

Singular stochastic PDE and Dynamical field theory models

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Plan of the presentation

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Singular stochastic PDE

SSPDE : Non-linear Langevin equations “whose solutions” are supposed to generate Markov processes with Euclidean field theory measures as invariant measures. This is also known as [stochastic quantisation](#).

A priori ill defined problem.

- ▶ EFT measures require renormalization to be defined. They are realized on spaces of distributions $\mathcal{D}'(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$.
- ▶ The SSPDE = Non-linear Langevin equations have coefficients inherited from EFT measures. These are typically distributions. Nonlinear \leftrightarrow pointwise products \leftrightarrow renormalization.
If “solutions” exist, Markov process is distribution valued.

What sort of solutions ?

Different points of view :

- **Probabilistic weak solutions/ Martingale solutions** : With cut-offs these are also strong solutions and Martingale solutions exist. This leads to Functional integral/RG point of view to construction of semigroup/invariant measures. We will elaborate on this later.

In $D = 2$, G. Jona-Lasinio and P. K. Mitter considered a modified non-linear Langevin equation and proved in the continuum limit the existence of a martingale (weak) solution with the finite volume continuum massive ϕ_2^4 measure as the unique invariant measure. This was the first mathematical result in this subject.

What sort of solutions ?

► Pathwise approaches

Pathwise solutions of dynamical super-renormalizable Euclidean Field theories. Counter-terms : those of super-renormalizable EFT's. (massive ϕ_d^4 $d = 2, 3$.)

a) $D = 2$: G. Da Prato and A. Debussche were the first to produce a unique strong solution with the finite volume continuum ϕ_2^4 measure as the invariant measure.

b) $D = 2, 3$: M. Hairer (theory of regularity structures), M. Gubinelli *et. al.* (paracontrolled distributions), Catelier and Chouk: These are theories of multiplication of distributions with counter-terms. Verified in low order perturbation theory. The remainder is controlled as a fixed point problem in a Banach space.

What sort of solutions ?

- c) Antti Kupiainen : **RG approach directly on the equation**

$D = 3$: UV cut-off noise, finite volume. Rescale to unit cut-off with enlarged volume. This gives rise to a sequence of effective equations with rescaling at each step \rightarrow Perturbative part + remainder. The limit of the sequence of remainders has been proved to exist by solving a Banach fixed point problem.

b), c) are : **short time solutions with the upper bound on time dependent on noise.**

Global solution, $D = 2$ by J.C. Mourrat and H.Weber.

BUT : Hairer gives optimal regularity for paths, and initial conditions.

An example

Scalar field theory in dimension D , $\phi(x)$, $x \in \mathbb{R}^D$.

$C_0(x - y)$ = Fourier transform of $\hat{C}_0(k) = \frac{1}{k^2 + m^2}$.

κ : UV cut-off

$$\hat{C}_\kappa(k) = \frac{1}{(k^2 + m^2)(1 + \frac{k^2}{\kappa^2})^p} \text{ for sufficiently large } p.$$

The random gaussian field ϕ in \mathbb{R}^d with covariance C_0 is a distributon for $D \geq 2$. For p sufficiently large, ϕ distributed according to C_κ is locally sufficiently differentiable.

$$d\mu_\kappa(\phi) = \frac{1}{Z_\kappa(\Lambda)} \int d\mu_{C_\kappa}(\phi) e^{-V_\kappa(\phi, \Lambda)}.$$

$$V_\kappa(\phi, \Lambda) = \int_\Lambda d^D x \left\{ \frac{\lambda}{4} : \phi^4 :_{C_\kappa}(x) + \text{counterterms}_\kappa \right\}$$

Λ_L : cube side, periodic b.c., $\Lambda_L = \mathbb{R}^D / (L\mathbb{Z}^D)$. C_κ : periodized covariance

Dynamical models

Nonlinear Langevin equations

Large class of equations available, such that if solutions exist, they have the same invariant measure.

Example : Let $0 < \rho \leq 1$.

$$\begin{aligned} d\tilde{\phi}_t &= dW_t - \frac{1}{2} \left(C_\kappa^{-\rho} \tilde{\phi}_t + \lambda C^{1-\rho} : \tilde{\phi}_t^3 :_{C_\kappa} \right) dt, \\ \tilde{\phi}_0 &= \phi, \end{aligned}$$

values in subspace of $\mathcal{D}'(\Lambda_L)$, (sufficiently differentiable functions).

f, g are test functions in $\mathcal{D}(\Lambda_L)$.

$$E(W_t(f), W_t(g)) = (f, C_\kappa^{1-\rho} g) \min(t, s).$$

[Canonical choice $\rho = 1$.] Counter-terms in the drift omitted.

Dynamical models

If the solutions exist, then there is a generator $L_{\kappa,p}$ and it is easy to see

$$L_{\kappa,p} = \frac{1}{2} \int dx dy C^{1-\rho}(x-y) \left(\frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} - \frac{\delta\mathcal{S}}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \right).$$

is symmetric with respect to $L^2(d\mu_{\kappa,\Lambda})$ and formally,

$$\int d\mu_{\kappa,\Lambda}(\phi) e^{tL_{\kappa,p}} F(\phi) = \int d\mu_{\kappa,\Lambda}(\phi) F(\phi).$$

F : bounded \mathcal{C}^2 cylindrical function.

Each choice of $\rho \in (0, 1]$ will, if equation can be solved, lead to Markov processes with same invariant measure $\mu_{\kappa,\Lambda}$.

Linear processes

$$\begin{aligned} d\phi_t &= dW_t - \frac{1}{2} C_\kappa^{-\rho} \phi_t dt, \\ \phi_t &= \phi \end{aligned}$$

This is an Ornstein-Uhlenbeck process/Langevin equation.

This has a unique solution

$$\phi_t = e^{-t/2} \phi_0 + \int_0^t e^{-\frac{1}{2}(t-s)} C_\kappa^{-\rho} dW_\kappa.$$

The O-U process has continuous sample paths.

Generator :

$$L_\kappa^{(0)} = \frac{1}{2} \int dx \int dy \left[C_\kappa^{1-\rho}(x-y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} - C_\kappa^{-\rho}(x-y) \phi(x) \frac{\delta}{\delta\phi(y)} \right].$$

Linear processes

Transition probability : $p_t(\phi, \mathcal{B}) = \mu_{C_{t,\kappa}} \left(B - e^{-\frac{t}{2} C_\kappa^{-\rho}} \phi \right),$
 \mathcal{B} : Borel set in $\mathcal{D}'(\Lambda_L).$

$$C_{t,\kappa} = (1 - e^{-t C_\kappa^{-\rho}}) C_\kappa.$$

μ_{C_κ} : invariant measure.

P_ϕ^{OU} : O-U measure on path space : $\mathcal{C}^0([0, \infty), \mathcal{D}'(\Lambda_L)).$

In terms of the linear process, the full process must solve the integral equation :

$$\tilde{\phi}_t = \phi_t - \frac{\lambda}{2} \int_0^t ds e^{-(t-s) C_\kappa^{-\rho}} C_\kappa^{1-\rho} : \tilde{\phi}_s^3 :_{C_\kappa}.$$

Probabilistic weak solution

Girsanov formula :

$$e^{t L_\kappa} F(\phi) = E_{\phi_0 = \phi}^{(w)} \left(F(\phi_t) e^{\xi_0^{t, \kappa}} \right) \quad (*)$$

O-U process : ϕ_t is a measurable function of w_t .

$$\xi^{0, (\kappa)} = -\frac{\lambda}{2} \int_0^t (:\phi_s^3 :_{C_\kappa}, dw_s) - \frac{\lambda^3}{8} \int_0^t ds (:\phi_s^3 :_{C_\kappa}, C^{1-\rho} : \phi_s^3 :_{C_\kappa}).$$

Because of the cut-off κ Ito's formula applied to $e^{\xi_0^{t, \kappa}}$ shows

$$E^{(w)}(e^{\xi_0^{t, \kappa}}) = 1. \quad (**)$$

Martingale property :

$$\begin{aligned} E^{(w)}(e^{\xi_s^{t, \kappa}} | \mathcal{F}_s) &= 1, \\ E^{(w)}(e^{\xi_0^{t, \kappa}} | \mathcal{F}_s) &= e^{\xi_0^{s, \kappa}}. \end{aligned}$$

Probabilistic weak solution

Because of $(**)$ in the previous slide, the representation $(*)$ defines a **semigroup** with transition probabilities,

$$\hat{p}_t^\kappa(\phi, \mathcal{B}) = E_\phi^{(w)} \left(\chi(\phi_t \in \mathcal{B}) e^{\xi_0^{t, \kappa}} \right),$$

and the family of transition probabilities gives a new probability measure \hat{P}_ϕ^κ on $C_0([0, T], \mathcal{D}'(\Lambda)) = \Omega_T$.

\hat{P}_ϕ^κ solves the non-linear integral equation in weak sense. Under \hat{P}_ϕ^κ

$$Z_t = \hat{\phi}_t + \frac{\lambda}{2} \int_0^t ds e^{-(t-s)} C_\kappa^{-\rho} C_\kappa^{1-\rho} : \hat{\phi}_s^3 :_{C_\kappa}, \quad \hat{\phi}_t \in \Omega_T$$

has probability distribution of an O-U process.

The drift is a gradient vector field.

Therefore we can get rid of the stochastic integral in the Girsanov exponent ξ_0^t , using Ito's formula.

The result is :

$$\xi_t = -\frac{1}{2} V_\kappa(\phi_t) + \frac{1}{2} V_\kappa(\phi_0) - \int_0^t ds \tilde{V}_\kappa(\phi_s),$$

$$\begin{aligned} \tilde{V}_\kappa(\phi_s) &= \frac{\lambda}{4} : (\phi_s^3, C_\kappa^{-\rho} \phi_s)_{L^2(\Lambda)} :_{C_\kappa} + \frac{\lambda^2}{8} (: \phi_s^3 :_{C_\kappa}, C_\kappa^{1-\rho} : \phi_s^3 :_{C_\kappa})_{L^2} \\ &\quad - \frac{3\lambda}{4} : (\phi_s, C_\kappa^{1-\rho} \phi_s)_{L^2(\Lambda)}, \end{aligned}$$

$$\left(e^{t L_\kappa} \right) (\phi) = E_{\phi_0=\phi} \left(F(\phi_t) e^{-\frac{1}{2} V_\kappa(\phi_t) + \frac{1}{2} V_\kappa(\phi_0) - \int_0^t ds \tilde{V}_\kappa(\phi_s)} \right).$$

$$\begin{aligned}
& \left(G, e^{t L_\kappa} F \right)_{L^2(d\mu_\kappa, \lambda, \Lambda_L)} \\
&= \int d\mu_{C_\kappa}(\phi) e^{-\frac{1}{2} V_\kappa(\phi)} G(\phi) E_{\phi_0=\phi} \left(F(\phi_t) e^{-\frac{1}{2} V_\kappa(\phi_t) - \int_0^t ds \tilde{V}_\kappa(\phi_s)} \right)
\end{aligned}$$

where C_κ is the cutoff covariance periodized on the torus Λ_L and the Λ_L dependence of V and \tilde{V} has been suppressed.

The previous formula actually gives an unitary transformation.
Define

$$\tilde{F}_\kappa(\phi) = e^{-\frac{1}{2} V_\kappa(\phi)} F(\phi)$$

and observe the isometry

$$\begin{aligned} L^2(d\mu_\kappa) &\rightarrow L^2(d\mu_{C_\kappa}) \\ \|\tilde{F}_\kappa\|_{L^2(d\mu_{C_\kappa})}^2 &= \|F\|_{L^2(d\mu_\kappa)}^2 \end{aligned}$$

The previous isometry leads to the representation

$$\left(G, e^{t L_\kappa} F \right)_{L^2(d\mu_\kappa)} = \left(\tilde{G}_\kappa, e^{t \tilde{L}_\kappa} \tilde{F}_\kappa \right)_{L^2(d\mu_{C_\kappa})}$$

where

$$e^{t \tilde{L}_\kappa} \tilde{F}(\phi) = E_{\phi_0=\phi} \left(F(\phi_t) e^{-\int_0^t ds \tilde{V}_\kappa(\phi_s)} \right)$$

ULTRAVIOLET CUT-OFF REMOVAL ($\kappa \rightarrow \infty$) .

$D = 2$ The weak solution of Jona-Lasinio, Mitter in the limit $\kappa \rightarrow \infty$ (1985)

Here the parameter $\rho : 0 < \rho \leq 1$ plays an important role. In other words we are studying a modified non-linear Langevin (Glauber) dynamics. The initial choice of ρ was very restricted for technical reasons : $0 < \rho < \frac{1}{10}$. Progressively, this restriction was removed by Rozovskii and Mikulevicius (1998) : $0 < \rho < 1$ and then $\rho = 1$. A strong solution was given by G. da Prato and Debussche (2003) for $\rho = 1$ in a very important work which introduced new technology (use of Besov spaces). Finally (2015) J-C Mourrat and H.Weber have extended da Prato's work to prove global in time (and also in space) pathwise solutions.

Estimates [J-L,M (CMP-1985), M(Spain-1986)], $\kappa \rightarrow \infty$

1) $D = 2$ $C_\kappa \xrightarrow{\kappa \rightarrow \infty} C$ (for massive covariance).

: $\phi^n :_C (f) \in L^p(d\mu_C), \forall p : 1 \leq p \leq \infty.$

2) The O-U transition probability $\mu_{C_t}(\phi, \cdot)$ is absolutely continuous with respect to $\mu_C(\cdot)$, μ_C a.e. in ϕ provided $\rho > 0$. The Radon-Nikodym derivative is then in $L^2(d\mu_C)$, μ_C a.e. in ϕ .

If $h \in L^{2p}(d\mu_C)$, then trivially,

$$E_\phi(|h(\phi_t)|^p) < \infty, \quad 1 \leq p < \infty.$$

So $h(\phi_t) \in L^p(dP_\phi, \Omega).$

3) ϕ_t is continuous in t , P_ϕ a.s., μ_C a.e. in ϕ .

Moreover the O-U semigroup is *hypercontractive*.

Using Riemann sum approximation, with $\kappa \rightarrow \infty$

$$\int_0^t ds h(\phi_s) \in L^p(dP_\phi, \Omega), \mu_C \text{ a.e. in } \phi.$$

Therefore provided $0 < \rho < 1$

$$E_\phi \left(\int_0^t ds (:\phi^3:_C, C^{1-\rho} : \phi^3 :_C)_{L^2} \right) < \infty,$$

μ_C a.e. in ϕ .

Ito isometry

$$E_{\phi} \left[\left(\int_0^t (:\phi_s^3:, dW_s) \right)^2 \right] = E_{\phi} \left(\int_0^t ds (:\phi^3:_C, C^{1-\rho} : \phi^3:_C)_{L^2} \right) < \infty,$$

μ_C a.e. in ϕ .

Because of the above, $\xi_0^{t,\infty}$ exists as a random variable.

4) We know

$$E_\phi(e^{\xi_0^{t,\kappa}}) = 1, \quad \mu_C \text{ a.e.}$$

By Fatou's lemma

$$E_\phi(e^{\xi_0^{t,\infty}}) \leq 1, \quad \mu_C \text{ a.e.}$$

For a martingale/weak solution we have to prove

$$E_\phi(e^{\xi_0^{t,\infty}}) \stackrel{?}{=} 1.$$

5)

$$e^{\xi_0^{t,\infty}} = e^{\xi_0^{t,\infty}} - e^{\xi_0^{t,\kappa}} + e^{\xi_0^{t,\kappa}}.$$

$$|E_\phi(e^{\xi_0^{t,\infty}}) - 1| \leq |E(e^{\xi_0^{t,\infty}} - e^{\xi_0^{t,\kappa}})| \quad (***)$$

$$\begin{aligned} |E(e^{\xi_0^{t,\infty}} - e^{\xi_0^{t,\kappa}})| &\leq E_\phi(|\xi_0^{t,\infty} - \xi_0^{t,\kappa}| |e^{\xi_0^{t,\infty}} + e^{\xi_0^{t,\kappa}}|) \\ &\leq (E_\phi(|\xi_0^{t,\infty} - \xi_0^{t,\kappa}|^2))^{\frac{1}{2}} \left[\left(E_\phi(e^{2\xi_0^{t,\infty}})\right)^{\frac{1}{2}} + \left(E_\phi(e^{2\xi_0^{t,\kappa}})\right)^{\frac{1}{2}} \right] \end{aligned}$$

It is easy to show $|\xi_0^{t,\infty} - \xi_0^{t,\kappa}| \rightarrow 0$ in $L^2(dP_\phi^W, \Omega)$, μ_C a.e. in ϕ .

6) Lemma

Suppose $0 < \rho < \frac{1}{10}$. Then

$$E_{\phi} \left(e^{2\xi_0^{t,\infty}} \right) < \infty, \quad \mu_C \text{ a.e. in } \phi.$$

$e^{2\xi_0^{t,\kappa}}$ is uniformly bounded in $L^2(dP_{\phi}^W, \Omega)$.

7) Therefore taking $\kappa \rightarrow \infty$ in $(***)$ we have

$$E_{\phi} \left(e^{\xi_0^{t,\infty}} \right) = 1.$$

Semigroup and Invariant measure

Because of 7) in previous frame we have an $L^\infty(d\mu_\Lambda)$ semigroup defined μ a.e. on bounded measurable functions

$$e^{tL_\infty} F(\phi) = \hat{E}_\phi(F(\hat{\phi}_t)) = E_\phi(F(\phi_t) e^{\xi_0^{t,\infty}})$$

where \hat{E} is the expectation with respect to a new path space measure \hat{P}_ϕ on $C^0([0, T], \mathcal{D}'(\Lambda)) = \Omega_T$. We have

$$e^{tL_\infty} 1 = \hat{E}_\phi(1) = E_\phi(e^{\xi_0^{t,\infty}}) = 1$$

The semigroup is also symmetric in $L^2(d\mu_\Lambda)$ restricted to the subspace of bounded measurable functions and is given by a probability measure. Therefore μ_Λ is an invariant measure:

$$\int d\mu_\Lambda e^{tL_\infty} F = \int d\mu_\Lambda F$$

where F is a bounded measurable function. The invariant measure is unique because the semigroup is positivity improving.

The above facts are enough to prove that the semigroup is actually a strongly continuous contraction on $L^p(d\mu_\Lambda)$ for $1 \leq p \leq \infty$.

Proof of the lemma

The idea is to undo the stochastic integral in $\xi_0^{t,\infty}$ since the drift perturbation is a gradient. To undo the stochastic integral we use the Ito's formula. Then we see that each term exists as a random variable in $L^2(dP_\phi^W, \Omega)$, μ_C a.e. in ϕ provided $0 < \rho < \frac{1}{2}$.

Then we use the method of [Nelson and Glimm](#) from the earliest days of constructive QFT. For the stability estimate to work we need to restrict : $0 < \rho < \frac{1}{10}$.

$$\int d\mu(\phi) E_{\phi}(e^{2\xi_0^{t,\infty}}) = \int d\mu_C(\phi) E_{\phi}\left(e^{-V(\phi_t) - 2\int_0^t ds \tilde{V}(\phi_s)}\right)$$

Where

$$\begin{aligned} 2\tilde{V}(\phi_s) &= \frac{\lambda}{2} : (\phi_s^3, C^{-\rho}\phi_s)_{L^2(\Lambda)} :_C + \frac{\lambda^2}{4} (: \phi_s^3 :_C, C^{1-\rho} : \phi_s^3 :_C)_{L^2} \\ &\quad - \frac{3\lambda}{2} : (\phi_s, C^{1-\rho}\phi_s) :_C \\ &\leq \left(\int d\mu_C(\phi) E_{\phi}(e^{-2V(\phi_t)}) \right)^{\frac{1}{2}} \underbrace{\left(\int d\mu_C(\phi) E_{\phi}(e^{-4\int_0^t ds \tilde{V}(\phi_s)}) \right)^{\frac{1}{2}}}_{(***)} \end{aligned}$$

1) The first factor in $(***)$ is easily proven to be finite.

$$\begin{aligned} \int d\mu_C(\phi) E_\phi(e^{-2V(\phi_t)}) &= \int d\mu_C(\phi) e^{tL_0} e^{-2V(\phi)} \\ &= \int d\mu_C(\phi) e^{-2V(\phi)} < \infty, \end{aligned}$$

by Nelson's estimate (have used that μ_C is invariant measure of O-U process).

2)

$$\int d\mu_C(\phi) E_{\phi}^{(w)}(e^{-4 \int_0^t ds \tilde{V}(\phi_s)}) \leq \int d\mu_C(\phi) e^{-4t \tilde{V}(\phi)}.$$

(ϕ_s a.s. continuous, Riemann sum approximation, Hölder's inequality)

$\tilde{V}(\phi)$: The non-local ϕ^6 term is a positive random variable, can be dropped.

The negative sign mass term is dominated by the Gaussian measure, for small λ .

We are left with the estimate provided by the following proposition.

Proposition

For $0 < \rho < \frac{1}{10}$

$$\int d\mu_C(\phi) e^{-pt\lambda G(\phi)} < \infty$$

$$G(\phi) =: (\phi^3, C^{-\rho}\phi)_{L^2} :_C, \quad \text{in } L^p(d\mu_C) \text{ for } 0 < \rho < \frac{1}{2}.$$

► Step 1

$$(\phi^3, C^{-\rho}\phi)_{L^2(\Lambda)} \geq \int_{\Lambda} d^2x \phi^4(x), \quad 0 < \rho < 1.$$

Proved using spectral representation and Young's inequality.

► Step 2

UV cut-off field (cut-off Fourier modes)

$$\phi_\kappa(x) = \int_{|k| \leq \kappa} \frac{d^2 k}{(2\pi)^2} e^{ik \cdot x} \hat{\phi}(k).$$

$C_\kappa(x, y)$ is μ_C covariance of ϕ_κ .

$$G_\kappa(\phi) = G(\phi_\kappa).$$

$$G_\kappa \rightarrow G \text{ in } L^p(d\mu_C) \text{ for } 0 < \rho < \frac{1}{2}.$$

Undo Wick ordering in G_κ . The Wick constants $\rightarrow \infty$ when $\kappa \rightarrow \infty$.

Using Step 1 and estimates of Wick constants get

$$G_\kappa \geq -\text{const} \times (\kappa^{4\rho} (\ln \kappa)^2)$$

Step 3

Define $\tilde{G}_\kappa = G - G_\kappa$.

Then :

$$\int d\mu_C |\tilde{G}_\kappa|^{2j} \leq (j!)^4 b^j ((\ln \kappa)^m \kappa^{-2+4\rho})^j \quad \forall j, \text{ some } m > 0.$$

Hypercontractivity to reduce $L^p(d\mu_C)$ estimates to $L^2(d\mu_C)$ estimates, then Feynman graph computation.

Step 4

$$\begin{aligned} \mu_C \left(\{G \leq -\text{const} (\kappa^{4\rho} (\ln \kappa)^2) - 1\} \right) &\leq \mu_C \left(\{|\tilde{G}_\kappa|^{2j} \geq 1\} \right) \\ &\leq \int d\mu_C |\tilde{G}_\kappa|^{2j} \leq (j!)^4 b^j (\ln \kappa)^m (\kappa^{-2+4\rho})^j \end{aligned}$$

Then Stirling's approximation and optimal κ -dependent choice of j gives

$$\begin{aligned} \mu_C \left(\{G \leq -\text{const} (\kappa^{4\rho} (\ln \kappa)^2) - 1\} \right) \\ \leq e^{-\text{const} \left(\kappa^{\frac{2-4\rho}{4}} (\ln \kappa)^{-\frac{m}{4}} \right)} \end{aligned}$$

Step 5

$$\kappa_n = 2^n \rightarrow \infty, C_{\kappa_n} = \text{const } \kappa_n^{4\rho} (\ln \kappa_n)^2.$$

$$\begin{aligned} \int d\mu_C e^{-G} &= \int d\mu_C e^{-G} (\mathcal{X}\{G > -C_{\kappa_0} - 1\} + \mathcal{X}\{G \leq -C_{\kappa_0} - 1\}) \\ &\leq e^{C_{\kappa_0} + 1} + \mathcal{I}_0 \end{aligned}$$

Step 5 continued

$$\begin{aligned}
\mathcal{I}_0 &= \int d\mu_C e^{-G} \mathcal{X}\{G \leq -C_{\kappa_0} - 1\} \\
&= \int d\mu_C e^{-G} \sum_{n=0}^{\infty} \mathcal{X}\{-C_{\kappa_{n+1}} - 1 < G \leq -C_{\kappa_n} - 1\} \\
&\leq \sum_{n=0}^{\infty} e^{C_{\kappa_{n+1}} + 1} \mu_C\{G \leq -C_{\kappa_n} - 1\} \\
&\leq \sum_{n=0}^{\infty} e^{C_{\kappa_{n+1}} + 1} e^{-\text{const} \left(\kappa^{\frac{2-4\rho}{4}} (\ln \kappa)^{-\frac{m}{4}} \right)},
\end{aligned}$$

and using the definition of C_{κ_n} , this series converges for $\rho < \frac{1}{10}$.



Stationary O-U process

\tilde{P} : measure on path space $\Omega = \mathcal{C}^0([0, \infty), \mathcal{D}')$.

$$\tilde{P}(B) = \int d\mu_{C_\kappa}(\phi) \tilde{P}_\phi(B), \quad \tilde{P}_\phi : \text{O-U measure.}$$

- Stationarity : $E^{\tilde{P}}(\phi_t(f) \phi_s(g)) = E^{\tilde{P}}(\phi_{t-s}(f) \phi_0(g)), \quad t > s.$
- Symmetry : $E^{\tilde{P}}(\phi_t(f) \phi_s(g)) = E^{\tilde{P}}(\phi_t(g) \phi_s(f)).$
- Covariance : $E^{\tilde{P}}(\phi_t(f) \phi_s(g)) = C_\kappa(f, e^{-\frac{t-s}{2}} C_\kappa^{-p} g).$

Because of stationarity : path space $\Omega \rightarrow \tilde{\Omega} = \mathcal{C}^0((-\infty, \infty), \mathcal{D}')$

All of this is true when $\kappa \rightarrow \infty$.

Euclidean formalism

A Euclidean formalism for the canonical choice $\rho = 1$.

$$x = (x_1, \dots, x_D) \in \mathbb{R}^D \quad x_0 \in \mathbb{R},$$

$$\tilde{x} = (x_0, x) \in \mathbb{R} \times \Lambda_L \subset \mathbb{R}^{D+1},$$

$\phi(\tilde{x}) = \phi(x_0, x)$, x_0 = “time” coordinate. “Time” is the time the process has run, which we call the **Langevin** time.

Define :

$$\tilde{C}_\kappa(\tilde{x}, \tilde{y}) = \int \frac{dk_0}{2\pi} \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik_0(x_0 - y_0) + i(k, x - y)_{\mathbb{R}^D}}}{k_0^2 + \hat{C}_\kappa^{-2}(k)}.$$

$\mu_{\tilde{C}_\kappa}$: Gaussian measure, covariance \tilde{C}_κ on $\mathcal{D}'(\mathbb{R} \times \Lambda_L)$.

Euclidean formalism

Consider fixed time test functions:

$$f_t(\tilde{x}) = f(x)\delta(x_0 - t), \quad t > 0$$

Take $t > s$. Then an easy computation shows

$$\begin{aligned} \int d\mu_{\tilde{C}_\kappa}(\phi) \phi(f_{\frac{t}{2}}) \phi(f_{\frac{s}{2}}) &= \left(f, C_\kappa e^{-\frac{|t-s|}{2}} C_\kappa^{-1} g \right)_{L^2} \\ &= E^{\tilde{P}^{OU}}(\phi_t(f) \phi_s(g)). \end{aligned}$$

Thus the analogy is given by a pull up

\tilde{C}_κ : “Euclidean covariance”.

C_κ : “Fock space covariance”.

“Time” reflection θ : The positive time subspace $\mathcal{D}_+(\mathbb{R} \times \Lambda_L)$ consists of test functions $f(\tilde{x})$ such that $f(x, t) = 0 : t < 0$. The time reflection operation on test functions in $\mathcal{D}(\mathbb{R} \times \Lambda_L)$ is

$$\theta f(x, t) = f(x, -t)$$

Reflection positivity : in the O-S inner product \langle, \rangle_+ the covariance \tilde{C} is positive definite. Let $f \in \mathcal{D}_+(\mathbb{R} \times \Lambda_L)$. Then computation shows:

$$\langle f, \tilde{C}f \rangle_+ = (\theta f, \tilde{C}f) \geq 0$$

$$\langle \phi(f), \phi(f) \rangle_+ = \int d\mu_{\tilde{C}} \phi(\theta f) \phi(f) = (\theta f, \tilde{C}f) \geq 0$$

The Osterwalder-Schrader Hilbert space

More generally, following Fröhlich or Glimm-Jaffe (see references) we consider the Hilbert space \mathcal{E} spanned by vectors in $L^2(d\mu_{\tilde{C}})$ of the form $F(\phi(f)) = \sum_{i=1}^n a_i e^{i\phi(f_i)}$ where the $f_i \in \mathcal{D}$ and the a_i are complex numbers. Restrict \mathcal{E} to the positive time subspace \mathcal{E}_+ obtained by restricting the test functions to \mathcal{D}_+ . Then computation shows that the Osterwalder Schrader inner product $\langle \cdot, \cdot \rangle_+$ satisfies for $f \in \mathcal{D}_+$

$$\langle F(\phi(f)), F(\phi(f)) \rangle_+ = \int d\mu_{\tilde{C}} \bar{F}(\phi(\theta f)) F(\phi(f)) \geq 0$$

Quotient by the null space \mathcal{N}_+ in the Osterwalder-Schrader inner product to get $\mathcal{E}_+/\mathcal{N}_+$. $\mathcal{E}_+/\mathcal{N}_+$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is the O-S Hilbert space \mathcal{H} .

The Osterwalder Schrader semigroup

The semigroup \tilde{T}_t is defined on the $O - S$ Hilbert space \mathcal{H} by

$$\langle F(\phi(f)), \tilde{T}_t G(\phi(f)) \rangle_+ = \int d\mu_{\tilde{C}} F(\phi(\theta f)) G(\phi(T_t f))$$

where on the right hand side T_t is the time translation operator.

The semigroup which acts on \mathcal{E}_+ acts on the physical Hilbert space \mathcal{H} since it preserves the null space \mathcal{N}_+ in the O-S inner product.

Then it can be shown (see Osterwalder-Schrader, Fröhlich and lectures of Glimm-Jaffe) that \tilde{T}_t is a self adjoint, contractive and strongly continuous operator on the Hilbert space \mathcal{H} , uniformly bounded in t . These considerations generalise when we add local interactions which do not upset reflection positivity.

The Euclidean Functional Integral

The partition function for stochastic quantisation in the torus Λ_L and finite “Time “ interval $[-T/2, T/2]$ is defined by

$$Z_{\kappa, T, \Lambda_L} = \int d\mu_{\tilde{C}_\kappa}(\phi) \exp - \left(\int_{-T/2}^{T/2} ds \tilde{V}_\kappa(\phi_s, \Lambda_L) \right)$$

where \tilde{V}_κ is as defined in an earlier frame. We have set $\rho = 1$. The corresponding probability measure is then given by

$$d\hat{P}_{\kappa, T, \Lambda_L} = \frac{1}{Z_{\kappa, T, \Lambda_L}} d\mu_{\tilde{C}_\kappa}(\phi) \exp - \left(\int_{-T/2}^{T/2} ds \tilde{V}_\kappa(\phi_s, \Lambda_L) \right)$$

This measure is **reflection positive** in the **Langevin** time.

The correlation functions are given by

$$\langle F(\phi(f))G(\phi(g)) \rangle_{\kappa, T, \Lambda_L} = \int d\hat{P}_{\kappa, T, \Lambda_L} \tilde{F}(\phi(f), \Lambda_L) \tilde{G}(\phi(g), \Lambda_L)$$

where

$$\tilde{F}(\phi(f), \Lambda_L) = e^{-\frac{1}{2} V_{\kappa}(\phi, \Lambda_L)} F(\phi(f))$$

The existence of the limit $T \rightarrow \infty$ can be proved. The Osterwalder-Schrader construction of the Hilbert space and semigroup follows. It is straightforward to prove that, in the presence of cutoffs when the functional integral is well defined, the Osterwalder-Schrader Hilbert space \mathcal{H} is isomorphic to the Hilbert space $L^2(d\mu_{\kappa, \Lambda})$ and the corresponding semigroups coincide. In particular there is a unique ground state.

The next step would be to apply rigorous RG methods in order to take the continuum ultraviolet cutoff ($\kappa \rightarrow \infty$) or equivalently the lattice spacing $\rightarrow 0$ in the lattice regularised theory. Much progress has been made in recent years, in particular the development and applications over the years of the RG based on finite range multiscale expansions due to David Brydges and collaborators. This method gets rid of the cluster expansion altogether at the level of the fluctuation integration and proofs of many results become easier.

For this purpose, and as a first step, we would need a multiscale finite range decomposition of the covariance \tilde{C}_κ or of the lattice version of \tilde{C}_∞ with good regularity properties. Such a non-pathwise program if successful would be quite robust and in particular might enable us to make progress which encompasses also dynamical models more singular than the superrenormalisable ones that are being presently considered.

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