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# Klt singularities of horospherical pairs

Boris Pasquier

September 30, 2015

## Abstract

Let  $X$  be a horospherical  $G$ -variety and let  $D$  be an effective  $\mathbb{Q}$ -divisor of  $X$  that is stable under the action of a Borel subgroup  $B$  of  $G$  and such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier. We prove, using Bott-Samelson resolutions, that the pair  $(X, D)$  is klt if and only if  $\lfloor D \rfloor = 0$ .

## 1 Introduction

Let  $X$  be a normal algebraic variety over  $\mathbb{C}$  and let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier. If the pair  $(X, D)$  has klt singularities (see Definition 2.1) then  $\lfloor D \rfloor = 0$  (ie  $D = \sum_{D_i \text{ irreducible}} a_i D_i$  with  $a_i \in [0, 1[$ ). The inverse implication is false in general. In [AB04], V. Alexeev and M. Brion proved that, if  $X$  is a spherical  $G$ -variety and  $D$  be an effective  $\mathbb{Q}$ -divisor of  $X$  such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier,  $\lfloor D \rfloor = 0$  and  $D = D_G + D_B$  where  $D_G$  is  $G$ -stable and  $D_B$  is stable under the action of a Borel subgroup  $B$  of  $G$ , then  $(X, D_G + D'_B)$  has klt singularities for general  $D'_B$  in  $|D_B|$ .

Here, we prove that, if  $X$  is a horospherical  $G$ -variety and  $D$  be an effective  $\mathbb{Q}$ -divisor of  $X$  such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier,  $\lfloor D \rfloor = 0$  and  $D$  is stable under the action of a Borel subgroup  $B$  of  $G$ , then the pair  $(X, D)$  has klt singularities.

The strategy of the proof is the following. In section 3, we recall the definitions and some properties of Bott-Samelson resolutions of any flag variety  $G/P$ . In particular, they are log resolutions and the klt singularity condition in the case of flag varieties becomes equivalent to some inequalities on the root systems of  $G$  and  $P \subset G$ , which we prove in section 5. And in section 4, we deduce the horospherical case from the case of flag varieties, using that any horospherical variety admits a desingularization that is a toric fibration over a flag variety (ie a fibration over a flag variety whose fiber is a smooth toric variety).

## 2 Notations and definitions

In all the paper, varieties are algebraic varieties over  $\mathbb{C}$ .

We first recall the definition of klt singularities.

**Definition 2.1.** Let  $X$  be a normal variety and let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $K_X + D$  is  $\mathbb{Q}$ -Cartier. The pair  $(X, D)$  is said to be klt (Kawamata log terminal) if for any resolution  $f : V \rightarrow X$  of  $X$  such that  $K_V = f^*(K_X + D) + \sum_{i \in \mathcal{E}} a_i E_i$  where the  $E_i$ 's are distinct irreducible divisors, we have  $a_i > -1$  for any  $i \in \mathcal{E}$ .

**Remark 2.2.** 1. In fact, it is enough to check the above property for one log-resolution to say that a pair  $(X, D)$  is klt. A log-resolution of  $(X, D)$  is a resolution  $f$  such that, the exceptional locus  $\text{Exc}(f)$  of  $f$  is of pure codimension one and the divisor  $f_*^{-1}(D) + \sum_{E \subset \text{Exc}(f)} E$  has simple normal crossings (where  $f_*^{-1}(D)$  is the strict transform of  $D$  by  $f$ ).

2. The condition " $a_i > -1$  for any  $i \in \mathcal{E}$ " can be replaced by:  $[D] = 0$  and for any  $i \in \mathcal{E}$  such that  $E_i$  is exceptional for  $f$ ,  $a_i > -1$ .

In all the paper,  $G$  denotes a connected reductive algebraic group over  $\mathbb{C}$ .

Let  $T$  be a maximal torus in  $G$  and let  $B$  be a Borel subgroup of  $G$  containing  $T$ . We denote by  $\mathcal{R}$  the root system of  $(G, B, T)$ , by  $\mathcal{R}^+$  the set of positive roots and by  $\mathcal{S}$  the set of simple roots. For any simple root  $\alpha \in \mathcal{S}$  we denote by  $s_\alpha$  the corresponding simple reflection of the Weyl group  $W = N_G(T)/T$ . By abuse of notation, for any  $w$  in  $W$ , we still denote by  $w$  one of its representative in  $G$ . We denote by  $w_0$  the longest element of  $W$ .

Let  $P$  be a parabolic subgroup of  $G$  that contains  $B$ . Denote by  $\mathcal{I}$  the set of simple roots of  $P$  (in particular, if  $P = B$  we have  $\mathcal{I} = \emptyset$  and, if  $P = G$  we have  $\mathcal{I} = \mathcal{S}$ ). Denote by  $W_P$  the subgroup of  $W$  generated by  $\{s_\alpha \mid \alpha \in \mathcal{I}\}$ . Also denote by  $W^P$  the quotient  $W/W_P$  and denote by  $w_0^P$  the longest element of  $W^P$ .

The Bruhat decomposition of  $G$  in  $B \times B$ -orbits gives the following decomposition of  $G/P$ :

$$G/P = \bigsqcup_{w \in W^P} BwP/P.$$

Moreover the dimension of a cell  $BwP/P$  equals the length of  $w$ . In particular, the length of  $w_0^P$  is the dimension of  $G/P$  and irreducible  $B$ -stable divisors of  $G/P$  are the closures of the cells  $Bs_\alpha w_0^P P/P$  with  $\alpha \in \mathcal{S} \setminus \mathcal{I}$ . We denote them by  $D_\alpha$ .

A horospherical variety  $X$  is a normal  $G$ -variety with an open  $G$ -orbit isomorphic to a torus fibration  $G/H$  over a flag variety  $G/P$  (ie  $P/H$  is a torus). The irreducible divisors of such  $X$  that are  $B$ -stable but not  $G$ -stable, are the closures in  $X$  of the inverse images in  $G/H$  of the Schubert divisors  $D_\alpha$  of  $G/P$  defined above. We still denote them by  $D_\alpha$ , with  $\alpha \in \mathcal{S} \setminus \mathcal{I}$ .

If  $X$  and  $Y$  are varieties such that a parabolic subgroup  $P$  have a right action on  $X$  and a left action on  $Y$ , we denote by  $X \times^P Y$  the quotient of the product  $X \times Y$  by the following equivalences:

$$\forall (x, y) \in X \times Y, \forall P \in P, (x, y) \sim (x \cdot p, p^{-1} \cdot y).$$

### 3 Bott-Samelson desingularizations and klt pairs of flag varieties

In that section, we prove the following result.

**Theorem 3.1.** *Let  $D = \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$  be a  $B$ -stable  $\mathbb{Q}$ -divisor of  $G/P$  such that  $\forall \alpha \in \mathcal{S} \setminus \mathcal{I}$ ,  $d_\alpha \in [0, 1[$ .*

*There exists a  $B$ -stable log-resolution  $\phi : Z/P \rightarrow G/P$  of  $(G/P, D)$ , where  $Z$  is a variety with a right action of  $P$  and a left action of  $B$ , such that the exceptional divisors of  $\phi$  are the quotient by  $P$  of irreducible divisors of  $Z$ , and such that  $K_{Z/P} - \pi^*(K_{G/P} + D) = \sum_{i \in \mathcal{E}} a_i E_i$  where for any  $i \in \mathcal{E}$ ,  $a_i > -1$  and  $E_i$  is an irreducible divisor of  $f$ .*

*In particular the pair  $(G/P, D)$  is klt.*

*Moreover, for any  $i \in \mathcal{E}$ ,  $E_i$  is the quotient of an exceptional  $B \times P$ -stable divisor  $F_i$  of  $Z$  by  $P$  (left action of  $B$  and right action of  $P$ ).*

**Remarks 3.2.** (i) In general,  $\sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} D_\alpha$  is not a simple normal crossing  $\mathbb{Q}$ -divisor of  $G/P$ .

Then, it is not enough to know that  $G/P$  is smooth to say that  $(G/P, D)$  is klt, when  $D \neq 0$ .

- (ii) Since  $D$  is globally generated, then  $(G/P, D')$  is klt for a general  $D'$  in  $|D|$  (consequence of [Laz04, Lemma 9.1.9]). We can generalize this remark to spherical pairs, see [AB04, Theorem 5.3].

To prove Theorem 3.1, we use a Bott-Samelson resolution of  $G/P$ . Bott-Samelson resolution of Schubert varieties of  $G/B$  have been introduced by M. Demazure in [Dem74]. Here, we use the easy (and well-known) generalization of his work to  $G/P$ . And we choose the equivalent definition of Bott-Samelson resolutions that is now used in almost all papers on the topic.

For any simple root  $\alpha$ , we denote by  $P_\alpha$  the minimal parabolic subgroup containing  $B$  such that  $\alpha$  is a simple root of  $P_\alpha$ .

**Definition 3.3.** Let  $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_N}$  be a reduced decomposition of  $w_0^P$  with  $\alpha_1, \dots, \alpha_N$  in  $\mathcal{S}$ . We define the Bott-Samelson variety  $BS$  to be the quotient of  $P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_N}$  by the right action of  $B^N$  given by,

$$(p_1, p_2, \dots, p_N) \cdot (b_1, b_2, \dots, b_N) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N b_N).$$

The map  $\phi' : BS \rightarrow G/P$  that sends  $(p_1, p_2, \dots, p_N)$  to  $p_1 p_2 \cdots p_N P/P$  is well-defined and birational (it is an isomorphism from the quotient of  $Bs_{\alpha_1}B \times Bs_{\alpha_2}B \times \cdots \times Bs_{\alpha_N}B$  by the right action of  $B^N$  to  $Bw_0^P/P$ ). (We can decompose this map by the usual map from  $V$  to the Schubert variety  $\overline{Bw_0^P B/B}$  of  $G/B$  and the projection map from  $G/B$  to  $G/P$ .)

Hence, to get  $Z$  as in Theorem 3.1, we define  $Z$  to be the quotient of  $P_{\alpha_1} \times \cdots \times P_{\alpha_{N-1}} \times P_{\alpha_N \cup \mathcal{I}}$  by the right action of  $B^{N-1}$  given by,

$$(p_1, \dots, p_N) \cdot (b_1, b_2, \dots, b_{N-1}) = (p_1 b_1, \dots, b_{N-1}^{-1} p_N).$$

Then, since  $P_{\alpha_N \cup \mathcal{I}}/P = P_{\alpha_N}P/P \simeq P_{\alpha_N}/B$ , the  $B$ -varieties  $Z/P$  and  $BS$  are isomorphic and  $\phi : Z/P \rightarrow G/P$  that sends  $(p_1, \dots, p_N)$  to  $p_1 \cdots p_N P/P$  is well-defined and birational.

The lines bundles and divisors Bott-Samelson varieties are well-known, so that we can describe the lines bundles of  $Z/P$ , and the divisors of  $Z/P$  and  $Z$ .

**Proposition 3.4.** For any  $i \in \{1, \dots, N-1\}$ , we define  $F_i$  to be the  $B \times P$ -stable divisor of  $Z$  defined by  $p_i \in B$ ; and we define  $F_N$  to be the  $B \times P$ -stable divisor of  $Z$  defined by  $p_N \in P$ .

Then, we can also define  $E_i$  to be the  $B$ -stable divisor  $F_i/P$  of  $Z/P$ . Moreover, the  $B$ -stable irreducible divisors of  $Z/P$  are the  $E_i$ 's with  $i \in \{1, \dots, N\}$ , and the family  $(E_i)_{i \in \{1, \dots, N\}}$  is a basis of the cone of effective divisors of  $Z/P$ .

First remark that the divisor  $\sum_{i=1}^N E_i$  is clearly a simple normal crossing divisor. Also, since  $G/P$  is smooth and by [Kol96, VI.1, Theorem 1.5], we know that the exceptional locus of  $\phi$  is of pure codimension one, so it is the union of the  $E_i$ 's contracted by  $\phi$ .

Now, let  $\lambda$  be a character of  $P$ . It defines a line bundle  $\mathcal{L}_{G/P}(\lambda)$  on  $G/P$  (where  $P$  acts on the fiber over  $P/P$  by the character  $\lambda$ ). And by pull-back by  $\phi$ , it defines a line bundle  $\mathcal{L}_{Z/P}(\lambda)$  on  $Z/P$ .

The total space of  $\mathcal{L}_{Z/P}(\lambda)$  is the quotient of  $P_{\alpha_1} \times \cdots \times P_{\alpha_{N-1}} \times P_{\alpha_N \cup \mathcal{I}} \times \mathbb{C}$  by the right action of  $B^{N-1} \times P$  given by,

$$(p_1, \dots, p_N, z) \cdot (b_1, \dots, b_{N-1}, p) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N p, \lambda(p)z).$$

By [Dem74, Section 2.5, Proposition 1] adapted to our notation and by induction on  $N$ , we have the following result.

**Proposition 3.5.** Let  $\lambda$  be a character of  $P$ . Then  $\mathcal{L}_{G/P}(\lambda)$  is the line bundle associated to the  $B$ -stable divisor  $D_\lambda := \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} \langle \lambda, \alpha^\vee \rangle D_\alpha$ .

Moreover,  $\phi^*(D_\lambda) = \sum_{i=1}^N \langle \lambda, \beta_i^\vee \rangle E_i$ , where for any  $i \in \{1, \dots, N\}$ ,  $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ .

If  $\mathcal{I} \subset \mathcal{S}$ , we denote by  $\mathcal{R}_{\mathcal{I}}^+$  the set of positive roots generated by simple roots of  $\mathcal{I}$ . Then we define  $\rho$  to be the half sum of positive roots, and  $\rho^P$  to be the half sum of positive roots that are not in  $\mathcal{R}_{\mathcal{I}}^+$  (in particular,  $\rho^B = \rho$ ).

It is well known that an anticanonical divisor of  $G/P$  is  $D_{2\rho^P}$ . Anticanonical divisors of Bott-Samelson resolutions are also well-known.

**Proposition 3.6.** ([Ram85, Proposition 2]) *An anticanonical divisor of  $Z/P$  is  $\phi^*(D_{\rho}) + \sum_{i=1}^N E_i$ .*

**Corollary 3.7.** *The pair  $(G/P, D)$  (with  $\lfloor D \rfloor = 0$  as in Theorem 3.1) is klt if and only if for any  $\beta$  in  $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ ,*

$$\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_{\alpha} \varpi_{\alpha}, \beta^{\vee} \rangle > 0.$$

*Proof.* By Propositions 3.5 and 3.6, we get

$$\begin{aligned} K_{Z/P} - \phi^*(K_{G/P} + D) &= -\phi^*(D_{\rho}) - \sum_{i=1}^N E_i + \phi^*(D_{2\rho^P}) - \phi^*(D) \\ &= \phi^*(D_{2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_{\alpha} \varpi_{\alpha}}) - \sum_{i=1}^N E_i \\ &= \sum_{i=1}^N (\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_{\alpha} \varpi_{\alpha}, \beta_i^{\vee} \rangle - 1) E_i. \end{aligned}$$

We conclude by remarking that, since  $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}$  is a reduced expression of  $w_0^P$ , the set  $\{\beta_i \mid i = 1 \cdots N\}$  is  $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ .  $\square$

The condition of Corollary 3.7 is always satisfied by Proposition 5.1 and the hypothesis that  $\lfloor D \rfloor = 0$ . Then Theorem 3.1 is proved.

## 4 Horospherical pairs

From the classification of horospherical  $G$ -varieties, the description of  $G$ -equivariant morphisms between horospherical  $G$ -varieties, the description of  $B$ -stable Cartier divisor of horospherical  $G$ -varieties and the description of a  $B$ -stable anticanonical divisor of horospherical  $G$ -varieties (see for example [Pas08]), we have the following result.

**Proposition 4.1.** *Let  $X$  be a horospherical  $G$ -variety with open  $G$ -orbit isomorphic to  $G/H$ , torus fibration over the flag variety  $G/P$ . Then, there exists a smooth toric  $P/H$ -variety  $Y$  and a  $G$ -equivariant birational morphism  $f$  from the smooth horospherical  $G$ -variety  $V := G \times^P Y$  to  $X$ , such that the exceptional locus of  $f$  is of pure codimension one.*

*Let  $D$  be a  $B$ -stable effective  $\mathbb{Q}$ -divisor of  $X$  such that  $\lfloor D \rfloor = 0$ . Write  $K_V - f^*(K_X + D) = -f_*^{-1}(D) + \sum_{i \in \mathcal{E}} a_i V_i$ . Then, for any  $i \in \mathcal{E}$ ,  $V_i$  is exceptional and  $G$ -stable, in particular there exists a  $P$ -stable divisor  $Y_i$  of  $Y$  such that  $V_i = G \times^P Y_i$ . Moreover,  $a_i > -1$  for any  $i \in \mathcal{E}$ .*

We do not want here to recall the long description and theory of horospherical varieties. To get more details, see for example [Pas08] or [Pas15].

*Proof.* With the description in terms of colored fans of horospherical  $G$ -varieties and  $G$ -equivariant morphisms between them,  $Y$  can be chosen as the toric  $P/H$ -variety associated to a smooth subdivision  $\mathbb{F}_Y$  of the fan associated to the colored fan  $\mathbb{F}_X$  of  $X$ . Then we clearly have that  $V := G \times^P Y$  is smooth and associated to the fan  $\mathbb{F}_Y$  considered as a colored fan without color. In particular, there exists a  $G$ -equivariant morphism from  $V := G \times^P Y$  to  $X$ .

Moreover, we can choose  $\mathbb{F}_Y$  such that:

- each image of a color of  $\mathbb{F}_X$  is in an edge of  $\mathbb{F}_Y$  and,
- each cone of  $\mathbb{F}_Y$  that is not a cone of  $\mathbb{F}_X$  contains an edge that is in  $\mathbb{F}_Y$  but not in  $\mathbb{F}_X$ .

These two conditions implies that the exceptional locus of  $f$  is of pure codimension one.

Any exceptional divisor  $V_i$  of  $f$  is  $G$ -stable and of the form  $G \times^P Y_i$  where  $Y_i$  is a  $P$ -stable divisor of  $Y$ .

It remains to prove that  $a_i > -1$  for any  $i \in \mathcal{E}$ . We use that  $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} a_\alpha D_\alpha$  where the  $X_i$ 's are the  $G$ -stable irreducible divisors of  $X$  and the  $a_\alpha$  are positive integers. Similarly, with our notation,  $K_V = -\sum_{i=1}^m X_i - \sum_{i \in \mathcal{E}} f_*^{-1}(X_i) - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} a_\alpha D_\alpha$ . In particular, by hypothesis on  $D$ , we remark that the divisor  $-K_X - D$  is strictly effective (ie,  $\sum_{i=1}^m b_i X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} b_\alpha D_\alpha$ , with  $b_i > 0$  for any  $i \in \{1, \dots, m\}$  and  $b_\alpha > 0$  for any  $\alpha \in \mathcal{S} \setminus \mathcal{I}$ ) and then, by the description of pull-backs of  $B$ -stable divisors of horospherical varieties,  $f^*(-K_X - D)$  is also strictly effective. Hence, we have  $a_i > -1$  for any  $i \in \mathcal{E}$ .  $\square$

**Theorem 4.2.** *Let  $X$  be a horospherical  $G$ -variety. Let  $D$  be any  $B$ -stable  $\mathbb{Q}$ -divisor  $D$  of  $X$  such that  $\lfloor D \rfloor = 0$ , then  $(X, D)$  has klt singularities.*

*Proof.* Let  $f$  be as in Proposition 4.1 and let  $Z$  be as in Theorem 3.1. Define  $V' := Z \times^P Y$  and let  $\pi : V' \rightarrow V$  the natural  $B$ -equivariant morphism defined from  $\phi$ .

We first prove that the  $B$ -equivariant morphism  $f \circ \pi : V' \rightarrow X$  is a log resolution of  $(X, D)$ . By composition, it is clearly a birational morphism and its exceptional locus is the union of the inverse images  $Z \times^P Y_i$  of the exceptional divisors of  $f$  and the exceptional divisors  $F_i \times^P Y$  of  $\pi$  (the exceptional locus of  $\pi$  is of pure codimension one because  $V$  is smooth).

The divisor  $(f \circ \pi)_*^{-1}(D) + \sum_{E \in \text{Exc}(f \circ \pi)} E$  is a  $B$ -stable divisor of  $V'$  and then has simple normal crossings. Indeed, a  $B$ -stable irreducible divisor of  $V'$  is either  $F_i \times^P Y$  where  $F_i$  is one of the  $B$ -stable irreducible divisors of  $Z$  described in Proposition 3.4, or  $Z \times^P Y_i$  where  $Y_i$  is a  $P$ -stable divisor of  $Y$ . (Recall that, any divisor of a smooth toric variety that is stable under the action of the torus has simple normal crossings, because such a variety is everywhere locally isomorphic to  $\mathbb{C}^n$  with the natural action of  $(\mathbb{C}^*)^n$ .)

Since  $D$  is  $B$ -stable, we have  $D = \sum_{i=1}^m d_i X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$  where the  $X_i$ 's are the  $G$ -stable irreducible divisors of  $X$ . We denote by  $D_B$  the  $B$ -stable but not  $G$ -stable part  $\sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$  of  $D$ . Then we decompose  $K_{V'} - (f \circ \pi)^*(K_X + D)$  as follows:

$$(K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))) + \pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D_B)).$$

By Proposition 4.1,  $K_V - f^*(K_X + D) + f_*^{-1}(D_B) = \sum_{i \in \mathcal{E}} a_i V_i$ , where for any  $i \in \mathcal{E}$ ,  $a_i > -1$  and  $V_i = G \times^P Y_i$  with some  $P$ -stable irreducible divisor  $Y_i$  of  $Y$ . We remark that the inverse image of  $V_i$  by  $\pi$  is the irreducible divisor  $Z \times^P V_i$  so that  $\pi^*(V_i) = Z \times^P Y_i$ . Hence,  $\pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D_B)) = \sum_{i \in \mathcal{E}} a_i Z \times^P Y_i$ .

To compute  $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))$ , we use the fibrations  $p : V = G \times^P Y \rightarrow G/P$  and  $p' : V' = Z \times^P Y \rightarrow Z/P$ , which have the same fiber. To summarize, we get the following commutative diagram.

$$\begin{array}{ccccc} V' = Z \times^P Y & \xrightarrow{\pi} & V = G \times^P Y & \xrightarrow{f} & X \\ \downarrow p' & & \downarrow p & & \\ Z/P & \xrightarrow{\phi} & G/P & & \end{array}$$

In particular, we have  $K_V = p^*(K_{G/P}) + K_p$  and  $K_{V'} = p^*(K_{Z/P}) + K_{p'}$ . Moreover, the relative canonical divisors  $K_{p'}$  and  $K_p$  satisfy  $K_{p'} = \pi^*(K_p)$ .

Moreover, for any  $B$ -stable irreducible divisor  $D$  of  $V$  that is not  $G$ -stable,  $D$  is the pull-back by  $p$  of a Schubert divisor of  $G/P$ , in particular  $D = p^*(p_*(D))$ .

Hence, we get

$$\begin{aligned} K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) &= p'^*(K_{Z/P}) + K_{p'} - \pi^*p'^*(K_{G/P}) - \pi^*(K_p) - \pi^*(f_*^{-1}(D_B)) \\ &= p'^*(K_{Z/P}) + \pi^*(K_p) - p'^*\phi^*(K_{G/P}) - \pi^*(K_p) \\ &\quad - \pi^*(p^*p_*(f_*^{-1}(D_B))) \\ &= p'^*(K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))). \end{aligned}$$

Remark that  $[p_*(f_*^{-1}(D_B))] = [D_B]$ , so that by Theorem 3.1, we get  $K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))) = \sum_{i \in \mathcal{E}' } a_i F_i / P$ , where for any  $i \in \mathcal{E}'$ , we have  $a_i > -1$  and  $F_i$  is a  $B \times P$ -stable irreducible divisor of  $Z$ .

Hence, we have  $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) = \sum_{i \in \mathcal{E}' } a_i F_i \times^P Y$ .

And finally, we have

$$K_{V'} - (f \circ \pi)^*(K_X + D) = \sum_{i \in \mathcal{E}' } a_i F_i \times^P Y + \sum_{i \in \mathcal{E} } a_i Z \times^P Y_i,$$

with, for any  $i \in \mathcal{E}' \cup \mathcal{E}$ ,  $a_i > -1$ . □

## 5 A result on root systems

In that independent section, we prove the result that permits to deduce Theorem 3.1 from Corollary 3.7. We keep notations of section 2 and we recall that, if  $\mathcal{I} \subset \mathcal{S}$ , we denote by  $\mathcal{R}_{\mathcal{I}}^+$  the set of positive roots generated by simple roots of  $\mathcal{I}$ ,  $\rho$  denotes the half sum of positive roots, and  $\rho^P$  denotes the half sum of positive roots that are not in  $\mathcal{R}_{\mathcal{I}}^+$ .

**Proposition 5.1.** *For any (proper) parabolic subgroup  $P$  of  $G$  containing  $B$ , and for any  $\beta$  in  $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ ,*

$$\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} \varpi_{\alpha}, \beta^{\vee} \rangle \geq 0. \quad (5.1.1)$$

Note that  $\rho = \sum_{\alpha \in \mathcal{S}} \varpi_{\alpha}$  and that  $2\rho^P = 2\rho - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma = 2 \sum_{\alpha \in \mathcal{S}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma$ . Hence, equation 5.1.1 is equivalent to

$$\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle \geq 0. \quad (5.1.2)$$

**Remark 5.2.** (i) If  $\mathcal{I} = \emptyset$  (ie if  $P = B$ ), equations 5.1.1 and 5.1.2 are trivially satisfied.

(ii) If  $\beta \in \mathcal{R}_{\mathcal{I}}^+$  then  $\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle = -\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, \beta^{\vee} \rangle$  and is negative.

**Lemma 5.3.** *Denote by  $w_{0,P}$  the longest element of  $W_P$ . Then, we have*

$$w_{0,P}(\sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}) = \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma.$$

*Proof.* First note that, for any character  $\lambda$  of  $T$  and for any  $w \in W_P$ ,  $w(\lambda) - \lambda$  is in the lattice  $\bigoplus_{\alpha \in \mathcal{I}} \mathbb{Z}\alpha$  (it can be easily proved by induction on the length of  $w$ ).

Moreover,  $\rho_P := \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma$  satisfies  $\langle \rho_P, \alpha^{\vee} \rangle = 2$  for any  $\alpha \in \mathcal{I}$  (same result as the well-known result:  $\langle \rho, \alpha^{\vee} \rangle = 2$  for any  $\alpha \in \mathcal{S}$ ). And, since  $w_{0,P}$  is the longest element of  $W_P$ , we that for any  $\alpha \in \mathcal{I}$ , the root  $w_{0,P}(\alpha)$  is the opposite of a simple root in  $\mathcal{I}$ .

Hence, if we denote by  $\lambda$  the character  $w_{0,P}(\sum_{\alpha \in \mathcal{I}} \varpi_\alpha) - \sum_{\alpha \in \mathcal{I}} \varpi_\alpha + \sum_{\gamma \in \mathcal{R}_\mathcal{I}^+} \gamma$ , we get  $\lambda \in \bigoplus_{\alpha \in \mathcal{I}} \mathbb{Z}\alpha$  and, for any  $\alpha \in \mathcal{I}$ , we have

$$\langle \lambda, \alpha^\vee \rangle = \left\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha, w_{0,P}(\alpha^\vee) \right\rangle - \left\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha, \alpha^\vee \right\rangle + \langle \rho_P, \alpha^\vee \rangle = -1 - 1 + 2 = 0.$$

Then, we deduce that  $\lambda = 0$ , which proves the lemma.  $\square$

*Proof of Proposition 5.1.* By Lemma 5.3, we get for any  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ ,

$$\left\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha - \sum_{\gamma \in \mathcal{R}_\mathcal{I}^+} \gamma, \beta^\vee \right\rangle = \left\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha, w_{0,P}(\beta^\vee) \right\rangle.$$

But, the positive roots that are sent to negative roots by  $w_{0,P}$  are exactly the roots in  $\mathcal{R}_\mathcal{I}^+$ . In particular, for any  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ ,  $w_{0,P}(\beta^\vee)$  is a positive coroot, and

$$\left\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha, w_{0,P}(\beta^\vee) \right\rangle \geq 0.$$

$\square$

**Remark 5.4.** Let  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ . Then  $\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha - \sum_{\gamma \in \mathcal{R}_\mathcal{I}^+} \gamma, \beta^\vee \rangle = 0$  if and only if  $w_{0,P}(\beta) \in \mathcal{R}_{\mathcal{S} \setminus \mathcal{I}}^+$ . In particular, if  $\mathcal{I} \neq \mathcal{S}$ , there exists  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$  such that  $\langle \sum_{\alpha \in \mathcal{I}} \varpi_\alpha - \sum_{\gamma \in \mathcal{R}_\mathcal{I}^+} \gamma, \beta^\vee \rangle = 0$ .

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