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1 **A HYBRID HIGH-ORDER METHOD FOR THE CAHN–HILLIARD**
2 **PROBLEM IN MIXED FORM***

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5 **Abstract.** In this work we develop a fully implicit Hybrid High-Order algorithm for the Cahn–
6 Hilliard problem in mixed form. The space discretization hinges on local reconstruction operators
7 from hybrid polynomial unknowns at elements and faces. The proposed method has several advan-
8 tageous features: (i) It supports fairly general meshes possibly containing polyhedral elements
9 and nonmatching interfaces; (ii) it allows arbitrary approximation orders; (iii) it has a moderate
10 computational cost thanks to the possibility of locally eliminating element-based unknowns by static
11 condensation. We perform a detailed stability and convergence study, proving optimal convergence
12 rates in energy-like norms. Numerical validation is also provided using some of the most common
13 tests in the literature.

14 *2010 Mathematics Subject Classification:* 65N08, 65N30, 65N12

15 *Keywords:* Hybrid High-Order, Cahn–Hilliard equation, phase separation, mixed formulation, dis-
16 crete functional analysis, polyhedral meshes

17 **1. Introduction.** Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a bounded connected convex
18 polyhedral domain with boundary $\partial\Omega$ and outward normal \mathbf{n} , and let $t_F > 0$. The
19 Cahn–Hilliard problem, originally introduced in [11, 10] to model phase separation in
20 a binary alloy, consists in finding the order-parameter $c : \Omega \times [0, t_F] \rightarrow \mathbb{R}$ and chemical
21 potential $w : \Omega \times [0, t_F] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} 22 \quad (1a) \quad & d_t c - \Delta w = 0 && \text{in } \Omega \times (0, t_F], \\ 23 \quad (1b) \quad & w = \Phi'(c) - \gamma^2 \Delta c && \text{in } \Omega \times (0, t_F], \\ 24 \quad (1c) \quad & c(0) = c_0 && \text{in } \Omega, \\ 25 \quad (1d) \quad & \partial_{\mathbf{n}} c = \partial_{\mathbf{n}} w = 0 && \text{on } \partial\Omega \times (0, t_F], \end{aligned}$$

27 where $c_0 \in H^2(\Omega) \cap L_0^2(\Omega)$ such that $\partial_{\mathbf{n}} c_0 = 0$ on $\partial\Omega$ denotes the initial datum, $\gamma > 0$
28 the interface parameter (usually taking small values), and Φ the free-energy such that

$$29 \quad (2) \quad \Phi(c) := \frac{1}{4}(1 - c^2)^2.$$

30 Relevant extensions of problem (1) (not considered here) include the introduction of
31 a flow which requires, in particular, to add a convective term in (1a); cf., e.g., [29, 5,
32 7, 8, 31, 30].

33 The discretization of the Cahn–Hilliard equation (1) has been considered in several
34 works. Different aspects of standard finite element schemes have been studied, e.g.,
35 in [22, 21, 14]; cf. also the references therein. Mixed finite elements are considered

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36 in [24]. In [35], the authors study a nonconforming method based on C^0 shape func-
 37 tions for the fourth-order primal problem obtained by plugging (1b) into (1a). Dis-
 38 continuous Galerkin (dG) methods have also received extensive attention. We can cite
 39 here [36], where a local dG method is proposed for a Cahn–Hilliard system modelling
 40 multi-component alloys, and a stability analysis is carried out; [23], where optimal
 41 error estimates are proved for a dG discretization of the Cahn–Hilliard problem in
 42 primal form; [30], which contains optimal error estimates for a dG method based on
 43 the mixed formulation of the problem including a convection term; [26], where a
 44 multi-grid approach is proposed for the solution of the systems of algebraic equations
 45 arising from a dG discretization of the Cahn–Hilliard equation. In all of the above
 46 references, standard meshes are considered. General polygonal meshes in dimension
 47 $d = 2$, on the other hand, are supported by the recently proposed C^1 -conforming Vir-
 48 tual Element (VE) method of [4] for the problem in primal formulation; cf. also [6] for
 49 VE methods with arbitrary regularity. Therein, the convergence analysis is carried
 50 out under the assumption that the discrete order-parameter satisfies a $C^0(L^\infty)$ -like a
 51 priori bound.

52 In this work, we develop and analyze a fully implicit Hybrid High-Order (HHO)
 53 algorithm for problem (1) where the space discretization is based on the HHO($k + 1$)
 54 variation proposed in [12] of the method of [19]. The method hinges on hybrid degrees
 55 of freedom (DOFs) located at mesh elements and faces that are polynomials of degree
 56 ($k + 1$) and k , respectively. The nonlinear term in (1b) is discretized by means of
 57 element unknowns only. For the second-order diffusive operators in (1a) and (1b), on
 58 the other hand, we rely on two key ingredients devised locally inside each element:
 59 (i) A potential reconstruction obtained from the solution of (small) Neumann problems
 60 and (ii) a stabilization term penalizing the lowest-order part of the difference between
 61 element- and face-based unknowns. See also [13, 34, 33] for related methods for second-
 62 order linear diffusion operators, each displaying a set of distinctive features. The
 63 global discrete problem is then obtained by a standard element-by-element assembly
 64 procedure. When using a first-order (Newton-like) algorithm to solve the resulting
 65 system of nonlinear algebraic equations, element-based unknowns can be statically
 66 condensed. As a result, the only globally coupled unknowns in the linear subproblems
 67 are discontinuous polynomials of degree k on the mesh skeleton for both the order-
 68 parameter and the chemical potential. With a backward Euler scheme to march in
 69 time, the $C^0(H^1)$ -like error on the order-parameter and the $L^2(H^1)$ -like error on the
 70 chemical potential are proved to optimally converge as $(h^{k+1} + \tau)$ (with h and τ
 71 denoting, respectively, the spatial and temporal mesh sizes) provided the solution has
 72 sufficient regularity.

73 The proposed method has several advantageous features: (i) It supports general
 74 meshes possibly including polyhedral elements and nonmatching interfaces (resulting,
 75 e.g., from nonconforming mesh refinement); (ii) it allows one to increase the spatial
 76 approximation order to accelerate convergence in the presence of (locally) regular
 77 solutions; (iii) it is (relatively) inexpensive. When $d = 2$, e.g., the number of globally
 78 coupled spatial unknowns for our method scales as $2 \text{card}(\mathcal{F}_h)(k + 1)$ (with $\text{card}(\mathcal{F}_h)$
 79 denoting the number of mesh faces) as opposed to $\text{card}(\mathcal{T}_h)(k + 3)(k + 2)$ (with $\text{card}(\mathcal{T}_h)$
 80 denoting the number of mesh elements) for a mixed dG method delivering the same
 81 order of convergence (i.e., based on broken polynomials of degree $k + 1$). Additionally,
 82 thanks to the underlying fully discontinuous polynomial spaces, the proposed method
 83 can accomodate abrupt variations of the unknowns in the vicinity of the interface
 84 between phases.

85 Our analysis adapts the techniques originally developed in [30] in the context of dG
 86 methods. Therein, the treatment of the nonlinear term in (1b) hinges on C^0 -in-time
 87 a priori estimates for various norms and seminorms of the discrete order-parameter.
 88 Instrumental in proving these estimates are discrete functional analysis results, includ-
 89 ing discrete versions of Agmon’s and Gagliardo–Nirenberg–Poincaré’s inequalities for
 90 broken polynomial functions on quasi-uniform matching simplicial meshes. Adapting
 91 these tools to hybrid polynomial spaces on general meshes entails several new ideas.
 92 A first key point consists in defining appropriate discrete counterparts of the Laplace
 93 and Green’s operators. To this end, we rely on a suitably tailored L^2 -like hybrid inner
 94 product which guarantees stability estimates for the former and optimal approxima-
 95 tion properties for the latter. Another key point consists in replacing the standard
 96 nodal interpolator used in the proofs of [30, Lemmas 2.2 and 2.3] by the L^2 -orthogonal
 97 projector which, unlike the former, is naturally defined for meshes containing polyhe-
 98 dral elements. We show that this replacement is possible thanks to the $W^{s,p}$ -stability
 99 and approximation properties of the L^2 -orthogonal projector on broken polynomial
 100 spaces recently presented in a unified setting in [15]; cf. also the references therein
 101 for previous results on this subject.

102 The material is organized as follows: In Section 2 we introduce the notation for
 103 space and time meshes and recall some key results on broken polynomial spaces; in
 104 Section 3 we introduce hybrid polynomial spaces and local reconstructions, and state
 105 the discrete problem; in Section 4 we carry out the stability analysis of the method,
 106 while the convergence analysis is detailed in Section 5; Section 6 contains an extensive
 107 numerical validation of the proposed algorithm; finally, in Appendix A we give proofs
 108 of the discrete functional analysis results used to derive stability bounds and error
 109 estimates.

110 **2. Discrete setting.** In this section we introduce the discrete setting and recall
 111 some basic results on broken polynomial spaces.

112 **2.1. Space and time meshes.** We recall here the notion of admissible spatial
 113 mesh sequence from [17, Chapter 1]. For the sake of simplicity, we will systematically
 114 use the term polyhedral also when $d = 2$. Denote by $\mathcal{H} \subset \mathbb{R}_*^+$ a countable set of
 115 spatial meshsizes having 0 as its unique accumulation point. We consider h -refined
 116 mesh sequences $(\mathcal{T}_h)_{h \in \mathcal{H}}$ where, for all $h \in \mathcal{H}$, \mathcal{T}_h is a finite collection of nonempty
 117 disjoint open polyhedral elements T of boundary ∂T such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$ and
 118 $h = \max_{T \in \mathcal{T}_h} h_T$ with h_T standing for the diameter of the element T .

119 A face F is defined as a planar closed connected subset of $\bar{\Omega}$ with positive $(d-1)$ -
 120 dimensional Hausdorff measure and such that (i) either there exist $T_1, T_2 \in \mathcal{T}_h$ such
 121 that $F \subset \partial T_1 \cap \partial T_2$ and F is called an interface or (ii) there exists $T \in \mathcal{T}_h$ such
 122 that $F \subset \partial T \cap \partial \Omega$ and F is called a boundary face. Mesh faces are collected in
 123 the set \mathcal{F}_h , and the diameter of a face $F \in \mathcal{F}_h$ is denoted by h_F . For all $T \in \mathcal{T}_h$,
 124 $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ denotes the set of faces lying on ∂T and, for all $F \in \mathcal{F}_T$,
 125 \mathbf{n}_{TF} is the unit normal to F pointing out of T . Symmetrically, for all $F \in \mathcal{F}_h$, we
 126 denote by \mathcal{T}_F the set of one (if $F \in \mathcal{F}_h^b$) or two (if $F \in \mathcal{F}_h^i$) elements sharing F .

127 ASSUMPTION 1 (Admissible spatial mesh sequence). *We assume that, for all $h \in \mathcal{H}$,*
 128 *\mathcal{T}_h admits a matching simplicial submesh \mathfrak{T}_h and there exists a real number $\varrho > 0$*
 129 *independent of h such that, for all $h \in \mathcal{H}$, the following properties hold: (i) Shape reg-*
 130 *ularity: For all simplex $S \in \mathfrak{T}_h$ of diameter h_S and inradius r_S , $\varrho h_S \leq r_S$; (ii) contact-*

131 regularity: For all $T \in \mathcal{T}_h$, and all $S \in \mathfrak{T}_h$ such that $S \subset T$, $\varrho h_T \leq h_S$.

132 To discretize in time, we consider a uniform partition $(t^n)_{0 \leq n \leq N}$ of the time interval
 133 $[0, t_F]$ with $t^0 = 0$, $t^N = t_F$ and $t^n - t^{n-1} = \tau$ for all $1 \leq n \leq N$ (the analysis can
 134 be adapted to nonuniform partitions). For any sufficiently regular function of time φ
 135 taking values in a vector space V , we denote by $\varphi^n \in V$ its value at discrete time t^n ,
 136 and we introduce the backward differencing operator δ_t such that, for all $1 \leq n \leq N$,

$$137 \quad (3) \quad \delta_t \varphi^n := \frac{\varphi^n - \varphi^{n-1}}{\tau} \in V.$$

138 In what follows, we often abbreviate by $a \lesssim b$ the inequality $a \leq Cb$ with a and b
 139 positive real numbers and $C > 0$ generic constant independent of both the meshsize
 140 h and the time step τ (named constants are used in the statements for the sake of
 141 easy consultation). Also, for a subset $X \subset \bar{\Omega}$, we denote by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ the usual
 142 $L^2(X)$ -inner product and norm, with the convention that we omit the index if $X = \Omega$.
 143 The same notation is used for the vector-valued space $L^2(X)^d$.

144 **2.2. Basic results on broken polynomial spaces.** The proposed method
 145 is based on local polynomial spaces on mesh elements and faces. Let an integer
 146 $l \geq 0$ be fixed. Let U be a subset of \mathbb{R}^d , H_U the affine space spanned by U , d_U its
 147 dimension, and assume that U has a non-empty interior in H_U . We denote by $\mathbb{P}^l(U)$
 148 the space spanned by d_U -variate polynomials on H_U of total degree l , and by π_U^l
 149 the L^2 -orthogonal projector onto this space. In the following sections, the set U will
 150 represent a mesh element or face. The space of broken polynomial functions on \mathcal{T}_h of
 151 degree l is denoted by $\mathbb{P}^l(\mathcal{T}_h)$, and π_h^l is the corresponding L^2 -orthogonal projector.

152 We next recall some functional analysis results on polynomial spaces. The following
 153 discrete trace and inverse inequalities are proved in [17, Chapter 1] (cf. in particular
 154 Lemmas 1.44 and 1.46): There is $C > 0$ independent of h such that, for all $T \in \mathcal{T}_h$,
 155 and all $\forall v \in \mathbb{P}^l(T)$,

$$156 \quad (4) \quad \|v\|_F \leq Ch_F^{-\frac{1}{2}} \|v\|_T \quad \forall F \in \mathcal{F}_T,$$

157 and

$$158 \quad (5) \quad \|\nabla v\|_T \leq Ch_T^{-1} \|v\|_T.$$

159 We will also need the following local direct and reverse Lebesgue embeddings (cf. [15,
 160 Lemma 5.1]): There is $C > 0$ independent of h such that, for all $T \in \mathcal{T}_h$, all $q, p \in$
 161 $[1, +\infty]$,

$$162 \quad (6) \quad \forall v \in \mathbb{P}^l(T), \quad C^{-1} \|v\|_{L^q(T)} \leq h_T^{\frac{d}{q} - \frac{d}{p}} \|v\|_{L^p(T)} \leq C \|v\|_{L^q(T)}.$$

The proof of the following results for the local L^2 -orthogonal projector can be found
 in [15, Appendix A.2]. For an open set U of \mathbb{R}^d , $s \in \mathbb{N}$ and $p \in [1, +\infty]$, we define the
 seminorm $|\cdot|_{W^{s,p}(U)}$ as follows: For all $v \in W^{s,p}(U)$,

$$|v|_{W^{s,p}(U)} := \sum_{\alpha \in \mathbb{N}^d, |\alpha|_{\ell^1} = s} \|\partial^\alpha v\|_{L^p(U)},$$

163 where $|\boldsymbol{\alpha}|_{\ell^1} := \alpha_1 + \dots + \alpha_d$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. For $s = 0$, we recover the usual
 164 Lebesgue spaces $L^p(U)$. The L^2 -orthogonal projector is $W^{s,p}$ -stable and has optimal
 165 $W^{s,p}$ -approximation properties: There is $C > 0$ independent of h such that, for all
 166 $T \in \mathcal{T}_h$, all $s \in \{0, \dots, l + 1\}$, all $p \in [1, +\infty]$, and all $v \in W^{s,p}(T)$, it holds,

$$167 \quad (7) \quad |\pi_T^l v|_{W^{s,p}(T)} \leq C |v|_{W^{s,p}(T)},$$

168 and, for all $m \in \{0, \dots, s\}$,

$$169 \quad (8) \quad |v - \pi_T^l v|_{W^{m,p}(T)} + h_T^{\frac{1}{p}} |v - \pi_T^l v|_{W^{m,p}(\mathcal{F}_T)} \leq C h_T^{s-m} |v|_{W^{s,p}(T)},$$

170 where $W^{m,p}(\mathcal{F}_T)$ denotes the set of functions that belong to $W^{m,p}(F)$ for all $F \in \mathcal{F}_T$.
 171 Finally, there is $C > 0$ independent of h such that it holds, for all $F \in \mathcal{F}_h$,

$$172 \quad (9) \quad \forall v \in H^1(F), \quad \|v - \pi_F^l v\|_F \leq C h |v|_{H^1(F)}.$$

173 In the proofs of Lemmas 3 and 8 below, we will make use of the following global
 174 inverse inequalities, which require mesh quasi-uniformity.

175 PROPOSITION 1 (Global inverse inequalities for Lebesgue norms of broken polynomi-
 176 als). *In addition to Assumption 1, we assume that the mesh is quasi-uniform, i.e.,*

$$177 \quad (10) \quad \forall T \in \mathcal{T}_h, \quad \varrho h \leq h_T.$$

178 Then, for all polynomial degree $l \geq 0$ and all $1 \leq p \leq q \leq +\infty$, it holds

$$179 \quad (11) \quad \forall w_h \in \mathbb{P}^l(\mathcal{T}_h), \quad \|w_h\|_{L^q(\Omega)} \leq C h^{\frac{d}{q} - \frac{d}{p}} \|w_h\|_{L^p(\Omega)},$$

180 with real number $C > 0$ independent of h .

181 *Proof.* Let $w_h \in \mathbb{P}^l(\mathcal{T}_h)$. We start by proving that, for all $p \in [1, +\infty]$,

$$182 \quad (12) \quad \forall w_h \in \mathbb{P}^l(\mathcal{T}_h), \quad \|w_h\|_{L^\infty(\Omega)} \lesssim h^{-\frac{d}{p}} \|w_h\|_{L^p(\Omega)},$$

which corresponds to (11) with $q = +\infty$. By the local reverse Lebesgue embed-
 183 dings (6), there is $C > 0$ independent of h such that

$$\forall T \in \mathcal{T}_h, \quad \|w_h\|_{L^\infty(T)} \leq C h_T^{-\frac{d}{p}} \|w_h\|_{L^p(T)} \leq C \rho^{-\frac{d}{p}} h^{-\frac{d}{p}} \|w_h\|_{L^p(\Omega)},$$

where we have used the mesh quasi-uniformity assumption (10) to conclude. Inequal-
 184 ity (12) follows observing that $\|w_h\|_{L^\infty(\Omega)} = \max_{T \in \mathcal{T}_h} \|w_h\|_{L^\infty(T)}$. Let us now turn to
 185 the case $1 \leq q < +\infty$. We have

$$\|w_h\|_{L^q(\Omega)}^q \leq \|w_h\|_{L^\infty(\Omega)}^{q-p} \|w_h\|_{L^p(\Omega)}^p \lesssim \left(h^{\frac{d}{q} - \frac{d}{p}} \|w_h\|_{L^p(\Omega)} \right)^q,$$

183 where the conclusion follows using (12). □

184 **3. The Hybrid High-Order method.** In this section we define hybrid spaces
 185 and state the discrete problem.

186 **3.1. Hybrid spaces.** The discretization of the diffusion operator hinges on the
 187 HHO method of [12] using polynomials of degree $(k + 1)$ inside elements and k on
 188 mesh faces (cf. Remark 6 for further insight on this choice). The global discrete space
 189 is defined as

$$190 \quad (13) \quad \underline{U}_h^k := \left(\prod_{T \in \mathcal{T}_h} \mathbb{P}^{k+1}(T) \right) \times \left(\prod_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right).$$

191 The restriction of \underline{U}_h^k to an element $T \in \mathcal{T}_h$ is denoted by \underline{U}_T^k . For a generic collection
 192 of DOFs in \underline{U}_h^k , we use the underlined notation $\underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h})$ and, for
 193 all $T \in \mathcal{T}_h$, we denote by $\underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T})$ its restriction to \underline{U}_T^k . Also, to keep the
 194 notation compact, we denote by v_h (no underline) the function in $\mathbb{P}^{k+1}(\mathcal{T}_h)$ such that

$$195 \quad v_h|_T = v_T \quad \forall T \in \mathcal{T}_h.$$

In what follows, we will also need the zero-average subspace

$$\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid (v_h, 1) = 0 \right\}.$$

196 The interpolator $\underline{I}_h^k : H^1(\Omega) \rightarrow \underline{U}_h^k$ is such that, for all $v \in H^1(\Omega)$,

$$197 \quad (14) \quad \underline{I}_h^k v := ((\pi_T^{k+1} v)_{T \in \mathcal{T}_h}, (\pi_F^k v)_{F \in \mathcal{F}_h}).$$

198 We define on \underline{U}_h^k the seminorm $\|\cdot\|_{1,h}$ such that

$$199 \quad (15) \quad \|\underline{v}_h\|_{1,h}^2 := \|\nabla_h v_h\|^2 + |\underline{v}_h|_{1,h}^2, \quad |\underline{v}_h|_{1,h}^2 := s_{1,h}(v_h, v_h),$$

200 where ∇_h denotes the usual broken gradient on $H^1(\mathcal{T}_h)$ and the stabilization bilinear
 201 form $s_{1,h}$ on $\underline{U}_h^k \times \underline{U}_h^k$ is such that

$$202 \quad (16) \quad s_{1,h}(v_h, z_h) := \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k(v_F - v_T), \pi_F^k(z_F - z_T))_F.$$

203 Using the stability and approximation properties of the L^2 -orthogonal projector ex-
 204 pressed by (7)–(8), one can prove that \underline{I}_h^k is H^1 -stable:

$$205 \quad (17) \quad \forall v \in H^1(\Omega), \quad \|\underline{I}_h^k v\|_{1,h} \lesssim \|v\|_{H^1(\Omega)}.$$

206 The following Friedrichs' inequalities can be proved using the arguments of [15,
 207 Lemma 7.2], where element DOFs of degree k are considered (cf. also [9, 16] for
 208 related results using dG norms): For all $r \in [1, +\infty)$ if $d = 2$, all $r \in [1, 6]$ if $d = 3$,

$$209 \quad (18) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k, \quad \|v_h\|_{L^r(\Omega)} \lesssim \|\underline{v}_h\|_{1,h}.$$

210 The case $r = 2$ corresponds to Poincaré's inequality. Finally, to close this section, we
 211 prove that $\|\cdot\|_{1,h}$ defines a norm on $\underline{U}_{h,0}^k$.

212 **PROPOSITION 2** (Norm $\|\cdot\|_{1,h}$). *The map $\|\cdot\|_{1,h}$ defines a norm on $\underline{U}_{h,0}^k$.*

213 *Proof.* We only have to show that $\|\underline{v}_h\|_{1,h} = 0 \implies \underline{v}_h = \underline{0}$. By (18), $\|\underline{v}_h\|_{1,h} = 0 \implies$
 214 $v_h \equiv 0$, i.e., $v_T \equiv 0$ for all $T \in \mathcal{T}_h$. Plugging this result into the definition (15) of $\|\cdot\|_{1,h}$,
 215 we get $\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F\|_F^2 = 0$, which implies in turn $v_F \equiv 0$ for all $F \in \mathcal{F}_h$. \square

216 **3.2. Diffusive bilinear form and discrete problem.** For all $T \in \mathcal{T}_h$, we
 217 define the potential reconstruction operator $p_T^{k+1} : \underline{U}_T^k \rightarrow \mathbb{P}^{k+1}(T)$ such that, for all
 218 $\underline{v}_T \in \underline{U}_T^k$, $p_T^{k+1} \underline{v}_T$ is the unique solution of the following Neumann problem:

$$219 \quad (19) \quad (\nabla p_T^{k+1} \underline{v}_T, \nabla z)_T = -(v_T, \Delta z)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla z \cdot \mathbf{n}_{TF})_F \quad \forall z \in \mathbb{P}^{k+1}(T),$$

220 with closure condition $(p_T^{k+1} \underline{v}_T, 1)_T = (v_T, 1)_T$. It can be proved that, for all $v \in$
 221 $H^1(T)$, denoting by \underline{I}_T^k the restriction of the reduction map \underline{I}_h^k defined by (14) to
 222 $H^1(T) \rightarrow \underline{U}_T^k$,

$$223 \quad (20) \quad (\nabla(p_T^{k+1} \underline{I}_T^k v - v), \nabla z)_T = 0 \quad \forall z \in \mathbb{P}^{k+1}(T),$$

224 which expresses the fact that $(p_T^{k+1} \circ \underline{I}_T^k)$ is the elliptic projector onto $\mathbb{P}^{k+1}(T)$ (and,
 225 as such, has optimal approximation properties in $\mathbb{P}^{k+1}(T)$). The diffusive bilinear
 226 form a_h on $\underline{U}_h^k \times \underline{U}_h^k$ is obtained by element-wise assembly setting

$$227 \quad (21) \quad a_h(\underline{v}_h, \underline{z}_h) := \sum_{T \in \mathcal{T}_h} (\nabla p_T^{k+1} \underline{v}_T, \nabla p_T^{k+1} \underline{z}_T)_T + s_{1,h}(\underline{v}_h, \underline{z}_h),$$

228 with stabilization bilinear form $s_{1,h}$ defined by (16). Denoting by $\|\cdot\|_{a,h}$ the seminorm
 229 defined by a_h on \underline{U}_h^k , a straightforward adaptation of the arguments used in [19,
 230 Lemma 4] shows that

$$231 \quad (22) \quad \forall \underline{v}_h \in \underline{U}_h^k, \quad \|\underline{v}_h\|_{1,h} \lesssim \|\underline{v}_h\|_{a,h} \lesssim \|\underline{v}_h\|_{1,h},$$

232 which expresses the coercivity and boundedness of a_h . Additionally, following the
 233 arguments in [19, Theorem 8], one can easily prove that the bilinear form a_h enjoys
 234 the following consistency property: For all $v \in H^{\max(2,l)}(\Omega) \cap L_0^2(\Omega)$, $l \geq 1$, such that
 235 $\partial_n v = 0$ on $\partial\Omega$,

$$236 \quad (23) \quad \sup_{\underline{z}_h \in \underline{U}_{h,0}^k, \|\underline{z}_h\|_{1,h}=1} \left| a_h(\underline{I}_h^k v, \underline{z}_h) + (\Delta v, z_h) \right| \lesssim h^{\min(k+1, l-1)} \|v\|_{H^l(\Omega)}.$$

237 **REMARK 1 (Consistency of a_h).** For sufficiently regular solutions (i.e., when $l = k + 2$),
 238 equation (23) shows that the consistency error scales as h^{k+1} . This is a consequence
 239 of the fact that both the potential reconstruction p_T^{k+1} (cf. (19)) and the stabilization
 240 bilinear form $s_{1,h}$ (cf. (16)) are consistent for exact solutions that are polynomials of
 241 degree $(k + 1)$ inside each element. In particular, a key point in $s_{1,h}$ is to penalize
 242 $\pi_F^k(v_F - v_T)$ instead of $(v_F - v_T)$. A similar stabilization bilinear form had been in-
 243 dependently suggested in the context of Hybridizable Discontinuous Galerkin methods
 244 in [32, Remark 1.2.4].

245 The discrete problem reads: For all $1 \leq n \leq N$, find $(\underline{c}_h^n, \underline{w}_h^n) \in \underline{U}_{h,0}^k \times \underline{U}_h^k$ such that

$$246 \quad (24a) \quad (\delta_t \underline{c}_h^n, \varphi_h) + a_h(\underline{w}_h^n, \varphi_h) = 0 \quad \forall \varphi_h \in \underline{U}_h^k,$$

$$247 \quad (24b) \quad (\underline{w}_h^n, \psi_h) = (\Phi'(\underline{c}_h^n), \psi_h) + \gamma^2 a_h(\underline{c}_h^n, \underline{\psi}_h) \quad \forall \underline{\psi}_h \in \underline{U}_h^k,$$

249 and $\underline{c}_h^0 \in \underline{U}_{h,0}^k$ solves

$$250 \quad (25) \quad a_h(\underline{c}_h^0, \varphi_h) = -(\Delta c_0, \varphi_h) \quad \forall \varphi_h \in \underline{U}_h^k.$$

251 We note, in passing, that the face DOFs in \underline{c}_h^0 are not needed to initialize the algorithm.

252 REMARK 2 (Static condensation). *Problem (24) is a system of nonlinear algebraic*
 253 *equations, which can be solved using an iterative algorithm. When first order (Newton-*
 254 *like) algorithms are used, element-based DOFs can be locally eliminated at each iter-*
 255 *ation by a standard static condensation procedure.*

256 **4. Stability analysis.** In this section we establish some uniform a priori bounds
 257 on the discrete solution. To this end, we need a discrete counterpart of Agmon's
 258 inequality; cf. [3, Lemma 13.2] and also [1, Theorem 3]. We define on \underline{U}_h^k the following
 259 L^2 -like inner product:

$$(26) \quad \begin{aligned} (\underline{v}_h, \underline{z}_h)_{0,h} &:= (v_h, z_h) + s_{0,h}(\underline{v}_h, \underline{z}_h), \\ s_{0,h}(\underline{v}_h, \underline{z}_h) &:= \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F (\pi_F^k(v_F - v_T), \pi_F^k(z_F - z_T))_F, \end{aligned}$$

261 and denote by $\|\cdot\|_{0,h}$ and $|\cdot|_{0,h}$ the norm and seminorm corresponding to the bilinear
 262 forms $(\cdot, \cdot)_{0,h}$ and $s_{0,h}$, respectively. For further insight on the role of $s_{0,h}$, cf. Re-
 263 mark 7. We introduce the discrete Laplace operator $\underline{L}_h^k : \underline{U}_h^k \rightarrow \underline{U}_h^k$ such that, for all
 264 $\underline{v}_h \in \underline{U}_h^k$,

$$(27) \quad -(\underline{L}_h^k \underline{v}_h, \underline{z}_h)_{0,h} = a_h(\underline{v}_h, \underline{z}_h) \quad \forall \underline{z}_h \in \underline{U}_h^k,$$

266 and we denote by $\underline{L}_h^k \underline{v}_h$ (no underline) the broken polynomial function in $\mathbb{P}^{k+1}(\mathcal{T}_h)$
 267 obtained from element DOFs in $\underline{L}_h^k \underline{v}_h$.

268 REMARK 3 (Restriction of \underline{L}_h^k to $\underline{U}_{h,0}^k \rightarrow \underline{U}_{h,0}^k$). *Whenever $\underline{v}_h \in \underline{U}_{h,0}^k$, $\underline{L}_h^k \underline{v}_h \in \underline{U}_{h,0}^k$.*
 269 *To prove it, it suffices to take $\underline{z}_h = \underline{I}_h^k \chi_\Omega$ in (27) (with χ_Ω characteristic function of*
 270 *Ω), and observe that the left-hand side satisfies $(\underline{L}_h^k \underline{v}_h, \underline{z}_h)_{0,h} = (\underline{L}_h^k \underline{v}_h, 1)$ while, by*
 271 *definition (21) of the bilinear form a_h , the right-hand side vanishes. In what follows,*
 272 *we keep the same notation for the (bijective) restriction of \underline{L}_h^k to $\underline{U}_{h,0}^k \rightarrow \underline{U}_{h,0}^k$.*

273 The following result, valid for $d \in \{2, 3\}$, will be proved in Appendix A.

274 LEMMA 3 (Discrete Agmon's inequality). *Assume mesh quasi-uniformity (10). Then, \blacksquare*
 275 *it holds with real number $C > 0$ independent of h ,*

$$(28) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k, \quad \|v_h\|_{L^\infty(\Omega)} \leq C \|\underline{v}_h\|_{1,h}^{\frac{1}{2}} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}}.$$

277 We also recall the following discrete Gronwall's inequality (cf. [28, Lemma 5.1]).

LEMMA 4 (Discrete Gronwall's inequality). *Let two reals $\delta, G > 0$ be given, and, for*
integers $n \geq 1$, let a^n, b^n , and χ^n denote nonnegative real numbers such that

$$a^N + \delta \sum_{n=1}^N b^n \leq \delta \sum_{n=1}^N \chi^n a^n + G \quad \forall N \in \mathbb{N}^*.$$

278 *Then, if $\chi^n \delta < 1$ for all n , letting $\varsigma^n := (1 - \chi^n \delta)^{-1}$, it holds*

$$(29) \quad a^N + \delta \sum_{n=1}^N b^n \leq \exp\left(\delta \sum_{n=1}^N \varsigma^n \chi^n\right) \times G \quad \forall N \in \mathbb{N}^*.$$

280 We are now ready to prove the a priori bounds.

281 LEMMA 5 (Uniform a priori bounds). *Under the assumptions of Lemma 3, and*
 282 *further assuming that $\tau \leq L$ for a given real number $L > 0$ independent of h and of τ*
 283 *(but depending on γ^2) and sufficiently small, there is a real number $C > 0$ independent*
 284 *of h and τ such that*

$$285 \quad \max_{1 \leq n \leq N} \left(\|\underline{c}_h^n\|_{a,h}^2 + (\Phi(c_h^n), 1) + \|w_h^n\|^2 + \|c_h^n\|_{L^\infty(\Omega)} + \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 \right) + \sum_{n=1}^N \tau \|\underline{w}_h^n\|_{a,h}^2 \leq C.$$

286 *Proof.* The proof is split into several steps.

287 (i) We start by proving that

$$288 \quad (30) \quad \max_{1 \leq n \leq N} \left(\|\underline{c}_h^n\|_{a,h}^2 + (\Phi(c_h^n), 1) \right) + \sum_{n=1}^N \tau \|\underline{w}_h^n\|_{a,h}^2 \lesssim 1.$$

289 Subtracting (24b) with $\underline{\psi}_h = \underline{c}_h^n - \underline{c}_h^{n-1}$ from (24a) with $\underline{\varphi}_h = \tau \underline{w}_h^n$, and using the
 290 fact that, for all $r, s \in \mathbb{R}$, $\Phi'(r)(r-s) \geq \Phi(r) - \Phi(s) - \frac{1}{2}(r-s)^2$, it is inferred, for all
 291 $1 \leq n \leq N$, that

$$292 \quad (31) \quad \gamma^2 a_h(\underline{c}_h^n, \underline{c}_h^n - \underline{c}_h^{n-1}) + \tau \|\underline{w}_h^n\|_{a,h}^2 + (\Phi(c_h^n), 1) \leq \frac{1}{2} \|\underline{c}_h^n - \underline{c}_h^{n-1}\|^2 + (\Phi(c_h^{n-1}), 1).$$

293 Notice that $(\Phi(c_h^n), 1) \geq 0$ for all $0 \leq n \leq N$ by definition (2) of Φ . Making $\underline{\varphi}_h =$
 294 $\tau(\underline{c}_h^n - \underline{c}_h^{n-1})$ in (24a) and using the Cauchy–Schwarz and Young’s inequalities, we
 295 infer that

$$296 \quad (32) \quad \|\underline{c}_h^n - \underline{c}_h^{n-1}\|^2 \leq \frac{\tau}{2} \|\underline{w}_h^n\|_{a,h}^2 + \frac{\tau}{2} \|\underline{c}_h^n - \underline{c}_h^{n-1}\|_{a,h}^2.$$

297 Additionally, recalling the following formula for the backward Euler scheme:

$$298 \quad (33) \quad 2x(x-y) = x^2 + (x-y)^2 - y^2,$$

299 it holds

$$300 \quad (34) \quad a_h(\underline{c}_h^n, \underline{c}_h^n - \underline{c}_h^{n-1}) = \frac{1}{2} \left(\|\underline{c}_h^n\|_{a,h}^2 + \|\underline{c}_h^n - \underline{c}_h^{n-1}\|_{a,h}^2 - \|\underline{c}_h^{n-1}\|_{a,h}^2 \right).$$

301 Plugging (32) and (34) into (31), we obtain

302

$$303 \quad \gamma^2 \|\underline{c}_h^n\|_{a,h}^2 + \left(\gamma^2 - \frac{\tau}{2} \right) \|\underline{c}_h^n - \underline{c}_h^{n-1}\|_{a,h}^2 + \frac{3\tau}{2} \|\underline{w}_h^n\|_{a,h}^2 + 2(\Phi(c_h^n), 1) \\ 304 \quad \leq \gamma^2 \|\underline{c}_h^{n-1}\|_{a,h}^2 + 2(\Phi(c_h^{n-1}), 1).$$

Provided $\tau < 2\gamma^2$, the bound (30) follows summing the above inequality over $1 \leq n \leq N$, and using the fact that $\gamma^2 \|\underline{c}_h^0\|_{a,h}^2 + 2(\Phi(c_h^0), 1) \lesssim 1$. To prove this bound, observe that

$$\begin{aligned} \gamma^2 \|\underline{c}_h^0\|_{a,h}^2 + 2(\Phi(c_h^0), 1) &\lesssim \gamma^2 \|\underline{c}_h^0\|_{1,h}^2 + 1 + \|c_h^0\|_{L^4(\Omega)}^4 + \|c_h^0\|^2 \\ &\lesssim \gamma^2 \|\underline{c}_h^0\|_{1,h}^2 + 1 + \|c_h^0\|_{1,h}^4 + \|c_h^0\|_{1,h}^2 \lesssim 1, \end{aligned}$$

306 where we have used the definition (2) of the free-energy Φ in the first line followed by
 307 the discrete Friedrichs’ inequality with $r = 4, 2$ in the second line and the first bound
 308 on the initial datum in (46) below to conclude.

309 (ii) We next prove that

$$310 \quad (35) \quad \sum_{n=1}^N \tau \|c_h^n\|_{L^\infty(\Omega)}^4 \lesssim 1.$$

The discrete Agmon's inequality (28) followed by the first inequality in (22) yields

$$\sum_{n=1}^N \tau \|c_h^n\|_{L^\infty(\Omega)}^4 \lesssim \sum_{n=1}^N \tau \|\underline{c}_h^n\|_{a,h}^2 \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 \lesssim \left(\max_{1 \leq n \leq N} \|\underline{c}_h^n\|_{a,h}^2 \right) \times \sum_{n=1}^N \tau \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2.$$

311 The first factor is $\lesssim 1$ owing to (30). Thus, to prove (35), it suffices to show that
 312 also the second factor is $\lesssim 1$. Using the definition (27) of \underline{L}_h^k followed by (24b) with
 313 $\underline{\psi}_h = \underline{L}_h^k \underline{c}_h^n$, we infer that

$$314 \quad (36) \quad \gamma^2 \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 = -\gamma^2 a_h(\underline{c}_h^n, \underline{L}_h^k \underline{c}_h^n) = (\Phi'(c_h^n), \underline{L}_h^k \underline{c}_h^n) - (w_h^n, \underline{L}_h^k \underline{c}_h^n).$$

Using again (27) for the second term in the right-hand side of (36) followed by the Cauchy–Schwarz and Young's inequalities, we obtain

$$\begin{aligned} \gamma^2 \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 &= (\Phi'(c_h^n), \underline{L}_h^k \underline{c}_h^n) + a_h(\underline{c}_h^n, \underline{w}_h^n) + s_{0,h}(\underline{L}_h^k \underline{c}_h^n, \underline{w}_h^n) \\ &\leq \frac{1}{2\gamma^2} \|\Phi'(c_h^n)\|^2 + \frac{\gamma^2}{2} \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 + \frac{\gamma^2}{2} \|\underline{c}_h^n\|_{a,h}^2 + \frac{1}{2\gamma^2} \|\underline{w}_h^n\|_{a,h}^2 + \frac{1}{2\gamma^2} |\underline{w}_h^n|_{0,h}^2. \end{aligned}$$

Hence, since $|\underline{w}_h^n|_{0,h} \leq h |\underline{w}_h^n|_{1,h} \lesssim \|\underline{w}_h^n\|_{a,h}$,

$$\gamma^2 \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 \lesssim \gamma^{-2} \|\Phi'(c_h^n)\|^2 + \gamma^2 \|\underline{c}_h^n\|_{a,h}^2 + \gamma^{-2} \|\underline{w}_h^n\|_{a,h}^2.$$

315 The fact that $\sum_{n=1}^n \tau \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^2 \lesssim 1$ then follows multiplying the above inequality by
 316 τ , summing over $1 \leq n \leq N$, using (30) to bound the second and third term in the
 317 right-hand side, and observing that

$$318 \quad (37) \quad \|\Phi'(c_h^n)\|^2 \leq \|c_h^n\|_{L^6(\Omega)}^6 + 2\|c_h^n\|_{L^4(\Omega)}^4 + \|c_h^n\|^2 \lesssim \|\underline{c}_h^n\|_{1,h}^6 + \|\underline{c}_h^n\|_{1,h}^4 + \|\underline{c}_h^n\|_{1,h}^2 \lesssim 1,$$

319 where we have used the definition (2) to obtain the first bound, Friedrichs' inequality
 320 (18) with $r = 6, 4, 2$ to obtain the second bound, and (30) together with the first
 321 inequality in (22) to conclude.

322 (iii) We proceed by proving that

$$323 \quad (38) \quad \max_{1 \leq n \leq N} \|w_h^n\|^2 + \gamma^2 \sum_{n=1}^N \tau \|\delta_t c_h^n\|^2 \lesssim 1.$$

324 Let $w_h^0 := \pi_h^{k+1}(\Phi'(c_h^0) - \gamma^2 \Delta c_0)$. Recalling (25), w_h^0 satisfies

$$325 \quad (39) \quad (w_h^0, \psi_h) = (\Phi'(c_h^0), \psi_h) + \gamma^2 a_h(\underline{c}_h^0, \underline{\psi}_h) \quad \forall \underline{\psi}_h \in \underline{U}_h^k.$$

For any $1 \leq n \leq N$, subtracting from (24b) at time step n (24b) at time step $(n-1)$
 if $n > 1$ or (39) if $n = 1$, and selecting $\underline{\psi}_h = \underline{w}_h^n$ as a test function in the resulting
 equation, it is inferred that

$$(w_h^n - w_h^{n-1}, w_h^n) = \tau \gamma^2 a_h(\delta_t \underline{c}_h^n, \underline{w}_h^n) + (\Phi'(c_h^n) - \Phi'(c_h^{n-1}), w_h^n).$$

326 Using (24a) with $\varphi_h = \tau\gamma^2\delta_t c_h^n$ to infer $\tau\gamma^2 a_h(\delta_t c_h^n, w_h^n) = -\tau\gamma^2 \|\delta_t c_h^n\|^2$, we get

$$327 \quad (40) \quad (w_h^n - w_h^{n-1}, w_h^n) + \tau\gamma^2 \|\delta_t c_h^n\|^2 = (\Phi'(c_h^n) - \Phi'(c_h^{n-1}), w_h^n).$$

328 From the fact that

$$329 \quad (41) \quad \Phi'(r) - \Phi'(s) = (r^2 + rs + s^2 - 1)(r - s),$$

330 followed by the Cauchy–Schwarz and Young’s inequalities, we infer

$$331 \quad (42) \quad |(\Phi'(c_h^n) - \Phi'(c_h^{n-1}), w_h^n)| \leq \frac{\tau\gamma^2}{2} \|\delta_t c_h^n\|^2 + \frac{\tau C^n}{2} \|w_h^n\|^2,$$

332 with $C^n := C(1 + \|c_h^n\|_{L^\infty(\Omega)}^4 + \|c_h^{n-1}\|_{L^\infty(\Omega)}^4)$ for a real number $C > 0$ independent of
 333 h and τ . Using (33) for the first term in the left-hand side of (40) together with (42)
 334 for the right-hand side, we get

$$335 \quad (43) \quad \|w_h^n\|^2 + \|w_h^n - w_h^{n-1}\|^2 + \tau\gamma^2 \|\delta_t c_h^n\|^2 \leq \tau C^n \|w_h^n\|^2 + \|w_h^{n-1}\|^2.$$

Summing (43) over $1 \leq n \leq N$, observing that, thanks to (35) and the second bound in (46) below, we can have $\tau C^n < 1$ for all $1 \leq n \leq N$ provided that we choose τ small enough, and using the discrete Gronwall’s inequality (29) (with $\delta = \tau$, $a^n = \|w_h^n\|^2$, $b^n = \gamma^2 \|\delta_t c_h^n\|^2$, $\chi^n = C^n$ and $G = \|w_h^0\|^2$), the estimate (38) follows if we can bound $\|w_h^0\|^2$. To this end, recalling the definition of w_h^0 and using the Cauchy–Schwarz inequality, one has

$$\|w_h^0\|^2 = (\Phi'(c_h^0), w_h^0) - \gamma^2 (\Delta c_0, w_h^0) \leq (\|\Phi'(c_h^0)\| + \gamma^2 \|c_0\|_{H^2(\Omega)}) \|w_h^0\|,$$

336 and the conclusion follows from the regularity of c_0 noting the first bound in (46)
 337 below and estimating the first term in parentheses as in (37).

338 (iv) We conclude by proving the bound

$$339 \quad (44) \quad \max_{1 \leq n \leq N} \left(\|c_h^n\|_{L^\infty(\Omega)} + \|\underline{L}_h^k c_h^n\|_{0,h}^2 \right) \lesssim 1.$$

340 Using the Cauchy–Schwarz and Young’s inequalities to bound the right-hand side
 341 of (36) followed by (18) with $r = 6, 4, 2$ and the first inequality in (22), we obtain, for
 342 all $1 \leq n \leq N$,

$$343 \quad (45) \quad \begin{aligned} \gamma^2 \|\underline{L}_h^k c_h^n\|_{0,h}^2 &\lesssim \gamma^{-2} (\|\Phi'(c_h^n)\|^2 + \|w_h^n\|^2) \\ &\lesssim \left(\|c_h^n\|_{L^6(\Omega)}^6 + \|c_h^n\|_{L^4(\Omega)}^4 + \|c_h^n\|^2 \right) + \|w_h^n\|^2 \\ &\lesssim (\|c_h^n\|_{a,h}^6 + \|c_h^n\|_{a,h}^4 + \|c_h^n\|_{a,h}^2) + \|w_h^n\|^2 \lesssim 1, \end{aligned}$$

where we have concluded using (30) multiple times for the terms in parentheses and (38) for $\|w_h^n\|^2$. Using the discrete Agmon’s inequality (28) followed by Young’s inequality and the first inequality in (22), we infer

$$\max_{1 \leq n \leq N} \|c_h^n\|_{L^\infty(\Omega)} \lesssim \max_{1 \leq n \leq N} \left(\|c_h^n\|_{a,h} + \|\underline{L}_h^k c_h^n\|_{0,h} \right) \lesssim 1,$$

344 where the conclusion follows using (30) for the first addend in the argument of the
 345 maximum and (45) for the second. \square

346 PROPOSITION 6 (Bounds for \underline{c}_h^0). Let $\underline{c}_h^0 \in \underline{U}_{h,0}^k$ be defined by (25) from an initial
 347 datum $c_0 \in H^2(\Omega) \cap L_0^2(\Omega)$ such that $\partial_n c_0 = 0$ on $\partial\Omega$. It holds, with real number
 348 $C > 0$ independent of h ,

$$349 \quad (46) \quad \|\underline{c}_h^0\|_{1,h} + \|c_h^0\|_{L^\infty(\Omega)} \leq C \|c_0\|_{H^2(\Omega)}.$$

Proof. To prove the first bound in (46), let $\varphi_h = \underline{c}_h^0$ in (25) and use the first inequality
 in (22), the Cauchy–Schwarz inequality and the discrete Poincaré’s inequality (18)
 with $r = 2$ to infer

$$\|\underline{c}_h^0\|_{1,h}^2 \lesssim a_h(\underline{c}_h^0, \underline{c}_h^0) = -(\Delta c_0, c_h^0) \leq \|\Delta c_0\| \|c_h^0\| \lesssim \|c_0\|_{H^2(\Omega)} \|\underline{c}_h^0\|_{1,h}.$$

To prove the second bound in (46), we start by noticing that, using the definition (27)
 of \underline{L}_h^k with $\underline{z}_h = -\underline{L}_h^k \underline{c}_h^0$,

$$\|\underline{L}_h^k \underline{c}_h^0\|_{0,h}^2 = -a_h(\underline{c}_h^0, \underline{L}_h^k \underline{c}_h^0) = (\Delta c_0, L_h^k \underline{c}_h^0) \leq \|c_0\|_{H^2(\Omega)} \|L_h^k \underline{c}_h^0\|,$$

hence $\|\underline{L}_h^k \underline{c}_h^0\|_{0,h} \leq \|c_0\|_{H^2(\Omega)}$. Combining the discrete Agmon’s inequality (28) with
 the latter inequality and the first bound in (46), one gets

$$\|c_h^0\|_{L^\infty(\Omega)} \leq \|\underline{c}_h^0\|_{1,h}^{\frac{1}{2}} \|\underline{L}_h^k \underline{c}_h^0\|_{0,h}^{\frac{1}{2}} \lesssim \|c_0\|_{H^2(\Omega)},$$

350 and the desired result follows. \square

351 **5. Error analysis.** In this section we carry out the error analysis of the method (24). \blacksquare

352 **5.1. Error equations.** Our goal is to estimate the difference between the dis-
 353 crete solution obtained solving (24) and the projections of the exact solution such
 354 that, for all $1 \leq n \leq N$, $\underline{w}_h^n = \underline{I}_h^k w^n$, while, for all $0 \leq n \leq N$, $\widehat{\underline{c}}_h^n \in \underline{U}_{h,0}^k$ solves

$$355 \quad a_h(\widehat{\underline{c}}_h^n, \varphi_h) = -(\Delta c^n, \varphi_h) \quad \forall \varphi_h \in \underline{U}_h^k,$$

356 and $(\widehat{\underline{c}}_h^n, 1) = 0$. We define, for all $1 \leq n \leq N$, the errors

$$357 \quad (47) \quad \underline{e}_{c,h}^n := \underline{c}_h^n - \widehat{\underline{c}}_h^n, \quad \underline{e}_{w,h}^n := \underline{w}_h^n - \widehat{\underline{w}}_h^n.$$

358 By definition (25), $\widehat{\underline{c}}_h^0 = \underline{c}_h^0$, which prompts us to set $\underline{e}_{c,h}^0 := \underline{0}$. Using Poincaré’s
 359 inequality (18) with $r = 2$ and the consistency (23) of a_h , the following estimate is
 360 readily inferred: For all $0 \leq n \leq N$, assuming the additional regularity $c^n \in H^{k+2}(\Omega)$,

$$361 \quad (48) \quad \|\widehat{\underline{c}}_h^n - \pi_h^{k+1} c^n\| \lesssim \|\widehat{\underline{c}}_h^n - \underline{I}_h^k c^n\|_{1,h} \lesssim h^{k+1} \|c^n\|_{H^{k+2}(\Omega)}.$$

362

363 REMARK 4 (Improved L^2 -estimate). We notice, in passing, that, using elliptic regu-
 364 larity (which holds since Ω is convex, cf., e.g., [25]), one can improve this result and
 365 show that $\|\widehat{\underline{c}}_h^n - \pi_h^{k+1} c^n\| \lesssim h^{k+2} \|c^n\|_{H^{k+2}(\Omega)}$.

366 Recalling (24), for all $1 \leq n \leq N$, the error $(\underline{e}_{c,h}^n, \underline{e}_{w,h}^n) \in \underline{U}_{h,0}^k \times \underline{U}_h^k$ solves

$$367 \quad (49a) \quad (\delta_t e_{c,h}^n, \varphi_h) + a_h(\underline{e}_{w,h}^n, \varphi_h) = \mathcal{E}(\varphi_h) \quad \forall \varphi_h \in \underline{U}_h^k,$$

$$368 \quad (49b) \quad (e_{w,h}^n, \psi_h) = (\Phi'(c_h^n) - \Phi'(c^n), \psi_h) + \gamma^2 a_h(\underline{e}_{c,h}^n, \underline{\psi}_h), \quad \forall \underline{\psi}_h \in \underline{U}_h^k,$$

370 where, in (49a), we have defined the consistency error

$$371 \quad (50) \quad \mathcal{E}(\underline{\varphi}_h) := -(\delta_t \widehat{c}_h^n, \varphi_h) - a_h(\widehat{w}_h^n, \underline{\varphi}_h),$$

while in (49b) we have combined the definitions of \widehat{w}_h^n and \widehat{c}_h^n with (1b) to infer

$$(\widehat{w}_h^n, \psi_h) - \gamma^2 a_h(\widehat{c}_h^n, \underline{\psi}_h) = (w^n + \Delta c^n, \psi_h) = (\Phi'(c^n), \psi_h).$$

372 5.2. Error estimate.

373 **THEOREM 7** (Error estimate). *Suppose that the assumptions of Lemma 5 hold true.*
 374 *Let (c, w) denote the solution to (1), for which we assume the following additional*
 375 *regularity:*

$$376 \quad (51) \quad c \in C^2([0, t_F]; L^2(\Omega)) \cap C^1([0, t_F]; H^{k+2}(\Omega)), \quad w \in C^0([0, t_F]; H^{k+2}(\Omega)).$$

377 *Then, the following estimate holds for the errors defined by (47):*

$$378 \quad (52) \quad \left(\max_{1 \leq n \leq N} \|\underline{e}_{c,h}^n\|_{a,h}^2 + \sum_{n=1}^N \tau \|\underline{e}_{w,h}^n\|_{a,h}^2 \right)^{\frac{1}{2}} \leq C(h^{k+1} + \tau),$$

379 *with real number $C > 0$ independent of h and τ .*

380 *Proof.* Let $1 \leq n \leq N$. Subtracting (49b) with $\underline{\psi}_h = \delta_t \underline{e}_{c,h}^n$ from (49a) with $\underline{\varphi}_h = \underline{e}_{w,h}^n$,
 381 we obtain

$$382 \quad (53) \quad \|\underline{e}_{w,h}^n\|_{a,h}^2 + \gamma^2 a_h(\underline{e}_{c,h}^n, \delta_t \underline{e}_{c,h}^n) = \mathcal{E}(\underline{e}_{w,h}^n) + (\Phi'(c^n) - \Phi'(c_h^n), \delta_t \underline{e}_{c,h}^n) := \mathfrak{T}_1 + \mathfrak{T}_2.$$

383 We proceed to bound the terms in the right-hand side.

(i) *Bound for \mathfrak{T}_1 .* Let $\underline{\varphi}_h \in \underline{U}_h^k$. Adding to (50) the quantity

$$(\mathrm{d}_t c^n - \Delta w^n, \varphi_h) + (\delta_t \pi_h^{k+1} c^n - \delta_t c^n, \varphi_h) = 0,$$

(use (1a) to prove that the first addend is 0 and the definition of the L^2 -orthogonal projector π_h^{k+1} to prove that the second is also 0), we can decompose $\mathcal{E}(\underline{\varphi}_h)$ as follows:

$$\begin{aligned} \mathcal{E}(\underline{\varphi}_h) &= (\mathrm{d}_t c^n - \delta_t c^n, \varphi_h) + (\delta_t (\pi_h^{k+1} c^n - \widehat{c}_h^n), \varphi_h) - \left(a_h(\widehat{w}_h^n, \underline{\varphi}_h) + (\Delta w^n, \varphi_h) \right) \\ &:= \mathfrak{T}_{1,1} + \mathfrak{T}_{1,2} + \mathfrak{T}_{1,3}. \end{aligned}$$

384 For the first term, we have

$$385 \quad (54) \quad |\mathfrak{T}_{1,1}| \leq \|\mathrm{d}_t c^n - \delta_t c^n\| \|\varphi_h\| \lesssim \tau \|c\|_{C^2([0, t_F]; L^2(\Omega))} \|\underline{\varphi}_h\|_{1,h} \lesssim \tau \|\underline{\varphi}_h\|_{1,h},$$

386 where we have used the Cauchy–Schwarz inequality, a classical estimate based on
 387 Taylor’s remainder, Poincaré’s inequality (18) with $r = 2$, and we have concluded
 388 using the regularity (51) for c . For the second term, on the other hand, using the
 389 Cauchy–Schwarz inequality followed by (48) together with the C^1 -stability of the
 390 backward differencing operator (3), Poincaré’s inequality, and the regularity (51) for
 391 c , we readily obtain

$$392 \quad (55) \quad |\mathfrak{T}_{1,2}| \leq \|\delta_t (\pi_h^{k+1} c^n - \widehat{c}_h^n)\| \|\varphi_h\| \lesssim h^{k+1} \|c^n\|_{C^1([0, t_F]; H^{k+2}(\Omega))} \|\varphi_h\| \lesssim h^{k+1} \|\underline{\varphi}_h\|_{1,h}.$$

393 Finally, recalling the consistency properties (23) of a_h , we get for the last term

$$394 \quad (56) \quad \begin{aligned} |\mathfrak{T}_{1,3}| &\lesssim h^{k+1} \|w^n\|_{H^{k+2}(\Omega)} \|\underline{\varphi}_h\|_{1,h} \leq h^{k+1} \|w\|_{C^0([0,t_F]; H^{k+2}(\Omega))} \|\underline{\varphi}_h\|_{1,h} \\ &\lesssim h^{k+1} \|\underline{\varphi}_h\|_{1,h}. \end{aligned}$$

395 Collecting the bounds (54)–(56), it is inferred that

$$396 \quad (57) \quad \mathfrak{S} := \sup_{\varphi_h \in \underline{U}_h^k, \|\varphi_h\|_{1,h}=1} \mathcal{E}(\underline{\varphi}_h) \lesssim h^{k+1} + \tau,$$

397 so that, for any real $\epsilon > 0$, denoting by $C_\epsilon > 0$ a real depending on ϵ but not on h or
398 τ , and using the second inequality in (22) to bound $\|\underline{e}_{w,h}^n\|_{1,h} \lesssim \|\underline{e}_{w,h}^n\|_{a,h}$,

$$399 \quad (58) \quad |\mathfrak{T}_1| \leq \mathfrak{S} \|\underline{e}_{w,h}^n\|_{1,h} \lesssim (h^{k+1} + \tau) \|\underline{e}_{w,h}^n\|_{1,h} \leq C_\epsilon (h^{k+1} + \tau)^2 + \epsilon \|\underline{e}_{w,h}^n\|_{a,h}^2.$$

400 (ii) *Bound for \mathfrak{T}_2 .* Set, for the sake of brevity, $Q^n := \Phi'(c_h^n) - \Phi'(c^n)$, and define the
401 DOF vector $\underline{z}_h \in \underline{U}_h^k$ such that

$$402 \quad (59) \quad z_T = \pi_T^{k+1} Q^n \quad \forall T \in \mathcal{T}_h, \quad z_F = \pi_F^k \{Q^n\}_F \quad \forall F \in \mathcal{F}_h^i, \quad z_F = \pi_F^k z_{T_F} \quad \forall F \in \mathcal{F}_h^b$$

403 where $\{\cdot\}_F$ denotes the usual average operator such that, for any function φ admitting
404 a possibly two-valued trace on $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$, $\{\varphi\}_F := \frac{1}{2}(\varphi|_{T_1} + \varphi|_{T_2})$, while, for a
405 boundary face $F \in \mathcal{F}_h^b$, T_F denotes the unique element in \mathcal{T}_h such that $F \in \mathcal{F}_{T_F}$.
406 We have, using the definition of π_T^{k+1} followed by (49a) with $\underline{\varphi}_h = \underline{z}_h$, (57), and the
407 second inequality in (22),

$$408 \quad (60) \quad \mathfrak{T}_2 = (z_h, \delta_t e_{c,h}^n) = \mathcal{E}(\underline{z}_h) - a_h(\underline{e}_{w,h}^n, \underline{z}_h) \lesssim (\mathfrak{S} + \|\underline{e}_{w,h}^n\|_{a,h}) \|\underline{z}_h\|_{1,h}.$$

409 By Proposition 9 below,

$$410 \quad (61) \quad \|\underline{z}_h\|_{1,h} \lesssim \|\underline{e}_{c,h}^n\|_{a,h} + h^{k+1},$$

411 hence, for any real $\epsilon > 0$, denoting by $C_\epsilon > 0$ a real number depending on ϵ but not
412 on h or τ , and recalling the bound (57) for \mathfrak{S} ,

$$413 \quad (62) \quad |\mathfrak{T}_2| \leq C_\epsilon (\|\underline{e}_{c,h}^n\|_{a,h}^2 + (h^{k+1} + \tau)^2) + \epsilon \|\underline{e}_{w,h}^n\|_{a,h}^2.$$

(iii) *Conclusion.* Using (58) and (62) with $\epsilon = \frac{1}{4}$ to bound the right-hand side of (53),
it is inferred

$$\|\underline{e}_{w,h}^n\|_{a,h}^2 + \gamma^2 a_h(\underline{e}_{c,h}^n, \delta_t \underline{e}_{c,h}^n) \lesssim (h^{k+1} + \tau)^2 + \|\underline{e}_{c,h}^n\|_{a,h}^2.$$

Multiplying by τ , summing over $1 \leq n \leq N$, using (33) for the second term in the
left-hand side, and recalling that, by definition, $\underline{e}_{c,h}^0 = \underline{0}$, we get

$$\gamma^2 \|\underline{e}_{c,h}^N\|_{a,h}^2 + \sum_{n=1}^N \tau \|\underline{e}_{w,h}^n\|_{a,h}^2 \leq \sum_{n=1}^N C \tau \|\underline{e}_{c,h}^n\|_{a,h}^2 + C(h^{k+1} + \tau)^2,$$

414 with $C > 0$ independent of h and τ . The error estimate (52) then follows from an
415 application of the discrete Gronwall's inequality (29) with $\delta = \tau$, $a^n = \gamma^2 \|\underline{e}_{c,h}^n\|_{a,h}^2$,
416 $b^n = \|\underline{e}_{w,h}^n\|_{a,h}^2$, $\chi^n = C$, and $G = C(h^{k+1} + \tau)^2$ assuming τ small enough. \square

REMARK 5 (BDF2 time discretization). *In Section 6, we have also used a BDF2 scheme to march in time, which corresponds to the backward differencing operator*

$$\delta_t^{(2)}\varphi := \frac{3\varphi^{n+2} - 4\varphi^{n+1} + \varphi^n}{2\tau},$$

used in place of (3). The analysis is essentially analogous to the backward Euler scheme, the main difference being that formula (33) is replaced by

$$2x(3x - 4y + z) = x^2 - y^2 + (2x - y)^2 - (2y - z)^2 + (x - 2y + z)^2.$$

417 *As a result, the right-hand side of (52) scales as $(h^{k+1} + \tau^2)$ instead of $(h^{k+1} + \tau)$.*

418 To prove the bound (61), we need discrete counterparts of the following Gagliardo–
419 Nirenberg–Poincaré’s inequalities valid for $p \in [2, +\infty)$ if $d = 2$, $p \in [2, 6]$ if $d = 3$, and
420 all $v \in H^2(\Omega) \cap L_0^2(\Omega)$:

$$421 \quad (63) \quad |v|_{W^{1,p}(\Omega)} \lesssim \|v\|^{1-\alpha} |v|_{H^2(\Omega)}^\alpha \lesssim |v|_{H^1(\Omega)}^{1-\alpha} |v|_{H^2(\Omega)}^\alpha, \quad \alpha := \frac{1}{2} + \frac{d}{2} \left(\frac{1}{2} - \frac{1}{p} \right),$$

422 where the first bound follows from [1, Theorem 3] and the second from Poincaré’s
423 inequality. The proof of the following Lemma will be given in Appendix A.

424 LEMMA 8 (Discrete Gagliardo–Nirenberg–Poincaré’s inequalities). *Under the as-*
425 *sumptions of Lemma 3, it holds for $p \in [2, +\infty)$ if $d = 2$, $p \in [2, 6]$ if $d = 3$ with*
426 *$C > 0$ independent of h and α defined as in (63),*

$$427 \quad (64) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k, \quad \|\nabla_h v_h\|_{L^p(\Omega)^d} \leq C \|\underline{v}_h\|_{1,h}^{1-\alpha} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^\alpha.$$

428 PROPOSITION 9 (Bound on $\|\underline{z}_h\|_{1,h}$). *With \underline{z}_h defined as in (59), the bound (61)*
429 *holds.*

430 *Proof.* Recalling the definition (15) of the $\|\cdot\|_{1,h}$ -norm, one has

$$431 \quad (65) \quad \|\underline{z}_h\|_{1,h}^2 = \|\nabla_h \pi_h^{k+1} Q^n\|^2 + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_h^i} h_F^{-1} \|\pi_F^k(\{Q^n\}_F - \pi_T^{k+1} Q^n)\|_F^2 := \mathfrak{T}_1^2 + \mathfrak{T}_2^2.$$

(i) *Bound for \mathfrak{T}_1 .* Using the H^1 -stability (7) of π_h^{k+1} , formula (41) to infer $Q^n = q^n(c_h^n - c^n)$ with $q^n := (c_h^n)^2 + c_h^n c^n + (c^n)^2 - 1$, the triangle and Hölder inequalities, we get, for all $T \in \mathcal{T}_h$,

$$\begin{aligned} |\mathfrak{T}_1| &\leq \|\nabla_h Q^n\| \leq \|q^n \nabla_h(c_h^n - c^n)\| + \|(c_h^n - c^n) \nabla_h q^n\| \\ &\leq \left(\|c_h^n\|_{L^\infty(\Omega)}^2 + \|c^n\|_{L^\infty(\Omega)}^2 + 1 \right) \|\nabla_h(c_h^n - c^n)\| \\ &\quad + \|c_h^n - c^n\|_{L^6(\Omega)} \left(\|c_h^n\|_{L^\infty(\Omega)} + \|c^n\|_{L^\infty(\Omega)} \right) \left(\|\nabla_h c_h^n\|_{L^3(\Omega)^d} + \|\nabla c^n\|_{L^3(\Omega)^d} \right). \end{aligned}$$

432 Noting the a priori bound (44) and the regularity assumption (51), both $\|c_h^n\|_{L^\infty(\Omega)}$ and
433 $\|c^n\|_{L^\infty(\Omega)}$ are $\lesssim 1$. Additionally, by the continuous Gagliardo–Nirenberg–Poincaré’s
434 inequality (63) with $p = 3$ and the regularity assumption (51), one has with $\alpha =$
435 $1/2 + d/12$, $\|\nabla c^n\|_{L^3(\Omega)^d} \lesssim |c^n|_{H^1(\Omega)}^{1-\alpha} \|c^n\|_{H^2(\Omega)}^\alpha \lesssim 1$. Similarly, the discrete Gagliardo–
436 Nirenberg–Poincaré’s inequality (64) with $p = 3$ combined with the a priori bounds (30)
437 and (44) yields $\|\nabla_h c_h^n\|_{L^3(\Omega)^d} \lesssim \|\underline{c}_h^n\|_{1,h}^{1-\alpha} \|\underline{L}_h^k \underline{c}_h^n\|_{0,h}^\alpha \lesssim 1$. Then, inserting $\pm(\widehat{c}_h^n - \pi_h^{k+1} c^n)$
438 and using the triangle inequality,

$$439 \quad (66) \quad |\mathfrak{T}_1| \lesssim \left(\|\nabla_h e_{c,h}^n\| + \|e_{c,h}^n\|_{L^6(\Omega)} \right) + \left(\|\nabla_h(\widehat{c}_h^n - \pi_h^{k+1} c^n)\| + \|\widehat{c}_h^n - \pi_h^{k+1} c^n\|_{L^6(\Omega)} \right) \\ + \left(\|\nabla_h(\pi_h^{k+1} c^n - c^n)\| + \|\pi_h^{k+1} c^n - c^n\|_{L^6(\Omega)} \right) := \mathfrak{T}_{1,1} + \mathfrak{T}_{1,2} + \mathfrak{T}_{1,3}.$$

440 Using the discrete Friedrichs' inequality (18) with $r = 6$ together with the defini-
 441 tion (15) of the $\|\cdot\|_{1,h}$ -norm and the first inequality in (22), it is readily inferred
 442 that $\mathfrak{T}_{1,1} \lesssim \|\underline{c}_{c,h}^n\|_{a,h}$. Again the Friedrichs' inequality (18) with $r = 6$ followed
 443 by the approximation properties (48) of \widehat{c}_h^n and the regularity (51) yields $\mathfrak{T}_{2,2} \lesssim$
 444 $h^{k+1} \|c^n\|_{H^{k+2}(\Omega)} \lesssim h^{k+1}$. Finally, using the approximation properties (8) of π_h^{k+1} , we
 445 have $\mathfrak{T}_{1,3} \lesssim h^{k+1} (\|c^n\|_{H^{k+2}(\Omega)} + \|c^n\|_{W^{k+1,6}(\Omega)}) \lesssim h^{k+1}$, where we have used the fact
 446 that $H^{k+2}(\Omega) \subset W^{k+1,6}(\Omega)$ for all $k \geq 0$ and $d \in \{2, 3\}$ on domains satisfying the
 447 cone condition (cf. [2, Theorem 4.12]). Gathering the previous bounds, we conclude
 448 that

$$449 \quad (67) \quad |\mathfrak{T}_1| \lesssim \|\underline{c}_{c,h}^n\|_{a,h} + h^{k+1}.$$

450 (ii) *Bound for \mathfrak{T}_2 .* For all interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$, we denote by $[\cdot]_F$ the usual
 451 jump operator such that, for every function φ with a possibly two-valued trace on
 452 F , $[\varphi]_F := \varphi|_{T_1} - \varphi|_{T_2}$ (the orientation is irrelevant). Let an element $T \in \mathcal{T}_h$ and an
 453 interface face $F \in \mathcal{F}_T \cap \mathcal{F}_{T^+}$ be fixed. Using the L^2 -stability of π_F^k , inserting $\pm Q_T^n$
 454 (with $Q_T^n := Q^n|_T$), and using the triangle inequality it holds,

$$455 \quad (68) \quad \begin{aligned} & \|\pi_F^k(\{Q^n\}_F - \pi_T^{k+1}Q_T^n)\|_F \leq \|\{Q^n\}_F - \pi_T^{k+1}Q_T^n\|_F \\ & \leq \frac{1}{2} \|[Q^n]_F\|_F + \|Q_T^n - \pi_T^{k+1}Q_T^n\|_F \\ & \lesssim \|[Q^n]_F\|_F + h_T^{\frac{1}{2}} \|\nabla Q_T^n\|_T, \end{aligned}$$

where we have used (8) for the last term. Let us bound the first term in the right-hand
 side. Observing that $[\Phi'(c^n)]_F = 0$ and recalling (41), it is inferred

$$\|[Q^n]_F\|_F = \|[\Phi'(c_h^n)]_F\|_F \leq \|c_h^n\|_F (|c_T|^2 + |c_{T^+}| + |c_{T^+}|^2 + 1).$$

Using this relation, and noticing the a priori bound (44), we get

$$\|[Q^n]_F\|_F \lesssim \left(\|c_h^n\|_{L^\infty(\Omega)}^2 + 1 \right) \|c_h^n\|_F \lesssim \|c_h^n\|_F = \|c_h^n - c^n\|_F,$$

456 where the conclusion follows observing that c^n has zero jumps across interfaces. In-
 457 sserting $\pm[\widehat{c}_h^n - \pi_h^{k+1}c^n]_F$ inside the norm and using the triangle inequality, we obtain

$$458 \quad (69) \quad \|[Q^n]_F\|_F \lesssim \|c_h^n - \widehat{c}_h^n\|_F + \|[\widehat{c}_h^n - \pi_h^{k+1}c^n]_F\|_F + \|[\pi_h^{k+1}c^n - c^n]_F\|_F.$$

459 Define on $H^1(\mathcal{T}_h)$ the jump seminorm $|v|_J^2 := \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \|[v]_F\|_F^2$. Let us prove that

$$460 \quad (70) \quad \forall \underline{v}_h \in \underline{U}_h^k, \quad |v_h|_J \lesssim \|\underline{v}_h\|_{1,h} \lesssim \|\underline{v}_h\|_{a,h}.$$

Inserting $\pm(\pi_F^k[v_h]_F - v_F)$ and using the triangle inequality, it is inferred that

$$|v_h|_J^2 \lesssim \sum_{F \in \mathcal{F}_h^i} \sum_{T \in \mathcal{T}_F} h_F^{-1} (\|v_T - \pi_F^k v_T\|_F^2 + \|\pi_F^k(v_T - v_F)\|_F^2) \lesssim \|\nabla_h v_h\|^2 + \|\underline{v}_h\|_{1,h}^2,$$

461 where we have used (9) followed by the discrete trace inequality (4) and the fact that
 462 $\text{card}(\mathcal{F}_T) \lesssim 1$ by mesh regularity for the first term, and the definition (15) of the
 463 $|\cdot|_{1,h}$ -seminorm for the second term. This proves the first bound in (70). The second
 464 bound follows from (22).

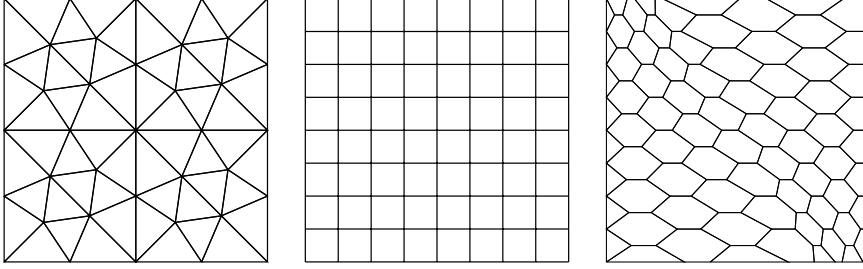


Fig. 1: Mesh families for the numerical tests

465 Multiplying (68) by $h_F^{-\frac{1}{2}}$, squaring, summing over $F \in \mathcal{F}_T \cap \mathcal{F}_h^i$ then over $T \in \mathcal{T}_h$, using
 466 mesh regularity to infer that $\text{card}(\mathcal{F}_T)$ is bounded uniformly in h , and noticing (69)
 467 yields

$$\begin{aligned}
 \mathfrak{I}_2^2 &\lesssim \|\nabla_h Q^n\|^2 + |c_h^n - \widehat{c}_h^n|_{\mathcal{J}}^2 + |\widehat{c}_h^n - \pi_h^{k+1} c^n|_{\mathcal{J}}^2 + |\pi_h^{k+1} c^n - c^n|_{\mathcal{J}}^2 \\
 468 \quad (71) \quad &\lesssim \|\nabla_h Q^n\|^2 + \|\underline{e}_{c,h}^n\|_{a,h}^2 + \|\widehat{c}_h^n - \underline{I}_h^k c^n\|_{a,h}^2 + |\pi_h^{k+1} c^n - c^n|_{\mathcal{J}}^2 \\
 &\lesssim \|\nabla_h Q^n\|^2 + \|\underline{e}_{c,h}^n\|_{a,h}^2 + (h^{k+1} \|c^n\|_{H^{k+2}(\Omega)})^2,
 \end{aligned}$$

469 where we have used (70) to pass to the second line and the approximation proper-
 470 ties (48) of \widehat{c}_h^n and (8) of π_h^{k+1} to conclude. Proceeding as in point (i) to bound the
 471 first term in the right-hand side of (71), and recalling the regularity assumptions (51)
 472 on c , we conclude

$$473 \quad (72) \quad |\mathfrak{I}_2| \leq \|\underline{e}_{c,h}^n\|_{a,h} + h^{k+1}.$$

474 (iii) *Conclusion.* Using (67) and (72) in (65), the estimate (61) follows. \square

475 **REMARK 6** (Polynomial degree for element DOFs). *The use of polynomials of degree*
 476 *($k + 1$) (instead of k) as elements DOFs in the discrete space (13) is required to infer*
 477 *an estimate of order h^{k+1} in (66) and for the last term in (71).*

478 **6. Numerical results.** In this section we provide numerical evidence to confirm
 479 the theoretical results.

480 **6.1. Convergence.** We start by a non-physical numerical test that demon-
 481 strates the orders of convergence achieved by our method. We solve the Cahn-Hilliard
 482 problem (49) on the unit square $\Omega = (0, 1)^2$ with $t_F = 1$, order-parameter

$$483 \quad (73) \quad c(\mathbf{x}, t) = t \cos(\pi x_1) \cos(\pi x_2),$$

484 and chemical potential w inferred from c according to (1b). The right-hand side of (1a)
 485 is also modified by introducing a nonzero source in accordance with the expression of
 486 c . The interface parameter γ is taken equal to 1.

487 We consider the triangular, Cartesian, and (predominantly) hexagonal mesh families
 488 of Figure 1. The two former mesh families were introduced in the FVCA5 bench-
 489 mark [27], whereas the latter was introduced in [20]. To march in time, we use the
 490 implicit Euler scheme. Since the order-parameter is linear in time, only the spatial

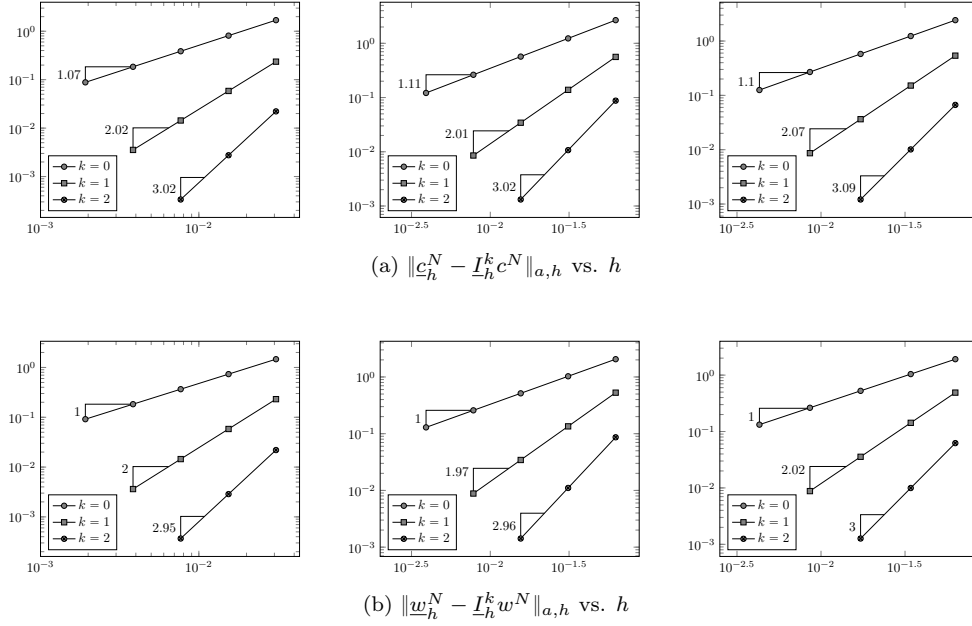


Fig. 2: Energy-errors at final time vs. h . From left to right: triangular, Cartesian and (predominantly) hexagonal mesh families; cf. Figure 1.

491 component of the discretization error is nonzero and the choice of the time step is
 492 irrelevant. The energy errors $\|c_h^N - I_h^k c^N\|_{a,h}$ and $\|w_h^N - I_h^k w^N\|_{a,h}$ at final time are
 493 depicted in Figure 2. For all mesh families, the convergence rate is $(k + 1)$, in accor-
 494 dance with Theorem 7. For the sake of completeness, we also display in Figure 3 the
 495 L^2 -errors $\|c_h^n - \pi_h^{k+1} c^n\|$ and $\|w_h^n - \pi_h^{k+1} w^n\|$, for which an optimal convergence rate
 496 of $(k + 2)$ is observed.

6.2. Evolution of an elliptic and a cross-shaped interfaces. The numerical
 examples of this section consist in tracking the evolution of initial data corresponding,
 respectively, to an elliptic and a cross-shaped interface between phases. For the elliptic
 interface test case of Figure 4, the initial datum is

$$c_0(\mathbf{x}) = \begin{cases} 0.95 & \text{if } 81(x_1 - 0.5)^2 + 9(x_2 - 0.5)^2 < 1, \\ -0.95 & \text{otherwise.} \end{cases}$$

For the cross-shaped interface test case of Figure 5, we take

$$c_0(\mathbf{x}) = \begin{cases} 0.95 & \text{if } 5(|(x_2-0.5) - \frac{2}{5}(x_1-0.5)| + |\frac{2}{5}(x_1-0.5) + (x_2-0.5)|) < 1 \\ & \text{or } 5(|(x_1-0.5) - \frac{2}{5}(x_2-0.5)| + |\frac{2}{5}(x_2-0.5) + (x_1-0.5)|) < 1, \\ -0.95 & \text{otherwise.} \end{cases}$$

497 In both cases, the space domain is the unit square $\Omega = (0, 1)^2$, and the interface
 498 parameter γ is taken to be $1 \cdot 10^{-2}$. We use a 64×64 uniform Cartesian mesh and
 499 $k = 1$ with time step $\tau = \gamma^2/10$.

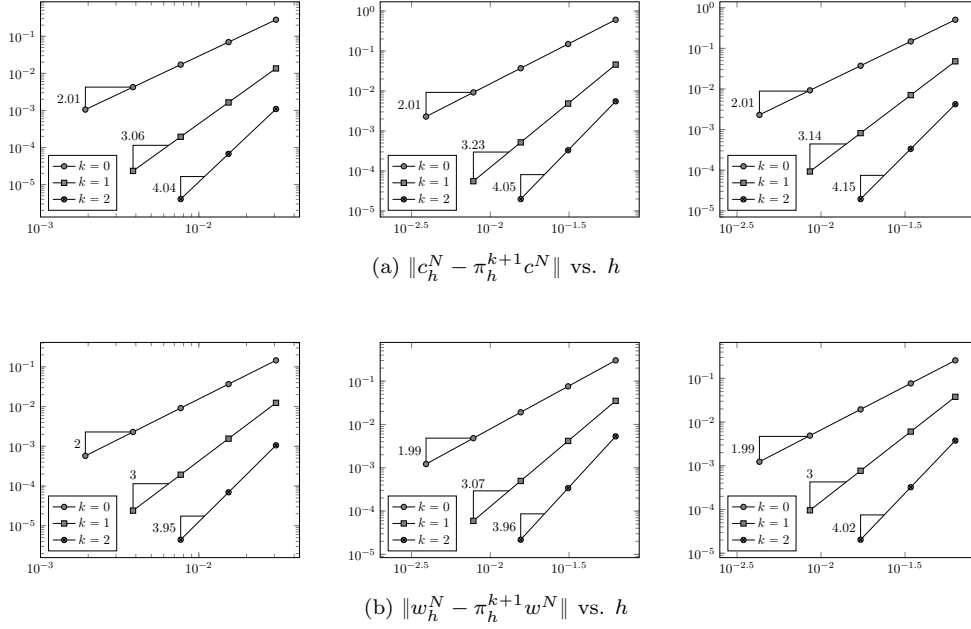


Fig. 3: L^2 -errors at final time vs. h . From left to right: triangular, Cartesian and (predominantly) hexagonal mesh families; cf. Figure 1.

500 In the test case of Figure 4, we observe evolution of the elliptic interface towards a
 501 circular interface and, as expected, mass is well preserved (+0.5% with respect to the
 502 initial ellipse). Similar considerations hold for the cross-shaped test case of Figure 5,
 503 which has the additional difficulty of presenting sharp corners.

504 **6.3. Spinodal decomposition.** Spinodal decomposition can be observed when
 505 a binary alloy is heated to a high temperature for a certain time and then abruptly
 506 cooled. As a result, phases are separated in well-defined high concentration areas. In
 507 Figure 6, we display the numerical solutions obtained on a 128×128 uniform Cartesian
 508 mesh for $k = 0$ and on a uniform 64×64 Cartesian mesh for $k = 1$. In both cases, we
 509 use the same initial conditions taking random values between -1 and 1 on a 32×32
 510 uniform Cartesian partition of the domain. The interface parameter is $\gamma = 1/100$,
 511 and we take $\tau = \gamma^2/10$. For $k = 0$, the time discretisation is based on the Backward
 512 Euler scheme while, for $k = 1$, we use the BDF2 formula to make sure that the spatial
 513 and temporal error contributions are equilibrated; cf. Remark 5.

514 The separation of the two components into two distinct phases happens over a very
 515 small time; see two leftmost panels of Figure 6 corresponding to times 0 and $5 \cdot 10^{-5}$,
 516 respectively. Later, the phases gather increasingly slowly until the interfaces develop
 517 a constant curvature; see the two rightmost panels of Figure 6, corresponding to times
 518 $1.25 \cdot 10^{-3}$ and $3.6 \cdot 10^{-2}$, respectively. At the latest stages, we can observe that the
 519 solution exhibits a (small) dependence on the mesh and/or the polynomial degree,
 520 and the high-concentration regions in Figures 6a and 6b are highly superposable but

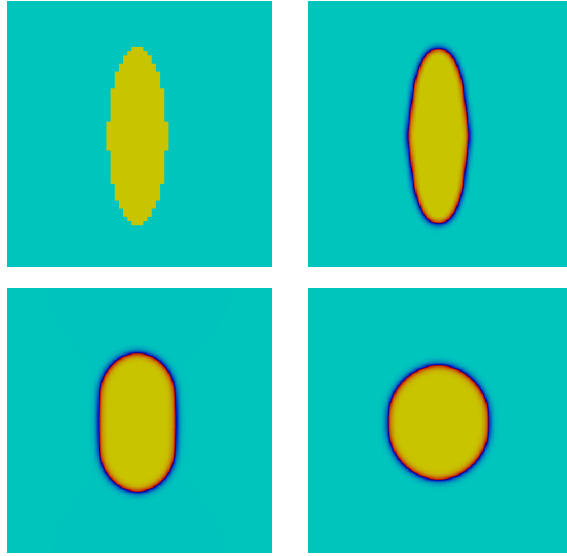


Fig. 4: Evolution of an elliptic interface (left to right, top to bottom). Displayed times are 0 , $3 \cdot 10^{-3}$, 0.3 , 1 .

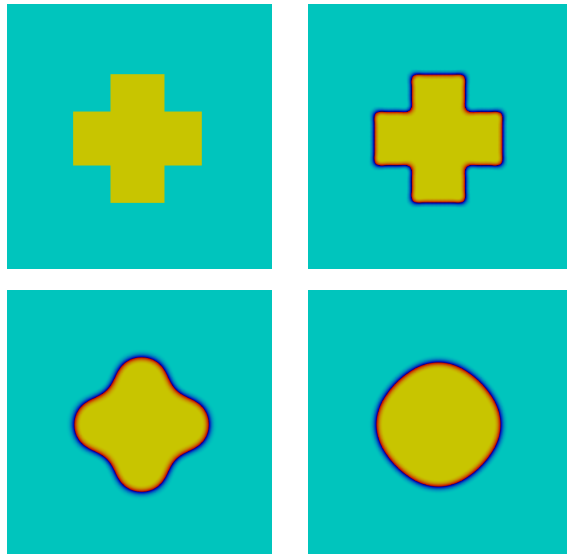


Fig. 5: Evolution of a cross-shaped interface (left to right, top to bottom). Displayed times are 0 , $5 \cdot 10^{-5}$, $1 \cdot 10^{-2}$, $8.17 \cdot 10^{-2}$.

521 not identical.

522 **Appendix A. Proofs of discrete functional analysis results.**

523 This section contains the proofs of Lemmas 3 and 8 preceded by the required pre-
524 liminary technical results.

525 PROPOSITION 10 (Estimates for \underline{L}_h^k). *Assuming mesh quasi-uniformity (10), it holds*

526 (74) $\forall \underline{v}_h \in \underline{U}_h^k, \quad \|\underline{L}_h^k \underline{v}_h\|_{0,h} \lesssim h^{-1} \|\underline{v}_h\|_{1,h},$

527 (75) $\forall \underline{v}_h \in \underline{U}_{h,0}^k, \quad \|\underline{L}_h^k \underline{v}_h\|_{H^{-1}(\Omega)} \lesssim \|\underline{v}_h\|_{1,h}.$

Proof. (i) *Proof of (74).* Let $\underline{v}_h \in \underline{U}_h^k$. Making $\underline{z}_h = -\underline{L}_h^k \underline{v}_h$ in the definition (27) of \underline{L}_h^k , we have

$$\|\underline{L}_h^k \underline{v}_h\|_{0,h}^2 = -a_h(\underline{v}_h, \underline{L}_h^k \underline{v}_h) \lesssim \|\underline{v}_h\|_{1,h} \|\underline{L}_h^k \underline{v}_h\|_{1,h} \lesssim \|\underline{v}_h\|_{1,h} h^{-1} \|\underline{L}_h^k \underline{v}_h\|_{0,h},$$

529 where we have used the continuity of a_h expressed by the second inequality in (22)
530 followed by the fact that, for all $\underline{z}_h \in \underline{U}_h^k$, $\|\underline{z}_h\|_{1,h} \lesssim h^{-1} \|\underline{z}_h\|_{0,h}$. This inequality
531 follows from the definition (15) of the $\|\cdot\|_{1,h}$ -norm using the inverse inequality (5) to
532 bound the first term and recalling mesh quasi-uniformity (10).

533 (ii) *Proof of (75).* Let $\underline{v}_h \in \underline{U}_{h,0}^k$. Observing that $\underline{L}_h^k \underline{v}_h$ has zero-average on Ω (cf.
534 Remark 3), we have

535 (76) $\|\underline{L}_h^k \underline{v}_h\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H^1(\Omega) \cap L_0^2(\Omega), \|\varphi\|_{H^1(\Omega)}=1} (\underline{L}_h^k \underline{v}_h, \varphi).$

Let now $\underline{\varphi}_h := \underline{L}_h^k \varphi$. Using the fact that $\underline{L}_h^k \underline{v}_h \in \mathbb{P}^{k+1}(\mathcal{T}_h)$ followed by the defini-
tions (27) of \underline{L}_h^k and (26) of $(\cdot, \cdot)_{0,h}$, one has

$$(\underline{L}_h^k \underline{v}_h, \varphi) = (\underline{L}_h^k \underline{v}_h, \pi_h^{k+1} \varphi) = -s_{0,h}(\underline{L}_h^k \underline{v}_h, \underline{\varphi}_h) - a_h(\underline{v}_h, \underline{\varphi}_h).$$

Hence, using the Cauchy–Schwarz inequality we get

$$\begin{aligned} |(\underline{L}_h^k \underline{v}_h, \varphi)| &\lesssim |\underline{L}_h^k \underline{v}_h|_{0,h} |\underline{\varphi}_h|_{0,h} + \|\underline{v}_h\|_{1,h} \|\underline{\varphi}_h\|_{1,h} \\ &\lesssim h^{-1} \|\underline{v}_h\|_{1,h} h |\underline{\varphi}_h|_{1,h} + \|\underline{v}_h\|_{1,h} \|\underline{\varphi}_h\|_{1,h} \\ &\lesssim \|\underline{v}_h\|_{1,h} \|\underline{\varphi}_h\|_{1,h} \lesssim \|\underline{v}_h\|_{1,h} \|\varphi\|_{H^1(\Omega)}, \end{aligned}$$

536 where we have used the second inequality in (22) in the first line, (74) together with
537 the fact that $|\underline{z}_h|_{0,h} \leq h |\underline{z}_h|_{1,h}$ for all $\underline{z}_h \in \underline{U}_h^k$ to pass to the second line, and the
538 H^1 -stability (17) of \underline{L}_h^k to conclude. To obtain (75), plug the above estimate into the
539 right-hand side of (76). \square

We introduce the continuous Green’s function $\mathcal{G} : L_0^2(\Omega) \rightarrow H^1(\Omega) \cap L_0^2(\Omega)$ such that,
for all $\varphi \in L_0^2(\Omega)$,

$$(\nabla \mathcal{G} \varphi, \nabla v) = (\varphi, v) \quad \forall v \in H^1(\Omega).$$

540 Owing to elliptic regularity (which holds since Ω is convex), we have $\mathcal{G} \varphi \in H^2(\Omega)$. Its
541 discrete counterpart $\underline{\mathcal{G}}_h^k : \underline{U}_{h,0}^k \rightarrow \underline{U}_{h,0}^k$ is defined such that, for all $\underline{\varphi}_h \in \underline{U}_{h,0}^k$,

542 (77) $a_h(\underline{\mathcal{G}}_h^k \underline{\varphi}_h, \underline{z}_h) = (\underline{\varphi}_h, \underline{z}_h)_{0,h} \quad \forall \underline{z}_h \in \underline{U}_{h,0}^k,$

with inner product $(\cdot, \cdot)_{0,h}$ defined by (26). We will denote by $\underline{\mathcal{G}}_h^k \underline{v}_h$ (no underline)
the broken polynomial function in $\mathbb{P}^{k+1}(\mathcal{T}_h)$ obtained from element DOFs in $\underline{\mathcal{G}}_h^k \underline{v}_h$.
We next show that $-\underline{\mathcal{G}}_h^k$ is the inverse of \underline{L}_h^k restricted to $\underline{U}_{h,0}^k \rightarrow \underline{U}_{h,0}^k$. Let $\underline{v}_h \in \underline{U}_{h,0}^k$.
Using (77) with $\underline{\varphi}_h = \underline{L}_h^k \underline{v}_h$ followed by (27), it is inferred, for all $\underline{z}_h \in \underline{U}_{h,0}^k$,

$$a_h(\underline{\mathcal{G}}_h^k \underline{L}_h^k \underline{v}_h, \underline{z}_h) = (\underline{L}_h^k \underline{v}_h, \underline{z}_h)_{0,h} = -a_h(\underline{v}_h, \underline{z}_h) \implies a_h(\underline{v}_h + \underline{\mathcal{G}}_h^k \underline{L}_h^k \underline{v}_h, \underline{z}_h) = 0.$$

543 Therefore, since $(\underline{v}_h + \underline{\mathcal{G}}_h^k \underline{I}_h^k \underline{v}_h) \in \underline{U}_{h,0}^k$ and a_h is coercive in $\underline{U}_{h,0}^k$ (cf. (22) and Propo-
544 sition 2), we conclude

$$545 \quad (78) \quad \underline{v}_h + \underline{\mathcal{G}}_h^k \underline{I}_h^k \underline{v}_h = \underline{0} \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k.$$

546 PROPOSITION 11 (Estimates for $\underline{\mathcal{G}}_h^k$). *It holds, for all $\underline{v}_h \in \underline{U}_{h,0}^k$,*

$$547 \quad (79) \quad \|\underline{\mathcal{G}}_h^k \underline{v}_h - \underline{I}_h^k \mathcal{G}v_h\|_{1,h} \lesssim h (|\underline{v}_h|_{0,h} + \|\mathcal{G}v_h\|_{H^2(\Omega)}) \lesssim h \|\underline{v}_h\|_{0,h}.$$

548 Moreover, using elliptic regularity, we have

$$549 \quad (80) \quad \|\underline{\mathcal{G}}_h^k \underline{v}_h - \pi_h^{k+1} \mathcal{G}v_h\| \lesssim h^2 (|\underline{v}_h|_{0,h} + \|\mathcal{G}v_h\|_{H^2(\Omega)}) \lesssim h^2 \|\underline{v}_h\|_{0,h}.$$

550 *Proof.* Let $\underline{v}_h \in \underline{U}_{h,0}^k$.

551 (i) *Proof of (79).* For all $\underline{z}_h \in \underline{U}_{h,0}^k$ we have, using the definition (77) of $\underline{\mathcal{G}}_h^k \underline{v}_h$ and
552 subtracting the quantity $(v_h + \Delta \mathcal{G}v_h, z_h) = 0$,

$$553 \quad (81) \quad a_h(\underline{\mathcal{G}}_h^k \underline{v}_h - \underline{I}_h^k \mathcal{G}v_h, \underline{z}_h) = \underbrace{(v_h, \underline{z}_h)_{0,h} - (v_h, z_h)}_{\mathfrak{T}_1} - \underbrace{a_h(\underline{I}_h^k \mathcal{G}v_h, \underline{z}_h) - (\Delta \mathcal{G}v_h, z_h)}_{\mathfrak{T}_2}.$$

554 Recalling the definition (26) of the inner product $(\cdot, \cdot)_{0,h}$, one has

$$555 \quad (82) \quad |\mathfrak{T}_1| = |s_{0,h}(v_h, \underline{z}_h)| \leq |\underline{v}_h|_{0,h} |\underline{z}_h|_{0,h} \leq h |\underline{v}_h|_{0,h} |\underline{z}_h|_{1,h}.$$

556 On the other hand, the consistency property (23) of the bilinear form a_h readily yields

$$557 \quad (83) \quad |\mathfrak{T}_2| \lesssim h \|\mathcal{G}v_h\|_{H^2(\Omega)} \|\underline{z}_h\|_{1,h}.$$

558 Making $\underline{z}_h = \underline{\mathcal{G}}_h^k \underline{v}_h - \underline{I}_h^k \mathcal{G}v_h$ in (81), and using the coercivity of a_h expressed by the first
559 inequality in (22) followed by the bounds (82)–(83), the first bound in (79) follows. To
560 prove the second bound in (79), use elliptic regularity to estimate $\|\mathcal{G}v_h\|_{H^2(\Omega)} \lesssim \|v_h\|$
561 and recall the definition of the $\|\cdot\|_{0,h}$ -norm.

562 (ii) *Proof of (80).* We follow the ideas of [19, Theorem 10] and [18, Theorem 11], to
563 which we refer for further details. Set, for the sake of brevity, $\underline{\varphi}_h := \underline{\mathcal{G}}_h^k \underline{v}_h - \underline{I}_h^k \mathcal{G}v_h$,
564 and let $z := \mathcal{G}\varphi_h$. By elliptic regularity, $z \in H^2(\Omega)$ and $\|z\|_{H^2(\Omega)} \lesssim \|\varphi_h\|$. Observing
565 that $-\Delta z = \varphi_h$, letting $\widehat{z}_h := \underline{I}_h^k z$, and using the definition (77) of $\underline{\mathcal{G}}_h^k$, we have

$$566 \quad (84) \quad \|\varphi_h\|^2 = \underbrace{-(\Delta z, \varphi_h) - a_h(\varphi_h, \widehat{z}_h)}_{\mathfrak{T}_1} + \underbrace{(v_h, \widehat{z}_h) - a_h(\underline{I}_h^k \mathcal{G}v_h, \widehat{z}_h)}_{\mathfrak{T}_2} + \underbrace{s_{0,h}(v_h, \widehat{z}_h)}_{\mathfrak{T}_3}.$$

567 Using the consistency (23) of a_h , it is readily inferred for the first term

$$568 \quad (85) \quad |\mathfrak{T}_1| \lesssim h \|z\|_{H^2(\Omega)} \|\underline{\varphi}_h\|_{1,h} \lesssim h^2 (|\underline{v}_h|_{0,h} + \|\mathcal{G}v_h\|_{H^2(\Omega)}) \|\varphi_h\|,$$

where we have used elliptic regularity to infer $\|z\|_{H^2(\Omega)} \lesssim \|\varphi_h\|$ and (79) to bound
 $\|\underline{\varphi}_h\|_{1,h}$. For the second term, upon observing that $(v_h, \widehat{z}_h) = -(\Delta \mathcal{G}v_h, z) = (\nabla \mathcal{G}v_h, \nabla z)$
since, by definition of, $-\Delta \mathcal{G}v_h = v_h \in \mathbb{P}^{k+1}(\mathcal{T}_h)$ and $\widehat{z}_h = \pi_h^{k+1} z$, recalling the def-
inition (21) of the bilinear form a_h and using the orthogonality property (20) of
 $(p_T^{k+1} \circ \underline{I}_T^k)$, we have

$$\mathfrak{T}_2 = \sum_{T \in \mathcal{T}_h} (\nabla(p_T^{k+1} \underline{I}_T^k \mathcal{G}v_h - \mathcal{G}v_h), \nabla(p_T^{k+1} \widehat{z}_h - z))_T + s_{1,h}(\underline{I}_h^k \mathcal{G}v_h, \widehat{z}_h).$$

569 By the approximation properties of $(p_T^{k+1} \circ \underline{I}_T^k)$ and of π_h^{k+1} , and bounding $\|z\|_{H^2(\Omega)}$
 570 and $\|\underline{\varphi}_h\|_{1,h}$ as before, we have

$$571 \quad (86) \quad |\mathfrak{T}_2| \lesssim h^2 (|\underline{v}_h|_{0,h} + \|\mathcal{G}v_h\|_{H^2(\Omega)}) \|\varphi_h\|.$$

572 Finally, for the last term, we write

$$573 \quad (87) \quad |\mathfrak{T}_3| \leq |\underline{v}_h|_{0,h} |\widehat{\underline{z}}_h|_{0,h} \lesssim |\underline{v}_h|_{0,h} h^2 \|z\|_{H^2(\Omega)} \lesssim h^2 |\underline{v}_h|_{0,h} \|\varphi_h\|,$$

574 where we have used the Cauchy–Schwarz inequality in the first bound, the approxima-
 575 tion properties (8) of π_h^{k+1} in the second bound, and elliptic regularity to conclude.
 576 Using (85)–(87) to estimate the right-hand side of (84) the first inequality in (80)
 577 follows. Using elliptic regularity to further bound $\|\mathcal{G}v_h\|_{H^2(\Omega)} \lesssim \|v_h\|$ and recalling
 578 the definition of the $\|\cdot\|_{0,h}$ -norm yields the second inequality in (80). \square

579 **REMARK 7** (Choice of $s_{0,h}$). *The choice (26) for the stabilisation bilinear form $s_{0,h}$ is*
 580 *crucial to have the right-hand side of (87) scaling as h^2 . Penalizing the full difference*
 581 *$(v_F - v_T)$ instead of the lowest-order part $\pi_F^k(v_F - v_T)$ would have lead to a right-hand*
 582 *side only scaling as h .*

583 We are now ready to prove Lemma 3.

584 *Proof of Lemma 3.* Let $\underline{v}_h \in \underline{U}_{h,0}^k$ and set $\underline{\varphi}_h := \underline{L}_h^k \underline{v}_h$. Recalling that, owing to (78),
 585 $v_h = -\mathcal{G}_h^k \underline{\varphi}_h$, it is inferred using the triangle inequality,

$$586 \quad (88) \quad \|v_h\|_{L^\infty(\Omega)} \leq \|\pi_h^{k+1} \mathcal{G} \varphi_h\|_{L^\infty(\Omega)} + \|\mathcal{G}_h^k \underline{\varphi}_h - \pi_h^{k+1} \mathcal{G} \varphi_h\|_{L^\infty(\Omega)} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

587 The L^∞ -stability of π_h^{k+1} (cf. (7)) followed by the continuous Agmon’s inequality
 588 readily yields for the first term

$$589 \quad (89) \quad \mathfrak{T}_1 \lesssim \|\mathcal{G} \varphi_h\|_{L^\infty(\Omega)} \lesssim \|\mathcal{G} \varphi_h\|_{H^1(\Omega)}^{\frac{1}{2}} \|\mathcal{G} \varphi_h\|_{H^2(\Omega)}^{\frac{1}{2}}.$$

590 Using a standard regularity shift (cf., e.g., [25]), recalling that $\varphi_h = L_h^k \underline{v}_h$, and using
 591 the H^{-1} -bound (75) for $L_h^k \underline{v}_h$, we have

$$592 \quad (90) \quad \|\mathcal{G} \varphi_h\|_{H^1(\Omega)} \lesssim \|\varphi_h\|_{H^{-1}(\Omega)} \lesssim \|\underline{v}_h\|_{1,h}, \quad \|\mathcal{G} \varphi_h\|_{H^2(\Omega)} \lesssim \|\varphi_h\| = \|L_h^k \underline{v}_h\|,$$

593 which plugged into (89) yields

$$594 \quad (91) \quad \mathfrak{T}_1 \lesssim \|\underline{v}_h\|_{1,h}^{\frac{1}{2}} \|L_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}}.$$

595 For the second term we have, on the other hand,

$$\begin{aligned} \mathfrak{T}_2 &\lesssim h^{-\frac{d}{2}} \|\mathcal{G}_h^k \underline{\varphi}_h - \pi_h^{k+1} \mathcal{G} \varphi_h\| \\ 596 \quad (92) \quad &\lesssim h^{\frac{3-d}{2}} (h \|L_h^k \underline{v}_h\|_{0,h})^{\frac{1}{2}} \|L_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}} \\ &\lesssim h^{\frac{3-d}{2}} \|\underline{v}_h\|_{1,h}^{\frac{1}{2}} \|L_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}} \lesssim \|\underline{v}_h\|_{1,h}^{\frac{1}{2}} \|L_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}}, \end{aligned}$$

597 where we have used the global inverse inequality (12) with $p = 2$ to obtain the first
 598 bound, the estimate (80) to obtain the second, (74) to obtain the third, and the fact
 599 that $d \leq 3$ together with $h \leq h_\Omega \lesssim 1$ (with h_Ω diameter of Ω) to conclude. The
 600 conclusion follows plugging (91) and (92) into (88). \square

601 REMARK 8 (Discrete Agmon's inequality in dimension $d = 2$). When $d = 2$, we have
 602 the following sharper form for the discrete Agmon's inequality:

$$603 \quad (93) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k, \quad \|v_h\|_{L^\infty(\Omega)} \lesssim \|\underline{v}_h\|_{0,h}^{\frac{1}{2}} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}}.$$

604 To obtain (93), the following modifications are required in the above proof: (i) The
 605 term \mathfrak{T}_1 is bounded as $\mathfrak{T}_1 \lesssim \|\mathcal{G}\varphi_h\|_{H^2(\Omega)}^{\frac{1}{2}} \|\mathcal{G}\varphi_h\|_{H^2(\Omega)}^{\frac{1}{2}} \lesssim \|v_h\|_{0,h}^{\frac{1}{2}} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}}$, where we have used
 606 $v_h = -\mathcal{G}\varphi_h$ (cf. (78)) for the first factor and (90) for the second; (ii) The third line
 607 of (92) becomes $\mathfrak{T}_2 \lesssim (h\|\underline{v}_h\|_{1,h})^{\frac{1}{2}} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}} \lesssim \|\underline{v}_h\|_{0,h}^{\frac{1}{2}} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^{\frac{1}{2}}$, where we have used
 608 the inverse inequality (5) and mesh quasi-uniformity to bound the first factor.

609 We next prove the discrete Gagliardo–Nirenberg–Poincaré's inequality of Lemma 8.

610 *Proof of Lemma 8.* Using the same notation as in the proof of Lemma 3, we have

$$611 \quad \|\nabla_h v_h\|_{L^p(\Omega)^d} \leq \|\nabla_h \pi_h^{k+1} \mathcal{G}\varphi_h\|_{L^p(\Omega)^d} + \|\nabla_h (\mathcal{G}_h^k \varphi_h - \pi_h^{k+1} \mathcal{G}\varphi_h)\|_{L^p(\Omega)^d} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

For the first term, we use the $W^{1,p}$ -stability of π_h^{k+1} (cf. (7)) followed by the continuous Gagliardo–Nirenberg–Poincaré's inequality (63), and (90) to infer

$$\mathfrak{T}_1 \lesssim |\mathcal{G}\varphi_h|_{W^{1,p}(\Omega)} \lesssim |\mathcal{G}\varphi_h|_{H^1(\Omega)}^{1-\alpha} \|\mathcal{G}\varphi_h\|_{H^2(\Omega)}^\alpha \lesssim \|\underline{v}_h\|_{1,h}^{1-\alpha} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^\alpha.$$

For the second term, on the other hand, we have

$$\begin{aligned} \mathfrak{T}_2 &\lesssim h^{d(\frac{1}{p}-\frac{1}{2})} \|\nabla_h (\mathcal{G}_h^k \varphi_h - \pi_h^{k+1} \mathcal{G}\varphi_h)\| \\ &\lesssim h^{d(\frac{1}{p}-\frac{1}{2})} \|\mathcal{G}_h^k \varphi_h - \underline{I}_h^k \mathcal{G}\varphi_h\|_{1,h}^{1-\alpha} \|\mathcal{G}_h^k \varphi_h - \underline{I}_h^k \mathcal{G}\varphi_h\|_{1,h}^\alpha \\ &\lesssim h^{\alpha+d(\frac{1}{p}-\frac{1}{2})} (h\|\underline{L}_h^k \underline{v}_h\|_{0,h})^{1-\alpha} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^\alpha \\ &\lesssim h^{\alpha+d(\frac{1}{p}-\frac{1}{2})} \|\underline{v}_h\|_{1,h}^{1-\alpha} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^\alpha \lesssim \|\underline{v}_h\|_{1,h}^{1-\alpha} \|\underline{L}_h^k \underline{v}_h\|_{0,h}^\alpha, \end{aligned}$$

where we have used the global reverse Lebesgue inequality (11) in the first line, the definition (15) of the $\|\cdot\|_{1,h}$ -norm to pass to the second line, the estimate (79) to pass to the third line, and (74) to pass to the fourth line. To obtain the second inequality in the fourth line, we observe that, recalling the definition (63) of α and the assumptions on p , it holds for the exponent of h ,

$$\alpha + d \left(\frac{1}{p} - \frac{1}{2} \right) = \frac{1}{2} - \frac{d}{2} \left(\frac{1}{2} - \frac{1}{p} \right) \geq 0,$$

612 and, since $h \leq h_\Omega \lesssim 1$, the conclusion follows. \square

613 REMARK 9 (Validity of the discrete Agmon's and Gagliardo–Nirenberg–Poincaré's
 614 inequalities). At the discrete level, the fact that the discrete Agmon's inequality (28)
 615 is valid only up to $d = 3$ and that the Gagliardo–Nirenberg–Poincaré's inequalities (64)
 616 are valid only for $p \in [2, +\infty)$ if $d = 2$, $p \in [2, 6]$ if $d = 3$ is reflected by the need to
 617 have nonnegative powers of h in the estimates of the terms \mathfrak{T}_2 to conclude in the
 618 corresponding proofs.

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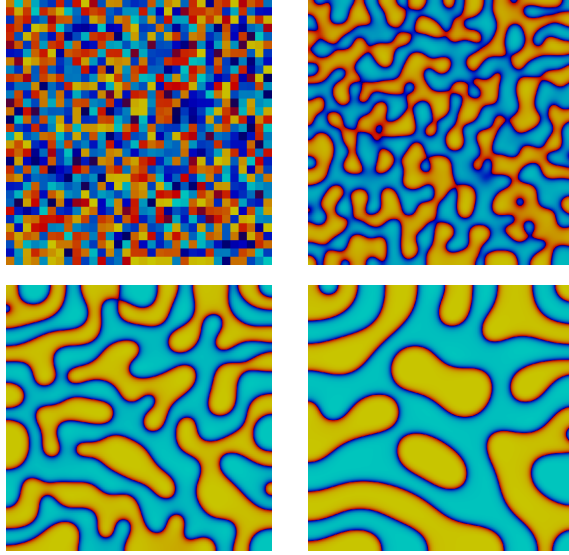
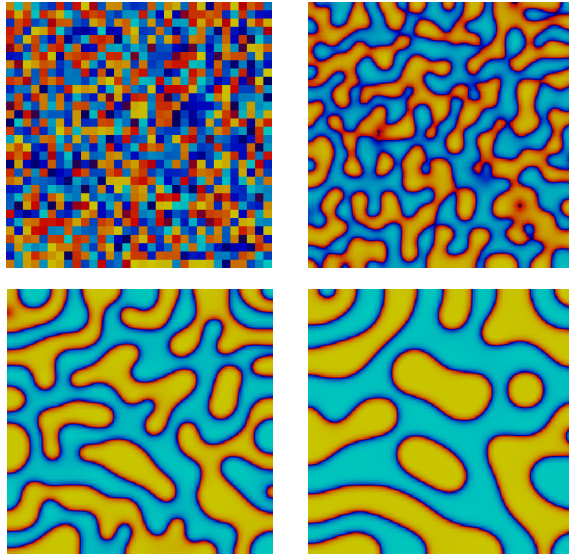
(a) 128×128 uniform Cartesian mesh, $k = 0$, BE(b) 64×64 uniform Cartesian mesh, $k = 1$, BDF2

Fig. 6: Spinoidal decomposition (left to right, top to bottom). In both cases, the same random initial condition is used. Displayed times are 0 , $5 \cdot 10^{-5}$, $1.25 \cdot 10^{-3}$, $3.6 \cdot 10^{-2}$.