mathbb{C})$.

Recall now that for any $(\tilde{c}_1, \ldots, \tilde{c}_n)$ in $(L\tilde{B})^\partial$, $a_1\tilde{c}_1 + \ldots + a_n\tilde{c}_n = 0$. Applied to the image under $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$ of the generic element of $(L\tilde{A})^\partial$, this relation implies that

$$a_1\rho(\alpha_1) + \ldots + a_n\rho(\alpha_n) = 0 \in End_{\mathbb{C}}((L\tilde{A})^\partial)$$

So $a_1R(\alpha_1)+\ldots+a_nR(\alpha_n)=0$ in $(Mat_f(\mathbb{C}))^e$. From the injectivity of $R\otimes 1$ on $D\otimes\mathbb{C}$ and the $\mathbb{Z}$-linear independence of the $a_i$, we derive that each $\alpha_i\in D$ vanishes, hence $\alpha=0$, proving the claim.

## 5.2 A counterexample

We conclude with the promised counterexample to the injectivity of $\mu_A\otimes 1$, namely Proposition 2.14.

### Construction of $A$

We will use Yves André's example of a simple traceless abelian variety $A$ over $\mathbb{C}(t)^{alg}$ with $0\neq U_A\subsetneq W_A$, cf. [7], just before Remark 3.10. Since $U_A\neq W_A$, this $A$ is not constant, but we will derive this property and the simplicity of $A$ from another argument, borrowed from [12], IX.6.

We start with a CM field $F$ of degree $2k$ over $\mathbb{Q}$, over a totally real number field $F_0$ of degree $k\geq 2$, and denote by $\{\sigma_1,\bar\sigma_1,...,\sigma_k,\bar\sigma_k\}$ the complex embeddings of $F$. We further fix the CM type $S:=\{\sigma_1,\bar\sigma_1,2\sigma_2,\ldots,2\sigma_k\}$. By [12], IX.6, we can attach to $S$ and to any $\tau\in\mathcal{H}$ (the Poincaré half-plane, or equivalently, the open unit disk) an abelian variety $A=A_\tau$ of dimension $g=2k$ and an embedding of $F$ into $End(A)\otimes\mathbb{Q}$ such that the representation $r$ of $F$ on $W_A$ is given by the type $S$. The representation $\rho$ of $F$ on $L\tilde{A}$ is then $r\oplus\bar r$, equivalent to twice the regular representation. (The notations used by [12] here read : $e_0=k,d=1,m=2,r_1=s_1=1,r_2=..=r_{e_0}=2,s_2=\ldots=s_{e_0}=0$, so, the product of the $\mathcal{H}_{r_i,s_i}$ of *loc. cit.* is just $\mathcal{H}$. Also, [12] considers the more standard "analytic" representation of $F$ on the Lie algebra $LA=L\tilde{A}/W_A$, which is $\bar r$ in our notation.)

From the bottom of [12], p. 271, one infers that the moduli space of such abelian varieties $A_\tau$ is an analytic curve $\mathcal{H}/\Gamma$. But Shimura has shown that it can be compactified to an algebraic curve $\mathcal{X}$, cf [12], p. 247. So, we can view the universal abelian variety $A_\tau=A$ of this moduli space as an abelian variety over $\mathbb{C}(\mathcal{X})$, hence as an abelian variety $A$ over $K=\mathbb{C}(t)^{alg}$. This will be our $A$ : it is by construction not constant - and it is a fourfold if we take, as we will in what follows, $k=2$.

Finally, since $A$ is the general element over $\mathcal{H}/\Gamma$, Theorem 9.1 of [12] and the hypothesis $k\geq 2$ imply that $End(A)\otimes\mathbb{Q}$ is *equal* to $F$. Therefore, $A$ is a simple abelian variety, necessarily traceless since it is not constant. We denote by $\mathcal{O}$ the order $End(A)$ of $F$.

**Action of $F$ and of $\nabla$ on $L\tilde{A}$**

For simplicity, we will now restrict to the case $k = 2$, but the general case (requiring $2k$ points) would work in exactly the same way. So, $F$ is a totally imaginary quadratic extension of a real quadratic field $F_0$, and $L\tilde{A}$ is 8-dimensional. As said in [7], and by definition of the CM-type $S$, the action $\rho$ of $F$ splits $L\tilde{A}$ into eigenspaces for its irreducible representations $\sigma$'s, as follows :

- $W_A = D_{\sigma_1} \oplus D_{\bar{\sigma}_1} \oplus P_{\sigma_2}$, where the $D$'s are lines , and $P_{\sigma_2}$ is a plane;
- $LA$ lifts to $L\tilde{A}$ into $D'_{\sigma_1} \oplus D'_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$, with same notations.

Since $\nabla := \nabla_{L\tilde{A}} = \partial_{L\tilde{A}}$ commutes with the action $\rho$ of $F$ and since $A$ is not constant, we infer that the maximal $\partial$-submodule of $W_A$ is

$$U_A = P_{\sigma_2} \ ,$$

while $W_A + \nabla(W_A) = \Pi_{\sigma_1} \oplus U_A \oplus \Pi_{\bar{\sigma}_1}$, with planes $\Pi_{\sigma_1} = D_{\sigma_1} \oplus D'_{\sigma_1}, \Pi_{\bar{\sigma}_1} = D_{\bar{\sigma}_1} \oplus D'_{\bar{\sigma}_1}$, each stable under $\nabla$ (just as is $P_{\bar{\sigma}_2}$, of course). In fact, for our proof, we only need to know that $P_{\sigma_2} \subset U_A$.

Now, let $\tilde{y} \in \tilde{A}(K)$ be a lift of a point $y \in A(K)$. Going into a complex analytic setting, we choose a logarithm $\tilde{x} \in L\tilde{A}(K^{diff})$ of $\tilde{y}$, locally analytic on a small disk in $\mathcal{X}(\mathbb{C})$. Let further $\alpha \in \mathcal{O}$, which canonically lifts to $End(\tilde{A})$. Then, $\rho(\alpha)\tilde{x}$ is a logarithm of $\alpha.\tilde{y} \in \tilde{A}(K)$, and thefore satisfies

$$\nabla(\rho(\alpha)\tilde{x}) = \partial \ell n_{\tilde{A}}(\alpha.\tilde{y}).$$

In fact, this appeal to analysis is not necessary : the formula just says that $\partial \ell n_{\tilde{A}}$ (and $\nabla$) commutes with the actions of $\mathcal{O}$. But once one $\tilde{y}$ and one $\tilde{x}$ are chosen, it will be crucial for the searched-for relation $(\mathfrak{R})$ following Proposition 2.14 that we take these $\rho(\alpha)\tilde{x}$ as solutions to the equations on the $\mathcal{O}$-orbit of $\tilde{y}$.

Concretely, if

$$\tilde{x} = x_{\sigma_2} \oplus x_{\sigma_1} \oplus x_{\bar{\sigma}_1} \oplus x_{\bar{\sigma}_2}$$

is the decomposition of $\tilde{x}$ in $L\tilde{A} = P_{\sigma_2} \oplus \Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$, then for any $\alpha \in \mathcal{O}$, we have

$$\rho(\alpha)(\tilde{x}) = \sigma_2(\alpha)x_{\sigma_2} \oplus \sigma_1(\alpha)x_{\sigma_1} \oplus \bar{\sigma}_1(\alpha)x_{\bar{\sigma}_1} \oplus \bar{\sigma}_2(\alpha)x_{\bar{\sigma}_2}.$$

**Conclusion**

Let $y \in A(K)$ be a non torsion point of the simple abelian variety $A$, for which we choose at will a lift $\tilde{y}$ to $\tilde{A}(K)$ and a logarithm $\tilde{x} \in L\tilde{A}(K^{diff})$. Let $\{\alpha_1, ..., \alpha_4\}$ be an integral basis of $F$ over $\mathbb{Q}$. We will consider the 4 points $y_i = \alpha_i.y$ of $A(K)$, $i = 1, \ldots, 4$. Since the action of $\mathcal{O}$ on $A$ is faithful, they are linearly independent over $\mathbb{Z}$. For each $i = 1, .., 4$, we consider the lift $\tilde{y}_i = \alpha_i \tilde{y}$ of $y_i$ to $L\tilde{A}(K)$, and set as above $\tilde{x}_i = \rho(\alpha_i)\tilde{x}$, which satisfies $\nabla(\tilde{x}_i) = \partial \ell n_{\tilde{A}} \tilde{y}_i$.

We claim that there exist complex numbers $a_1, .., a_4$, not all zero, such that

$$u := a_1\tilde{x}_1 + \ldots + a_4\tilde{x}_4 = \big(a_1\rho(\alpha_1) + \ldots + a_4\rho(\alpha_4)\big)(\tilde{x})$$

lies in $U_A(K^{diff})$, i.e. such that in the decomposition above of $L\tilde{A} = P_{\sigma_2} \oplus \Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$, the components of $u = u_{\sigma_2} \oplus u_{\sigma_1} \oplus u_{\bar{\sigma}_1} \oplus u_{\bar{\sigma}_2}$ on the last three planes vanish.

The whole point is that the complex representation $\hat{\sigma}^{\oplus 2}$ of $F$ which $\rho$ induces on $\Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$ is twice the representation $\hat{\sigma} := \sigma_1 \oplus \bar{\sigma}_1 \oplus \bar{\sigma}_2$ of $F$ on $\mathbb{C}^3$, and so, does not contain the full regular representation of $F$. More concretely, the 4 vectors $\hat{\sigma}(\alpha_1), ..., \hat{\sigma}(\alpha_4)$ of $\mathbb{C}^3$ are of necessity linearly dependent over $\mathbb{C}$, so, there exists a non trivial linear relation

$$a_1\hat{\sigma}(\alpha_1) + \ldots + a_4\hat{\sigma}(\alpha_4) = 0 \text{ in } \mathbb{C}^3$$

(where the complex numbers $a_i$ lie in the normal closure of $F$). Therefore, *any* element $\tilde{x}_{\hat{\sigma}} = (x_{\sigma_1}, x_{\bar{\sigma}_1}, x_{\bar{\sigma}_2})$ of $\Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$ satisfies :

$$(a_1\hat{\sigma}^{\oplus 2}(\alpha_1) + \ldots + a_4\hat{\sigma}^{\oplus 2}(\alpha_4))\tilde{x}_{\hat{\sigma}} = 0 \text{ in } \Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$$

(view each $\hat{\sigma}^{\oplus 2}(\alpha_i)$ as a $(6 \times 6)$ diagonal matrice inside the $(8 \times 8)$ diagonal matrix $\rho(\alpha_i)$), i.e. the 3 plane-components $u_{\sigma_1}, u_{\bar{\sigma}_1}, u_{\bar{\sigma}_2}$ of $u$ all vanish, and $u$ indeed lies in $P_{\sigma_2}$, and so in $U_A$.

The existence of such a point $u = a_1\tilde{x}_1 + \ldots + a_4\tilde{x}_4$ in $U_A(K^{diff})$ establishes relation $(\mathfrak{R})$ of §2.4, and concludes the proof of Proposition 2.14.

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*AMS 2010 Classification numbers*: 12H05, 14K05, 03C60, 34M15, 11J95 .
*Key-words* : differential Galois theory, algebraic independence over function fields, semi-abelian schemes, Manin maps.