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► **To cite this version:**

Mohammed El Azhari. Functional continuity of unital  $B_0$ -algebras with orthogonal bases. *Le Matematiche*, 2017, 72 (1), pp.97-102. hal-01205148v3

**HAL Id: hal-01205148**

**<https://hal.science/hal-01205148v3>**

Submitted on 13 Mar 2016 (v3), last revised 16 Nov 2019 (v4)

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## Functional continuity of unital $B_0$ -algebras with orthogonal bases

M. El Azhari

**Abstract.** Let  $A$  be a unital  $B_0$ -algebra with an orthogonal basis, then every multiplicative linear functional on  $A$  is continuous. This gives an answer to a problem posed by Z. Sawon and Z. Wronski.

*2010 Mathematics Subject Classification:* 46H40.

*Keywords and phrases:*  $B_0$ -algebra, orthogonal basis, multiplicative linear functional.

### I. Preliminaries

A topological algebra is a complex associative algebra which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra is a topological algebra whose topology is determined by a family of seminorms. A complete metrizable locally convex algebra is called a  $B_0$ -algebra. The topology of a  $B_0$ -algebra  $A$  may be given by a countable family  $(\|\cdot\|_i)_{i \geq 1}$  of seminorms such that  $\|x\|_i \leq \|x\|_{i+1}$  and  $\|xy\|_i \leq \|x\|_{i+1}\|y\|_{i+1}$  for all  $i \geq 1$  and  $x, y \in A$ . A multiplicative linear functional on a complex algebra  $A$  is an algebra homomorphism from  $A$  to the complex field. Let  $A$  be a topological algebra.  $M^*(A)$  denotes the set of all nonzero multiplicative linear functionals on  $A$ .  $M(A)$  denotes the set of all nonzero continuous multiplicative linear functionals on  $A$ . A seminorm  $p$  on  $A$  is lower semicontinuous if the set  $\{x \in A : p(x) \leq 1\}$  is closed in  $A$ .

Let  $A$  be a topological algebra. A sequence  $(e_n)_{n \geq 1}$  in  $A$  is a basis if for each  $x \in A$  there is a unique sequence  $(\alpha_n)_{n \geq 1}$  of complex numbers such that  $x = \sum_{n=1}^{\infty} \alpha_n e_n$ . Each linear functional  $e_n^* : A \rightarrow \mathbb{C}$ ,  $e_n^*(x) = \alpha_n$ , is called a coefficient functional. If each  $e_n^*$  is continuous, the basis  $(e_n)_{n \geq 1}$  is called a Schauder basis. A basis  $(e_n)_{n \geq 1}$  is orthogonal if  $e_i e_j = \delta_{ij} e_i$  where  $\delta_{ij}$  is the Kronecker symbol. If  $(e_n)_{n \geq 1}$  is an orthogonal basis, then each  $e_n^*$  is a multiplicative linear functional on  $A$ . Let  $A$  be a topological algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . If  $A$  has a unity  $e$ , then  $e = \sum_{n=1}^{\infty} e_n$ . Let  $(x_k)_k$  be a net in  $A$  converging to 0, since the multiplication is separately continuous,  $e_n x_k = e_n^*(x_k) e_n \rightarrow_k 0$  and so  $e_n^*(x_k) \rightarrow_k 0$ . Then each orthogonal basis in a topological algebra is a Schauder basis. Let  $f$  be a multiplicative linear functional on  $A$ . If  $f(e_{n_0}) \neq 0$  for some  $n_0 \geq 1$ , then  $f(x)f(e_{n_0}) = f(xe_{n_0}) = f(e_{n_0}^*(x)e_{n_0}) = e_{n_0}^*(x)f(e_{n_0})$  for all  $x \in A$  and therefore  $f = e_{n_0}^* \in M(A)$ . This shows that  $M(A) = \{e_n^* : n \geq 1\}$ .

Here we consider unital  $B_0$ -algebras with orthogonal bases. These algebras were investigated in [3] where we can find examples of such algebras.

## II. Results

**Proposition II.1.** Let  $(A, (\|\cdot\|_i)_{i \geq 1})$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . Then there exists  $x \in A$  such that the sequence  $(e_n^*(x))_{n \geq 1}$  is not bounded.

**Proof.** Suppose that  $\sup_{n \geq 1} |e_n^*(x)| < \infty$  for all  $x \in A$ . For each  $x \in A$ , let  $\|x\| = \sup_{n \geq 1} |e_n^*(x)|$ ,  $\|\cdot\|$  is a lower semicontinuous norm on  $A$  since  $(e_n)_{n \geq 1}$  is a Schauder basis. Let  $\tau_A$  be the topology on  $A$  determined by the family  $(\|\cdot\|_i)_{i \geq 1}$  of seminorms. We define a new topology  $\tau$  on  $A$  described by the norm  $\|\cdot\|$  and the family  $(\|\cdot\|_i)_{i \geq 1}$  of seminorms. The topology  $\tau$  is stronger than the topology  $\tau_A$ . By Garling's completeness theorem [1, Theorem 1],  $(A, \tau)$  is complete. The topologies  $\tau_A$  and  $\tau$  are homeomorphic by the open mapping theorem. Then there exist  $i_0 \geq 1$  and  $M > 0$  such that  $\|x\| \leq M\|x\|_{i_0}$  for all  $x \in A$ , hence  $1 = |e_n^*(e_n)| \leq \|e_n\| \leq M\|e_n\|_{i_0}$  for all  $n \geq 1$ . This contradicts the fact that  $e_n \rightarrow_n 0$ .

**Proposition II.2.** Let  $A$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . If  $x = \sum_{n=1}^{\infty} t_n e_n \in A$  such that  $t_n \in \mathbb{R}$ ,  $t_n \leq t_{n+1}$  for  $n \geq 1$  and  $t_n \rightarrow \infty$ , then  $f(x) \in \mathbb{R}$  for all  $f \in M^*(A)$ .

**Proof.** If  $f \in M(A)$ ,  $f = e_n^*$  for some  $n \geq 1$ , then  $f(x) = t_n \in \mathbb{R}$ . If  $f \in M^*(A) \setminus M(A)$ , then  $f(e_n) = 0$  for all  $n \geq 1$ . Suppose that  $f(x) \notin \mathbb{R}$ ,  $f(x) = \alpha + i\beta$  with  $\beta \neq 0$ . Since  $t_n \rightarrow \infty$ , there exists  $n_0 \geq 1$  such that  $t_n \geq \alpha + |\beta|$  for all  $n \geq n_0$ . We define the sequence  $(s_n)_{n \geq 1}$  by  $s_n = t_{n_0}$  for  $1 \leq n \leq n_0$  and  $s_n = t_n$  for  $n \geq n_0 + 1$ . It is clear that  $y = \sum_{n=1}^{\infty} s_n e_n \in A$  such that  $s_n \geq \alpha + |\beta|$ ,  $s_n \leq s_{n+1}$  for  $n \geq 1$  and  $s_n \rightarrow \infty$ . Since  $f(e_n) = 0$  for all  $n \geq 1$ ,  $f(y) = f(x) = \alpha + i\beta$ . We have  $f(|\beta|^{-1}y) = |\beta|^{-1}\alpha + i|\beta|^{-1}\beta$ , then  $f(|\beta|^{-1}y - |\beta|^{-1}\alpha e) = i|\beta|^{-1}\beta$ . Set  $z = |\beta|^{-1}y - |\beta|^{-1}\alpha e = \sum_{n=1}^{\infty} \frac{s_n - \alpha}{|\beta|} e_n$ . The real sequence  $(\frac{s_n - \alpha}{|\beta|})_{n \geq 1}$  is positive increasing and  $\frac{s_n - \alpha}{|\beta|} \rightarrow \infty$ , then  $z^{-1} = \sum_{n=1}^{\infty} \frac{|\beta|}{s_n - \alpha} e_n \in A$  by [3, Theorem 0.1] and  $f(z^{-1}) = -i|\beta|\beta^{-1}$ , so  $f(z + z^{-1}) = 0$ . Set  $v = z + z^{-1} = \sum_{n=1}^{\infty} (\frac{s_n - \alpha}{|\beta|} + \frac{|\beta|}{s_n - \alpha}) e_n$  and  $v_n = \frac{s_n - \alpha}{|\beta|} + \frac{|\beta|}{s_n - \alpha}$  for all  $n \geq 1$ . Since the map  $g : [1, \infty) \rightarrow \mathbb{R}$ ,  $g(x) = x + \frac{1}{x}$ , is increasing and the sequence  $(\frac{s_n - \alpha}{|\beta|})_{n \geq 1} \subset [1, \infty)$  is increasing, it follows that  $(v_n)_{n \geq 1}$  is a positive increasing sequence and  $v_n \rightarrow \infty$ . By [3, Theorem 0.1],  $v^{-1} = \sum_{n=1}^{\infty} \frac{1}{v_n} e_n \in A$  and therefore  $f(e) = f(v)f(v^{-1}) = 0$ . This contradicts the fact that  $f$  is nonzero.

The following two results are due to Sawon and Wronski [3], the proofs are given for completeness.

**Theorem II.3**[3, Theorem 2.1]. Let  $A$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . If  $x = \sum_{n=1}^{\infty} t_n e_n \in A$  such that  $t_n \in \mathbb{R}$ ,  $t_n \leq t_{n+1}$  for  $n \geq 1$  and  $t_n \rightarrow \infty$ , then every multiplicative linear functional on  $A$  is continuous.

**Proof.** Suppose that  $M^*(A) \setminus M(A)$  is nonempty. Let  $f \in M^*(A) \setminus M(A)$ , then  $f(e_n) = 0$  for all  $n \geq 1$ . Put  $f(x) = \alpha$ , then  $\alpha \in \mathbb{R}$  by Proposition II.2. Since  $t_n \rightarrow \infty$ , there exists  $n_0 \geq 1$  such that  $t_n > \alpha$  for  $n \geq n_0$ . Consider  $y = \sum_{n=1}^{\infty} \lambda_n e_n \in A$  such that  $\lambda_n = t_{n_0}$  for  $1 \leq n \leq n_0$  and  $\lambda_n = t_n$  for  $n \geq n_0 + 1$ . Since  $f(e_n) = 0$  for all  $n \geq 1$ , it follows that  $f(x) = f(y) = \alpha$ . We have  $y - \alpha e = \sum_{n=1}^{\infty} v_n e_n \in A$  where  $v_n = \lambda_n - \alpha$  for all  $n \geq 1$ . It is clear that  $v_n > 0$ ,  $v_n \leq v_{n+1}$  for  $n \geq 1$  and  $v_n \rightarrow \infty$ . By [3, Theorem 0.1],  $(y - \alpha e)^{-1} = \sum_{n=1}^{\infty} \frac{1}{v_n} e_n \in A$ , so  $y - \alpha e$  is invertible and  $f(y - \alpha e) = 0$ , a contradiction.

**Proposition II.4**[3, p.109]). Let  $(A, (\|\cdot\|_i)_{i \geq 1})$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . Then the set  $N$  of all positive integers can be split into two disjoint subsets  $N_1$  and  $N_2$  such that by putting  $A_1 = \overline{\text{span}}(e_n)_{n \in N_1}$  and  $A_2 = \overline{\text{span}}(e_n)_{n \in N_2}$ , we have

- (1)  $A = A_1 \oplus A_2$ ;
- (2) if  $f$  is a multiplicative linear functional on  $A$  such that  $f \notin M(A)$ , then  $f|_{A_1} = 0$ .

**Proof.** By Proposition II.1, there is  $x = \sum_{n=1}^{\infty} t_n e_n \in A$  such that the sequence  $(t_n)_{n \geq 1}$  is not bounded. Then there exists a subsequence  $(t_{k_n})_{n \geq 1}$  of  $(t_n)_{n \geq 1}$  such that  $|t_{k_n}| \geq n^2$  for all  $n \geq 1$ . For each  $i \geq 1$ , there is  $M_i > 0$  such that  $\|t_n e_n\|_i \leq M_i$  for all  $n \geq 1$ . Let  $i \geq 1$  and  $n \geq 1$ ,  $n^2 \|e_{k_n}\|_i \leq |t_{k_n}| \|e_{k_n}\|_i = \|t_{k_n} e_{k_n}\|_i \leq M_i$ , then  $\|e_{k_n}\|_i \leq n^{-2} M_i$ . This implies that  $\sum_{n=1}^{\infty} e_{k_n}$  is absolutely convergent. Let  $A_1 = \overline{\text{span}}\{e_{k_n} : n \geq 1\}$ ,  $A_1$  is a unital  $B_0$ -algebra with an orthogonal basis  $(e_{k_n})_{n \geq 1}$  and  $\sum_{n=1}^{\infty} n^{\frac{1}{2}} e_{k_n} \in A_1$  since  $n^{\frac{1}{2}} \|e_{k_n}\|_i \leq n^{-\frac{3}{2}} M_i$  for all  $i \geq 1$  and  $n \geq 1$ . Set  $N_1 = \{k_n : n \geq 1\}$ ,  $N_2 = N \setminus N_1$  and  $A_2 = \overline{\text{span}}\{e_n : n \in N_2\}$ .  $A_2$  is a  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \in N_2}$  and the unity  $u_2 = e - u_1$  where  $u_1 = \sum_{n=1}^{\infty} e_{k_n}$  is the unity of  $A_1$ . Let  $x \in A$ ,  $x = x e = x(u_1 + u_2) = x u_1 + x u_2 \in A_1 + A_2$ , then  $A = A_1 \oplus A_2$ . If  $f$  is a multiplicative linear functional on  $A$  such that  $f \notin M(A)$ ,  $f|_{A_1}$  is a multiplicative linear functional on  $A_1$  such that  $f|_{A_1}(e_{k_n}) = 0$  for all  $n \geq 1$ . Since  $\sum_{n=1}^{\infty} n^{\frac{1}{2}} e_{k_n} \in A_1$ ,  $f|_{A_1}$  is continuous on  $A_1$  by Theorem II.3 and therefore  $f|_{A_1} = 0$ .

Sawon and Wronski [3, p.109] posed the following problem:

**Problem.** Let  $A$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . Does there exist a maximal subalgebra  $A'_1 = \overline{\text{span}}\{e_n : n \in N'_1\}$  ( $N'_1 \subset N$ ) of  $A$  for which (1) and (2) hold?

**Proposition II.5.** Let  $A$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . Then the following assertions are equivalent:

- (i)  $A = \overline{\text{span}}\{e_n : n \in N\}$  is a maximal subalgebra of itself for which (1) and (2) hold;
- (ii) every multiplicative linear functional on  $A$  is continuous.

**Proof.** (i)  $\Rightarrow$  (ii) : Let  $f$  be a multiplicative linear functional on  $A$  such that  $f \notin M(A)$ ,  $f$  is zero on  $A$  by (i). Then every multiplicative linear functional on  $A$  is continuous.

(ii)  $\Rightarrow$  (i) : It is clear that  $A$  satisfies (1). Let  $f$  be a multiplicative linear functional on  $A$  such that  $f \notin M(A)$ , then  $f$  is zero on  $A$  since  $f$  is continuous, hence  $A$  satisfies (2).

**Proposition II.6.** Let  $(t_n)_{n \geq n_0}$  be a complex sequence, the following assertions are equivalent:

(i)  $\sum_{n=n_0}^{\infty} |t_n - t_{n+1}| < \infty$ ;

(ii) there exists  $M > 0$  such that  $|t_q| + \sum_{n=p}^{q-1} |t_n - t_{n+1}| \leq M$  for all  $q > p \geq n_0$ .

**Proof.** (i)  $\Rightarrow$  (ii) : Let  $\epsilon > 0$ , there exists  $n_1 \geq n_0$  such that  $\sum_{k=n}^{\infty} |t_k - t_{k+1}| \leq \epsilon$  for every  $n \geq n_1$ . let  $m > n \geq n_1$ ,  $|t_n - t_m| \leq |t_n - t_{n+1}| + \dots + |t_{m-1} - t_m| \leq \epsilon$ . Then the sequence  $(t_n)_{n \geq n_0}$  converges, so there is  $M_0 > 0$  such that  $|t_n| \leq M_0$  for all  $n \geq n_0$ . Let  $q > p \geq n_0$ ,  $|t_q| + \sum_{n=p}^{q-1} |t_n - t_{n+1}| \leq M_0 + \sum_{n=n_0}^{\infty} |t_n - t_{n+1}|$ . (ii)  $\Rightarrow$  (i) : Let  $p \geq n_0$  and  $q = p+1$ ,  $\sum_{n=n_0}^p |t_n - t_{n+1}| \leq |t_q| + \sum_{n=n_0}^{q-1} |t_n - t_{n+1}| \leq M$ . Then the sequence  $(\sum_{n=n_0}^p |t_n - t_{n+1}|)_{p \geq n_0}$  is positive increasing and bounded, so it is convergent i.e.  $\sum_{n=n_0}^{\infty} |t_n - t_{n+1}| < \infty$ .

**Proposition II.7.** Let  $(A, (\|\cdot\|_i)_{i \geq 1})$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . If  $(t_n)_{n \geq n_0}$  is a complex sequence such that  $\sum_{n=n_0}^{\infty} |t_n - t_{n+1}| < \infty$ , then  $\sum_{n=n_0}^{\infty} t_n e_n \in A$ .

**Proof.** Let  $q > p \geq n_0$ , by using the equality  $t_n = t_q + \sum_{k=n}^{q-1} (t_k - t_{k+1})$  for every  $p \leq n < q$ , we obtain that  $\sum_{n=p}^{q-1} t_n e_n = t_q(e_p + \dots + e_{q-1}) + \sum_{k=p}^{q-1} (t_k - t_{k+1})(e_p + \dots + e_k)$ . Let  $i \geq 1$ ,  $\|\sum_{n=p}^{q-1} t_n e_n\|_i \leq |t_q| \|e_p + \dots + e_{q-1}\|_i + \sum_{k=p}^{q-1} |t_k - t_{k+1}| \|e_p + \dots + e_k\|_i \leq (|t_q| + \sum_{k=p}^{q-1} |t_k - t_{k+1}|) \sup_{p \leq k \leq q} \|e_p + \dots + e_k\|_i \leq M \sup_{p \leq k \leq q} \|e_p + \dots + e_k\|_i$  by Proposition II.6. Let  $\epsilon > 0$ , since  $e = \sum_{n=1}^{\infty} e_n \in A$ , there is  $n_1 \geq n_0$  such that  $\|e_p + \dots + e_k\|_i \leq \epsilon M^{-1}$  for  $n_1 \leq p \leq k$ , hence  $\sup_{p \leq k \leq q} \|e_p + \dots + e_k\|_i \leq \epsilon M^{-1}$  for  $n_1 \leq p < q$ . Then  $\|\sum_{n=p}^{q-1} t_n e_n\|_i \leq \epsilon$  for  $n_1 \leq p < q$ . This shows that  $\sum_{n=n_0}^{\infty} t_n e_n$  is convergent in  $A$ .

**Theorem II.8.** Let  $(A, (\|\cdot\|_i)_{i \geq 1})$  be a unital  $B_0$ -algebra with an orthogonal basis  $(e_n)_{n \geq 1}$ . Then every multiplicative linear functional on  $A$  is continuous.

**Proof.** By Proposition II.7,  $x = \sum_{n=1}^{\infty} \frac{1}{n} e_n \in A$ . Let  $f \in M^*(A)$  and  $\alpha = f(x)$ , then  $f(\alpha e - x) = 0$ . We have  $\alpha e - x = \sum_{n=1}^{\infty} (\alpha - \frac{1}{n}) e_n = \sum_{n=1}^{\infty} \frac{\alpha n - 1}{n} e_n$ . Let  $\mathbb{N}$  be the set of all positive integers. Put  $I_\alpha = \{n \in \mathbb{N} : n \neq \alpha^{-1}\}$ ,  $I_\alpha = \mathbb{N}$  if  $\alpha^{-1} \notin \mathbb{N}$  and  $I_\alpha = \mathbb{N} \setminus \{\alpha^{-1}\}$  if  $\alpha^{-1} \in \mathbb{N}$ . Let  $m_\alpha = \inf\{n \in \mathbb{N} : |\alpha|n - 1 > 0\}$  and consider the complex sequence  $(\frac{n}{\alpha n - 1})_{n \geq m_\alpha}$ . Let  $n \geq m_\alpha$ ,  $\frac{n}{\alpha n - 1} - \frac{n+1}{\alpha(n+1) - 1} = \frac{1}{(\alpha n - 1)(\alpha(n+1) - 1)}$ . We have  $|\alpha n - 1| \geq |\alpha|n - 1$  and  $|\alpha(n+1) - 1| \geq |\alpha|(n+1) - 1 = |\alpha|n + |\alpha| - 1 \geq |\alpha|n - 1$ . Let  $n \geq m_\alpha$ ,  $|\alpha|n - 1 > 0$ , hence  $\frac{1}{|\alpha n - 1|} \leq \frac{1}{|\alpha|n - 1}$  and  $\frac{1}{|\alpha(n+1) - 1|} \leq \frac{1}{|\alpha|n - 1}$ . Consequently  $|\frac{n}{\alpha n - 1} - \frac{n+1}{\alpha(n+1) - 1}| =$

$\frac{1}{|\alpha n - 1| |\alpha(n+1) - 1|} \leq \frac{1}{(|\alpha|n - 1)^2}$ . Then  $\sum_{n=m_\alpha}^{\infty} \frac{n}{\alpha n - 1} e_n \in A$  by Proposition II.7 and therefore  $\sum_{n \in I_\alpha} \frac{n}{\alpha n - 1} e_n \in A$ . If  $\alpha^{-1} \notin \mathbb{N}$  i.e.  $I_\alpha = \mathbb{N}$ ,  $(\alpha e - x) \sum_{n=1}^{\infty} \frac{n}{\alpha n - 1} e_n = e$ , then  $\alpha e - x$  is invertible, a contradiction. If  $\alpha^{-1} \in \mathbb{N}$ , put  $n_\alpha = \alpha^{-1}$ , then  $(\alpha e - x) \sum_{n \in I_\alpha} \frac{n}{\alpha n - 1} e_n = \sum_{n \in I_\alpha} e_n$ , hence  $\sum_{n \in I_\alpha} e_n \in \text{Ker}(f)$  since  $\alpha e - x \in \text{Ker}(f)$ . If  $e_{n_\alpha} \in \text{Ker}(f)$ , then  $e = e_{n_\alpha} + \sum_{n \in I_\alpha} e_n \in \text{Ker}(f)$ , a contradiction. Finally  $f(e_{n_\alpha}) \neq 0$  and therefore  $f = e_{n_\alpha}^*$ .

**Remark.** Proposition II.5 and Theorem II.8 give an answer to Sawon and Wronski's problem.

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