Non-localization of eigenfunctions for Sturm-Liouville operators

Thibault Liard, Pierre Lissy, Yannick Privat

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Thibault Liard* Pierre Lissy† Yannick Privat‡§

Abstract

In this article, we investigate a non-localization property of the eigenfunctions of Sturm-Liouville operators $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$, where $a(\cdot)$ runs over the bounded nonnegative potential functions on the interval $(0, L)$ with $L > 0$. More precisely, we address the extremal spectral problem of minimizing the L^2-norm of a function $e(\cdot)$ on a measurable subset ω of $(0, L)$, where $e(\cdot)$ runs over all eigenfunctions of A_a, at the same time with respect to all subsets ω having a prescribed measure and all nonnegative L^∞ potential functions $a(\cdot)$ having a prescribed essentially upper bound. We provide some existence and qualitative properties of the minimizers, as well as precise lower and upper estimates on the optimal value. Numerous consequences in control and stabilization theory are then highlighted, both theoretically and numerically.

Keywords: Sturm-Liouville operators, eigenfunctions, extremal problems, calculus of variations, control theory, wave equation.

AMS classification: 34B24, 49K15, 47A75, 49J20, 93B07, 93B05

1 Introduction and main results

1.1 Localization/Non-localization of Sturm-Liouville eigenfunctions

In a recent survey article concerning the Laplace operator ([8]), D. Grebenkov and B.T. Nguyen introduce, recall and gather many possible definitions of the notion of *localization of eigenfunctions*. In particular, they consider, in the section 7.7 of their article, the Dirichlet-Laplace operator on a given bounded open set Ω of $\mathbb{R}^n$, a Hilbert basis of eigenfunctions $(u_j)_{j\in\mathbf{N}^*}$ in $L^2(\Omega)$ and use as a measure of localization of the eigenfunctions on a measurable subset $V \subset \Omega$ the following criterion

$$C_p(V) = \inf_{j\in\mathbf{N}^*} \frac{\|u_j\|_{L^p(V)}}{\|u_j\|_{L^p(\Omega)}},$$

where $p \geqslant 1$. For instance, evaluating this quantity for different choices of subdomains V if Ω is a ball or an ellipse allows to illustrate the so-called *whispering galleries* or *bouncing ball* phenomena. At the opposite, when Ω denotes the d-dimensional box $(0, \ell_1) \times \cdots \times (0, \ell_d)$ (with $\ell_1, \ldots, \ell_d > 0$),

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it is recalled that $C_p(V) > 0$ for any $p \geqslant 1$ and any measurable subset $V \subset \Omega$ whenever the ratios $(\ell_i/\ell_j)^2$ are not rational numbers for every $i \neq j$.

Many other notions of localization have been introduced in the literature. Regarding the Dirichlet/Neumann/Robin Laplacian eigenfunctions on a bounded open domain Ω of $\mathbb{R}^n$ and using a semi-classical analysis point of view, the notions of *quantum limit* or *entropy* have been widely investigated (see e.g. [1, 3, 4, 7, 11, 17]) and provide an account for possible strong concentrations of eigenfunctions. Notice that the properties of $C_p(V)$ are intimately related to the behavior of high-frequency eigenfunctions and especially to the set of quantum limits of the sequence of eigenfunctions considered. Identifying such limits is a great challenge in quantum physics ([4, 7, 34]) and constitute a key ingredient to highlight non-localization/localization properties of the sequence of eigenfunctions considered.

Given a nonzero integer p, the *non-localization property* of a sequence $(u_j)_{j\in\mathbb{N}^*}$ of eigenfunctions means that the real number $C_p(V)$ is positive for every measurable subset $V \subset \Omega$. Regarding the one-dimensional Dirichlet-Laplace operator on $\Omega = (0, \pi)$, it has been highlighted in the case where $p = 2$ (for instance in [10, 21, 30]) that

$$\inf_{|V|=r\pi} C_2(V) = \inf_{|V|=r\pi} \inf_{j\in\mathbb{N}^*} \frac{2}{\pi} \int_V \sin(jx)^2 \, dx > 0,$$

for every $r \in (0, 1)$.

Motivated by these considerations, the present work is devoted to studying similar issues in the case $p = 2$, for a general family of one-dimensional Sturm-Liouville operators of the kind $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ is a nonnegative bounded potential defined on an interval $(0, L)$ with $L > 0$. More precisely, we aim at providing lower quantitative estimates of the quantity $C_2(V)$, where $(u_j)_{j\in\mathbb{N}^*}$ denotes now a sequence of eigenfunctions of A_a, in terms of the measure of V and the essential supremum of $a(\cdot)$ by minimizing this criterion at the same time with respect to V and $a(\cdot)$, over the class of subsets V having a prescribed measure and over a well-chosen class of potentials $a(\cdot)$, relevant from the point of view of applications. Independently of its intrinsic interest, the choice "$p = 2$" is justified by the fact that the quantity $C_2(V)$ plays a crucial role in many control, stabilization or inverse problems, as highlighted in the sequel.

Let us mention that using the Liouville transform (see [5, Page 292]), the result obtained for the Sturm-Liouville operator A_a can be easily transferred to the operator $-\partial_x(w(x)\partial_x)$ for some weight function w belonging to an appropriate class of functions.

The article is organized as follows: in Section 1.2, the extremal problem we will investigate is introduced. The main results of this article are stated in Section 1.3: a comprehensive analysis of the extremal problem is performed, reducing in some sense (that will be made precise in the sequel) this infinite-dimensional problem to a finite one. We moreover provide very simple lower and upper estimates of the optimal value. The whole section 2 is devoted to the proofs of the main and intermediate results. Finally, consequences and applications of our main results for observation and control theory and several numerical illustrations and investigations are gathered in Section 3.

1.2 The extremal problem

Let L be a positive real number and $a(\cdot)$ be an essentially nonnegative function belonging to $L^\infty(0, L)$. We consider the operator

$$A_a := -\partial_{xx} + a(\cdot)\,\mathrm{Id},$$

defined on $\mathcal{D}(A_a) = H_0^1(0, L) \cap H^2(0, L)$. As a self-adjoint operator, A_a admits a Hilbert basis of $L^2(0, L)$ made of eigenfunctions denoted $e_{a,j} \in \mathcal{D}(A_a)$ and there exists a sequence of increasing positive real numbers $(\lambda_{a,j})_{j\in\mathbb{N}^*}$ such that $e_{a,j}$ solves the eigenvalue problem

$$\begin{cases} -e''_{a,j}(x) + a(x)e_{a,j}(x) = \lambda^2_{a,j} e_{a,j}(x), \ x \in (0,L), \\ \qquad e_{a,j}(0) = e_{a,j}(L) = 0. \end{cases} \tag{1}$$

By definition, the normalization condition

$$\int_0^L e^2_{a,j}(x)\, dx = 1 \tag{2}$$

is satisfied and we will also impose in the sequel that $e'_{a,j}(0) > 0$, so that the function $e_{a,j}$ is uniquely defined.

With regards to the explanations of Section 1.1, we are interested in the non-localization property of the sequence of eigenfunctions $(e_{a,j})_{j\in\mathbf{N}^*}$. The quantity of interest, denoted $J(a,\omega)$, is defined by

$$J(a,\omega) = \inf_{j\in\mathbf{N}^*} \frac{\int_\omega e^2_{a,j}(x)\, dx}{\int_0^L e^2_{a,j}(x)\, dx} = \inf_{j\in\mathbf{N}^*} \int_\omega e^2_{a,j}(x)\, dx, \tag{3}$$

where ω denotes a measurable subset of $(0,L)$ of positive measure.

The real number $J(a,\omega)$ is the equivalent for the Sturm-Liouville operators A_a of the quantity $C_2(V)$ introduced in Section 1.1 for the one-dimensional Dirichlet-Laplace operator.

It is natural to assume the knowledge of *a priori* informations about the subset ω and the potential function $a(\cdot)$. Indeed, we will choose them in some classes that are small enough to make the minimization problems we will deal with non-trivial, but also large enough to provide "explicit" (at least numerically) values of the criterion for a large family of potential.

Hence, in the sequel, we will assume that:

- the measure (or at least an upper bound of the measure) of the subset ω is given. Indeed, we will show in the sequel that prescribing either the measure of ω or an upper bound drives to the same solution of the optimal design problem we will consider;
- the potential function $a(\cdot)$ is nonnegative and essentially bounded.

We believe that such conditions are physically relevant and also adapted to the context of control or inverse problems.

Fix $M > 0$, $r \in (0,1)$, α and β two real numbers such that $\alpha < \beta$. Let us introduce the class of admissible observation subsets

$$\Omega_r(\alpha,\beta) = \{\text{Lebesgue measurable subset } \omega \text{ of } (\alpha,\beta) \text{ such that } |\omega| \leqslant r(\beta-\alpha)\}, \tag{4}$$

as well as the class of admissible potentials

$$\mathcal{A}_M(\alpha,\beta) = \{a \in L^\infty(\alpha,\beta) \text{ such that } 0 \leqslant a \leqslant M \text{ a.e. on } (\alpha,\beta)\}, \tag{5}$$

in which the potential function $a(\cdot)$ will be chosen in the sequel.

Let us now introduce the optimal design problem we will investigate.

Extremal spectral problem. *Let $M > 0$, $r \in (0,1)$ and $L > 0$ be fixed. We consider*

$$m(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J(a,\omega), \tag{$\mathcal{P}_{L,r,M}$}$$

where the functional J is defined by (3), $\Omega_r(0,L)$ and $\mathcal{A}_M(0,L)$ are respectively defined by (4) and (5).

In the sequel, we will call *minimizer of the problem* ($\mathcal{P}_{L,r,M}$) a triple $(a,\omega,j) \in \mathcal{A}_M(0,L) \times \Omega_r(0,L) \times \mathbb{N}^*$ (whenever it exists) such that

$$\inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J(a,\omega) = \int_\omega e_{a,j}(x)^2\,dx.$$

1.3 Main results and comments

Let us state the main results of this article. The next theorems are devoted to the analysis of the optimal design problems ($\mathcal{P}_{L,r,M}$).

We also stress on the fact that the estimates of $J(a,\omega)$ in the following result are valuable for every measurable subset ω of prescribed measure and that we do not need to make additional topological assumption on it.

Theorem 1. *Let $r \in (0,1)$ and $M \in \mathbb{R}_+^*$.*

i Problem ($\mathcal{P}_{L,r,M}$) has a solution (j_0,ω^*,a^*). In particular, there holds

$$m(L,M,r) \quad = \quad \min_{a\in\mathcal{A}_M(0,L)} \min_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j_0}(x)^2\,dx,$$

and the solution a^* of Problem ($\mathcal{P}_{L,r,M}$) is bang-bang, equal to 0 or M a.e. in $(0,L)$.

ii Assume that $M \in (0,\pi^2/L^2]$. Then, ω^* is the union of j_0+1 intervals, and a^* has at most $3j_0-1$ and at least j_0 switching points[1]*.*

Moreover, one has the estimate

$$\gamma r^3 \leqslant m(L,M,r) \leqslant r - \frac{\sin(\pi r)}{\pi}, \tag{6}$$

with $\gamma = \frac{7\sqrt{3}}{8}(3-2\sqrt{2}) \simeq 0.2600$.

The estimate (6) is in fact inferred from a more precise estimate for the optimal design problem

$$m_j(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx, \tag{$\mathcal{P}_{j,L,r,M}$}$$

with $j \in \mathbb{N}^*$. Because of its intrinsic interest, we state this estimate in the following proposition, which constitutes therefore an essential ingredient for the proof of the last point of Theorem 1.

Proposition 1. *Let $r \in (0,1)$ and let us assume that $M \in (0,\pi^2/L^2]$. Then, there holds*

$$m_j(L,M,r) \geqslant r^3 \underline{m}_j, \tag{7}$$

for every $j \in \mathbb{N}^*$, where the sequence $(\underline{m}_j)_{j\in\mathbb{N}^*}$ is defined by

$$\underline{m}_j = \begin{cases} \frac{1}{2} & \text{if } j=1, \\ \dfrac{(2j^2-1)(j^2-1)^{\frac{3}{2}}\left(\sqrt{\frac{j^2}{j^2-2}}-1\right)^2}{3j^3\left(\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}}-1\right)^2} & \text{if } j \geqslant 2. \end{cases}$$

[1] Recall that a switching point of a *bang-bang* function is a point at which this function is not continuous.

In the following result, we highlight the necessity of imposing a pointwise upper bound on the potentials functions $a(\cdot)$ to get the existence of a minimizer. Indeed, we have the following non-existence result when the pointwise constraint "$a \leqslant M$" is not imposed anymore.

Theorem 2. *Let $r \in (0,1)$ and $j \in \mathbb{N}^*$. Then, the optimal design problem*

$$\inf_{a \in \mathcal{A}_\infty(0,L)} \inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,j}(x)^2 \, dx, \tag{$\mathcal{P}_{j,L,r,\infty}$}$$

where $\mathcal{A}_\infty(0,L) = \cup_{M>0} \mathcal{A}_M(0,L)$, has no solution.

We conclude this section by a series of remarks and comments on the main results we have just stated.

Remark 1. As we will see later, the proofs of point (i), the first part of point (ii) are quite classical and rests upon classical optimization tools. The main difficulty was to prove the left inequality of (6), which comes from the more refine estimates (7).

Remark 2. Let us highlight the interest of Theorem 1 for numerical investigations. Indeed, in view of providing numerical lower bounds of the quantity $J(a,\omega)$, Theorem 1 enables us to reduce the solving of the infinite-dimensional problems ($\mathcal{P}_{L,r,M}$) to the solving of finite ones, since one has just to choose the optimal $3j_0^* - 1$ switching points defining the best potential function a^* and to remark that necessarily ω^* is uniquely defined once a^* is defined (since it is defined in terms of a precise level set of e_{a^*,j_0^*}, see Proposition 2). We will strongly use this remark in Section 3.2.1, where illustrations and applications of Problem ($\mathcal{P}_{L,r,M}$) are developed.

Remark 3. As it will be emphasized in the sequel, the restriction on the range of values of the parameter M in the second point of Theorem 1 makes each eigenfunction $e_{a,j}$ convex or concave on each nodal domain. It is a crucial element of the proofs of Theorem 1 and Proposition 1, that allows to compare each quantity $\int_\omega e_{a,j}(x)^2 \, dx$ with the integral of the square of well-chosen piecewise affine functions.

Remark 4. According to Section 3.2.1, numerical simulations seem to indicate, at least in the case where $M = \frac{\pi^2}{L^2}$, that there exists a triple (j_0^*, ω^*, a^*) such that $j_0^* = 1$, the set ω^* and the graph of a^* are symmetric with respect to $L/2$ and a^* is a *bang-bang* function equal to 0 or M a.e. in $(0,L)$ having exactly two switching points, solving Problem ($\mathcal{P}_{L,r,M}$). We were unfortunately unable to prove this assertion.

A less ambitious issue would consist in finding an upper estimate of the optimal index j_0^*. Even this question appears difficult in particular since it is not obvious to compare the real numbers $m_j(L,M,r)$ for different indexes j. One of the reasons of that comes from the fact that the cost functional we considered does not write as the minimum of an energy function, making the comparison between eigenfunctions of different orders intricate.

Remark 5. It can be noticed that the lower bound in Proposition 1 is independent of the parameter L. This can be justified by using a rescaling argument allowing in particular to restrict our investigations to the case where $L = \pi$.

Lemma 1. *Let $j \in \mathbb{N}^*$, $r \in (0,1)$, $M > 0$ and $L > 0$. Then, there holds*

$$\inf_{a \in \mathcal{A}_M(0,\pi)} \inf_{\omega \in \Omega_r(0,\pi)} \int_\omega e_{a,j}(x)^2 \, dx = \inf_{a \in \mathcal{A}_{\frac{M\pi^2}{L^2}}(0,L)} \inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,j}(x)^2 \, dx, \tag{8}$$

The proof of this lemma rests upon the simple change of variable $x = \pi y / L$.

Remark 6. The estimate (6) can be considered as sharp with respect to the parameter r, at least for r small enough (which is the most interesting case, at least in view of the applications). Indeed, there holds $\frac{\pi r - \sin(\pi r)}{\pi} \sim \frac{\pi^2}{6} r^3$ as r tends to 0, which shows that the power of r in the left-hand side cannot be improved. The graphs of the quantities appearing in the left and right-hand sides of (6) with respect to r are plotted on Figure 1 below.

An interesting issue would thus consist in determining the optimal bounds in the estimate (6), namely

$$\ell_- = \inf_{r\in(0,1)} \frac{m(L,M,r)}{r^3} \qquad \text{and} \qquad \ell_+ = \sup_{r\in(0,1)} \frac{m(L,M,r)}{r^3}.$$

It is likely that investigating this issue would rely on a refined study of the optimality conditions of the problems above, but also of the problem ($\mathcal{P}_{L,r,M}$). According to (6), we know that $\ell_- \in [\gamma, 1]$ and $\ell_+ \in [\gamma, \pi^2/6]$.

Remark 7 (*Behavior of the sequence* $(\underline{m}_j)_{j\in\mathbf{N}^*}$*.*)**.** One can prove by using the explicit formula of $\underline{m}_j$that this sequence is increasing. Moreover, straightforward computations show that it converges to $2/3$ as j tends to $+\infty$.

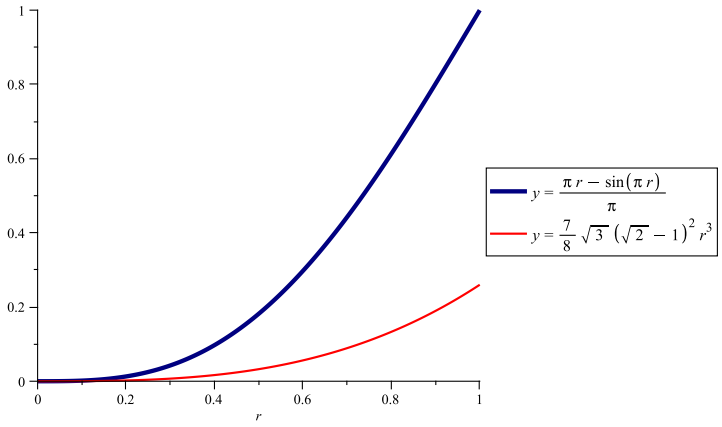

Figure 1: Plots of $r \mapsto \gamma r^3$ (thin line) and $x \mapsto (\pi r - \sin(\pi r))/\pi$ (bold line).

2 Proofs of Theorem 1, Proposition 1, and Theorem 2

2.1 Preliminary material: existence results and optimality conditions

We gather in this section several results we will need in the sequel to prove Theorem 1, Proposition 1, and Theorem 2.

Let us first investigate the following simpler optimal design problem, where the potential $a(\cdot)$ is now assumed to be fixed, and which will constitute an important ingredient in the proof.

Auxiliary problem: fixed potential. *For a given $j \in \mathbb{N}^*$, $M > 0$, $r \in (0,1)$ and $a \in \mathcal{A}_M(0,L)$, we investigate the optimal design problem*

$$\inf_{\chi\in\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\, dx, \tag{Aux-Pb}$$

where the set $\mathcal{U}_r$ *is defined by*

$$\mathcal{U}_r = \left\{ \chi \in L^\infty(0,L) \mid 0 \leqslant \chi \leqslant 1 \text{ a.e. in } (0,L) \text{ and } \int_0^L \chi(x)\,dx = rL \right\}. \tag{9}$$

We provide a characterization of the solutions of Problem (Aux-Pb).

Proposition 2. *Let* $r \in (0,1)$. *The optimal design problem Problem* (Aux-Pb) *has a unique solution that writes as the characteristic function of a measurable set* ω^* *of Lebesgue measure* rL. *Moreover, there exists a positive real number* τ *such that* ω^* *is a solution of Problem* (Aux-Pb) *if and only if*

$$\omega^* = \{e_{a,j}(x)^2 < \tau\}, \tag{10}$$

up to a set of zero Lebesgue measure.

In other words, any optimal set, solution of Problem (Aux-Pb), is characterized in terms of the level set of the function $e_{a,j}(\cdot)^2$.

A proof of this result can be found in [28, Theorem 1] or [33, Chapter 1]. It yields the existence of a positive real number τ such that ω^* is a solution of Problem (Aux-Pb) if and only if

$$\chi_{\{e_{a,j}(x)^2<\tau\}} \leqslant \chi_{\omega^*}(x) \leqslant \chi_{\{e_{a,j}(x)^2\leqslant\tau\}}(x),$$

for almost every $x \in (0,L)$, by using a standard argument of decreasing rearrangement. We also mention [13, 19] as references on rearrangements and symmetrization of functions and their use in shape optimization. The conclusion follows by noting that for every $c > 0$, the set $\{e_{a,j}^2 = c\}$ has zero Lebesgue measure, by using standard properties of eigenfunctions of Sturm-Liouville operators.

Let us now investigate the continuity of the mapping $a \mapsto e_{a,j}$.

Lemma 2. *Let* $M \in \mathbb{R}_+^*$ *and* $j \in \mathbb{N}^*$. *Let us endow the space* $\mathcal{A}_M(0,L)$ *with the weak-*$\star$ *topology of* $L^\infty(0,L)$ *and the space* $H_0^1(0,L)$ *with the standard strong topology inherited from the Sobolev norm* $\|\cdot\|_{H_0^1}$. *Then the function* $a \in \mathcal{A}_M(0,L) \mapsto e_{a,j} \in H_0^1(0,L)$ *is continuous.*

The principle of the proof of this lemma is standard, and is roughly recalled for the sake of completeness in Appendix A. The following existence result is a direct consequence of this lemma.

Proposition 3. *Let* $M \in \mathbb{R}_+^*$, $j \in \mathbb{N}^*$ *and* $r \in (0,1)$. *The optimal design Problem* ($\mathcal{P}_{j,L,r,M}$) *has at least one solution* (a_j^*, ω_j^*).

Proof of Proposition 3. Let us consider a relaxed version of the optimal design Problem ($\mathcal{P}_{j,L,r,M}$), where the characteristic function of ω has been replaced by a function χ in $\mathcal{U}_r$ where $\mathcal{U}_r$ is defined by (9). This relaxed version of ($\mathcal{P}_{j,L,r,M}$) writes

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\,dx.$$

Let us endow $\mathcal{A}_M(0,L)$ and $\mathcal{U}_r$ with the weak-$\star$ topology of $L^\infty(0,L)$. Thus, both sets are compact. Moreover, according to Lemma 2 and since it is linear in the variable χ, the functional $(a,\chi) \in \mathcal{A}_M(0,L)\times\mathcal{U}_r \mapsto \int_0^L \chi(x) e_{a,j}(x)^2\,dx$ is continuous. The existence of a solution (a_j^*, χ_j^*) follows for the relaxed problem. Finally, noting that

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\,dx = \inf_{\chi\in\mathcal{U}_r} \int_0^L \chi(x) e_{a_j^*,j}(x)^2\,dx,$$

there exists a measurable set ω_j^* of measure rL such that $\chi_j^* = \chi_{\omega_j^*}$, by Proposition 2.

The existence of a solution of Problem ($\mathcal{P}_{j,L,r,M}$) is then proved and there holds

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\,dx = \inf_{a\in\mathcal{A}_M(0,L)} \int_{\omega^*} e_{a,j}(x)^2\,dx.$$

■

We now state necessary first order optimality conditions that enable us to characterize every critical point (a_j^*, ω_j^*) of the optimal design problem ($\mathcal{P}_{j,L,r,M}$).

Proposition 4. *Let $j \in \mathbb{N}^*$, $r \in (0,1)$ and $M > 0$. Let (a_j^*, ω_j^*) be a solution of the optimal design problem ($\mathcal{P}_{j,L,r,M}$) and let*

$$0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j \tag{11}$$

be the $j+1$ zeros[2] of the j-th eigenfunction $e_{a_j^*,j}$.

For $i \in \{1 \dots j\}$, let us denote by $a_{j,i}^*$ the restriction of the function a_j^* to the interval (x_j^{i-1}, x_j^i) and by $\omega_{j,i}^*$ the set $\omega_j^* \cap (x_j^{i-1}, x_j^i)$. Then, necessarily, there exists $\tau \in \mathbb{R}_+^*$ such that

- *one has $\omega_j^* = \{e_{a_j^*,j}(x)^2 < \tau\}$ up to a set of zero Lebesgue measure,*
- *one has*

$$M\chi_{\{p_j(x)e_{a_j^*,j}(x)>0\}} \leqslant a_j^*(x) \leqslant M\chi_{\{p_j(x)e_{a_j^*,j}(x)\geqslant 0\}}(x), \tag{12}$$

for almost every $x \in (0,L)$, where p_j is defined piecewisely as follows: for $i \in \{1 \dots j\}$, the restriction of p_j to the interval (x_j^{i-1}, x_j^i) is denoted $p_{j,i}$, and $p_{j,i}$ is the (unique) solution of the adjoint system

$$\begin{cases} -p_{j,i}''(x) + (a_{j,i}^*(x) - \lambda_{a_j^*,j}^2) p_{j,i}(x) = (q_{j,i} - \tilde{c_{j,i}}) e_{a_j^*,j}, \quad x \in (x_j^{i-1}, x_j^i), \\ p_{j,i}(x_j^{i-1}) = p_{j,i}(x_j^i) = 0, \end{cases} \tag{13}$$

verifying moreover

$$\int_{x_j^{i-1}}^{x_j^i} p_{j,i}(x) e_{a_{j,i}^*,j}(x)\,dx = 0, \tag{14}$$

where $q_{j,i}$ and $\tilde{c_{j,i}}$ are given by

$$\tilde{c_{j,i}} = \frac{\int_{x_j^{i-1}}^{x_j^i} q_{j,i}(x) e_{a_j^*,j}^2(x)\,dx}{\int_{x_j^{i-1}}^{x_j^i} e_{a_j^*,j}^2(x)\,dx}, \qquad \text{and} \qquad q_{j,i} = \chi_{\omega_{j,i}^*}. \tag{15}$$

In other words, any optimal set solution of Problem ($\mathcal{P}_{j,L,r,M}$) is characterized in terms of a level set of the function $e_{a_j^*,j}(\cdot)^2$ and the optimal potential function is characterized in terms of a level set of the function $p_j(\cdot)e_{a_j^*,j}(\cdot)$.

Remark 8. i Note that, according to Fredholm's alternative (see for example [12]), System (13)-(14) has a unique solution.

[2] Recall that the family $\{x_j^k\}_{0\leqslant k\leqslant j}$ is the set of nodal points of the eigenfunction $e_{a^*,j}$ that are known to be simple.

ii Another presentation of the first order optimality conditions gathered in Proposition 4 could have been obtained by applying the so-called Pontryagin Maximum Principle (see e.g. [22]).

Before proving this proposition, let us state a technical lemma about the differentiability of the eigenfunctions $e_{a,j}$ with respect to a.

Lemma 3. *Let us endow the space $\mathcal{A}_M(0,L)$ with the weak-$\star$ topology of $L^\infty(0,L)$ and the space $H_0^1(0,L)$ with the standard strong topology inherited from the Sobolev norm $\|\cdot\|_{H^1}$. Let $a \in \mathcal{A}_M(0,L)$ and $\mathcal{T}_{a,\mathcal{A}_M(0,L)}$ the tangent cone[3] to the set $\mathcal{A}_M(0,L)$ at point a. For every $h \in \mathcal{T}_{a,\mathcal{A}_M(0,L)}$, the mapping $a \mapsto e_{a,j}$ is Gâteaux-differentiable in the direction h, and its derivative, denoted $\dot{e}_{a,j}$, is the (unique) solution of*

$$\begin{cases} -\dot{e}''_{a,j}(x) + a(x)\dot{e}_{a,j}(x) + h(x)e_{a,j}(x) = \dot{\lambda}^2_{a,j}e_{a,j}(x) + \lambda^2_{a,j}\dot{e}_{a,j}(x), \ x \in (0,L), \\ \dot{e}_{a,j}(0) = \dot{e}_{a,j}(L) = 0, \end{cases} \tag{16}$$

with $\dot{\lambda}^2_{a,j} = \int_0^L h(x)e_{a,j}(x)^2\,dx$.

The proof of the differentiability is completely standard and is based on the fact that the eigenvalues $\lambda_{a,j}$ are simple. For this reason, we skip it and refer to [18, pages 375 and 425].

Proof of Proposition 4. The first point results from Proposition 2.

Let us show the second point.

Let us now compute the Gâteaux-derivative of the cost functional $a \mapsto J_{\omega_j^*}(a)$, where

$$J_{\omega_j^*}(a) = J(a,\omega_j^*) = \int_{\omega_j^*} e_{a,j}(x)^2\,dx,$$

at $a = a_j^*$ in the direction $h_j \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$. We denote it $\langle dJ_{\omega_j^*}(a_j^*), h\rangle$ and there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = 2\int_{\omega^*} \dot{e}_{a_j^*,j}(x)e_{a_j^*,j}(x)\,dx,$$

where $\dot{e}_{a_j^*,j}$ is the solution of (16).

Let us write this quantity in a more convenient form to state the necessary first order optimality conditions. Let h_j be an element of the tangent cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$. Let us write $h_j = \sum_{i=1}^{j-1} h_{j,i}$ where $h_{j,i} = h_j\chi_{(x_j^{i-1},x_j^i)}$ for all $i \in \{1\dots j\}$. Hence, $h_{j,i} \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$ and there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = \sum_{i=1}^{j}\langle dJ_{\omega_j^*}(a_j^*), h_{j,i}\rangle.$$

It follows that it is enough to consider perturbations with compact support contained in each nodal domain in order to compute the Gâteaux-derivative of $J_{\omega_j^*}$. We will use for that purpose the adjoint state p_j piecewisely defined by (13)-(14).

Fix $i \in \{1\dots j\}$ and let $h_{j,i}$ be an element of the tangent cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(x_j^{i-1},x_j^i)}$. Let us multiply the first line of (13) by $\dot{e}_{a_{j,i}^*,1}$ and then integrate by parts. We get

$$\int_{x_j^{i-1}}^{x_j^i} \dot{e}'_{a_{j,i}^*,1}(x)p'_{j,i}(x) + (a_{j,i}^*(x) - \lambda^2_{a_{j,i}^*,1})\dot{e}_{a_{j,i}^*,1}(x)p_{j,i}(x)\,dx = \frac{1}{2}\langle dJ(a_j^*), h_{j,i}\rangle. \tag{17}$$

[3] That is the set of functions $h \in L^\infty(0,L)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(0,L)$ converging to h as $n \to +\infty$, and $a + \varepsilon_n h_n \in \mathcal{A}_M(0,L)$ for every $n \in \mathbb{N}$ (see for instance [13, chapter 7]).

Similarly, let us multiply the first line of (16) by $p_{j,i}$ and then integrate by parts. We get

$$\begin{aligned}&\int_{x_j^{i-1}}^{x_j^i} \dot{e}'_{a^*_{j,i},1}(x)p'_{j,i}(x) + (a^*_{j,i}(x) - \lambda^2_{a^*_{j,i},1})\dot{e}_{a_{j,i},1}(x)p_{j,i}(x)\,dx \\ &\quad = \quad \dot{\lambda}^2_{a^*_j,j}\int_{x_j^{i-1}}^{x_j^i} e_{a_{j,i},1}(x)p_{j,i}(x)\,dx - \int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)e_{a_{j,i},1}(x)p_{j,i}(x)\,dx. \qquad (18)\end{aligned}$$

Combining (17) with (18) and using the condition (14) yields

$$\langle dJ_{\omega^*_j}(a^*_j), h_{j,i}\rangle = -2\int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)e_{a^*_{j,i},1}(x)p_{j,i}(x)\,dx. \qquad (19)$$

As a result, for a general $h_j \in \mathcal{T}_{a^*_j,\mathcal{A}_M(0,L)}$, there holds

$$\langle dJ_{\omega^*_j}(a^*_j), h_j\rangle = -2\sum_{i=1}^{j}\int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)p_{j,i}(x)e_{a^*_{j,i},1}\,dx = -2\int_0^L h_j(x)e_{a^*_j,j}(x)p_j(x)\,dx.$$

Let us state the first order optimality conditions. For every perturbation h_j in the cone $\mathcal{T}_{a^*_j,\mathcal{A}_M(0,L)}$, there holds $\langle dJ(a^*_j), h_j\rangle \geqslant 0$, which writes

$$-2\int_0^L h_j(x)e_{a^*_j,j}(x)p_j(x)\,dx \geqslant 0. \qquad (20)$$

The analysis of such optimality condition is standard in optimal control theory (see for example [22]) and permits to recover easily (12). This ends the proof. ∎

2.2 Proof of Theorem 1

Step 1: existence and bang-bang property of the minimizers (first point of Theorem 1). Notice first that each of the infima defining Problem ($\mathcal{P}_{L,r,M}$) can be inverted with each other. As a result, and according to Proposition 3, there exists an optimal pair (a^*_j, ω^*_j) such that

$$\begin{aligned}m(L,M,r) &= \inf_{a\in\mathcal{A}_M(0,L)}\inf_{\omega\in\Omega_r(0,L)} J(a,\omega) = \inf_{j\in\mathbb{N}^*}\inf_{a\in\mathcal{A}_M(0,L)}\inf_{\omega\in\Omega_r(0,L)}\int_\omega e_{a,j}(x)^2\,dx \\ &= \inf_{j\in\mathbb{N}^*}\int_{\omega^*_j} e_{a^*_j,j}(x)^2\,dx.\end{aligned}$$

It remains then to prove that the last infimum is reached by some index $j_0 \in \mathbb{N}^*$. We will use the two following lemmas.

Lemma 4. *Let $M > 0$ and $(a_j)_{j\in\mathbb{N}^*}$ be a sequence of $\mathcal{A}_M(0,L)$. The sequence $(e^2_{a_j,j})_{j\in\mathbb{N}^*}$ converges weakly-$\star$ in $L^\infty(0,L)$ to the constant function $\frac{1}{L}$.*

The proof of Lemma 4 is standard and is postponed to Appendix B for the sake of clarity. The proof of the next lemma can be found in [25, 29].

Lemma 5. *Let $L > 0$ and $V_0 \in (0,L)$. There holds*

$$\inf_{\substack{\rho\in L^\infty(0,L;[0,1])\\ \int_0^L \rho(x)\,dx = V_0}}\int_0^L \rho(x)\sin^2\left(\frac{j\pi}{L}x\right)\,dx = \frac{1}{2}\left(V_0 - \frac{L}{\pi}\sin\left(\frac{\pi}{L}V_0\right)\right),$$

for every $j \in \mathbb{N}^*$. Moreover, this problem has a unique solution ρ that writes as the characteristic function of a measurable subset ω_j of $(0,L)$.

As a consequence of Lemma 4 and Lemma 5 (the weak-$\star$ convergence being used with the "test" function $\xi_{\omega_j^*} \in L^1(0,L)$), and by minimality of $m(L,M,r)$ (note that $0 \in \mathcal{A}_M(0,L)$), there holds

$$\begin{aligned} m(L,M,r) &= \inf_{j\in\mathbf{N}^*} \int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx = \inf_{\omega\in\Omega_r(0,L)} \inf_{j\in\mathbf{N}^*} \int_{\omega} e_{a_j^*,j}(x)^2\,dx \\ &\leqslant \inf_{\omega\in\Omega_r(0,L)} \inf_{j\in\mathbf{N}^*} \int_{\omega_j^*} e_{0,j}(x)^2\,dx = \frac{2}{L} \inf_{\omega\in\Omega_r(0,L)} \inf_{j\in\mathbf{N}^*} \int_{\omega^*} \sin^2\left(\frac{j\pi}{L}x\right)\,dx \\ &= r - \frac{1}{\pi}\sin(r\pi) < r = \lim_{j\to+\infty} \int_{\omega^*} e_{a_j^*,j}(x)^2\,dx. \end{aligned}$$

As a consequence, the infimum $\inf_{j\in\mathbf{N}^*} \int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx$ is reached by a finite integer j_0^*. The existence result follows.

From now on and for the sake of clarity, we will denote by (j_0,ω^*,a^*) instead of $(j_0,\omega_{j_0}^*,a_{j_0}^*)$ a solution of Problem $(\mathcal{P}_{L,r,M})$. We now prove that the solution a^* of Problem $(\mathcal{P}_{L,r,M})$ is *bang-bang*. Let us write the necessary first order optimality conditions of Problem $(\mathcal{P}_{L,r,M})$. To simplify the notations, the adjoint state introduced in Proposition 4 will be denoted by p (resp. p_i) instead of p_{j_0} (resp. $p_{j_0,i}$). For $0<\alpha<\beta<L$, introduce the sets

- $\mathcal{I}_{0,a^*}(\alpha,\beta)$: any element of the class of subsets of $[\alpha,\beta]$ in which $a^*(x)=0$ a.e.;
- $\mathcal{I}_{M,a^*}(\alpha,\beta)$: any element of the class of subsets of $[\alpha,\beta]$ in which $a^*(x)=M$ a.e.;
- $\mathcal{I}_{\star,a^*}(\alpha,\beta)$: any element of the class of subsets of $[\alpha,\beta]$ in which $0<a^*(x)<M$ a.e., that writes also

$$\mathcal{I}_{\star,a^*}(\alpha,\beta) = \bigcup_{k=1}^{+\infty} \left\{ x\in(\alpha,\beta) : \frac{1}{k} < a^*(x) < M - \frac{1}{k} \right\} =: \bigcup_{k=1}^{+\infty} \mathcal{I}_{\star,a^*,k}(\alpha,\beta).$$

We will prove that the set $\mathcal{I}_{\star,a^*,k}(0,L) = \bigcup_{i_0=1}^{j} \mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ has zero Lebesgue measure, for every nonzero integer k. Let us argue by contradiction, assuming that one of these sets $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ is of positive measure. Let $x_0 \in \mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ and let $(G_{k,n})_{n\in\mathbf{N}}$ be a sequence of measurable subsets with $G_{n,k}$ included in $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ and containing x_0, the perturbations a^*+th and a^*-th are admissible for t small enough. Choosing $h=\chi_{G_{k,n}}$ and letting t go to 0, it follows that

$$\langle dJ(a^*),h\rangle = \int_{x_j^{i_0-1}}^{x_j^{i_0}} h(x)\left(-e_{a_j^*,j}(x)p_{i_0}(x)\right)\,dx = 0 \Longleftrightarrow \int_{G_{k,n}} \left(-e_{a_j^*,j}(x)p_{i_0}(x)\right)\,dx = 0.$$

Dividing the last equality by $|G_{k,n}|$ and letting $G_{k,n}$ shrink to $\{x_0\}$ as $n\to+\infty$ shows that $e_{a_j^*,j}(x_0)p_{i_0}(x_0)=0$ for almost every $x_0 \in \mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$, according to the Lebesgue density Theorem. Since $e_{a_j^*,j}$ does not vanish on $(x_j^{i_0-1},x_j^{i_0})$ we then infer that $p_{i_0}(x)=0$ for almost every $x \in \mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$. Let us prove that such a situation cannot occur. The variational formulation of System (13)-(14) writes: find $p_{i_0} \in H_0^1(x_j^{i_0-1},x_j^{i_0})$ such that for every test function $\varphi \in H_0^1(x_j^{i_0-1},x_j^{i_0})$, there holds

$$-\int_{x_j^{i_0-1}}^{x_j^{i_0}} p_{i_0}(x)\varphi''(x) + (a_{i_0}(x) - \lambda_{a_{i_0},1}^2)p_{i_0}(x)\varphi(x)\,dx = \int_{x_j^{i_0-1}}^{x_j^{i_0}} (q_{i_0} - c_{i_0,j}^{\sim})e_{a_j^*,j}\varphi(x)\,dx.$$

Since $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1}, x_j^{i_0})$ is assumed to be of positive measure, let us choose test functions φ whose support is contained in $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1}, x_j^{i_0})$. There holds

$$\int_0^L (q_{i_0} - \tilde{c_{i_0,j}}) e_{a_j^*,j}(x)\varphi(x)\,dx = 0,$$

for such a choice of test functions, whence the contradiction. We then infer that $|\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1}, x_j^{i_0})| = 0$ and necessarily, a^* is bang-bang.

Step 2: counting the switching points of a^* and the number of connected components of ω^* (first part of the second point of Theorem 1). For the sake of simplicity, we first give the argument in the case where the infimum $m(L, M, r)$ is reached at $j_0 = 1$ and we will then generalize our analysis to any $j_0 \in \mathbb{N}^*$.

At this step, we know that a_1^* is *bang-bang* and ω_1^* is characterized in terms of the level set of the function $e_{a_1^*,1}(\cdot)^2$. According to (20), the number of switching points of a_1^* corresponds to the number of zeros of the function $x \mapsto p_1(x)e_{a_1^*,1}(x)$. Let us evaluate this number.

Since $M \leqslant \pi^2/L^2$, there holds $\|a_1^*\|_\infty \leqslant \frac{\pi^2}{L^2}$. As a consequence and using (1), one deduces that the eigenfunction $e_{a_1^*,1}$ is strictly concave and reaches its maximum at a unique point $x_{\max} \in (0, L)$. Moreover, since $e_{a_1^*,1}$ is increasing on $(0, x_{\max})$ and decreasing on $(x_{\max}, L)$ with $e_{a_1^*,1}(0) = e_{a_1^*,1}(\pi) = 0$, from Proposition 2, there exists $(\alpha, \beta) \in (0, L)^2$ such that $\alpha < \beta$ and

$$q^* := \chi_{\omega_1^*} = 1 - \chi_{(\alpha,\beta)}, \tag{21}$$

with $\alpha < x_{\max} < \beta$.

Let us provide a precise description of the function p_1. One readily check by differentiating two times the function $p_1/e_{a_1^*,1}$ that the function p_1 may be written as

$$p_1(x) = f(x)e_{a_1^*,1}(x) \tag{22}$$

for every $x \in (0, L)$, where the function f is defined by

$$f(x) = -\int_0^x g(t)dt + f(0), \qquad \text{with} \qquad f(0) = \int_0^L \left(\int_0^x g(t)dt\right) e_{a_1^*,1}^2(x)\,dx \tag{23}$$

and the function g is defined by

$$g(t) = \frac{\int_0^t (q^*(s) - \tilde{c}) e_{a_1^*,1}^2(s)\,ds}{e_{a_1^*,1}^2(t)}, \tag{24}$$

where here and in the rest of the proof, we will call by $\tilde{c}$ the number $\tilde{c}_{1,1}$ (whose definition was given in (15)). In the following result, we provide a description of the function g.

Lemma 6. *The function g defined by (24) verifies*

$$g(0) = g(L) = 0, \tag{25}$$

$$\text{there exists a unique real number } o_g \text{ in } (0, L) \text{ such that } g(o_g) = 0, \tag{26}$$

$$g > 0 \text{ in } (0, o_g) \text{ and } g < 0 \text{ in } (o_g, L), \tag{27}$$

$$g \text{ is decreasing on } (\alpha, \min(o_g, x_{max})) \text{ and } (\max(o_g, x_{max}), \beta). \tag{28}$$

Proof of Lemma 6. Let us first prove (26). We consider the function $\tilde{g}$ defined by

$$\tilde{g}(t) = \int_0^t (q^*(s) - \tilde{c}) e^2_{a_1^*,1}(s)\, ds, \tag{29}$$

so that the function g writes

$$g = \frac{\tilde{g}}{e_{a_1^*,1}(\cdot)^2}. \tag{30}$$

According to (21) and remarking that $0 < \tilde{c} < 1$ according to (15), the function $\tilde{g}$ is strictly increasing on $(0, \alpha)$ and (β, L), and strictly decreasing on (α, β). Besides, using (15), there holds

$$\tilde{g}(0) = \tilde{g}(L) = 0. \tag{31}$$

Hence, using the variations of $\tilde{g}$ given before and (31) that $\tilde{g}$ (and hence g) has a unique zero on $(0, L)$ that we call o_g from now on. Moreover, clearly $\tilde{g} > 0$ on $(0, o_g)$ and $\tilde{g} < 0$ on (o_g, L), hence, using (30), we deduce the same property for g and (27) is proved.

Equality (25) is readily obtained by making a Taylor expansion of $e^2_{a_1^*,1}$ and $\tilde{g}$ around 0 and L and using (30). Indeed, there holds

$$e^2_{a_1^*,1}(x) \sim x^2 {e^2_{a_1^*,1}}'(0) \quad \text{and} \quad \tilde{g}(x) \sim (q^*(0) - \tilde{c}) x^3 {e^2_{a_1^*,1}}'(0)/3 \qquad \text{as } x \to 0,$$
$$e^2_{a_1^*,1}(x) \sim \frac{(x-\pi)^2}{2} {e^2_{a_1^*,1}}'(\pi) \quad \text{and} \quad \tilde{g}(x) \sim (q^*(\pi) - \tilde{c}) x^3 {e^2_{a_1^*,1}}'(\pi)/3 \qquad \text{as } x \to \pi.$$

To conclude, it remains to prove (28). From (24), one observes that g is differentiable almost everywhere on $(0, L)$ and

$$g'(x) = q^*(x) - \tilde{c} - \frac{2 e'_{a_1^*,1}(x) g(x)}{e_{a_1^*,1}(x)}, \tag{32}$$

for almost every $x \in (0, L)$. Using the variations of $e_{a_1^*,1}$, (21) and (32), we infer that g' is negative almost everywhere on $(\alpha, \min(o_g, x_{\max})) \cup (\max(o_g, x_{\max}), \beta)$, from which we deduce (28).

The proof of Lemma 6 is then complete. ∎

As a consequence of (27) and (23), f is strictly decreasing on $(0, o_g)$ and strictly increasing on (o_g, L) where o_g is defined in (26). We conclude that f has at most two zeros in $(0, L)$. Moreover, thanks to (14) and (22), f has at least one zero in $(0, L)$. Since $e_{a_1^*,1}(\cdot)$ does not vanish in $(0, L)$, one infers that a_1^* has at least 1 and at most 2 switching points.

To generalize our argument to any order $j \geqslant 2$, notice that using the notations of Proposition 4 and its proof, one has $(-1)^{i_0+1} e^*_{a_{j,i_0},1}(x) > 0$ for all $x \in (x_j^{i_0-1}, x_j^{i_0})$ with $i_0 \in \{1 \cdots j\}$. Then, mimicking the argument above in the particular case where $j_0 = 1$, one shows that the function a_j^* has at most two switching points in $(x_j^{i_0-1}, x_j^{i_0})$ and at least one. Besides, since the nodal points $\{x_j^i\}_{i \in \{1, \cdots j-1\}}$ can also be switching points, we conclude that the function a_j^* has at most $3j_0 - 1$ and at least j_0 switching points in $(0, L)$.

Step 3: proof of the estimate (6) (last part of the second point of Theorem 1). Let us first show the easier inequality, in other word the right one. It suffices to write that $m(L, M, r)$ is bounded from above by $\inf_{\omega \in \Omega_r(0,L)} \inf_{j \in \mathbb{N}^*} J(0, \omega)$. Inverting the two infima and using Lemma 5 leads immediately to the desired estimate.

The left inequality appears more intricate to establish. It rests upon the difficult result stated in Proposition 1, whose proof is quite long and technical but the method is elementary and interesting. For this reason, we will temporarily admit this result and postpone its proof to Section 2.3.

Let us remark that $\underline{m}_2 = \frac{7}{8}\sqrt{3}(3-2\sqrt{2})$. Then, it just remains to prove that $\underline{m}_j \geqslant \underline{m}_2$ for every $j \in \mathbb{N}\backslash\{0,1\}$. For that purpose, introduce the function F defined on $[2,+\infty)$ by

$$F(x) = \frac{(2x^2-1)(x^2-1)^{3/2}\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right)^2}{x^3\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)^2}.$$

Notice that $F(j) = 3\underline{m}_j$ for every $j \in \mathbb{N}^*$. Let us write $F(x) = u(x)v(x)$ with

$$u(x) = \frac{(2x^2-1)\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right)^2}{\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)^2} \quad \text{and} \quad v(x) = \frac{(x^2-1)^{3/2}}{x^3}.$$

Let us show that $u(x) \geqslant u(2)$ for every $x \geqslant 2$. This comes to show that $\psi(x) \leqslant 0$, where

$$\psi(x) = \gamma\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right) - \sqrt{2x^2-1}\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right),$$

with $\gamma = \sqrt{u(2)} = \sqrt{7}(\sqrt{2}-1)$, for every $x \geqslant 2$. The derivative of ψ writes

$$\psi'(x) = \gamma\left(\frac{x^2}{x^2-2}\right)^{x/2} w(x) - \frac{2x}{\sqrt{2x^2-1}}\left(\sqrt{\frac{x^2}{x^2-2}}-1\right) - \frac{\sqrt{2x^2-1}}{(x^2-2)^{3/2}},$$

with $w(x) = \ln\left(\sqrt{\frac{x^2}{x^2-2}}\right) - \frac{1}{x^2-2}$. The derivative of w writes

$$w'(x) = \frac{4}{x(x^2-2)^2}$$

and therefore, the function w is increasing and negative since $\lim_{x\to+\infty} w(x) = 0$. One then infers that $\psi'(x)$ writes as the sum of three negative terms and is thus negative on $[2,+\infty)$. Hence, the function ψ decreases on this interval and therefore, $\psi(x) \leqslant \psi(2) = 0$ for every $x \in [2,+\infty)$. The expected result on u follows.

Let us now show that $v(x) \geqslant v(2)$ for every $x \in [2,+\infty)$. The derivative of v writes

$$v'(x) = \frac{3\sqrt{x^2-1}}{x^4},$$

is therefore positive on $[2,+\infty)$, and the expected conclusion follows.

2.3 Proof of Proposition 1

The proof is quite technical. As previously, we will first consider the case where $j=1$, in other words we will provide a lower estimate of the quantity $m_1(L,r)$ by using a convexity argument. We will then deduce the general result holding for any $j \in \mathbb{N}^*$ by using that the j-th eigenfunction $e_{a,j}$ of A_a coincides with the first eigenfunction of the restriction of A_a on each nodal domain.

Notice that, proving that the estimate (7) holds for every $M \in (0, \pi^2/L^2]$ is equivalent to showing it for the particular value

$$M = \pi^2/L^2,$$

which will be assumed from now on. Let a be a generic element of $\mathcal{A}_M(0,L)$.

First step: proof of Proposition 1 in the case "$j = 1$". We will need the following result, that yields an estimate of the maximum of $e_{a,1}$ on $(0, L)$ in terms of the L^2 norm of $e_{a,1}$ and the derivatives of $e_{a,1}$ at $x = 0$ and $x = L$.

Lemma 7. *With the assumptions of Proposition 1, there holds*

$$e_{a,1}^2(x_{max}) \geqslant \max\left\{ \frac{3}{2L}\int_0^L e_{a,1}^2(x)\,dx, \frac{L^2}{2\pi^2}\max\{e'_{a,1}(0)^2, e'_{a,1}(L)^2\}\right\}. \tag{33}$$

Proof of Lemma 7. Since $e'_{a,1}(x_{max}) = 0$, multiplying Equation (1) by e'_1 and integrating on (x, x_{max}) (with possibly $x > x_{max}$) leads to

$$e'_{a,1}(x)^2 = \int_x^{x_{max}} (\lambda_{a,j}^2 - a(x))\frac{d}{dx}(e_{a,1}^2(x))\,dx \tag{34}$$

for every $x \in (0, L)$. Besides, according to the Courant-Fischer minimax principle, one has

$$0 \leqslant \lambda_{a,1}^2 - a(\cdot) \leqslant \frac{2\pi^2}{L^2} \qquad \text{in } (0, L). \tag{35}$$

Therefore, combining (34) and (35) yields

$$e'_{a,1}(x)^2 \leqslant \frac{2\pi^2}{L^2}(e_{a,1}^2(x_{max}) - e_{a,1}^2(x)), \tag{36}$$

for every $x \in (0, L)$. In particular, applying (36) at $x = L$ and $x = 0$, we obtain

$$\max\{e'_{a,1}(L)^2, e'_{a,1}(0)^2\} \leqslant \frac{2\pi^2}{L^2}e_{a,1}^2(x_{max}), \tag{37}$$

Moreover, by integrating (36) between 0 and L, we obtain

$$\int_0^L e'_{a,1}(x)^2\,dx + \frac{2\pi^2}{L^2}\int_0^L e_{a,1}^2(x)\,dx \leqslant \frac{2\pi^2}{L^2}e_{a,1}^2(x_{max})L. \tag{38}$$

We obtain (33) from (37) and by combining (38) with Poincaré's inequality

$$\int_0^L e_{a,1}(x)^2\,dx \leqslant \frac{L^2}{\pi^2}\int_0^L e'_{a,1}(x)^2\,dx.$$

■

According to (33), and assuming since the eigenfunction $e_{a,1}$ is normalized in $L^2(0, L)$, there holds

$$e_{a,1}^2(x_{max}) \geqslant \frac{3}{2L}. \tag{39}$$

Since $e_{a,1}$ is concave and according to (39), one has the successive inequalities

$$e_{a,1}(x) \geqslant Tr_{a,1}(x) \geqslant \triangle_1(x), \tag{40}$$

for every $x \in [0, L]$, where $Tr_{a,1}$ and $\triangle_1$ denote the piecewise affine functions defined by

$$Tr_{a,1}(x) = \begin{cases} \frac{e_{a,j}(x_{max})x}{x_{max}} & \text{on } (0, x_{max}) \\ \frac{e_{a,j}(x_{max})(L-x)}{L-x_{max}} & \text{on } (x_{max}, L) \end{cases} \quad \text{and} \quad \triangle_1(x) = \begin{cases} \frac{\sqrt{3}x}{\sqrt{2L}x_{max}} & \text{on } (0, x_{max}) \\ \frac{\sqrt{3}(L-x)}{\sqrt{2L}(L-x_{max})} & \text{on } (x_{max}, L). \end{cases}$$

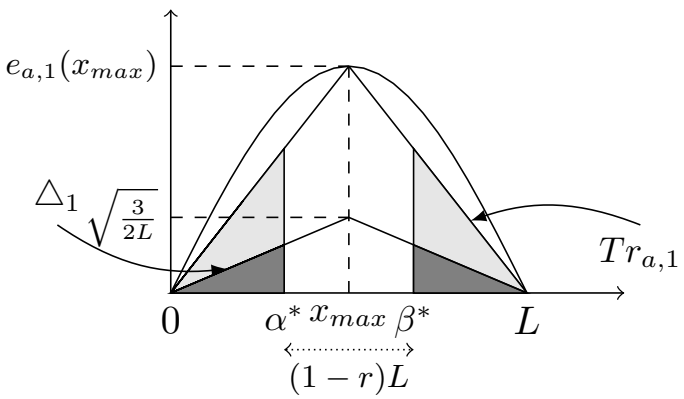

Figure 2: Graphs of the functions $e_{a,1}$, $Tr_{a,1}$ and $\triangle_1$.

Combining (39) with (40) and according to Proposition 2, we readily obtain

$$\inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,1}(x)^2\,dx \geqslant \inf_{\omega\in\Omega_r(0,L)} \int_\omega \triangle_1(x)^2 dx = \int_{\widehat{\omega}} \triangle_1(x)^2 dx, \tag{41}$$

with $\widehat{\omega} = (0,\alpha^*)\cup(\beta^*,L)$ verifying

$$\triangle_1(\alpha^*) = \triangle_1(\beta^*) \qquad \text{and} \qquad |\widehat{\omega}| = L - \beta^* + \alpha^* = rL.$$

It follows that $\alpha^* = rx_{max}$, $\beta^* = (1-r)L + rx_{max}$ and one obtains

$$\int_{\widehat{\omega}} \triangle_1(x)^2\,dx = \int_0^{rx_{max}} \triangle_1(x)^2\,dx + \int_{(1-r)L+rx_{max}}^{L} \triangle_1(x)^2\,dx = \frac{r^3}{2} = r^3\underline{m}_1. \tag{42}$$

We have then proved Proposition 1 in the case where $j=1$.

Second step: proof of Proposition 1 in the general case. In what follows, we assume that $j \geqslant 2$.
Let $0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j$ be the $j+1$ zeros of the j-th eigenfunction $e_{a,j}$. Introduce $\gamma = (\gamma_1,\cdots,\gamma_{j-1}) \in (0,1)^j$ such that

$$\gamma_i = \int_{x_j^i}^{x_j^{i+1}} e_{a,j}^2(x)\,dx, \qquad i = 1,\ldots,j-1. \tag{43}$$

Note that, because of the normalization condition on the function $e_{a,j}$, there holds

$$\sum_{i=0}^{j-1} \gamma_i = 1. \tag{44}$$

In the sequel, we will use the following notations, for all $i \in \{0,\cdots,j-1\}$,

$$\Omega_i = (x_j^i, x_j^{i+1}), \qquad \eta_i = \frac{|\Omega_i|}{L} \in (0,1), \qquad \text{and} \qquad e_{a,j}(x_{max}^i) = \max_{x\in\Omega i} e_{a,j}(x). \tag{45}$$

In the sequel, we will distinguish between several cases, depending on the value of the first integer $i_0 \in \{0, \cdots, j-1\}$ (that exists thanks to (44)) such that

$$\gamma_{i_0} \geqslant \frac{1}{j}. \tag{46}$$

First case: assume that $i_0 = 1$. Since the function $e_{a,j}(\cdot)$ is concave, we claim that

$$\int_\omega e_{a,j}(x)^2\, dx \geqslant \int_\omega Tr_{a,j}(x)^2 dx, \tag{47}$$

for every $\omega \in \Omega_r(0,L)$ where the function $Tr_{a,j}$ is piecewise affine, defined on each interval (x_j^i, x_j^{i+1}), with $i \in \{1, \ldots, j-1\}$, by

$$Tr_{a,j}(x) = \begin{cases} \frac{x-x_j^i}{x_{max}^i - x_j^i} e_{a,j}(x_{max}^i) & \text{on } (x_j^i, x_{max}^i), \\ \frac{x_j^{i+1}-x}{x_j^{i+1}-x_{max}^i} e_{a,j}(x_{max}^i) & \text{on } (x_{max}^i, x_j^{i+1}), \end{cases}$$

for every $x \in (x_j^i, x_j^{i+1})$.

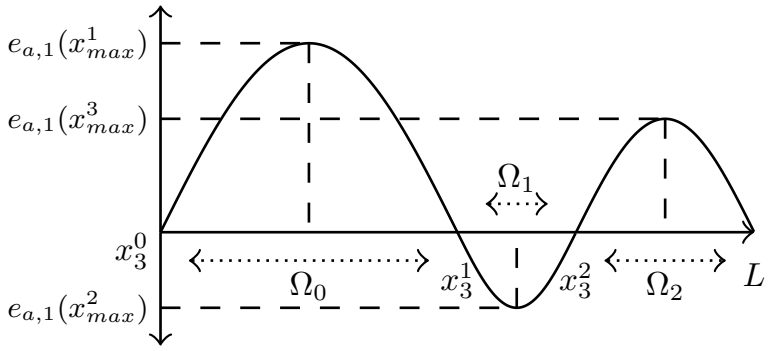

Figure 3: Illustration of the case "$i_0 = 1$" with 3 nodal domains ($j = 3$).

Since the j-th eigenfunction $e_{a,j}$ coincides with the first eigenfunction of $-\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet conditions on (x_j^i, x_j^{i+1}), the method of the first step can be adapted. By reproducing the proof of Lemma 7 to show (33), we obtain

$$e_{a,j}(x_{max}^i)^2 \geqslant \max\left\{ \frac{\frac{2\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}}{\left(\frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}\right)|\Omega_i|} \int_{x_j^i}^{x_j^{i+1}} e_{a,j}(x)^2\, dx, \frac{1}{\frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}} \max\{e'_{a,j}(x_j^i)^2, e'_{a,j}(x_j^{i+1})^2\} \right\}. \tag{48}$$

One derives from the equivalent of (36) in this case the estimates

$$\frac{\pi^2}{|\Omega_i|^2} - \frac{pi^2}{L^2} \leqslant \frac{e'_{a,j}(x_j^i)^2}{e_{a,j}(x_{max}^i)^2} \leqslant \frac{\pi^2}{|\Omega_i|^2} + \frac{pi^2}{L^2} \quad \text{and} \quad \frac{\pi^2}{|\Omega_i|^2} - \frac{pi^2}{L^2} \leqslant \frac{e'_{a,j}(x_j^{i+1})^2}{e_{a,j}(x_{max}^i)^2} \leqslant \frac{\pi^2}{|\Omega_i|^2} + \frac{pi^2}{L^2}. \tag{49}$$

Applying (48) for $i = 0$ and using (46) yields

$$e_{a,j}(x_{max}^0)^2 \geqslant A_0 \quad \text{with} \quad A_0 = \frac{\frac{2\pi^2}{|\Omega_0|^2} + \frac{\pi^2}{L^2}}{j\left(\frac{\pi^2}{|\Omega_0|^2} + \frac{\pi^2}{L^2}\right)|\Omega_0|} = \frac{1}{Lj}\left(\frac{2+\eta_0^2}{\eta_0(1+\eta_0^2)}\right). \tag{50}$$

Let us now provide an estimate of $e_{a,1}(x^1_{max})^2$. Combining the inequalities (48) with $i=1$ and (49) with $i=0$, we get

$$e_{a,1}(x^1_{max})^2 \geqslant \frac{e'_{a,j}(x^1_j)^2}{\frac{\pi^2}{|\Omega_1|^2}+\frac{\pi^2}{L^2}} \geqslant \frac{(\frac{\pi^2}{|\Omega_0|^2}-\frac{\pi^2}{L^2})e_{a,1}(x^0_{max})^2}{\frac{\pi^2}{|\Omega_1|^2}+\frac{\pi^2}{L^2}}. \tag{51}$$

Combining (50) and (51) yields

$$e_{a,j}(x^1_{max})^2 \geqslant A_1 \quad \text{with} \quad A_1 = \frac{(L^2-|\Omega_0|^2)|\Omega_1|^2}{(L^2+|\Omega_1|^2)|\Omega_0|^2}A_0 = \frac{(1-\eta_0^2)\eta_1^2}{(1+\eta_1^2)\eta_0^2}A_0.$$

By induction, it follows that

$$e_{a,j}(x^i_{max})^2 \geqslant A_i \quad \text{with} \quad A_i = \left(\prod_{k=1}^{i}\frac{(1-\eta_{k-1}^2)\eta_k^2}{(1+\eta_k^2)\eta_{k-1}^2}\right)A_0, \tag{52}$$

for every $i \in \{1,\cdots,j-1\}$. Hence, (52) together with (47) allows us to write

$$\int_\omega e_{a,j}(x)^2\,dx \geqslant \int_\omega \triangle_j(x)^2\,dx,$$

where $\triangle_j$ is the piecewise affine function defined on $(0,L)$ by

$$\triangle_j(x) = \begin{cases} \frac{(x-x^i_j)}{(x^i_{max}-x^i_j)}\sqrt{A_i} & \text{on } (x^i_j, x^i_{max}), \\ \frac{(x^{i+1}_j-x)}{(x^{i+1}_j-x^i_{max})}\sqrt{A_i} & \text{on } (x^i_{max}, x^{i+1}_j), \end{cases}$$

for every $i \in \{0,\cdots,j-1\}$ and $x \in (x^i_j, x^{i+1}_j)$.

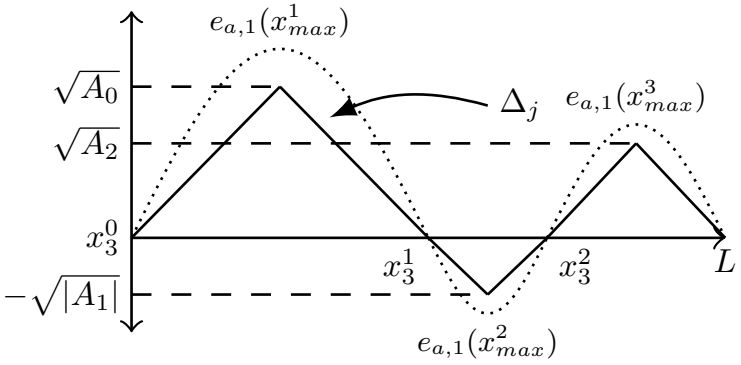

Figure 4: Graphs of the functions $e_{a,1}$ and $\triangle_j$.

According to Proposition 2, we obtain

$$\inf_{\omega\in\Omega_r(0,L)}\int_\omega e_{a,j}(x)^2\,dx \geqslant \inf_{\omega\in\Omega_r(0,L)}\int_\omega \triangle_j(x)^2\,dx = \int_{\hat\omega}\triangle_j(x)^2\,dx, \tag{53}$$

where $\hat\omega = \{\triangle_j(x)^2 < \tau\}$ up to a set of zero Lebesgue measure and $|\hat\omega| = rL$. Let $i \in \{0,\cdots,j-1\}$ and let us introduce $\omega_i = \hat\omega \cap (x^i_j, x^{i+1}_j)$. There exist α_i^*, β_i^* and $r_i \in (0,1)$ such that $\omega_i =$

$(x_j^i, \alpha_i^*) \cup (\beta_i^*, x_j^{i+1})$, $|\omega_i| = r_i(x_j^{i+1} - x_j^i)$, and therefore

$$\sum_{i=0}^{j-1} r_i(x_j^{i+1} - x_j^i) = rL. \tag{54}$$

By definition of $\hat{\omega}$, one has $\triangle_j(\alpha_i^*) = \triangle_j(\beta_i^*) = \triangle_j(\alpha_{i+1}^*)$, consequently there holds

$$\alpha_i^* = r_i(x_{max}^i - x_j^i) + x_j^i \quad , \quad \beta_i^* = x_j^{i+1} - r_i(x_j^{i+1} - x_{max}^i), \tag{55}$$

and

$$r_{i+1}^2 A_{i+1} = r_i^2 A_i = \cdots = r_0^2 A_0. \tag{56}$$

As a result, one obtains

$$\int_{\omega^*} \triangle_j(x)^2 \, dx = \sum_{i=0}^{j-1} \int_{\omega_i} \triangle_j(x)^2 \, dx = \frac{1}{3} \sum_{i=0}^{j-1} r_i^3 |\Omega_i| A_i. \tag{57}$$

To compute the numbers r_i, we use (56) together with (52), which yields to

$$r_i = \sqrt{\frac{A_0}{A_i}} r_0 = \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}} \, r_0. \tag{58}$$

Since $\sum_{i=0}^{j-1} r_i \frac{|\Omega_i|}{L} = \sum_{i=0}^{j-1} r_i \eta_i = r$, one infers

$$\begin{aligned} r_0 &= \frac{r}{\eta_0 + \sum_{i=1}^{j-1} \eta_i \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}} \\ &= \frac{r}{\eta_0 \left(1 + \sum_{i=1}^{j-1} \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)}{(1-\eta_{k-1}^2)}}\right)}. \end{aligned} \tag{59}$$

Besides, using (58), there holds

$$\sum_{i=0}^{j-1} r_i^3 |\Omega_i| A_i = r_0^2 A_0 \sum_{i=0}^{j-1} r_i |\Omega_i| = r_0^2 A_0 r L. \tag{60}$$

We conclude by combining (60) with (57) that

$$\int_{\hat{\omega}} \triangle_j(x)^2 \, dx = \frac{1}{3} A_0 r_0^2 r L, \tag{61}$$

where A_0 and r_0 are respectively given by (50) and (59).

Since our goal is to estimate $\int_{\hat{\omega}} \triangle_j(x)^2 \, dx$ from below, regarding (61), we need to find a lower bound on r_0 and consequently on the numbers $|\Omega_i|$ according to (60). We will use the following Lemma.

Lemma 8. *For every $i \in \{0, \cdots, j-1\}$, there holds*

$$\frac{L}{\sqrt{j^2+1}} \leqslant |\Omega_i| \leqslant \frac{L}{\sqrt{j^2-1}}. \tag{62}$$

Proof of Lemma 8. According to the Courant-Fischer minimax principle, one has

$$\frac{j\pi}{L} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{j\pi}{L}\right)^2 + \frac{\pi^2}{L^2}}.$$

Since the j-th eigenfunction $e_{a,j}$ is also the first eigenfunction of the operator $-\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions on Ω_i, we also have

$$\frac{\pi}{|\Omega_i|} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{\pi}{|\Omega_i|}\right)^2 + \frac{\pi^2}{L^2}}.$$

We then infer

$$\frac{\pi}{\sqrt{\left(\frac{j\pi}{L}\right)^2 + \frac{\pi^2}{L^2}}} \leqslant |\Omega_i| \leqslant \frac{\pi}{\sqrt{\left(\frac{j\pi}{L}\right)^2 - \frac{\pi^2}{L^2}}}.$$

■

It follows from Lemma 8 that $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ and therefore

$$\sum_{i=1}^{j-1} \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)}{(1-\eta_{k-1}^2)}} \leqslant g_1(j),$$

where

$$g_1(j) = \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}} = \frac{\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}{\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}} - 1}.$$

According to (59), one has

$$r_0 \geqslant \frac{r}{\eta_0(1+g_1(j))}. \tag{63}$$

Combining (50), (61) and (63), we obtain

$$\int_{\hat{\omega}} \triangle_j(x)^2\,dx \geqslant r^3 \inf_{\eta_0 \in \left(\frac{1}{\sqrt{j^2+1}}, \frac{1}{\sqrt{j^2-1}}\right)} g_2(\eta_0, j), \tag{64}$$

with

$$g_2(\eta_0, j) = \frac{1}{3j}\left(\frac{2+\eta_0^2}{\eta_0(1+\eta_0^2)}\right)\left(\frac{1}{\eta_0 + \eta_0 g_1(j)}\right)^2.$$

Since for every $\eta_0 > 0$, we have

$$\frac{\partial g_2}{\partial \eta_0}(\eta_0, j) = -\frac{\left(\sqrt{\frac{j^2}{j^2-1}} - 1\right)^2 \left(11\eta_0^2 + 3\eta_0^4 + 6\right)}{\left((\frac{j^2}{j^2-1})^{\frac{j}{2}} - 1\right)^2 \eta_0^4 (1+\eta_0^2)^2 j} \leqslant 0,$$

the function $\eta_0 \mapsto g_2(\eta_0, j)$ is decreasing, so that (64) becomes

$$\int_{\hat{\omega}} \triangle_{a,j}(x)^2\,dx \geqslant r^3 g_2\left(\sqrt{\frac{1}{j^2-1}}, j\right) = r^3 \underline{w}_j, \tag{65}$$

and the expected result is proved.

Second case: assume now that $i_0 = 2$. We will prove that the estimate choosing $i_0 = 1$ is worst than the estimate that we obtain with $i_0 = 2$. Using (48) with $i = 1$, we have

$$e_{a,j}^2(x_{max}^1) \geqslant A_1 \quad \text{with} \quad A_1 = \frac{1}{Lj}\left(\frac{2+\eta_1^2}{\eta_1(1+\eta_1^2)}\right). \tag{66}$$

Combining the inequality (48) with $i = 0$, (49) with $i = 1$ and (66) we get

$$A_0 = \frac{(1-\eta_1^2)\eta_0^2}{(1+\eta_0^2)\eta_1^2} A_1. \tag{67}$$

Using (48) with $i = 2$, (49) with $i = 1$ and (66) we have

$$e_{a,j}^2(x_{max}^2) \geqslant A_2 \quad \text{with} \quad A_2 = \frac{(1-\eta_1^2)\eta_2^2}{(1+\eta_2^2)\eta_1^2} A_1.$$

By induction, for every $i \in \{2, \cdots, j-1\}$ we have

$$e_{a,j}^2(x_{max}^i) \geqslant A_i \quad \text{with} \quad A_i = \left(\prod_{k=2}^{i} \frac{(1-\eta_{k-1}^2)\eta_k^2}{(1+\eta_k^2)\eta_{k-1}^2}\right) A_1 \tag{68}$$

Since $r_i^2 A_i = r_1^2 A_1$, $\sum_{i=0}^{j-1} r_i \eta_i = r$, one computes by using (68) and (67)

$$\begin{aligned} r_1 &= \frac{r}{\eta_1 + \eta_0\sqrt{\frac{A_1}{A_0}} + \sum_{i=2}^{j-1} \eta_i \sqrt{\frac{A_1}{A_i}}} \\ &= \frac{r}{\eta_1 + \eta_0\sqrt{\frac{1+\eta_0^2}{1-\eta_1^2}} + \sum_{i=2}^{j-1} \eta_i \sqrt{\prod_{k=2}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}}. \end{aligned}$$

Moreover,

$$\sum_{i=2}^{j-1} \eta_i \sqrt{\prod_{k=2}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}} = \eta_1 \sum_{i=2}^{j-1} \sqrt{\prod_{k=2}^{i} \frac{1+\eta_k^2}{1-\eta_{k-1}^2}}$$

and since $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ according to Lemma 8, there holds

$$r_1 \geqslant \frac{r}{\eta_1 + \eta_1\sqrt{\frac{j^2}{j^2-2}} + \eta_1 \sum_{i=2}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i-1}{2}}}.$$

Since $j \geqslant 2$, we have

$$\sqrt{\frac{j^2}{j^2-2}} + \sum_{i=2}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i-1}{2}} - \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}} = \sqrt{\frac{j^2}{j^2-2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{j-1}{2}} \leqslant 0,$$

and it follows that

$$r_1 \geqslant \frac{r}{\eta_1 + \eta_1 \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}}}.$$

As a consequence, using the same approach as the one used for the case where $i_0 = 1$, we infer that

$$\inf_{\omega\in\Omega_r(0,L)} \int_\omega \triangle_j(x)^2\,dx \geqslant r^3 \inf_{\eta_1\in\left(\frac{1}{\sqrt{j^2+1}},\frac{1}{\sqrt{j^2-1}}\right)} g_2(\eta_1,j)$$

with

$$g_2(\eta_1,j) = \frac{1}{3j}\left(\frac{2+\eta_1^2}{\eta_1(1+\eta_1^2)}\right)\left(\frac{1}{\eta_1+\eta_1 g_1(j)}\right)^2 \quad \text{and} \quad g_1(j) = \frac{\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}{\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}} - 1}.$$

One gets the same conclusion as in the first case.

Finally, mimicking this proof and adapting it for every $j_0 \in \{3,\cdots,j-1\}$, we prove that the estimate with $i_0 = 1$ is the worst one. We then obtain the same conclusion. ■

2.4 Proof of Theorem 2

We argue by contradiction, assuming that the optimal design problem ($\mathcal{P}_{j,L,r,\infty}$) has a solution $a^* \in L^\infty(0,L)$. Then, there exists M_0 such that a^* is a solution of the problem

$$\inf_{a\in\mathcal{A}_\infty(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx = \inf_{a\in\mathcal{A}_{M_0}(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx.$$

We will use the notations of Proposition 4 and Section 2.2.

The contradiction will be obtained by constructing a perturbation a_n^* of a^* such that

$$J(a_n^*) < J(a^*).$$

According to Proposition 4, a^* is non-trivial and *bang-bang*, equal to 0 and M_0 almost everywhere in $(0,L)$ so that there exists $i_0 \in \{1,\cdots,j\}$ such that the set $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})$ is measurable of positive measure.

Thanks to the regularity of the Lebesgue measure, there exists an increasing sequence of compact sets $(K_n)_{n\in\mathbf{N}}$ strictly included in $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})$ satisfying

$$\lim_{n\to\infty} |K_n| = |\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})|,$$

where $|\cdot|$ denotes the Lebesgue measure. In what follows, we will use the notation I^c to denote the complement of any set $I \subset [0,L]$ in $[0,L]$. We introduce

$$a_n^*(x) = \begin{cases} M_0 + \frac{1}{\varphi(n)} & \text{on } K_n, \\ 0 & \text{on } K_n^c, \end{cases} \quad \text{with} \quad \varphi(n) = |\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0}) \cap K_n^c|.$$

Let us remark that

$$a_n^*(x) - a(x) = \begin{cases} \frac{1}{\varphi(n)} & \text{on } K_n, \\ -M_0 & \text{on } \mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0}) \cap K_n^c, \\ 0 & \text{on } \mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})^c. \end{cases}$$

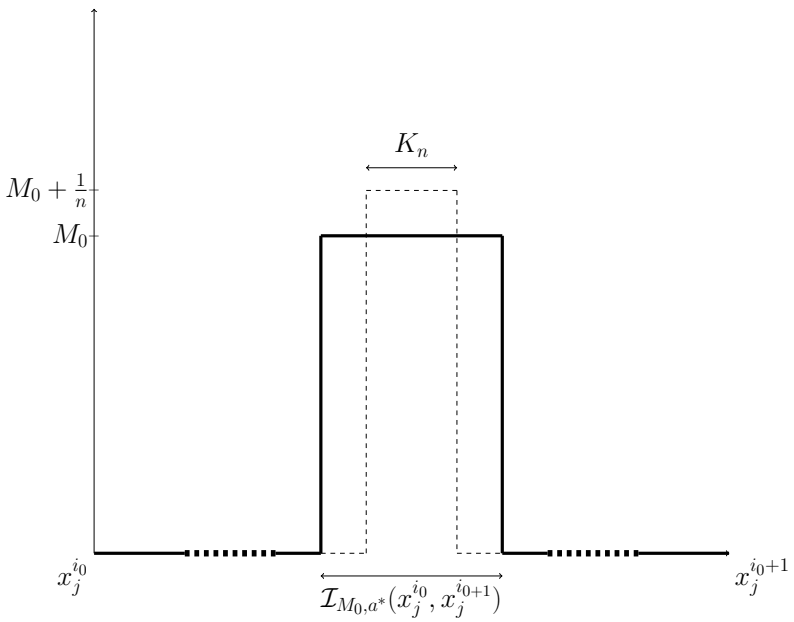

Figure 5: perturbation $a_n^*(\dots)$ of $a^*(-)$.

Hence, we get

$$\begin{aligned}\langle dJ(a), \varphi(n)(a_n^* - a^*)\rangle &= -2\int_{x_j^{i_0-1}}^{x_j^{i_0}} \varphi(n)(a_n^*(x) - a^*(x)) e_{a_{i_0},1}(x) p_{i_0}(x)\, dx \\ &= 2M_0\varphi(n) \int_{\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c} e_{a_{i_0},1}(x) p_{i_0}(x)\, dx \\ &\quad - \int_{K_n} e_{a_{i_0},1}(x) p_{i_0}(x)\, dx,\end{aligned}$$

for $n \in \mathbb{N}$. Using (12), we have $e_{a_{i_0},1} p_{i_0} \geqslant 0$ on $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c$ and $e_{a_{i_0},1} p_{i_0} > 0$ on K_n for all $n \in \mathbb{N}$. Since $\lim_{n\to+\infty} \varphi(n) = 0$ and according to the Lebesgue density theorem,

$$\lim_{n\to+\infty} 2M_0\varphi(n) \int_{\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c} e_{a_{i_0},1}(x) p_{i_0}(x)\, dx = 0.$$

As a consequence, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\langle dJ(a), \varphi(n)(a_n^* - a^*)\rangle < 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ verifying $J(a_{n_1}^*) < J(a^*)$, whence the contradiction. We then infer that the optimal design problem ($\mathcal{P}_{j,L,r,\infty}$) has no solution.

3 Applications and numerical investigations

3.1 Controllability issues for the wave equation

3.1.1 The cost of the control in large time

Let us fix $T > 0$ and consider the one dimensional wave equation with potential

$$\begin{aligned} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0, \qquad & (t,x) \in (0,T)\times(0,L), \\ \varphi(t,0) = \varphi(t,L) = 0, \qquad & t \in [0,T], \\ (\varphi(0,x), \partial_t\varphi(0,x)) = (\varphi_0(x), \varphi_1(x)), \qquad & x \in [0,L], \end{aligned} \tag{69}$$

where the potential $a(\cdot)$ is a nonnegative function belonging to $L^\infty(0,L)$. It is well known that for every initial data $(\varphi_0,\varphi_1) \in H_0^1(0,L)\times L^2(0,L)$, there exists a unique solution φ in $C^0(0,T;H_0^1(0,L)) \cap C^1(0,T;L^2(0,L))$ of the Cauchy problem (69).

Let ω be a given measurable subset of $(0,L)$ of positive Lebesgue measure. The equation (69) is said to be *observable* on ω in time T if there exists a positive constant C such that

$$C\int_0^L \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right)\,dx \leqslant \int_0^T\int_\omega \partial_t\varphi(t,x)^2\,dxdt \tag{70}$$

for all $(\varphi_0,\varphi_1) \in H_0^1(0,L)\times L^2(0,L)$. We denote by $C_{T,\mathrm{obs}}(a,\omega)$ the largest constant in the previous inequality, that is

$$C_{T,\mathrm{obs}}(a,\omega) = \inf_{\substack{(\varphi_0,\varphi_1)\in H_0^1(0,L)\times L^2(0,L)\\(\varphi_0,\varphi_1)\neq(0,0)}} \frac{\int_0^T\int_\omega \partial_t\varphi(t,x)^2\,dxdt}{\int_0^L \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right)\,dx}. \tag{71}$$

This constant can be interpreted a quantitative measure of the well-posed character of the inverse problem of reconstructing the solutions from measurements over $[0,T]\times\omega$. Moreover, this constant also plays a crucial role in the frameworks of control theory. Indeed, consider the internally controlled wave equation on $(0,L)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_{tt}y(t,x) - \partial_{xx}y(t,x) + a(x)y(t,x) = h_{a,\omega}(t,x), \qquad & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = y(t,\pi) = 0, \qquad & t \in [0,T], \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), \qquad & x \in (0,L), \end{cases} \tag{72}$$

where $h_{a,\omega}$ is a control supported by $[0,T]\times\omega$ and ω is a Lebesgue measurable subset of $(0,L)$. Recall that for every initial data $(y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)$ and every $h_{a,\omega} \in L^2((0,T)\times(0,L))$, the problem (72) has a unique solution y verifying moreover $y \in C^0(0,T;L^2(0,L)) \cap C^1(0,T;H^{-1}(0,L))$. This problem is said to be *null controllable at time* T if and only if for every initial data $(y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)$, one can find a control $h_{a,\omega} \in L^2((0,T)\times(0,L))$ supported by $[0,T]\times\omega$ such that the solution y of (72) verifies $y(T,\cdot) = \partial_t y(T,\cdot) = 0$.

Let us assume that (72) is null controllable. At fixed $(y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)$, since the set of all controls $h_{a,\omega}$ steering (y^0,y^1) to $(0,0)$ is a closed vector space of $L^2((0,T)\times(0,L))$, there exists a unique control of minimal $L^2((0,T)\times\omega)-$norm that we denote $h_{a,\omega}^{opt}$, which can be constructed "explicitly" as the minimum of a functional according to the Hilbert Uniqueness Method (HUM, see [23]). Thus, we can define the HUM operator $\Gamma_{a,\omega}^T$ by

$$\begin{array}{rccc} \Gamma_{a,\omega}^T : & H_0^1(0,L)\times L^2(0,L) & \longrightarrow & L^2((0,T)\times(0,L)) \\ & (y^0,y^1) & \longmapsto & h_{a,\omega}^{opt}. \end{array}$$

$\Gamma^T_{a,\omega}$ is linear and continuous and we define its norm

$$\|\Gamma^T_{a,\omega}\| = \sup\left\{ \frac{\|h_{a,\omega}\|_{L^2((0,T)\times(0,L)}}{\|(y^0,y^1)\|_{L^2(0,L)\times H^{-1}(0,L)}} \mid (y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)\setminus\{(0,0)\}\right\},$$

which is called *the cost of the control at time* T (because it measures the minimal energy needed to bring an initial condition to $(0,0)$). Using a standard duality argument, it can be showed that (72) is null controllable if and only if (69) is observable, and in this case the cost of the control is

$$\|\Gamma^T_{a,\omega}\| = C_{T,\text{obs}}(a,\omega)^{-1},$$

with $C_{T,\text{obs}}(\omega)^{-1}$ the optimal constant in the observability inequality (70), defined by (71).

The dependence of $C_{T,\text{obs}}(a,\omega)^{-1}$ with respect to different parameters (the observability time T, the potential a, the observability set ω) has been studied by many authors (see [35], where an application to the controllability of semilinear wave equations is given, [6] for some results in the multi-dimensional case obtained thanks to Carleman estimates and [10] for precise lower bounds obtained through different methods) but its exact behavior is not known.

In the following result, one provides several estimates of $C_{T,\text{obs}}(a,\omega)$ (and then $\|\Gamma^T_{a,\omega}\|$) and constitutes another justification of the interest of the problems introduced in Section 1.2, in particular of the issue of obtaining a lower bound estimate of the quantity $J(a,\omega)$.

Theorem 3. *Let $L>0$ and let a be a nonnegative function in $L^\infty(0,L)$.*

i There holds

$$C_{T,\text{obs}}(a,\omega) \sim TJ(a,\omega) \qquad as \quad T\to+\infty.$$

ii Let $a\in\mathcal{A}_M(0,L)$ with $M<3\pi^2/L^2$ and define $T(a)=\frac{2\pi}{\gamma_a}$ with $\gamma_a=\frac{\frac{3\pi^2}{L^2}-M}{\frac{2\pi}{L}+\sqrt{\frac{\pi^2}{L^2}+M}}$. For all $T>T(a)$, there holds

$$0<c_1(T,\gamma_a)\leqslant \frac{C_{T,\text{obs}}(a,\omega)}{J(a,\omega)} \leqslant c_2(T,\gamma_a),$$

with $c_1(T,\gamma_a)=\frac{2}{\pi}\left(T-\frac{4\pi^2}{\gamma_a^2 T}\right)$ and $c_2(T,\gamma_a)=\frac{20T}{\min\{2\pi,\gamma_a T\}}$.

The proof of this theorem is postponed to Section C.

3.1.2 Randomized observability constant

Let us provide another interpretation of the quantity $J(a,\omega)$. It corresponds also to an averaged version of the observability constant $C_{T,\text{obs}}(a,\omega)$ defined by (71), over random initial data. More precisely, let $(\beta^\nu_{1,j})_{j\in\mathbb{N}^*}$ and $(\beta^\nu_{2,j})_{j\in\mathbb{N}^*}$ be two sequences of independent Bernoulli random variables on a probability space $(\mathcal{X},\mathcal{A},\mathbb{P})$, satisfying

$$\mathbb{P}(\beta^\nu_{1,j}=\pm1)=\mathbb{P}(\beta^\nu_{2,j}=\pm1)=\frac{1}{2} \qquad \text{and} \qquad \mathbb{E}(\beta^\nu_{1,j}\beta^\nu_{2,k})=0,$$

for every j and k in $\mathbb{N}^*$ and every $\nu\in\mathcal{X}$. Here, the notation $\mathbb{E}$ stands for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$.

Then, $J(a,\omega)$ is the largest constant C for which the inequality

$$C\int_0^L \left(\varphi_1(x)^2+\varphi_0'(x)^2+a(x)\varphi_0(x)^2\right)dx \leqslant \mathbb{E}\left(\int_0^T\int_\omega |\partial_t\varphi_\nu(t,x)|^2\,dx\,dt\right), \tag{73}$$

holds for all $(\varphi_0, \varphi_1) \in H_0^1(0, L; \mathbb{C}) \times L^2(0, L; \mathbb{C})$, where φ_ν is defined by

$$\varphi_\nu(t, x) = \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_{a,j} t} + \beta_{2,j}^\nu b_j e^{-i\lambda_{a,j} t} \right) e_{a,j}(x),$$

with

$$\begin{aligned} a_j &= \frac{1}{2} \left(\int_0^L \varphi_0(x) e_{a,j}(x)\, dx - \frac{i}{\lambda_{a,j}} \int_0^L \varphi_1(x) e_{a,j}(x)\, dx \right), \\ b_j &= \frac{1}{2} \left(\int_0^L \varphi_0(x) e_{a,j}(x)\, dx + \frac{i}{\lambda_{a,j}} \int_0^L \varphi_1(x) e_{a,j}(x)\, dx \right), \end{aligned} \tag{74}$$

for every $j \in \mathbb{N}^*$.

In other words, φ_ν denotes the solution of the wave equation (69) with the random initial data $\varphi_{0,\nu}(\cdot)$ and $\varphi_{1,\nu}(\cdot)$ determined by their Fourier coefficients $a_j^\nu = \beta_{1,j}^\nu a_j$ and $b_j^\nu = \beta_{2,j}^\nu b_j$.

In this context, the quantity $J(a, \omega)$ is called *randomized observability constant* and we refer to [27, Section 2.3] and [31, Section 2.1] for further explanations on its use in inverse problems. Moreover, a deterministic interpretation of this quantity is provided in [32].

3.1.3 Decay rate for a damped wave equation

From the estimates of the observability constant $C_{T,\mathrm{obs}}(a, \omega)$, we can also deduce estimates of the rate at which energy decays in a damped string. Consider the damped wave equation on $(0, \pi)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_{tt} y(t, x) - \partial_{xx} y(t, x) + a(x) y(t, x) + 2k\chi_\omega(x) \partial_t y(t, x) = 0, & (t, x) \in (0, T) \times (0, L), \\ y(t, 0) = y(t, \pi) = 0, & t \in [0, T], \\ (y(0, x), \partial_t y(0, x)) = (y^0(x), y^1(x)), & x \in (0, L), \end{cases} \tag{75}$$

with $k > 0$. Recall that for all initial data $(y_0, y_1) \in H_0^1(0, \pi) \times L^2(0, \pi)$, the problem (75) is well posed and its solution y belongs to $C^0(0, T; H_0^1(0, \pi)) \cap C^1(0, T; L^2(0, \pi))$.

The energy associated to System (75) is defined by

$$E_{a,\omega}(t) = \int_0^\pi \left(\partial_t y(t, x)^2 + \partial_x y(t, x)^2 + a(x) y(t, x)^2 \right) dx.$$

According to Theorem 3 and to [10, Section 3.3], by using the same notations as in the statement of Theorem 3, if ω is a measurable subset of $(0, \pi)$, $a \in \mathcal{A}_M(0, \pi)$ with $M < 3\pi^2/L^2$, there holds for every $(y_0, y_1) \in H_0^1(0, \pi) \times L^2(0, \pi)$ and $t \geqslant 2T(a)$,

$$E_{a,\omega}(t) \leqslant E_{a,\omega}(0) e^{-\delta(a,\omega) t},$$

with

$$\ln \left(\frac{1 + (1 + T(a)^2) c_1(T, \gamma_a) J(a, \omega)}{(1 + T(a)^2) c_1(T, \gamma_a) J(a, \omega)} \right) \leqslant 2T(a)\delta(a, \omega) \leqslant \ln \left(\frac{1 + (1 + T(a)^2) c_2(T, \gamma_a) J(a, \omega)}{(1 + T(a)^2) c_2(T, \gamma_a) J(a, \omega)} \right).$$

3.2 Numerical investigations

In the whole section, we will consider that $L = \pi$ according to Lemma 1, and two given numbers $r \in (0, 1)$ and $M \in (0, 1)$.

3.2.1 The toy case "$j_0 = 1$" and $M \in (0, 1]$

In the case "$j_0 = 1$", according to Theorem 1, there exist at most two switching points in $(0, \pi)$ denoted o_1 and o_2 such that

$$0 \leqslant o_1 \leqslant o_2 \leqslant \pi \qquad \text{and} \qquad a(x) = M\chi_{(0,o_1)\cup(o_2,\pi)}. \tag{76}$$

Therefore, the issue of determining the optimal potential $a(\cdot)$ comes to minimize the function $(0,\pi)^2 \ni (o_1, o_2) \mapsto \inf_{\substack{\omega\subset(0,\pi) \\ \text{s.t. } |\omega|=r\pi}} \int_\omega e_{a,1}(x)^2\,dx$, where $a(\cdot)$ is given by (76). Fixing $\tau_a = \sqrt{\lambda_{a,1}^2 - M}$, one computes

$$e_{a,1}(x) = \begin{cases} \sin(\tau_a x) & x \in (0, o_1), \\ \sin(\tau_a o_1)\cos(\lambda_{a,1}(x-o_1)) + \frac{\tau_a}{\lambda_{a,1}}\cos(\tau_a o_1)\sin(\lambda_{a,1}(x-o_1)) & x \in (o_1, o_2), \\ \frac{\sin(\tau_a o_1)\cos(\lambda_{a,1}(o_2-o_1)) + \frac{\tau_a}{\lambda_{a,1}}\cos(\tau_a o_1)\sin(\lambda_{a,1}(o_2-o_1))}{\sin(\tau_a(\pi-o_2))}\sin(\tau_a(\pi-x)) & x \in (o_2, \pi), \end{cases}$$

up to a multiplicative normalization constant, where the eigenvalue $\lambda_{a,1}$ solves the transcendental equation

$$\lambda_{a,j}\tau_a \frac{\tan(\tau_a(\pi - o_2 + o_1))}{\tan(\lambda(o_2 - o_1))} - \tau_a^2 = M\frac{\sin(\tau_a(\pi - o_2))\sin(\tau_a o_1)}{\cos(\tau_a(\pi - o_2 + o_1))}.$$

This last equation is solved numerically by using a Newton method. Since $M \in (0, 1]$, the eigenfunction $e_{a,1}$ is concave on $(0, \pi)$. As a consequence, the optimal set ω, as level set of the function $e_{a,1}^2$, writes $\omega = (0, \alpha) \cup (\beta, \pi)$ with $\alpha < \beta$, according to Proposition 2. In that case, we determined α and β with the help of a Newton method, using that $\beta = (1 - r)\pi + \alpha$ and $e_{a,1}(\alpha)^2 = e_{a,1}(\beta)^2$.

These considerations allow to rewrite the extremal problem ($\mathcal{P}_{L,r,M}$) as a two-dimensional optimization problem, that we solved numerically by using a Nelder-Mead simplex search method on a standard desktop machine.

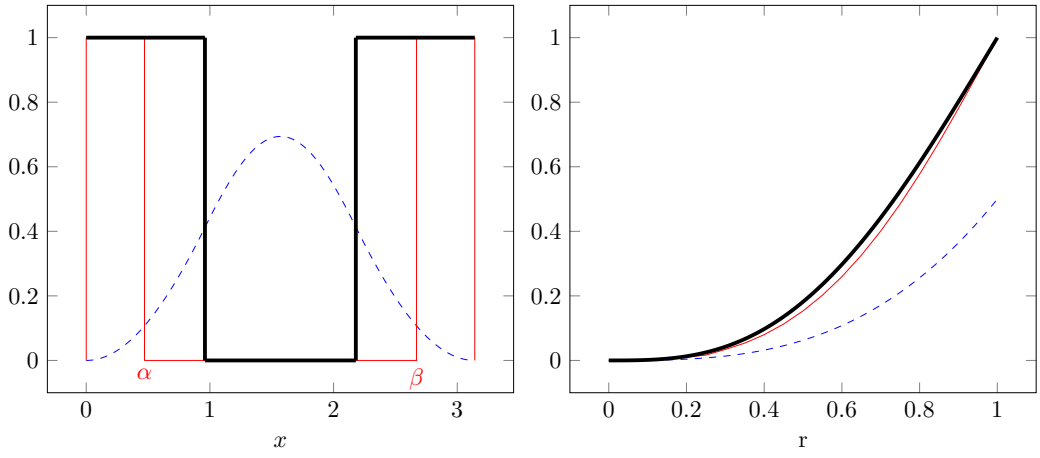

Figure 6: $L = \pi$ and $M = 1$. Left: plots of the optimal set $\omega(-)$, a(-) and $e_{a_1^*,1}^2\,(\dots)$ with respect to the space variable with $r = 0.3$. Right: plot of $r \mapsto m_1(L, r)\,(-)$, $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ (-) and $r \mapsto r^3/2\,(\cdots)$.

3.2.2 Numerical solving of $(\mathcal{P}_{j,L,r,M})$ for for $j \geqslant 2$.

Let $j \in \mathbb{N}\backslash\{0,1\}$ and $M = (0,1]$. We fix $o_0 = 0$ and $o_{3j} = \pi$. According to Theorem 1, we reduce the solving of the infinite-dimensional problem $(\mathcal{P}_{j,L,r,M})$ to a $(3j-1)$-dimensional one. In other terms, we minimize the function

$$(0,\pi)^{3j-1} \ni o = (o_1, \cdots, o_{3j-1}) \longmapsto \inf_{\substack{\omega \subset (0,\pi) \\ \text{s.t. } |\omega| = r\pi}} \int_\omega e_{a,1}(x)^2 \, dx,$$

where $a(\cdot)$ is a potential function defined on $(0,\pi)$ such that,

$$a(x) = \begin{cases} M & \text{on } (o_i, o_{i+1}), \\ 0 & \text{on } (o_{i+1}, o_{i+2}), \end{cases}$$

for every even integer $i \in \{0, \cdots, 3j-2\}$ and every $x \in (o_i, o_{i+2})$. Notice that, when $3j-2$ is a odd number then $a(x) = M$ on (o_{3j-1}, o_{3j}). Thus, given the switching points $o \in (0,\pi)^{3j-1}$,

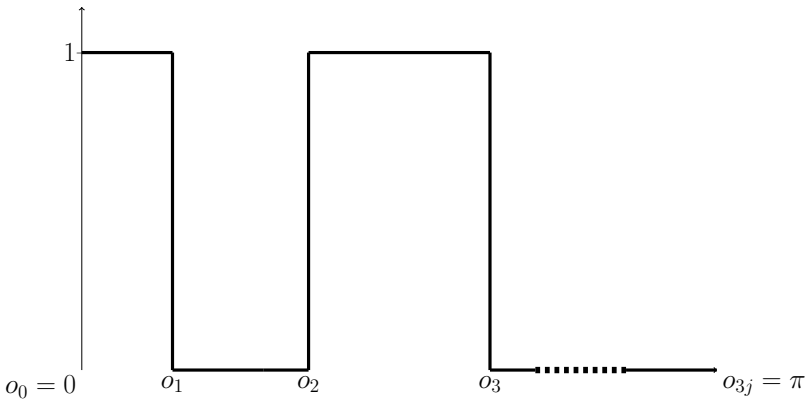

Figure 7: Construction of $a(\cdot)$.

one computes the eigenfunction $e_{a,j}(\cdot)$ by using a shooting method combined with a Runge-Kutta method. The eigenvalue λ is determined by solving $e_{a,\lambda}(\pi) = 0$ with a Newton method.

According to Proposition 2, the set ω coincides with $\{e_{a,j}^2 \leqslant \tau\}$ for some parameter τ chosen in such a way that $|\omega| = r\pi$. We are then driven to find an estimate of τ, which is done by computing the decreasing rearrangement ${e_{a,j}^*}^2$ of $e_{a,j}^2$ (see, e.g., [13, 19, 33]) and using that $\tau = {e_{a,j}^*}^2(r\pi)$.

These considerations allow to rewrite the cost functional as a function of $(3j-1)$ variables. The resulting finite-dimensional problem is then solved numerically by using a Nelder-Mead simplex search method on a standard desktop machine.

Figures 9 and 10 illustrate the research of the optimal potential. The parameter r is running over the interval $[0,1]$. On Figure 9, the optimal value of the criterion (w.r.t. r) is compared to the estimate obtained in Theorem 1 for the parameter values $j \in \{2,3,4\}$. Recall that the numbers $\underline{m}_j$ are defined in Proposition 1.

On Figure 10, the graph of the optimal value with respect to r is plotted for the parameter values $j \in \{1,2,6\}$. Notice that the mapping $j \mapsto m_j(L,M,r)$ seems to be increasing, although we did not manage to prove it.

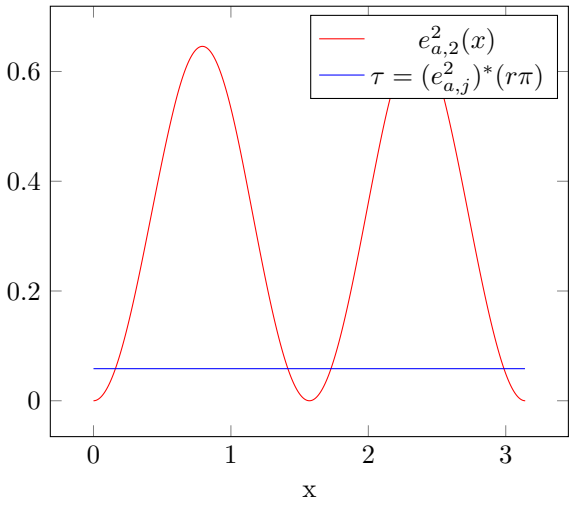

Figure 8: $j = 2$, $L = \pi$, $M = 1$ and $r = 0.3$. Graph of $e^2_{a,2}$ and determination of the Lagrange parameter τ.

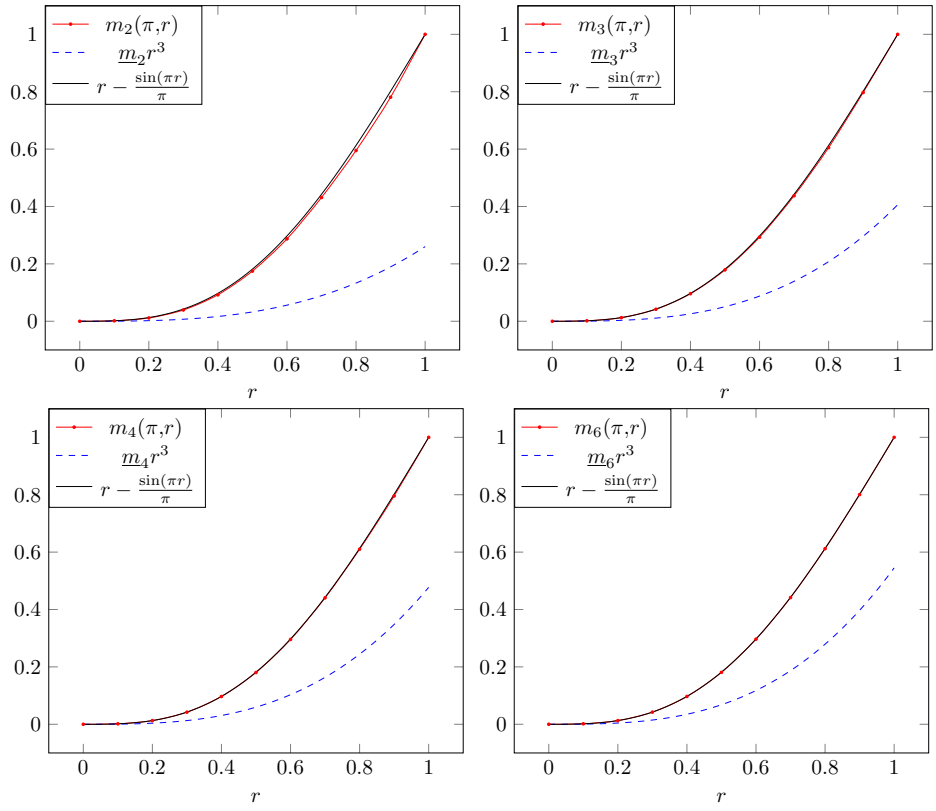

Figure 9: $L = \pi$ and $M = 1$. Plots of m_j $(-o-)$, $\underline{m}_j r^3$ $(\cdots)$ and $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ $(-)$ with respect to r for $j \in \{2, 3, 4\}$.

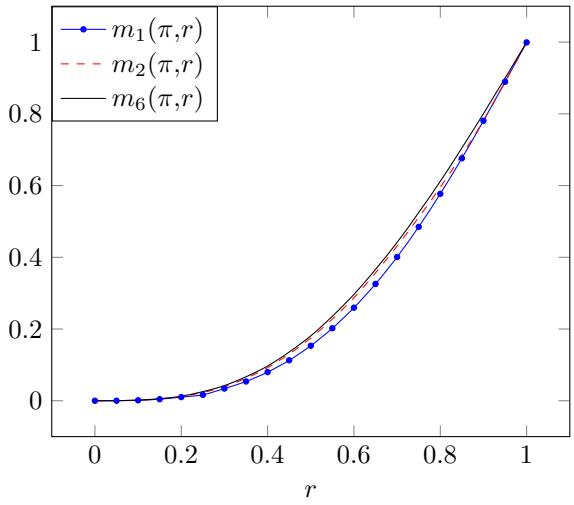

Figure 10: $L = \pi$ and $M = 1$. Plots of $m_j(\pi,r)$ for $j = 1$(o), $j = 2$(- -) and $j = 6$ with respect to r.

Appendix

A Proof of Lemma 2

One refers for instance to [12] for a survey of the tools used to prove the continuity of eigenvalues and eigenfunctions with respect to several parameters. Nevertheless, we provide here the main lines of the proof for the sake of completeness.

We first investigate the continuity of the first eigenfunction, in other words, the case "$j = 1$". Let $(a_n)_{n\in\mathbf{N}^*}$ be a sequence of $\mathcal{A}_M(0,L)$ converging for the weak-$\star$ topology of $L^\infty(0,L)$ to a. Then, using that $a_n \leqslant M$ in $(0,L)$ for all $n \in \mathbb{N}^*$ and according to the Courant-Fischer minimax principle (see [5, Page 405]), one has

$$\lambda_{a_n,1}^2 = \min_{u\in H_0^1(0,L)\setminus\{0\}} \frac{\int_0^L u'(x)^2 + a_n(x)u(x)^2\,dx}{\int_0^L u(x)^2\,dx} \leqslant \frac{\pi^2}{L^2} + M.$$

It follows that, up to a subsequence, the sequence $(\lambda_{a_n,1}^2)_{n\in\mathbf{N}^*}$ converges to a real nonnegative number denoted λ^2. The variational formulation of the eigenproblem (1) writes: find $e_{a_n,1}$ in $H_0^1(0,L)$ such that for every test function $\varphi \in H_0^1(0,L)$, one has

$$\int_0^L e'_{a_n,1}(x)\varphi'(x) + a_n(x)e_{a_n,1}(x)\varphi(x)\,dx = \lambda_{a_n,1}^2 \int_0^L e_{a_n,1}(x)\varphi(x)\,dx. \tag{77}$$

Taking as test function $\varphi = e_{a_n,1}$ and using the fact that $\int_0^L e_{a_n,1}(x)^2\,dx = 1$ for every $n \in \mathbb{N}^*$ yields easily that the sequence $(e_{a_n,1})_{n\in\mathbf{N}^*}$ is uniformly bounded in $H_0^1(0,L)$. From Rellich-Kondrachov Theorem, it converges up to a subsequence weakly in $H^1(0,L)$ and strongly in $L^2(0,L)$ to an element f of $H_0^1(0,L)$. Passing into the limit in (77) yields that there exists some $j \in \mathbb{N}^*$ such that $f = e_{a,j}$ and $\lambda = \lambda_{a,j}$. Moreover, since

$$\int_0^L e'_{a_n,1}(x)^2\,dx = \lambda_{a_n,1} - \int_0^L a_n(x)e_{a_n,1}(x)^2\,dx,$$

the sequence $(\|e_{a_n,1}\|_{H^1(0,L)})_{n\in\mathbb{N}^*}$ converges to $\|e_{a,j}\|_{H^1(0,L)}$. Combining this fact with the weak convergence of $(e_{a_n,1})_{n\in\mathbb{N}^*}$ to $e_{a,j}$ in $H^1_0(0,L)$ proves that this convergence is in fact strong. Finally, using the embedding $C^0([0,L]) \hookrightarrow H^1(0,L)$ and passing to the limit in the inequality $e_{a_n,1} \geqslant 0$ in $(0,L)$ yields that $e_{a,j}$ is of constant sign. Therefore, there holds necessarily $j = 1$ and the conclusion follows.

To adapt this result to higher orders, it is enough to make the same reasoning on each nodal domain. Without loss of generality, consider the case $j = 2$. Using the same notations as previously, the function $e_{a_n,2}$ has two nodal domains $(0,\xi_n)$ and (ξ_n,L). Moreover, the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges, up to a subsequence to some $\xi^* \in [0,L]$. Applying the previous arguments to $e_{a_n,2}\chi_{(0,\xi_n)}$ and $e_{a_n,2}\chi_{(\xi_n,L)}$ proves that each of these quantities converges strongly in $H^1(0,L)$, respectively to $e_{a,j}\chi_{(0,\xi^*)}$ and $e_{a,j}\chi_{(\xi^*,L)}$. Moreover, one shows as previously that each of these functions is of constant sign and that $\xi^* \in (0,L)$ by passing to the limit in the equality

$$\int_0^L e_{a_n,1}(x)e_{a_n,2}(x)\,dx = 0,$$

which proves that the sign of the function $e_{a,j}$ is non constant. Hence, the function $e_{a,j}$ has two nodal domains, whence the result.

To generalize this result to other eigenfunctions, it is enough to use an induction argument and the L^2 orthogonality of $e_{a,j}$ with $e_{a,j-1}$ that we do not present in details here. The conclusion follows.

B Proof of Lemma 4

Let us define $\phi_j = \frac{e_{a_j,j}(\cdot)}{e'_{a_j,j}(0)}$. The function ϕ_j solves the Cauchy system

$$\begin{cases} -\phi_j''(x) + a_j(x)\phi_j(x) = \lambda_j^2\phi_j(x), & x \in (0,L), \\ \phi_j(0) = 0, \quad \phi_j'(0) = 1. \end{cases}$$

Let us notice that, according to the Courant-Fischer minimax principle, there holds $\lambda_{a_j,j} \geqslant \frac{\pi}{L}$ for every $j \in \mathbb{N}^*$ and $\lim_{j\to+\infty} \lambda_{a_j,j} = +\infty$. According to [26, Chapter 1, Theorem 3] and using a rescaling argument, we infer

$$\phi_j(x) = \frac{\sin(\lambda_{a_j,j}x)}{\lambda_{a_j,j}} + \mathrm{O}\left(\frac{1}{\lambda^2_{a_j,j}}\right).$$

As a consequence, there holds

$$\phi_j^2(x) = \frac{\sin^2(\lambda_{a_j,j}x)}{\lambda^2_{a_j,j}} + \mathrm{O}\left(\frac{1}{\lambda^3_{a_j,j}}\right), \tag{78}$$

where the remainder term does not depend on x. Therefore, using the Riemann-Lebesgue lemma, one gets that

$$\int_0^L \phi_j^2(x)dx = \frac{L}{2\lambda^2_{a,j}} + o\left(\frac{1}{\lambda^2_{a,j}}\right), \tag{79}$$

and since $e_{a_j,j} = \frac{\phi_j}{\|\phi_j\|_2}$, the combination of (78) and (79) yields

$$e^2_{a_j,j}(x) = \frac{2}{L}\sin^2(\lambda_{a_j,j}x) + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}}\right). \tag{80}$$

Let $\varphi \in L^1(0,L)$. Using (80), one shows that

$$\int_0^L e_{a_j,j}(x)^2 \varphi(x)\,dx = \frac{2}{L}\int_\alpha^\beta \sin^2(\lambda_{a_j,j}x)\varphi(x)\,dx + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}}\right).$$

By linearizing $\sin^2(\lambda_{a_j,j}x)$ and using the real version of the Riemann-Lebesgue lemma, the expected result follows. ∎

C Proof of Theorem 3

Before proving this theorem, let us recall some basic facts on Ingham's inequality (see [14]), an inequality for nonharmonic Fourier series much used in control theory.

Proposition 5. *For every $\gamma > 0$ and every $T > \frac{2\pi}{\gamma}$, there exist two positive constants $C_1(T,\gamma)$ and $C_2(T,\gamma)$ such that for every sequence of real numbers $(\mu_n)_{n\in\mathbb{N}^*}$ satisfying*

$$\forall n \in \mathbb{N}^* \quad |\mu_{n+1} - \mu_n| \geqslant \gamma, \tag{81}$$

there holds

$$C_1(T,\gamma)\sum_{n\in\mathbb{Z}^*}|a_n|^2 \leqslant \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\mu_n t}\right|^2 dt \leqslant C_2(T,\gamma)\sum_{n\in\mathbb{Z}^*}|a_n|^2, \tag{82}$$

for every $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$.

Denoting by $C_1(T,\gamma)$ and $C_2(T,\gamma)$ the optimal constants in (82), several explicit estimates of these constants are provided in [14]. For example, it is proved in the article cited above that

$$C_1(T,\gamma) \geqslant 2\left(\frac{T}{\pi} - \frac{4\pi}{\gamma^2 T}\right) \quad \text{and} \quad C_2(T,\gamma) \leqslant \frac{20T}{\min\{2\pi, \gamma T\}}.$$

The idea to use Ingham inequalities in control theory is a long story (see for instance [2, 9, 15, 16, 20]).

Notice that, up to our knowledge, the best constants in [14] are not known. In the particular case where $\mu_n = \pi n/L$ for every $n \in \mathbb{N}^*$, one shows easily that for every $T > 2L$, $C_1(T,\gamma) = \pi\left\lfloor\frac{T}{2\pi}\right\rfloor$ and $C_2(T,\gamma) = C_1(T,\gamma) + 1$, the bracket notation standing for the integer floor.

The following result on the asymptotic as $T \to +\infty$ of optimal constants Ingham's inequalities will be a crucial tool to prove Theorem 3.

Proposition 6. *Assume that the sequence $(\mu_n)_{n\in\mathbb{N}^*}$ satisfies (81). Then, there holds*

$$\lim_{T\to+\infty}\frac{C_1(T,\gamma)}{T} = \lim_{T\to+\infty}\frac{C_2(T,\gamma)}{T} = 1.$$

Proof of Proposition 6. Let $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$ be such that $\|a\|_{\ell^2} = 1$. Introduce the quantity

$$Q_T(a,\mu) = \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\mu_n t}\right|^2 dt.$$

We write

$$\begin{aligned} Q_T(a,\mu) &= \int_0^T \sum_{n\in\mathbb{Z}^*} |a_n|^2\,dt + \int_0^T \sum_{n\neq m} a_n\bar{a}_m e^{i(\mu_n-\mu_m)t}\,dt \\ &= T - i\sum_{n\neq m} \frac{a_n\bar{a}_m(e^{i(\mu_n-\mu_m)T}-1)}{\mu_n-\mu_m} \\ &= T - i\sum_{n\neq m} \frac{b_n\bar{b}_m}{\mu_n-\mu_m} + i\sum_{n\neq m} \frac{a_n\bar{a}_m}{\mu_n-\mu_m}, \end{aligned}$$

with $b_n = a_n e^{i\mu_n T}$ for every $n\in\mathbb{Z}^*$.

According to [24, Theorem 2], one has

$$\left|\sum_{n\neq m} \frac{b_n\bar{b}_m}{\mu_n-\mu_m}\right| \leqslant \frac{\pi}{\gamma}\|a\|_{\ell^2}^2 \quad \text{and} \quad \left|\sum_{n\neq m} \frac{a_n\bar{a}_m}{\mu_n-\mu_m}\right| \leqslant \frac{\pi}{\gamma}\|a\|_{\ell^2}^2,$$

where $\gamma = \inf_{n\in\mathbb{Z}^*} \mu_{n+1}-\mu_n$. Then, it follows that

$$1 - \frac{\pi}{\gamma T} \leqslant \frac{1}{T}\inf_{a\in\ell^2\,|\,\|a\|_{\ell^2}=1} Q_T(a,\mu) \leqslant \frac{1}{T}\sup_{a\in\ell^2\,|\,\|a\|_{\ell^2}=1} Q_T(a,\mu) \leqslant 1 + \frac{\pi}{\gamma T},$$

whence the result. ∎

Decomposing the solution φ of (69) in the spectral basis $\{e_{a,j}\}_{j\in\mathbf{N}^*}$ allows to write that

$$\varphi(t,x) = \sum_{j=1}^{+\infty} \left(\alpha_j\cos(\lambda_{a,j}t) + \beta_j\sin(\lambda_{a,j}t)\right) e_{a,j}(x), \tag{83}$$

where

$$\alpha_j = \int_0^L \varphi_0(x)e_{a,j}(x)\,dx, \qquad \beta_j = \frac{1}{\lambda_{a,j}}\int_0^L \varphi_1(x)e_{a,j}(x)\,dx, \tag{84}$$

for every $j\in\mathbb{N}^*$. We are now ready to prove Theorem 3.

Proof of Theorem 3. Introduce the spectral gap

$$\gamma = \inf_{j\in\mathbb{N}^*} \lambda_{a,j+1} - \lambda_{a,j}. \tag{85}$$

It is well-known that $\gamma > 0$ for every $a\in L^\infty(0,L)$. Let us first prove point (i). Using (82), we get that, for $T \geqslant 2\pi/\gamma$,

$$\begin{aligned} \int_0^T\int_\omega |\partial_t\varphi(t,x)|^2\,dxdt &= \frac{1}{2}\int_0^T\int_\omega \left|\sum_{k\in\mathbb{Z}^*} i\,\mathrm{sgn}(k)\lambda_{a,|k|}\sqrt{a_{|k|}^2+b_{|k|}^2}\,e^{i\,\mathrm{sgn}(k)(\lambda_{a,|k|}t-\theta_{|k|})}e_{a,|k|}(x)\right|^2 dxdt \\ &\geqslant \frac{C_1(T,\gamma)}{2}\sum_{k\in\mathbb{Z}^*}(a_{|k|}^2+b_{|k|}^2)\lambda_{a,|k|}^2\int_\omega e_{a,|k|}(x)^2\,dx \\ &= C_1(T,\gamma)\sum_{j=1}^{+\infty}(a_j^2+b_j^2)\lambda_{a,j}^2\int_\omega e_{a,j}(x)^2\,dx, \end{aligned} \tag{86}$$

where $(\theta_j)_{j\in\mathbb{N}^*}$ denotes the sequence defined by $e^{i\theta_j} = \frac{a_j+ib_j}{\sqrt{a_j^2+b_j^2}}$ for every $j \in \mathbb{N}^*$. Similarly, one gets

$$\int_0^T \int_\omega |\partial_t \varphi(t,x)|^2 \, dxdt \leqslant C_2(T,\gamma) \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \lambda_{a,j}^2 \int_\omega e_{a,j}(x)^2 \, dx. \tag{87}$$

Using the energy identity

$$\int_0^L \left(\varphi_t^2(t,x) + \varphi_x^2(t,x) + a(x)\varphi^2(t,x)\right) \, dx = \sum_{j=1}^{+\infty} \lambda_{a,j}^2 (a_j^2 + b_j^2), \tag{88}$$

and combining it with (86), (87) and the asymptotic of optimal constants in Ingham's inequalities stated in Proposition 6 leads to the desired result. Let us now prove point ii. According to (86), (87), and (88), there holds

$$0 < C_1(T,\gamma) \leqslant \frac{C_{T,\mathrm{obs}}(\omega)}{\inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}(x)^2 \, dx} \leqslant C_2(T,\gamma), \tag{89}$$

with $C_1(T,\gamma) \geqslant 2\left(\frac{T}{\pi} - \frac{4\pi}{\gamma^2 T}\right)$ and $C_2(T,\gamma) \leqslant \frac{20T}{\min\{2\pi,\gamma T\}}$. To conclude, it remains to provide an estimate of the spectral gap γ defined by (85).

Lemma 9. *Let $a \in \mathcal{A}_M(0,L)$ with $M \in (0, 3\pi^2/L^2)$. There holds*

$$\lambda_{a,j+1} - \lambda_{a,j} \geqslant \frac{\frac{3\pi^2}{L^2} - M}{\frac{2\pi}{L} + \sqrt{\frac{\pi^2}{L^2} + M}},$$

for every $j \in \mathbb{N}^*$.

Proof. The Courant-Fischer minimax principle writes

$$\lambda_{a,j}^2 = \min_{\substack{V \subset H_0^1(0,\pi) \\ \dim V = j}} \max_{u \in V\setminus\{0\}} \frac{\int_0^\pi (u'(x)^2 + a(x)u(x)^2)dx}{\int_0^\pi u(x)^2 \, dx}.$$

Using that $0 \leqslant a(x) \leqslant M$ for almost every $x \in (0,L)$ yields

$$\frac{j\pi}{L} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{j\pi}{L}\right)^2 + M}, \tag{90}$$

for every $j \in \mathbb{N}^*$. It suffices indeed to compare $\lambda_{a,j}^2$ with the j-th eigenvalue of a Sturm-Liouville operator with constant coefficients. We infer

$$\begin{aligned} \lambda_{a,j+1} - \lambda_{a,j} &\geqslant \frac{(j+1)\pi}{L} - \sqrt{\left(\frac{j\pi}{L}\right)^2 + M}, \\ &= \frac{(1+2j)\frac{\pi^2}{L^2} - M}{(j+1)\frac{\pi}{L} + \sqrt{\left(\frac{j^2\pi^2}{L^2}\right) + M}}. \end{aligned}$$

for every $j \in \mathbb{N}^*$. The sequence $j \mapsto \frac{(1+2j)\frac{\pi^2}{L^2} - M}{(j+1)\frac{\pi}{L} + \sqrt{\left(\frac{j^2\pi^2}{L^2}\right) + M}}$ being increasing, the expected estimate follows. □

Therefore, according to Lemma 9, one has $\gamma \geqslant \gamma_a$. Hence, choosing $\gamma = \gamma_a$ in (89) for all $T > T(a) = \frac{2\pi}{\gamma_a}$ yields the expected conclusion. ■

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