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# INCOMPATIBILITY-GOVERNED SINGULARITIES IN LINEAR ELASTICITY WITH DISLOCATIONS

NICOLAS VAN GOETHEM

ABSTRACT. The purpose of this paper is to prove the relation  $\text{incc}\epsilon = \text{Curl } \kappa$  relating the elastic strain  $\epsilon$  and the contortion tensor  $\kappa$ , related to the density tensor of mesoscopic dislocations. Here, the dislocations are given by a finite family of closed Lipschitz curves in  $\Omega \subset \mathbb{R}^3$ . Moreover the fields are singular at the dislocations, and in particular the strain is non square integrable. Moreover, the displacement fields shows a constant jump around each isolated dislocation loop. This relation is called after E. Kröner who first derived the same formula for smooth fields at the macroscale.

## 1. INTRODUCTION

Let  $\Omega$  be a simply-connected smooth and bounded subset of  $\mathbb{R}^3$ . Let  $\mathcal{L}$  be a set of dislocation lines in  $\Omega$ , and the dislocation density  $\Lambda_{\mathcal{L}} \in \mathcal{M}(\Omega, \mathbb{M}^3)$  be given as a Radon measure concentrated in  $\mathcal{L}$ , defined as

$$\Lambda_{\mathcal{L}} := \tau \otimes B\mathcal{H}_{|\mathcal{L}}^1,$$

with  $\tau$ , the tangent vector to  $\mathcal{L}$  and  $\mathcal{H}_{|\mathcal{L}}^1$  the one-dimensional Hausdorff measure concentrated in  $\mathcal{L}$ , and where  $B$  stand for the Burgers vector of the line, constant for a given line.

It is well known that as soon as dislocations are present, i.e. as soon as their density is nonvanishing, the strain can not be a symmetric gradient of a vector field.

At the macroscopic scale, that is, at a scale where the fields are assumed smooth, Kröner has indeed shown that the incompatibility of the elastic strain  $\epsilon$  is related to the curl of the contortion tensor  $\kappa$ . Here the contortion is a symmetric tensor related to the dislocation density  $\Lambda$  by the relation  $\kappa = \Lambda - \frac{\mathbb{I}_2}{2} \text{tr}\Lambda$ , with  $\Lambda$  the macroscopic dislocation density. Kröner's identity reads

$$\text{incc}\epsilon = \text{Curl } \kappa.$$

This relation was to the knowledge of the author first introduced in [7] though it strictly spoken appeared first in [11] in a simple geometrical setting. The geometrical meaning of the contortion tensor in a differential geometry approach to dislocations is also to be emphasized, as discussed in e.g., [5, 7, 8, 10, 15].

However, the concept of dislocation line is related to another scale of matter description, namely the mesoscale, where it appears as the set of singularity for the elastic strain  $\epsilon$  and stress field  $\sigma = \mathbb{A}\epsilon$ , with  $\mathbb{A} = 2\mu\mathbb{I}_4 + \lambda\mathbb{I}_2$ , the elasticity tensor. It is indeed well known that these fields are not square integrable at this scale. Proving a Kröner identity at the mesoscale such as

$$\text{incc}\epsilon = \text{Curl } \kappa_{\mathcal{L}}, \quad \kappa_{\mathcal{L}} := \Lambda_{\mathcal{L}} - \frac{\mathbb{I}_2}{2} \text{tr}\Lambda_{\mathcal{L}},$$

was carried on by the author in a series of works [16–18] for some simple families of lines. Though, a proof of such relations for general lines was still missing. It is the purpose of this paper to propose a proof by studying pointwise and distributional properties of fields which

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posses a jump around dislocation lines, and are thus understood by means of functions of bounded variation.

**Notations and conventions.** Let  $\mathbb{M}^3$  denote the space of square 3-matrices, and  $\mathbb{S}^3$  that of symmetric 3-matrices. Let  $E \in \mathbb{S}^3$  and  $\beta \in \mathbb{M}^3$ .

$$E = \mathbb{E}(\sigma) \Leftrightarrow \operatorname{div}(\mathbb{A}E) = 0, \quad \sigma = \mathbb{A}E, \quad (1)$$

$$E = \mathbb{D}(\kappa_{\mathcal{L}}) \Leftrightarrow \operatorname{inc}E = \operatorname{Curl} \kappa_{\mathcal{L}}, \quad (2)$$

$$\beta = \mathbb{B}(\Lambda_{\mathcal{L}}) \Leftrightarrow \operatorname{Curl} \beta = \Lambda_{\mathcal{L}}^t. \quad (3)$$

The divergence and curl of a tensor  $E$  are defined componentwise as  $(\operatorname{div}E)_i := \partial_j E_{ij}$  and  $(\operatorname{Curl} E)_{ij} := \epsilon_{jkl} \partial_k E_{il}$ , respectively. The incompatibility of a tensor  $E$  is the symmetric<sup>1</sup> tensor defined componentwise as follows:

$$(\operatorname{inc}E)_{ij} := (\operatorname{Curl} \operatorname{Curl}^t E)_{ij} = \epsilon_{ikm} \epsilon_{jln} \partial_k \partial_l E_{mn}, \quad (4)$$

where subscript  $t$  stands for the transpose of a matrix.

The symmetric and skew-symmetric parts of a tensor  $\mathbb{M}$  are denoted by  $\mathbb{M}^S$  and  $\mathbb{M}^A$ , respectively. Similarly, the symmetric and skew-symmetric parts of a gradient  $\nabla u$  are denoted by  $\nabla^S u$  and  $\nabla^A u$ , respectively.

The functional space of (finite) vector-valued Radon measures,  $\mathcal{M}(\Omega, \mathbb{R}^3)$ , is defined as the dual space of  $\mathcal{C}_c(\Omega, \mathbb{R}^3)$ , that of tensor-valued Radon measures,  $\mathcal{M}(\Omega, \mathbb{M}^3)$ , as the dual space of  $\mathcal{C}_c(\Omega, \mathbb{M}^3)$ . A function  $u$  is said of bounded variation if  $u \in L^1(\Omega)$  and if its distributional gradient  $Du$  is a Radon measure. Moreover, one writes

$$u \in SBV(\Omega)$$

to mean that  $u$  is of bounded variation and that  $Du$  is decomposed additively in two terms, the first which is absolutely continuous w.r.t. Lebesgue measure on  $\Omega$ , and the second which is concentrated on the jump set of  $u$ . Moreover,

$$\|\Lambda\|_{\mathcal{M}} := \sup_{\substack{\varphi \in \mathcal{C}_c(\Omega): \\ \|\varphi\|_{\infty} \leq 1}} |\langle \Lambda, \varphi \rangle_{\mathcal{M}}|,$$

where  $\langle \Lambda, \cdot \rangle_{\mathcal{M}}$  stands for the duality pairing. We refer to [1] for an introduction to the mathematical properties of these functions.

## 2. PRELIMINARY RESULTS

The aim of this section is to prove that in the presence of a dislocation line in linear elasticity, there exists a strain  $E$  such that (2) and (3) are satisfied. To this aim, a series of results about fields of bounded variation and deformation must be proved.

**Lemma 1.** *Let  $\mathcal{L}$  be a Lipschitz closed curve in  $\mathbb{R}^3$  and  $S$  a bounded Lipschitz surface with boundary  $\mathcal{L}$  and unit normal  $N$ . Let  $B \in \mathbb{R}^3$ . The solution of*

$$\begin{cases} \operatorname{div}(\mathbb{A}\nabla w) = 0 & \text{in } \mathbb{R}^3 \setminus S \\ \llbracket w \rrbracket := w^+ - w^- = B & \text{on } S \\ \llbracket (\mathbb{A}\nabla w)N \rrbracket := ((\mathbb{A}\nabla w)N)^+ - ((\mathbb{A}\nabla w)N)^- = 0 & \text{on } S \end{cases} \quad (5)$$

is given componentwise by

$$w_i(x) = -B_j \int_S (\mathbb{A}\nabla\Gamma(y - \cdot)N(y))_{ij} d\mathcal{H}^2(x'), \quad (6)$$

for  $x \in \mathbb{R}^3 \setminus S$ , where  $\Gamma$  is the solution in  $\mathbb{R}^3$  of  $\operatorname{div}(\mathbb{A}\nabla\Gamma) = \delta_0 \mathbb{I}$ .

<sup>1</sup>Symmetry is intended with  $(\operatorname{inc}E)$  seen as a distribution tensor.

*Proof.* Let  $S \subset \hat{\Omega}$  be a smooth surface of discontinuity bounded by  $\mathcal{L}$ . Let  $S^- \neq S$  be another smooth surface bounded by  $\mathcal{L}$  and staying below  $S$ . Let  $V$  be the volume comprised between  $S$  and  $S^-$  and  $S_V := S \cup S^-$  with outer unit normal  $N$  be such that  $\partial V := S_V$ . Supposing that  $w$  is smooth enough, we have the following identities in  $V$ :

$$\int_V \partial'_k (\partial'_l w_j(x') \Gamma_{ip}(x' - x)) dx' = \int_{S_V} \partial'_l w_j(x') \Gamma_{ip}(x' - x) N_k(x') d\mathcal{H}^2(x')$$

and

$$\int_V \partial'_l (w_j(x') \partial'_k \Gamma_{ip}(x' - x)) dx' = \int_{S_V} w_j(x') \partial'_k \Gamma_{ip}(x' - x) N_l(x') d\mathcal{H}^2(x').$$

Thus by subtraction it holds

$$\begin{aligned} & \int_V \partial'_k \partial'_l w_j(x') \Gamma_{ip}(x' - x) dx' - \int_V w_j(x') \partial'_k \partial'_l \Gamma_{ip}(x' - x) dx' \\ &= \int_{S_V} (\partial'_l w_j(x'))^- \Gamma_{ip}(x' - x) N_k(x') d\mathcal{H}^2(x') - \int_{S_V} w_j^-(x') \partial'_k \Gamma_{ip}(x' - x) N_l(x') d\mathcal{H}^2(x'). \end{aligned}$$

Moreover, the same identities in  $\mathbb{R}^3 \setminus \bar{V}$  yield

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus \bar{V}} \partial'_k \partial'_l w_j(x') \Gamma_{ip}(x' - x) dx' - \int_{\mathbb{R}^3 \setminus \bar{V}} w_j(x') \partial'_k \partial'_l \Gamma_{ip}(x' - x) dx' \\ &= - \int_{S_V} (\partial'_l w_j(x'))^+ \Gamma_{ip}(x' - x) N_k(x') d\mathcal{H}^2(x') + \int_{S_V} w_j^+(x') \partial'_k \Gamma_{ip}(x' - x) N_l(x') d\mathcal{H}^2(x'), \end{aligned}$$

and hence, by summing,

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus S_V} \partial'_k \partial'_l w_j(x') \Gamma_{ip}(x' - x) dx' - \int_{\mathbb{R}^3 \setminus S_V} w_j(x') \partial'_k \partial'_l \Gamma_{ip}(x' - x) dx' \\ &= - \int_{S_V} \llbracket \partial'_l w_j(x') \rrbracket \Gamma_{ip}(x' - x) N_k(x') d\mathcal{H}^2(x') + \int_{S_V} \llbracket w_j(x') \rrbracket \partial'_k \Gamma_{ip}(x' - x) N_l(x') d\mathcal{H}^2(x'). \end{aligned}$$

Contracting with  $\mathbb{A}_{ljk i}$  yields

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus S_V} (\operatorname{div}(\mathbb{A} \nabla w)_i(x') \Gamma_{ip}(x' - x)) dx' - \int_{\mathbb{R}^3 \setminus S_V} w_j(x') (\operatorname{div}(\mathbb{A} \nabla \Gamma)_{jp}(x' - x)) dx' \\ &= - \int_{S_V} \llbracket \mathbb{A} \nabla' w(x') N \rrbracket_i \Gamma_{ip}(x' - x) d\mathcal{H}^2(x') + \int_{S_V} \llbracket w_j(x') \rrbracket (\mathbb{A} \nabla' \Gamma(x' - x) N)_{jp} d\mathcal{H}^2(x'), \end{aligned} \tag{7}$$

that is, for  $x \in \mathbb{R}^3 \setminus S_V$ ,

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus S_V} (\operatorname{div}(\mathbb{A} \nabla w)_i(x') \Gamma_{ip}(x' - x)) dx' - w_p(x) \\ &= - \int_{S_V} \llbracket \mathbb{A} \nabla' w(x') N \rrbracket_i \Gamma_{ip}(x' - x) d\mathcal{H}^2(x') \\ &+ \int_{S_V} \llbracket w_j(x') \rrbracket (\mathbb{A} \nabla' \Gamma(x' - x) N)_{jp} d\mathcal{H}^2(x'). \end{aligned} \tag{8}$$

Taking the particular

$$w = \int_S (\mathbb{A} \nabla \Gamma(y - \cdot)) N(y) B d\mathcal{H}^2(y),$$

$w$  satisfies  $\operatorname{div}(\mathbb{A}\nabla w)(x) = 0$  for  $x \in \mathbb{R}^3 \setminus S$ , and hence, by (8) and for  $x \in \mathbb{R}^3 \setminus S_V$ ,

$$\begin{aligned} w_p(x) &= \int_{S_V} \llbracket \mathbb{A}\nabla' w(x')N \rrbracket_i \Gamma_{ip}(x' - x) d\mathcal{H}^2(x') \\ &\quad - \int_{S_V} \llbracket w_j(x') \rrbracket (\mathbb{A}\nabla'\Gamma(x' - x)N)_{jp} d\mathcal{H}^2(x') H^2(x'). \end{aligned}$$

Consider now any smooth tensor test function  $\varphi$  with compact support in place of the tensor  $\Gamma$ . By (7), it holds

$$\begin{aligned} &\int_{\mathbb{R}^3 \setminus S_V} w_j(x') (\operatorname{div}(\mathbb{A}\nabla\varphi)_{jp}(x')) dx' = \int_{\mathbb{R}^3} w_j(x') (\operatorname{div}(\mathbb{A}\nabla\varphi)_{jp}(x')) dx' \\ &= \int_{S_V} \llbracket \mathbb{A}\nabla' w(x')N \rrbracket_i \varphi_{ip}(x') d\mathcal{H}^2(x') - \int_{S_V} \llbracket w_j(x') \rrbracket (\mathbb{A}\nabla'\varphi(x')N)_{jp} d\mathcal{H}^2(x'). \end{aligned} \tag{9}$$

Define the distribution  $\gamma_B$  concentrated on  $S$  as

$$\langle \gamma_B, \varphi \rangle := - \int_S \partial_N \varphi B(y) d\mathcal{H}^2(y).$$

By definition,  $w(x) = - \int_S \partial_N \Gamma(x - y) B d\mathcal{H}^2(y) = - \langle \gamma_B, \Gamma(x - \cdot) \rangle$ . Observe that

$$\operatorname{div}(\mathbb{A}\nabla w) = -\gamma_B \tag{10}$$

holds in the distribution sense, since for any smooth test function with compact support  $\varphi$ , by definition of the convolution between distributions [14], one has

$$\begin{aligned} \langle \operatorname{div}(\mathbb{A}\nabla w), \varphi \rangle = \langle w, \operatorname{div}(\mathbb{A}\nabla\varphi) \rangle &= - \langle \langle \gamma_B, \Gamma(x - \cdot) \rangle, \operatorname{div}(\mathbb{A}\nabla\varphi)(x) \rangle \\ &= - \langle \gamma_B, \langle \operatorname{div}(\mathbb{A}\nabla\Gamma)(x - \cdot), \varphi(x) \rangle \rangle \\ &= - \langle \gamma_B, \varphi \rangle. \end{aligned} \tag{11}$$

Subtracting (11) from (9) yields

$$\begin{aligned} 0 &= \int_{S_V} \llbracket \mathbb{A}\nabla' w(x')N \rrbracket_i \varphi_{ip}(x') d\mathcal{H}^2(x') - \int_S \llbracket w_j(x') - B \rrbracket (\mathbb{A}\nabla'\varphi(x')N)_{jp} d\mathcal{H}^2(x') \\ &\quad - \int_{S^-} \llbracket w_j(x') \rrbracket (\mathbb{A}\nabla'\varphi(x')N)_{jp} d\mathcal{H}^2(x'), \end{aligned} \tag{12}$$

which since it holds for any test function  $\varphi$ , yields (5) by (10), achieving the proof.  $\square$

Remark that taking an arbitrary  $\partial_N \varphi$  on  $S^-$  while  $\partial_N \varphi = \varphi = 0$  on  $S$  in (12) yields the continuity of  $w$  on  $S^-$ . By (6), it holds

$$\partial_k w_i(x) = -B_j \int_S \partial_k (\mathbb{A}\nabla\Gamma(y - \cdot)N(y))_{ij} d\mathcal{H}^2(x'). \tag{13}$$

More results on this topic can be found in [4].

**Lemma 2.** *Let  $\mathcal{L} \subset \Omega$  be the union of a finite number of smooth dislocation (i.e., Lipschitz and closed curves) and  $S \subset \Omega$  a smooth surface enclosed by  $\mathcal{L}$ . Referring to Lemma 1, let  $w$  be the solution of*

$$-\operatorname{div}(\mathbb{A}\nabla w) = 0 \quad \text{in } \mathbb{R}^3 \setminus S, \quad \llbracket w \rrbracket = B, \quad \llbracket (\mathbb{A}\nabla w)N \rrbracket = 0 \quad \text{on } S.$$

Then  $w \in SBV(\Omega, \mathbb{R}^3)$ ,  $\nabla w \in L^p(\Omega, \mathbb{R}^3)$  for  $1 \leq p < 2$  and

$$-\operatorname{Curl} \bar{\nabla} w = \Lambda_{\mathcal{L}}^t,$$

in the distribution sense, where  $\overline{\nabla}w$  is the absolutely continuous part of the distributional derivative  $Dw$  in  $\Omega$  (that is,  $\nabla w = \overline{\nabla}w$  almost everywhere). Moreover  $-\operatorname{div}(\mathbb{A}\nabla w) = 0$  in  $\mathbb{R}^3 \setminus \mathcal{L}$ ,  $w \in C^\infty(\Omega \setminus \mathcal{L}, \mathbb{R}^3)$  and it holds

$$|\nabla w(x)| \leq c|B|(\|c_{\mathcal{L}}\|_{L^\infty(\mathcal{L})}^2|\mathcal{L}| + \frac{1}{d(x, \mathcal{L})}), \quad (14)$$

with  $c_{\mathcal{L}}$  the line curvature, and  $|\mathcal{L}|$  its length.

*Proof.* The second part of the statement, namely (14), is proven as in Lemma 4 of [13] by estimating  $|\partial_i u(x)|$  by means of formula (13), and up to a positive factor given by the uniform bound of  $\mathbb{A}$ . Let us now prove the first part of the statement in the case of a smooth  $\mathcal{L}$ . Let  $u$  be a solution to (5). By (14),  $\nabla w \in L^p(\Omega, \mathbb{R}^{3 \times 3})$  for  $p < 2$ . It has been shown that  $w$  is smooth outside  $S$  where it has a jump of amplitude  $b := |B|$ . In particular  $w$  belongs to  $SBV(\Omega, \mathbb{R}^3)$  and its distributional derivative is given by

$$\langle Dw, \varphi \rangle := -\langle w, \operatorname{div} \varphi \rangle = S(\varphi) + \langle \nabla w, \varphi \rangle, \quad (15)$$

for all  $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^{3 \times 3})$ , where  $S$  denotes the distribution  $S(\varphi) = -\int_S N_j B_i \varphi_{ij} d\mathcal{H}^2$ .

Let us prove that  $-\operatorname{Curl} \nabla w = \mathcal{L} \otimes B$ . To this aim let us take  $\psi \in \mathcal{D}(\Omega, \mathbb{R}^{3 \times 3})$  and write

$$\begin{aligned} -\langle \operatorname{Curl} \nabla w, \psi \rangle &:= -\langle \nabla w, \operatorname{Curl} \psi \rangle = -\langle Dw, \operatorname{Curl} \psi \rangle + S(\operatorname{Curl} \psi) \\ &= \int_C \tau_j B_i \psi_{ij} d\mathcal{H}^1, \end{aligned} \quad (16)$$

where the second equality follows from (15) with  $\varphi = \operatorname{Curl} \psi$ , and the third one by Stokes theorem. We now prove that  $\operatorname{Div} \nabla w = 0$ . Let  $\hat{S} \supset S$  such that  $\hat{S}$  separates  $\Omega$  in two parts  $\Omega^-$  and  $\Omega^+$ . Then for every test function  $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$  it holds

$$\begin{aligned} \int_\Omega \nabla w \nabla \varphi dx &= \int_{\Omega^+} \nabla w \nabla \varphi dx + \int_{\Omega^-} \nabla w \nabla \varphi dx = \\ &= -\int_{\Omega^+} \operatorname{Div} \nabla w \varphi dx - \int_{\Omega^-} \operatorname{Div} \nabla w \varphi dx + \int_{\hat{S}^+} \partial_N w^+ \varphi dx - \int_{\hat{S}^-} \partial_N w^- \varphi dx = 0, \end{aligned}$$

achieving the proof.  $\square$

### 3. MAIN RESULT: KRÖNER RELATION

In the following theorem we first prove Kröner relation  $\operatorname{incc} \epsilon = \operatorname{Curl} \kappa_{\mathcal{L}}$ . The condition  $\epsilon \in L^p(\Omega)$ , with  $1 \leq p < 2$ , yields a-priori that  $\operatorname{incc} \epsilon \in W^{-2,p}(\Omega)$ . We also prove that the sharper result  $\operatorname{incc} \epsilon \in W^{-2,p}(\Omega)$ ,  $1 \leq p < 3/2$  holds true, due to the regularity of  $\kappa_{\mathcal{L}}$  and Kröner's relation.

**Theorem 1.** *Under the hypotheses of Lemma 2, there exists  $\bar{u} \in SBV(\Omega, \mathbb{R}^3)$  such that  $\overline{\nabla} \bar{u} \in L^p(\Omega, \mathbb{M}^3)$  for  $1 \leq p < 2$  and satisfying  $\overline{\nabla}^S \bar{u} = \mathbb{E}(\sigma) \in L^p(\Omega, \mathbb{S}^3)$  and  $\overline{\nabla} \bar{u} = \mathbb{B}(\Lambda_{\mathcal{L}})$ . As a consequence,  $\overline{\nabla}^S \bar{u} = \mathbb{D}(\kappa_{\mathcal{L}})$ . It also holds that  $\kappa_{\mathcal{L}} \in W^{-1,p}(\Omega)$  with  $1 < p \leq 3/2$ .*

*Proof.* Let  $w$  be the vector of Lemma 2. Then

$$-\operatorname{Curl} \overline{\nabla} w = \Lambda_{\mathcal{L}}^t.$$

Let  $v \in W^{1,p}(\Omega, \mathbb{R}^3)$  solution to

$$-\operatorname{div}(\mathbb{A}\nabla^S v) = -f \quad \text{in } \Omega, \quad (\mathbb{A}\nabla v)N = -g - (\mathbb{A}\nabla w)N \quad \text{on } \partial\Omega.$$

Then,  $\bar{u} := -(v + w)$  satisfies

$$-\operatorname{div}(\mathbb{A}\overline{\nabla}^S \bar{u}) = f \quad \text{in } \Omega \setminus S, \quad (\mathbb{A}\overline{\nabla} \bar{u})N = g \quad \text{on } \partial\Omega \setminus S. \quad (17)$$

Remark that if instead, one poses  $v = -w$  on  $\partial\Omega$ , then  $\bar{u} = 0$  on  $\partial\Omega$ . Since  $[[\bar{u}]] = -B$  on  $S$  and  $[[A\bar{\nabla}\bar{u}N]] = 0$ , one has

$$\text{Curl } \bar{\nabla}\bar{u} = -\text{Curl } \bar{\nabla}w = \Lambda_{\mathcal{L}}^t,$$

with  $\bar{\nabla}\bar{u} \in L^1(\Omega)$  by virtue of Theorem 2, and recalling that  $\nabla v = Dv$  is intended in the distribution sense, and  $[[v]] = 0$ . Moreover,  $\bar{\nabla}^S\bar{u} \in L^p(\Omega)$ . Now, by identity  $\bar{\nabla}\bar{u} = \bar{\nabla}^S\bar{u} + \bar{\nabla}^A\bar{u}$ , one has

$$\text{Curl } \bar{\nabla}^S\bar{u} = \Lambda_{\mathcal{L}}^t - \text{Curl } \bar{\nabla}^A\bar{u} = \Lambda_{\mathcal{L}}^t - \nabla^t\omega + \mathbb{I}\text{div}\omega,$$

where one has componentwise  $(\bar{\nabla}^A\bar{u})_{ij} = \epsilon_{ijk}\omega_k$  and  $\omega_i = -\frac{1}{2}\epsilon_{ikl}(\bar{\nabla}\bar{u})_{kl}$ . Note that  $\omega_i \in L^1(\Omega, \mathbb{R}^3)$  and hence  $\nabla\omega = D\omega$  is intended in the distribution sense. Moreover,  $\text{div}\omega = -\frac{1}{2}\text{tr Curl } \bar{\nabla}\bar{u} = -\frac{1}{2}\text{tr}\Lambda_{\mathcal{L}}$ . Thus,

$$\text{Curl } \bar{\nabla}^S\bar{u} = \kappa_{\mathcal{L}}^t - \nabla^t\omega \quad (18)$$

and hence

$$\text{inc}\bar{\nabla}^S\bar{u} = \text{Curl Curl } {}^t\bar{\nabla}^S\bar{u} = \text{Curl } (\kappa_{\mathcal{L}} - D\omega) = \text{Curl } \kappa_{\mathcal{L}}.$$

Let  $\varphi \in W_0^{2,p'}(\Omega)$  with  $p' \geq 3$ . Then by Sobolev embedding,  $\nabla\varphi \in \mathcal{C}^0(\Omega)$ . Let  $r$  be the radial variable such that  $r = 0$  corresponds to points of  $\mathcal{L}$ , and  $\theta$  the azimuthal angle in the planar section at  $x \in \mathcal{L}$ . Then, taking  $\varphi \in W^{2,p'}(\Omega) \cap \mathcal{C}_0^\infty(\Omega)$ , it holds for some  $\mathbb{K} \in L^\infty(\mathcal{L}, \mathbb{M}^3)$ , that

$$\begin{aligned} |\langle \kappa_{\mathcal{L}}, \nabla\varphi \rangle| &= \left| \int_{\mathcal{L}} \mathbb{K} \nabla\varphi(\xi) d\mathcal{H}^1(\xi) \right| = \left| \int_{\mathcal{L}} \mathbb{K} d\mathcal{H}^1(\xi) \int_0^R \partial_r \nabla\varphi(r, \theta, \xi) dr \right| \\ &= \left| \int_{\mathcal{L}} \mathbb{K} d\mathcal{H}^1(\xi) \right| \frac{1}{\pi} \int_0^\pi d\theta \int_0^R \frac{1}{r} \partial_r \nabla\varphi(r, \theta, \xi) r dr \\ &\leq c \|D^2\varphi\|_{L^{p'}(\Omega)} \|d(\cdot, \mathcal{L})^{-1}\|_{L^p(\Omega)} \leq C \|\varphi\|_{W^{2,p'}(\Omega)}, \end{aligned}$$

for  $R$  large enough and where the constant in the RHS is finite since  $p \leq \frac{3}{2}$  where  $1/p + 1/p' = 1$  with  $p' \geq 3$ . By density the result holds in  $W_0^{2,p'}(\Omega)$ , and thus it has been established that

$$\text{div}\kappa_{\mathcal{L}} \in W^{-2,p}(\Omega), \quad 1 < p \leq \frac{3}{2}. \quad (19)$$

On the other hand, let  $\varphi \in W_0^{2,p'}(\Omega)$ , then

$$|\langle \text{Curl } \kappa_{\mathcal{L}}, \varphi \rangle| = |\langle \bar{\nabla}^S\bar{u}, \text{inc}\varphi \rangle| \leq C \|\bar{\nabla}^S\bar{u}\|_{L^p(\Omega)} \|\varphi\|_{W^{2,p'}(\Omega)},$$

proving that

$$\text{Curl } \kappa_{\mathcal{L}} \in W^{-2,p}(\Omega). \quad (20)$$

We now claim that  $\kappa_{\mathcal{L}} \in W^{-1,p}(\Omega)$  for  $1 < p \leq \frac{3}{2}$ . In fact, following [6], for every  $\varphi \in W_0^{1,p'}(\Omega)$ , one has  $\varphi = \nabla\psi + \text{Curl } W$  with  $\psi$  a vector in  $W^{2,p'}(\Omega)$  and  $W$  a solenoidal tensor-valued  $W^{2,p'}(\Omega)$ . Therefore, by (19) and (20), one has

$$\begin{aligned} |\langle \kappa_{\mathcal{L}}, \varphi \rangle| &= |\langle \kappa_{\mathcal{L}}, \nabla\psi + \text{Curl } W \rangle| \leq |\langle \text{div}\kappa_{\mathcal{L}}, \psi \rangle| + |\langle \text{Curl } \kappa_{\mathcal{L}}, W \rangle| \\ &\leq C \left( \|\psi\|_{W^{2,p'}(\Omega)} + \|W\|_{W^{2,p'}(\Omega)} \right) \leq C \left( \|\nabla\psi\|_{W^{1,p'}(\Omega)} + \|\text{Curl } W\|_{W^{1,p'}(\Omega)} \right) \\ &\leq C \|\varphi\|_{W^{1,p'}(\Omega)}, \end{aligned}$$

where in the 2nd inequality we made use of Friedrich-Poincaré-type inequalities in bounded simply connected domains [6], taking into account that  $\psi = W \times N = 0$  on  $\partial\Omega$ .

The proof is achieved.  $\square$

**Corollary 1.** *Under the hypotheses of Theorem 1, there exists a flux  $j \in L^p(\Omega, \mathbb{R}^{3 \times 3 \times 3})$  with  $1 \leq p < 2$  such that*

$$\Lambda_{\mathcal{L}}^t = \operatorname{div} j \quad \text{and} \quad \|\Lambda_{\mathcal{L}}\|_{W^{-1,p}} = \|j\|_{L^p} \leq \|\Lambda_{\mathcal{L}}\|_{\mathcal{M}}, \quad 1 \leq p \leq 3/2. \quad (21)$$

Moreover, there exists  $\beta \in L^p(\Omega, \mathbb{M}^3)$  with  $1 \leq p < 2$  such that

$$\Lambda_{\mathcal{L}}^t = \operatorname{Curl} \beta \quad \text{and} \quad \|\beta\|_{L^p} \leq C \|\Lambda_{\mathcal{L}}\|_{\mathcal{M}}, \quad 1 \leq p \leq 3/2. \quad (22)$$

*Proof.* One has  $\Lambda_{\mathcal{L}} = \kappa_{\mathcal{L}} + \operatorname{tr} \kappa_{\mathcal{L}} \in W^{-1,p}(\Omega, \mathbb{M}^3)$  by Theorem 1. It is a classical result that by Riesz theorem there exists  $f \in W_0^{1,p}(\Omega, \mathbb{M}^3), j := \nabla f \in L^p(\Omega, \mathbb{R}^{3 \times 3 \times 3})$  such that  $\Lambda_{\mathcal{L}}^t = f + \operatorname{div} j$ , and satisfying  $\|\Lambda_{\mathcal{L}}\|_{W^{-1,p}}^p = \|f\|_{L^p}^p + \|j\|_{L^p}^p$ . Moreover, letting  $\varphi \in H_0^1(\Omega, \mathbb{M}^3) \cap \mathcal{C}(\Omega, \mathbb{M}^3)$  be such that its support is away from  $\mathcal{L}$  and on which  $\varphi = 1$  by a partition of unity, one has  $0 = \langle \Lambda_{\mathcal{L}}, \varphi \rangle = \langle f, \varphi \rangle$ , since  $\Lambda_{\mathcal{L}}$  is concentrated in  $\mathcal{L}$ , and hence  $f = 0$ . Moreover

$$\|\Lambda_{\mathcal{L}}\|_{W^{-1,p}} = \sup_{\substack{\varphi \in W_0^{1,p'}(\Omega): \\ \|\varphi\|_{W^{1,p'}} \leq 1}} |\langle \Lambda_{\mathcal{L}}, \varphi \rangle| \leq \sup_{\substack{\varphi \in \mathcal{C}^1(\Omega): \\ \|\varphi\|_{W^{1,p'}} \leq 1}} |\langle \Lambda_{\mathcal{L}}, \varphi \rangle| \leq \|\Lambda_{\mathcal{L}}\|_{\mathcal{M}} \|\varphi\|_{\infty} \leq \|\Lambda_{\mathcal{L}}\|_{\mathcal{M}},$$

by Morrey embedding, since  $1 \leq p \leq 3/2$  and hence  $p' \geq 3$ . This achieves the proof, since the second claim is proved in Theorem 1 with  $\beta = \bar{\nabla} \bar{u}$ . Note that the bound in the RHS of (22) was proved in [12], recalling that  $\Lambda_{\mathcal{L}}^t$  is divergence free. Another reference for this bound can be found in [3].  $\square$

#### 4. CONCLUDING REMARKS

Kröner relation is often mentioned in the literature but a complete proof was missing. By means of this formula, it was the aim of this paper to make the link between functions of bounded variation, viz., the displacement field  $\bar{u}$ , and dislocations at the mesoscopic scale. This formula shows several important features. First, the role of the contortion, in place, or in parallel, of the dislocation density. It turns out that the contortion has a clear geometrical meaning related to the metric torsion in the presence of dislocations [8, 15]. Second, it shows the crucial role of the incompatibility operator. Indeed, this operator is related to the Beltrami decomposition of symmetric tensors, namely  $\epsilon = \nabla^S \bar{u} + \operatorname{inc} F$  (see, e.g., [9]), where  $\operatorname{inc} F$  is the part of the elastic strain, which is incompatible. Note that once such a relation is proved, the strain satisfies

$$\operatorname{inc} \epsilon = \operatorname{inc} \operatorname{inc} F = \operatorname{Curl} \kappa,$$

putting light on a special 4th-order operator,  $\operatorname{inc} \operatorname{inc}$ , whose mathematical properties, among which coercivity (that is, ellipticity) were studied in [2].

Lastly, this formula teaches us that under the assumption of linearized elasticity, where the skewsymmetric part of  $\bar{\nabla} \bar{u}$  (recall that  $\bar{\nabla} \bar{u}$  stands for the absolutely continuous part of  $D\bar{u}$  w.r.t. Lebesgue measure in  $\Omega$ ) is not taken into account, the relation between deformation and dislocation density might be given by the incompatibility of  $\nabla^S \bar{u}$ , precisely by Kröner's formula, in place of the classical  $\operatorname{Curl} \bar{\nabla} \bar{u} = \Lambda_{\mathcal{L}}^t$ , valid for finite as well as for infinitesimal elastic strains, which would require to also consider the skewsymmetric part, for which no Poincaré-Korn-types of bounds do exist.

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