



HAL
open science

Quadratic dissipation inequalities for nonlinear systems using event-triggered controllers

Romain Postoyan

► **To cite this version:**

Romain Postoyan. Quadratic dissipation inequalities for nonlinear systems using event-triggered controllers. 54th IEEE Conference on Decision and Control, Dec 2015, Osaka, Japan. hal-01204828

HAL Id: hal-01204828

<https://hal.science/hal-01204828>

Submitted on 8 Dec 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Quadratic dissipation inequalities for nonlinear systems using event-triggered controllers

Romain Postoyan

Abstract—We present a method to design event-triggered controllers that ensure the satisfaction of quadratic dissipation inequalities for nonlinear sampled-data systems. We follow an emulation approach for this purpose. We first assume that a static feedback law is designed in continuous-time to guarantee such a dissipativity property. We then take into account sampling and we synthesize a triggering rule to preserve dissipativity. The parameters of the sampling law can be adjusted to approximately recover the quadratic terms of the initial supply rate with any desired accuracy by solving a linear matrix inequality, which can always be satisfied. We then tailor our results to specific quadratic dissipativity properties, namely (strict-)passivity and \mathcal{L}_2 -stability. Our results cover periodic sampling as a particular case, for which we provide new explicit bounds on the maximum allowable sampling period.

I. INTRODUCTION

Event-triggered control consists in updating the input of a control system only at time instants determined by a state-dependent criterion. This paradigm is attractive for systems subject to communication or computation constraints, as it may significantly reduce transmissions between the plant and the controller as well as the number of controller executions compared to periodic sampling. Various event-triggered control techniques are available for stabilization (e.g., [12], [25]), consensus (e.g., [6], [23], [27]), or estimation (e.g., [2], [16], [28]). On the other hand, the design of event-triggered controllers to ensure dissipativity properties remains largely unaddressed, despite its importance in a wide range of control problems. This is probably due to the fact that it is difficult to guarantee the existence of a uniform minimum time between any two transmissions in this context, which is an important requirement in practice. It is interesting to note that several event-triggered controllers of the literature require a dissipativity property to hold for the closed-loop system in the absence of sampling. For instance in [32], output feedback passivity properties of the plant and the controller are used to construct output-feedback event-triggered controllers. Another example is the work in [24] where the event-triggered coordination among a network of agents is investigated and where it is assumed that the agents satisfy a strict-passivity property. In a number of cases, dissipativity is not an intrinsic property of the system but is ensured by a feedback law. It is therefore important to construct event-triggered controllers to preserve this property. Results for \mathcal{L}_p -stability and input-to-state stability can be

found in [1], [7], [8], [31] for instance. Recently, the authors of [33] have designed event-triggering conditions to preserve a specific type of quadratic dissipativity for interconnected nonlinear systems, and a model-based approach for ensuring quadratic dissipativity properties is investigated in [17] for discrete-time systems.

In this paper, we propose a method to design event-triggered controllers to ensure quadratic dissipation inequalities for nonlinear systems. Compared to [33], we study a more general type of dissipativity, we present a different approach, and we provide sufficient conditions for the existence of a minimum amount of time between any two transmissions, which is not the case in [33]. We adopt an emulation approach for this purpose. We assume that a feedback law is designed to guarantee a quadratic dissipativity inequality in the absence of sampling. We then take into account the communication constraints and we derive a suitable triggering rule to preserve dissipativity. The triggering condition is based on an auxiliary variable that we design, whose dynamics depends (in general) on the state of the plant. The idea to introduce extra variables to construct the transmission law was suggested in [25], and has been pursued in [7], [9], [24]. We show that the system in closed-loop with the event-triggered controller verifies a quadratic dissipativity property. The supply rate consists of extra terms compared to the one in the absence of sampling. These terms can be arbitrarily adjusted by appropriately tuning the triggering law. In other words, we can fix any admissible performance degradation compared to the initial supply rate in continuous-time, it is then easy to adapt the triggering law accordingly, by solving a linear matrix inequality, which is always feasible. There is a trade-off between the performance requirement (in terms of supply rate) and the amount of transmissions, in the sense that to ask the supply rate to be close to the one in continuous-time typically leads to a triggering condition which generates more transmissions. Similar observations are made in [1], [7] where robust event-triggered control is studied. We also present results tailored to (strict-)passivity and \mathcal{L}_2 -stability as particular cases.

We model the event-triggered control system as a hybrid system using the formalism of [10]. We use the definition of dissipativity proposed in [20], which suggests to consider different supply rates on flows and at jumps, as in e.g., [11], [30]. As mentioned above, we provide sufficient conditions to ensure the existence of a uniform minimum time between two sampling instants (thus ruling out Zeno phenomenon). We explain how to apply our results to linear time-invariant systems, in which case the assumptions are written as linear

R. Postoyan is with the Université de Lorraine, CRAN, UMR 7039 and the CNRS, CRAN, UMR 7039, France romain.postoyan@univ-lorraine.fr. His work was partially supported by the ANR under the grant COMPACS (ANR-13-BS03-0004-02).

matrix inequalities. We also address the problem of preserving strict-passivity for a pendulum with an event-triggered controller, which is useful for coordination problems, see [3], [24].

It is important to notice that our results cover periodic sampling as a particular case. In this context, we provide an explicit estimate of the maximum allowable sampling period (MASP) to ensure dissipativity with the proposed emulation-based controllers. We expect the bound to be less conservative than those derived in [15]. These results complement the literature on dissipativity for nonlinear sampled-data systems, see e.g. [15], [18], [19], [29].

The proofs are omitted for space reasons.

II. PRELIMINARIES

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{R}_{> 0} := (0, \infty)$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$, and $\mathbb{Z}_{> 0} := \{1, 2, \dots\}$. A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K}_{∞} if it is continuous, zero at zero and unbounded. Let $(x, y) \in \mathbb{R}^{n+m}$, (x, y) stands for $[x^T, y^T]^T$. The notation \mathbb{I} denotes the identity matrix, whose dimensions depend on the context. In matrices, the symbol \star stands for the symmetric block component. Let $P \in \mathbb{R}^{n \times n}$ be a real, symmetric matrix, we respectively denote its maximum and minimum eigenvalues by $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$. For two real, symmetric matrices M and N , we write that $M \leq N$ when $N - M$ is positive semi-definite.

We study hybrid systems of the form ([5], [10])

$$\begin{cases} x \in \mathcal{C} & \dot{x} = F(x, v) \\ x \in \mathcal{D} & x^+ = G(x), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}^m$ is the input, F is the flow map, G is the jump map, \mathcal{C} is the flow set and \mathcal{D} is the jump set. We assume that F and G are continuous on $\mathcal{C} \times \mathbb{R}^m$ and \mathcal{D} , respectively, and that \mathcal{C} and \mathcal{D} are closed.

We recall some definitions related to [5], [10]. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, \dots, J\}) = \bigcup_{j \in \{0, 1, \dots, J-1\}} ([t_j, t_{j+1}), j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_J$. A function $\phi : E \rightarrow \mathbb{R}^n$ is a *hybrid signal* if E is a hybrid time domain. A hybrid signal $v : E \rightarrow \mathbb{R}^m$ is called a *hybrid input* if $v(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each $j \in \mathbb{Z}_{\geq 0}$. A hybrid signal $x : E \rightarrow \mathbb{R}^n$ is called a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous for each $j \in \mathbb{Z}_{\geq 0}$. A hybrid arc $x : E \rightarrow \mathbb{R}^n$ and a hybrid input $v : E \rightarrow \mathbb{R}^m$ is a *solution pair* (x, v) to (1) if: (i) $x(0, 0) \in \mathcal{C} \cup \mathcal{D}$; (ii) for any $j \in \mathbb{Z}_{\geq 0}$, $x(t, j) \in \mathcal{C}$ and $\frac{d}{dt}x(t, j) = F(x(t, j), v(t, j))$ for almost all $t \in I^j$ where $I^j := \{t : (t, j) \in E\}$; (iii) for every $(t, j) \in E$ such that $(t, j+1) \in E$, $x(t, j) \in \mathcal{D}$ and $x(t, j+1) = G(x(t, j))$.

Let v be a hybrid input, for any $(t, j) \in \text{dom } v$ [5]

$$\|v\|_{(t, j)} := \max \left\{ \begin{array}{l} \text{ess.sup}_{(t', j') \in \text{dom } v \setminus \Gamma(v), t' + j' \leq t + j} |v(t', j')|, \\ \sup_{(t', j') \in \Gamma(v), t' + j' \leq t + j} |v(t', j')| \end{array} \right\}, \quad (2)$$

where $\Gamma(v)$ is the set of all $(t', j') \in \text{dom } v$ such that $(t', j' + 1) \in \text{dom } v$.

We adapt below the definition of uniform semiglobal dwell-times proposed in [25] for autonomous hybrid systems to hybrid systems with inputs.

Definition 1: The solutions to (1) have a *uniform semiglobal dwell-time* if for any $\Delta \geq 0$, there exists $\tau(\Delta) > 0$ such that for any solution pair (x, v) to (1) with $|x(0, 0)| \leq \Delta$ and $\|v\|_{(t', j')} \leq \Delta$ for any $(t', j') \in \text{dom } v$, and any $(s, i), (t, j) \in \text{dom } x = \text{dom } v$ with $s + i \leq t + j$, $j - i \leq (t - s)/\tau(\Delta) + 1$. \square

We use the definition below of dissipativity, see¹ Definition 2 in [20].

Definition 2: System (1) is dissipative with the *pair of supply rates* $(p^c(x, v), p^d(x, v))$ with p^c, p^d continuous, if there exists a *storage function* $V : \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0}$ which is continuously differentiable on an open set containing \mathcal{C} and continuous on \mathbb{R}^n such that

$$\begin{aligned} \langle \nabla V(x), F(x, v) \rangle &\leq p^c(x, v) & \forall (x, v) \in \mathcal{C} \times \mathbb{R}^m \\ V(G(x)) - V(x) &\leq p^d(x, v) & \forall x \in \mathcal{D}. \end{aligned} \quad (3)$$

\square

III. PROBLEM STATEMENT

Consider the system

$$\dot{x} = f(x, u), \quad y = h(x), \quad (4)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is an output, $n, m \in \mathbb{Z}_{> 0}$. We focus on the case where the control input is given by a static feedback law of the form

$$u = k(x) \quad (5)$$

where $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable. The control law may depend on the full state x or on some output of plant (4), which is not necessarily y .

We consider the scenario where a digital channel is used to ensure the communications between controller (5) and plant (4). The sequence of transmission instants is denoted by t_i , $i \in \mathcal{I} \subseteq \mathbb{Z}_{\geq 0}$, and is generated by an event-triggering condition to be designed, which is co-located with the controller. The closed-loop dynamics becomes, for any $i \in \mathcal{I}$ and almost all $t \in [t_i, t_{i+1})$,

$$\begin{aligned} \dot{x} &= f(x, k(x) + e) \\ y &= h(x), \end{aligned} \quad (6)$$

where $e \in \mathbb{R}^m$ is the sampling-induced error on the control input which is defined by

$$e(t) := k(x(t_i)) - k(x(t)) \text{ for almost all } t \in [t_i, t_{i+1}). \quad (7)$$

The objective is to design k and the sequence of transmission instants to ensure a dissipativity property for system (6). We follow for this purpose an emulation approach. We assume that controller (5) is designed to guarantee a dissipativity property for system (4) in the absence of

¹We define the storage function V on $\mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D})$ in Definition 2 and not on \mathbb{R}^n as in Definition 2 in [20].

sampling. We then take into account sampling and design a triggering condition to preserve dissipativity for the closed-loop system. The approach is presented in details in the following.

IV. ASSUMPTIONS

We assume that controller (5) is designed such that the assumption below holds.

Assumption 1: There exist a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$, $H : \mathbb{R}^n \rightarrow \mathbb{R}^m$, real matrices P, Q, R, S with P, Q, R symmetric and P positive definite, such that the following holds for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$

$$\langle \nabla V(x), f(x, k(x) + v) \rangle \leq -\rho(x) - H(x)^T P H(x) + v^T R v + y^T Q y + 2y^T S v. \quad (8)$$

□

Assumption 1 means that the closed-loop system $\dot{x} = f(x, k(x) + v)$ with input v and output y is dissipative with supply rate $p(v, x) = -\rho(x) - H(x)^T P H(x) + v^T R v + y^T Q y + y^T S v + v^T S^T y$ and storage function V . The quadratic terms in p are justified by the fact that (strict-)passivity, \mathcal{L}_2 -stability and sector-bounded systems lead to such terms as shown in Chapter 10.7 in [13], see also Section VI. Note that ρ is not required to be positive definite, neither is V . The term $H(x)$ can be seen as an output to system (4), and the corresponding term $-H(x)^T P H(x)$ in (8) is useful to compensate for the sampling effect, like in [21]. It can be noted that the results still apply when H also depend on the input v . There exist various techniques to construct controller (5) such that (8) holds, see e.g., [4], [22], [26].

We make the assumptions below on the dynamics of the sampling-induced error between two successive transmission instants.

Assumption 2: There exist real matrices M, N with M symmetric and positive definite, continuous functions $L, g_1, g_2 : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ such that, the following holds.

- (i) $M \leq P$, where P comes from Assumption 1.
- (ii) Let $W : e \mapsto e^T M e$. For any $x \in \mathbb{R}^n$ and $e, v \in \mathbb{R}^m$,

$$\langle \nabla W(e), -\nabla k(x) f(x, k(x) + e + v) \rangle \leq L(x, e) W(e) + 2e^T M (g_1(x, e) H(x) + g_2(x, e) N v). \quad (9)$$

□

Assumption 2 provides information on the growth of the sampling-induced error e on flows, through the function W . When L, g_1 and g_2 are constant, the conditions are similar to those assumed in [21] for the stabilization of nonlinear sampled-data systems with time-triggered transmissions. The underlying idea is that the e -system is affected by the x -system and the exogenous input v through the ‘output’ $H(x)$ and Nv , respectively. To let L, g_1 and g_2 being dependent on x and e as we do in Assumption 2 (and like in Section V.B in [25]) is more general and is justified in many cases, see Section VIII-B for an example. There may not be a unique choice for M, L, g_1 and g_2 in (9). We will investigate

this point and its impact on the performances of the event-triggered controlled system for specific classes of systems in future work.

V. TRIGGERING CONDITION AND HYBRID MODEL

The triggering condition we propose is inspired by those developed in [25], [24] for the stabilization and the coordination of nonlinear systems, respectively. The idea is trigger transmissions using an auxiliary variable $\phi \in \mathbb{R}$. After a transmission, ϕ is reset to $b > 0$ and the next triggering instant occurs when ϕ is equal to $a \in [0, b)$. The constants a, b are design parameters, we explain later how to tune them. Between two triggering instants, the dynamics of ϕ is given by

$$\dot{\phi} = -L(x, e)\phi - (g_1(x, e)^2 + \eta g_2(x, e)^2)\phi^2 - \psi, \quad (10)$$

where L, g_1, g_2 come from Assumption 2, and η, ψ are design parameters. The constants η and ψ offer a trade-off between the speed of decrease of ϕ on flows (which is related to the amount of transmissions) and the accuracy with which the quadratic terms of the supply rate in Assumption 1 are preserved, as explained in more details in Section VI-A. While η can take any value in $\mathbb{R}_{>0}$, ψ is selected such that there exist real, symmetric, positive definite matrices F and G which verify the linear matrix inequality (LMI) below

$$\begin{pmatrix} R - \psi M & R & S^T \\ \star & -F & 0 \\ \star & \star & -G \end{pmatrix} < 0. \quad (11)$$

Notice that (11) holds for any such matrices F and G by taking ψ sufficiently big. On the other hand, the bigger ψ , the faster ϕ decreases between two transmission instants in view of (10), which typically leads to more transmissions.

The overall system is described by the hybrid model below

$$\dot{q} = F(q, v) \quad \text{for } q \in \mathcal{C} \quad q^+ = G(q) \quad \text{for } q \in \mathcal{D}, \quad (12)$$

with $q := (x, e, \phi) \in \mathbb{R}^{n_q}$, $\mathcal{C} := \{q : \phi \in [a, b]\}$, $\mathcal{D} := \{q : \phi = a\}$, $F(q, v) := (f(x, k(x) + e + v), -\nabla k(x) f(x, k(x) + e + v), -L(x, e)\phi - (g_1^2(x, e) + \eta g_2^2(x, e))\phi^2 - \psi)$, $G(q) := (x, 0, b)$, and $n_q := n + m + 1$.

VI. MAIN RESULTS

A. Dissipativity

We are ready to state the main result.

Theorem 1: Let $U(q) := V(x) + \phi W(e)$ for $q \in \mathcal{C} \cup \mathcal{D}$ and suppose Assumptions 1-2 are satisfied. System (12) is dissipative with storage function U and the pair of supply rates $(p^c(x, v), 0)$ where $p^c(x, v) = -\rho(x) + v^T (R + F + \frac{1}{\eta} N^T M N) v + y^T (Q + G) y + 2y^T S v$. □

Compared to the dissipativity property in the absence of sampling (see Assumption 1), the quadratic terms in p^c consists of extra terms, namely $v^T (F + \frac{1}{\eta} N^T M N) v + y^T G y$, which depend on the designed parameters F, G and η . We are free to select these parameters as we wish. We can therefore fix any admissible ‘degradation’ of the quadratic part of the initial supply rate in Assumption 1 and select

the matrices F , G and the constant η accordingly. Then, we choose ψ in (10) such that (11) holds. Again, (11) always holds by taking ψ sufficiently big, however this leads to a fast decrease of ϕ in view of (10) and therefore, potentially, to frequent transmissions. To mitigate this effect, we propose to minimize ψ subject to the linear constraint (11), which can be efficiently done by numerical solvers. In that way, the triggering condition directly adapts to the desired admissible degradation of the supply rate. It is interesting to note that the constants a and b have no influence on the supply rate. We can thus take $a = 0$ and b big, in order to (heuristically) increase the inter-transmissions times. On the other hand, the pair (a, b) has an impact on the storage function U as $\phi \in [a, b]$ according to (12).

B. Passivity & \mathcal{L}_2 -stability

We mentioned above that inequality (8) covers (strict-)passivity and \mathcal{L}_2 -stability as particular cases. We tailor the previous result to these specific properties.

The corollary below allows preserving (strict-)passivity for system (12).

Corollary 1 ((Strict-)passivity): Suppose the following holds.

- (i) Assumption 1 is verified with $R = 0$, $Q = 0$, $S = \frac{1}{2}\mathbb{I}$ and $\rho(x) \geq \bar{\rho}(x) + y^T E y$ for any $x \in \mathbb{R}^n$, where E is a real symmetric and positive definite matrix and $\bar{\rho} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$.
- (ii) Assumption 2 is verified with $N = 0$.

Select $\psi > 0$ in (12) such that

$$\begin{pmatrix} -\psi M & \frac{1}{2}\mathbb{I} \\ \star & -E \end{pmatrix} < 0. \quad (13)$$

System (12) is dissipative with storage function U defined in Theorem 1 and the pair of supply rates $(p^c(x, v), 0)$ where $p^c(x, v) = -\bar{\rho}(x) + y^T v$. \square

Corollary 1 provides sufficient conditions to preserve the passivity of system (4)-(5) (as implied by item (i) of Corollary 1) using event-triggered controllers. LMI (13) corresponds to (11) with $R = 0$, $S = \frac{1}{2}\mathbb{I}$ and $G = E$.

The result below focuses on \mathcal{L}_2 -stability. Its proof also follows from the proof of Theorem 1.

Corollary 2 (\mathcal{L}_2 -stability): Suppose the following holds.

- (i) Assumption 1 is verified with $R = \gamma^2\mathbb{I}$, $Q = -\mathbb{I}$, $S = 0$ and $\gamma \in \mathbb{R}$.
- (ii) Assumption 2 is verified.

Select $\psi > 0$ in (12) such that, for $\varepsilon > 0$,

$$\begin{pmatrix} \gamma^2\mathbb{I} - \psi M & \gamma^2\mathbb{I} \\ \star & -\varepsilon \end{pmatrix} < 0. \quad (14)$$

System (12) is dissipative with storage function U defined in Theorem 1 and the pair $(p^c(x, v), 0)$ where $p^c(x, v) = -\rho(x) - y^T y + v^T \left((\gamma^2 + \varepsilon)\mathbb{I} + \frac{1}{\eta} N^T M N \right) v$. \square

As before, LMI (14) is always verified by taking ψ sufficiently big. We note that the original \mathcal{L}_2 -gain γ becomes with the event-triggered controller $\sqrt{\gamma^2 + \varepsilon + \frac{1}{\eta} \lambda_{\max}(N^T M N)}$ and that this gain can take any value in (γ, ∞) by adjusting

the design parameters η and ε (through an appropriate choice of ψ in (14)).

C. Time-triggered control

When Assumption 2 holds with constant L , g_1 and g_2 , the time it takes for ϕ to decrease from b to a is constant. We therefore obtain a time-triggered policy. The previous results can be used to compute an estimate of the MASP with which dissipativity is preserved.

Let F , G and η be fixed. We assume that these are selected according to the desired supply rate under sampling (in agreement with Assumption 1). Select ψ to be the minimum value such that (11) holds. Then the time it takes ϕ to decrease from b to a is equal to

$$T(a, b) := \begin{cases} \frac{2}{r} \left(\arctan\left(\frac{2b\ell+L}{r}\right) - \arctan\left(\frac{2a\ell+L}{r}\right) \right) & \text{when } L < 2\sqrt{\ell\psi} \\ \frac{1}{a\ell+\frac{L}{2}} - \frac{1}{b\ell+\frac{L}{2}} & \text{when } L = 2\sqrt{\ell\psi} \\ \frac{1}{r} \left(\ln\left(\frac{2b\ell+L-r}{2a\ell+L-r}\right) - \ln\left(\frac{2b\ell+L+r}{2a\ell+L+r}\right) \right) & \text{when } L > 2\sqrt{\ell\psi}, \end{cases} \quad (15)$$

where $r := \sqrt{|-L^2 + 4\psi\ell|}$ and $\ell := g_1^2 + \eta g_2^2$. Since a, b can be arbitrarily selected such that $0 \leq a < b$, we derive that the MASP denoted by T^* has to be such that

$$\begin{aligned} T^* &< \lim_{(a,b) \rightarrow (0,\infty)} T(a, b) \\ &= \begin{cases} \frac{2}{r} \arctan\left(\frac{r}{L}\right) & \text{when } L < 2\sqrt{\ell\psi} \\ \frac{2}{r} & \text{when } L = 2\sqrt{\ell\psi} \\ \frac{2}{r} \operatorname{arctanh}\left(\frac{r}{L}\right) & \text{when } L > 2\sqrt{\ell\psi}. \end{cases} \quad (16) \end{aligned}$$

Remark 1: System (12) generates periodic sampling when Assumption 2 holds with constant L , g_1 and g_2 . These results can be easily extended to cover time-varying sampling where the sequence of transmission instants t_i , $i \in \mathbb{Z}_{\geq 0}$, verifies $v \leq t_{i+1} - t_i \leq T^*$ where $v \in (0, T^*]$ models the minimum allowable transmission intervals imposed by the set-up under consideration, as done in Section V in [24] for example. \square

VII. EXISTENCE OF DWELL-TIMES

Theorem 1 and the subsequent corollaries show that dissipativity is preserved using the proposed event-triggered controllers. However, these results do not inform us about the existence of a minimum amount of time between two transmissions. It is clear from (12) that, after each jump, the solution will flow for some time as long as x and e do not explode in finite (hybrid) time. Indeed, the time between two successive jumps corresponds to the time it takes for ϕ to decrease from b to a ; time which is always strictly positive in this case, since L , g_1 and g_2 are continuous. In practice, we often have a stronger requirement that is that there exists a *uniform* minimum amount of time between two jumps. We have already seen in Section VI-C that this is the case when Assumption 2 is verified with constant L , g_1 and g_2 , but transmissions are periodic in this case. We present a different set of assumptions below to ensure the existence of uniform dwell-times with event-triggered transmissions.

Theorem 2: Consider system (12) and suppose the following holds.

- (i) Assumption 1 is verified with V and ρ such that for any $x \in \mathbb{R}^n$

$$\begin{aligned} \underline{\alpha}(|x|_{\mathcal{A}}) &\leq V(x) \\ -\rho(x) + y^T(Q + E)y &\leq -\alpha(V(x)), \end{aligned} \quad (17)$$

where $\underline{\alpha}, \alpha \in \mathcal{K}_\infty$, $\mathcal{A} \subseteq \mathbb{R}^n$ and E is a real symmetric positive definite matrix.

- (ii) Assumption 2 holds and for any $(x, e) \in \mathbb{R}^{n+m}$

$$\max\{L(x, e), g_1(x, e)^2, g_2(x, e)^2\} \leq \chi(|x|_{\mathcal{A}}, |e|) \quad (18)$$

where $\chi: \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$ is continuous.

Select $\psi > 0$ in (12) such that (11) holds with $G = E$. Then the solutions have a uniform semiglobal dwell-times. \square

The first inequality in (17) holds when V is positive definite for instance, in which case $\mathcal{A} = \{0\}$. The second inequality in (17) essentially means that the dissipativity property in Assumption 1 has to be ‘strict’, see Chapter 6 in [14]. Item (ii) of Theorem 2 is a boundedness conditions on the functions L , g_1 and g_2 of Assumption 2. Finally, we note that we can always select ψ such that (11) is verified with $G = E$ in view of the explanations after (11). Examples of systems that verify the conditions of Theorem 2 are provided in the next section.

VIII. CASE STUDIES

A. Linear time-invariant systems

Consider the system

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad (19)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$. We make the following assumption.

Assumption 3: Let R, Q, S be real matrices and R and Q be symmetric. There exist a controller matrix K , and real, symmetric, positive definite matrices Π and J such that the following holds

$$C^TQC < J \quad (20)$$

and

$$\begin{pmatrix} (A + BK)^T \Pi + \Pi(A + BK) + J - C^TQC & \star \\ B^T \Pi - S^T C & -R \end{pmatrix} < 0. \quad (21)$$

\square

Inequality (20) always holds and conditions to ensure (21) can be found in Chapter 6 in [14] for instance depending on the considered triplet (R, Q, S) .

The proposition below shows that the conditions of Theorem 2 are verified under Assumption 3.

Proposition 1: Suppose Assumption 3 is satisfied and that controller (5) is given by $u = Kx$, then the following holds.

- 1) Assumption 1 is verified with $V(x) = x^T \Pi x$, $\rho(x) = -x^T (J - (A + BK)^T K^T M K (A + BK)) x$ where M is specified below, $H(x) = -K(A + BK)x$ for $x \in \mathbb{R}^n$, $P = M$, and R, Q, S as in Assumption 3.

- 2) Assumption 2 is verified with $W(e) = e^T M e$, L sufficiently big such that $LM \geq -2MKB$, $g_1(x, e) = g_2(x, e) = 1$, $N = -KB$ and M such that

$$(A + BK)^T K^T M K (A + BK) + C^T Q C < J. \quad (22)$$

- 3) Equation (17) holds with any $\underline{\alpha} \in \mathcal{K}_\infty$, E real, symmetric, positive definite such that $E < J - (A + BK)^T K^T M K (A + BK) - C^T Q C$, $\alpha(s) = \epsilon s^2$ for some $\epsilon > 0$, for any $s \in \mathbb{R}_{\geq 0}$, and $\mathcal{A} = \mathbb{R}^n$.

- 4) Equation (18) holds with $\chi(s_1, s_2) = \max\{L, 1\}$ for any $s_1, s_2 \in \mathbb{R}_{\geq 0}$. \square

Condition (22) can always be satisfied (it suffices to take $M = \epsilon \mathbb{I}$ with sufficiently small $\epsilon > 0$ for instance, since $C^T Q C < J$ from Assumption 3). The existence of the matrix E in item 3) of Proposition 1 follows from (22). A consequence of Proposition 1 is that the conclusions of Theorems 1 and 2 apply to linear systems, provided Assumption 3 holds. We note that transmissions are periodic in this case as Assumption 2 holds with constant L, g_1, g_2 (see Section VI-C).

B. Pendulum

It is explained in [3] how to design control laws to ensure coordination among networks of systems of the form

$$\dot{x}_1 = h(x_2), \quad \dot{x}_2 = f(x_2, u), \quad (23)$$

with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the states, $u \in \mathbb{R}^{n_1}$ is the control input, and where the x_2 -system with output $h(x_2)$ is strictly passive. It is shown in [24] how to emulate the controllers of [3] for event-triggered control. In some cases, the strict passivity of the x_2 -system is ensured by feedback, which is ignored in [24]. We show below how to ensure strict passivity using event-triggered controllers for a particular type of systems of the form (23), namely for the pendulum. Hence, we consider the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\theta_1 \sin(x_1) - \theta_2 x_2 + u, \end{aligned} \quad (24)$$

where $x_1 \in \mathbb{R}$ is the position, $x_2 \in \mathbb{R}$ is the velocity, $\theta_1 > 0$ is the ratio of the acceleration due to gravity over the length of the rod and $\theta_2 > 0$ is the ratio of the friction coefficient over the mass of the bob. We design the feedback law (5) as $u = \theta_1 \sin(x_1)$. The dynamics in (24) becomes, once the input is sampled,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\theta_2 x_2 + e, \quad (25)$$

where e is defined as in (7).

We now verify that the conditions of Corollary 1 and Theorem 2 hold. Let $W(e) = M e^2$ for $e \in \mathbb{R}$, with $M = P = \frac{1}{2} \theta_2$. For $x_1, x_2, e \in \mathbb{R}$,

$$\langle \nabla W(e), -\theta_1 \cos(x_1) x_2 \rangle = 2M e (-\theta_1 \cos(x_1) x_2), \quad (26)$$

hence Assumption 2 is verified with $L = g_2 = N = 0$, $g_1(x_1) = -\theta_1 \cos(x_1)$ and $H(x_2) = x_2$, and item (ii) of

Corollary 1 holds. We also note that (18) holds with $\chi = \theta_1^2$. On the other hand, for $V(x_2) = \frac{1}{2}x_2^2$, for any $x_1, x_2, v \in \mathbb{R}$,

$$\langle \nabla V(x_2), -\theta_2 x_2 + v \rangle = -\theta_2 x_2^2 + x_2 v. \quad (27)$$

Thus, Assumption 1 holds with $\rho(x_2) = \frac{1}{2}\theta_2 x_2^2$, $P = M$, $R = 0$, $Q = 0$ and $S = \frac{1}{2}$. Noting that $\rho(x_2) = \frac{1}{2}y^2$ with $y = x_2$, we have proved that the conditions of Corollary 1 are verified with $E = \frac{1}{4}\theta_2$ and $\bar{\rho}(x_2) = \frac{1}{4}\theta_2 x_2^2$, $x_2 \in \mathbb{R}$. We are left with proving that (17) is guaranteed. The first inequality in (17) is trivially satisfied by taking $\mathcal{A} = \mathbb{R}^2$. The second inequality of (17) holds with $E = \frac{1}{4}\theta_2$ and $\alpha(s) = \frac{1}{2}\theta_2 s$ for $s \in \mathbb{R}_{\geq 0}$. We have proved that the conditions of Corollary 1 and Theorem 2 are guaranteed.

As a conclusion, the proposed approach allows ensuring the strict passivity of the x_2 -system in (24) with output x_2 using event-triggered controllers.

IX. CONCLUSIONS

We have presented a method to ensure quadratic dissipation inequalities for nonlinear systems using emulation-based event-triggered controllers. The triggering condition is designed using an auxiliary variable. A feature of the triggering rule is that it adapts to the desired accuracy with which the quadratic terms of the initial supply rate are preserved by appropriately selecting its parameters. New results have also been presented for time-triggered control, in particular new MASP bounds are provided. Finally, we have shown how the approach can be applied to linear time-invariant systems, and that it can be used to construct event-triggered controllers which guarantee a strict passivity property for the pendulum. We think that the presented results will be relevant in a variety of control problems where dissipativity plays a key role, like coordination, robustness analysis, and stabilization problems.

ACKNOWLEDGEMENTS

The author would like to thank Claudio de Persis and Dragan Nešić for helpful discussions.

REFERENCES

- [1] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić. Input-to-state stabilization of nonlinear systems using event-triggered output feedback controllers. In *European Control Conf., Linz, Austria*, 2015.
- [2] V. Andrieu, M. Nadri, U. Serres, and J.-C. Vivalda. Continuous discrete observer with updated sampling period. In *IFAC Symposium on Nonlinear Control Systems, Toulouse, France*, volume 9, pages 439–444, 2013.
- [3] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8):1380–1390, 2007.
- [4] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and Control*. Springer-Verlag, London, 2nd edition, 2007.
- [5] C. Cai and A.R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58:47–53, 2009.
- [6] D.V. Dimarogonas, E. Frazzoli, and K.H. Johansson. Distributed event-triggered control for multi-agent systems. *IEEE Transactions on Automatic Control*, 57(5):1291–1297, 2012.
- [7] V.S. Dolk, D.P. Borgers, and W.P.M.H. Heemels. Dynamic event-triggered control: Tradeoffs between transmission intervals and performance. In *IEEE Conference on Decision and Control, Los Angeles, U.S.A.*, pages 2764–2769, 2014.
- [8] M.C.F. Donkers and W.P.M.H. Heemels. Output-based event-triggered control with guaranteed \mathcal{L}_∞ -gain and improved and decentralized event-triggering. *IEEE Transactions on Automatic Control*, 57(6):1362–1376, 2012.
- [9] A. Girard. Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control (available on-line)*, 2015.
- [10] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid dynamical systems*. Princeton University Press, Princeton, U.S.A., 2012.
- [11] W.M. Haddad, V. Chellaboina, and S.G. Nersisov. *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton University Press, 2006.
- [12] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *CDC (IEEE Conference on Decision and Control) Maui, U.S.A.*, 2012.
- [13] A. Isidori. *Nonlinear Control Systems II*. Springer-Verlag, London, 1999.
- [14] H.K. Khalil. *Nonlinear systems*. Prentice-Hall, Englewood Cliffs, New Jersey, U.S.A., 3rd edition, 2002.
- [15] D. S. Laila, D. Nešić, and A.R. Teel. Open and closed loop dissipation inequalities under sampling and controller emulation. *European Journal of Control*, 18:109–125, 2002.
- [16] M. Lemmon. Event-triggered feedback in control, estimation, and optimization. In *Networked Control Systems*, Lecture Notes in Control and Information Sciences, A. Bemporad and W.P.M.H. Heemels and M. Johansson (Eds), pages 293–358. Springer-Verlag, London, 2010.
- [17] M.J. McCourt, E. Garcia, and P.J. Antsaklis. Model-based event-triggered control of nonlinear dissipative systems. In *American Control Conference, Portland, U.S.A., 2014*, pages 5355–5360, 2014.
- [18] S. Monaco and D. Normand-Cyrot. On the conditions of passivity and losslessness in discrete-time. In *European Control Conference, Brussels, Belgium*, 1997.
- [19] S. Monaco, D. Normand-Cyrot, and F. Triefensee. Sampled-data stabilization; a PBC approach. *IEEE Transactions on Automatic Control*, 56(4):907–912, 2011.
- [20] R. Naldi and R.G. Sanfelice. Passivity-based control for hybrid systems with applications to mechanical systems exhibiting impacts. *Automatica*, 49(5):1104–1116, 2013.
- [21] D. Nešić, A.R. Teel, and D. Carnevale. Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. *IEEE Trans. on Automatic Control*, 54(3):619–624, 2009.
- [22] R. Ortega, A. Loria, P.J. Nicklasson, and H. Sira-Ramírez. *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Springer-Verlag, London, 1998.
- [23] C. De Persis and P. Frasca. Robust self-triggered coordination with ternary controllers. *IEEE Trans. on Aut. Control*, 58:3024–3038, 2013.
- [24] C. De Persis and R. Postoyan. A Lyapunov redesign of coordination algorithms for cyberphysical systems. In *arXiv 1404.0576*, 2014.
- [25] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996, 2015.
- [26] R. Sepulchre, M. Jankovic, and P.V. Kokotović. *Constructive Nonlinear Control*. Springer-Verlag, London, 1997.
- [27] G.S. Seyboth, D.V. Dimarogonas, and K.H. Johansson. Event-based broadcasting for multi-agent average consensus. *Automatica*, 49:245–252, 2012.
- [28] D. Shi, T. Chen, and L. Shi. Event-triggered maximum likelihood state estimation. *Automatica*, 50(1):247–254, 2014.
- [29] S. Stramigioli, C. Secchi, A.J. van der Schaft, and C. Fantuzzi. A novel theory for sampled data system passivity. In *IEEE/RSJ International Conference on Intelligent Robots and System, Lausanne, Switzerland*, volume 2, pages 1936–1941, 2002.
- [30] A.R. Teel. Asymptotic stability for hybrid systems via decomposition, dissipativity, and detectability. In *IEEE Conference on Decision and Control, Atlanta, U.S.A.*, pages 7419–7424, 2010.
- [31] X. Wang and M.D. Lemmon. Self-triggered feedback control systems with finite-gain \mathcal{L}_2 stability. *IEEE Transactions on Automatic Control*, 45:452–467, 2009.
- [32] H. Yu and P.J. Antsaklis. Event-triggered output feedback control for networked control systems using passivity: Achieving \mathcal{L}_2 stability in the presence of communication delays and signal quantization. *Automatica*, 49:30–38, 2012.
- [33] F. Zhu, M. Xia, and P.J. Antsaklis. Passivity analysis and passivation of interconnected event-triggered feedback systems using passivity indices. In *IFAC World Congress, Cape Town, South Africa*, pages 11177–11182, 2014.