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Klt singularities of horospherical pairs

Boris Pasquier

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Abstract

Let X be a horospherical G -variety and let D be an effective \mathbb{Q} -divisor of X that is stable under the action of a Borel subgroup B of G and such that $D + K_X$ is \mathbb{Q} -Cartier. We prove, using Bott-Samelson resolutions, that the pair (X, D) is klt if and only if $\lfloor D \rfloor = 0$.

1 Introduction

Let X be a normal algebraic variety over \mathbb{C} and let D be an effective \mathbb{Q} -divisor such that $D + K_X$ is \mathbb{Q} -Cartier. If the pair (X, D) has klt singularities (see Definition 2.1) then $\lfloor D \rfloor = 0$ (ie $D = \sum_{D_i \text{ irreducible}} a_i D_i$ with $a_i \in [0, 1[$). The inverse implication is false in general. In [AB04], V. Alexeev and M. Brion proved that, if X is a spherical G -variety and D be an effective \mathbb{Q} -divisor of X such that $D + K_X$ is \mathbb{Q} -Cartier, $\lfloor D \rfloor = 0$ and $D = D_G + D_B$ where D_G is G -stable and D_B is stable under the action of a Borel subgroup B of G , then $(X, D_G + D'_B)$ has klt singularities for general D'_B in $|D_B|$.

Here, we prove that, if X is a horospherical G -variety and D be an effective \mathbb{Q} -divisor of X such that $D + K_X$ is \mathbb{Q} -Cartier, $\lfloor D \rfloor = 0$ and D is stable under the action of a Borel subgroup B of G , then the pair (X, D) has klt singularities.

The strategy of the proof is the following. In section 3, we recall the definitions and some properties of Bott-Samelson resolutions of any flag variety G/P . In particular, they are log resolutions and the klt singularity condition in the case of flag varieties becomes equivalent to some inequalities on the root systems of G and $P \subset G$, which we prove in section 5. And in section 4, we deduce the horospherical case from the case of flag varieties, using that any horospherical variety admits a desingularization that is a toric fibration over a flag variety (ie a fibration over a flag variety whose fiber is a smooth toric variety).

2 Notations and definitions

In all the paper, varieties are algebraic varieties over \mathbb{C} .

We first recall the definition of klt singularities.

Definition 2.1. Let X be a normal variety and let D be an effective \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier. The pair (X, D) is said to be klt (Kawamata log terminal) if for any resolution $f : V \rightarrow X$ of X such that $K_V = f^*(K_X + D) + \sum_{i \in \mathcal{E}} a_i E_i$ where the E_i 's are distinct irreducible divisors, we have $a_i > -1$ for any $i \in \mathcal{E}$.

Remark 2.2. 1. In fact, it is enough to check the above property for one log-resolution to say that a pair (X, D) is klt. A log-resolution of (X, D) is a resolution f such that, the exceptional locus $\text{Exc}(f)$ of f is of pure codimension one and the divisor $f_*^{-1}(D) + \sum_{E \subset \text{Exc}(f)} E$ has simple normal crossings (where $f_*^{-1}(D)$ is the strict transform of D by f).

2. The condition " $a_i > -1$ for any $i \in \mathcal{E}$ " can be replaced by: $[D] = 0$ and for any $i \in \mathcal{E}$ such that E_i is exceptional for f , $a_i > -1$.

In all the paper, G denotes a connected reductive algebraic group over \mathbb{C} .

Let T be a maximal torus in G and let B be a Borel subgroup of G containing T . We denote by \mathcal{R} the root system of (G, B, T) , by \mathcal{R}^+ the set of positive roots and by \mathcal{S} the set of simple roots. For any simple root $\alpha \in \mathcal{S}$ we denote by s_α the corresponding simple reflection of the Weyl group $W = N_G(T)/T$. By abuse of notation, for any w in W , we still denote by w one of its representative in G . We denote by w_0 the longest element of W .

Let P be a parabolic subgroup of G that contains B . Denote by \mathcal{I} the set of simple roots of P (in particular, if $P = B$ we have $\mathcal{I} = \emptyset$ and, if $P = G$ we have $\mathcal{I} = \mathcal{S}$). Denote by W_P the subgroup of W generated by $\{s_\alpha \mid \alpha \in \mathcal{I}\}$. Also denote by W^P the quotient W/W_P and denote by w_0^P the longest element of W^P .

The Bruhat decomposition of G in $B \times B$ -orbits gives the following decomposition of G/P :

$$G/P = \bigsqcup_{w \in W^P} BwP/P.$$

Moreover the dimension of a cell BwP/P equals the length of w . In particular, the length of w_0^P is the dimension of G/P and irreducible B -stable divisors of G/P are the closures of the cells $Bs_\alpha w_0^P P/P$ with $\alpha \in \mathcal{S} \setminus \mathcal{I}$. We denote them by D_α .

A horospherical variety X is a normal G -variety with an open G -orbit isomorphic to a torus fibration G/H over a flag variety G/P (ie P/H is a torus). The irreducible divisors of such X that are B -stable but not G -stable, are the closures in X of the inverse images in G/H of the Schubert divisors D_α of G/P defined above. We still denote them by D_α , with $\alpha \in \mathcal{S} \setminus \mathcal{I}$.

If X and Y are varieties such that a parabolic subgroup P have a right action on X and a left action on Y , we denote by $X \times^P Y$ the quotient of the product $X \times Y$ by the following equivalences:

$$\forall (x, y) \in X \times Y, \forall P \in P, (x, y) \sim (x \cdot p, p^{-1} \cdot y).$$

3 Bott-Samelson desingularizations and klt pairs of flag varieties

In that section, we prove the following result.

Theorem 3.1. *Let $D = \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$ be a B -stable \mathbb{Q} -divisor of G/P such that $\forall \alpha \in \mathcal{S} \setminus \mathcal{I}$, $d_\alpha \in [0, 1[$.*

There exists a B -stable log-resolution $\phi : Z/P \rightarrow G/P$ of $(G/P, D)$, where Z is a variety with a right action of P and a left action of B , such that the exceptional divisors of ϕ are the quotient by P of irreducible divisors of Z , and such that $K_{Z/P} - \pi^(K_{G/P} + D) = \sum_{i \in \mathcal{E}} a_i E_i$ where for any $i \in \mathcal{E}$, $a_i > -1$ and E_i is an irreducible divisor of f .*

In particular the pair $(G/P, D)$ is klt.

Moreover, for any $i \in \mathcal{E}$, E_i is the quotient of an exceptional $B \times P$ -stable divisor F_i of Z by P (left action of B and right action of P).

Remarks 3.2. (i) In general, $\sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} D_\alpha$ is not a simple normal crossing \mathbb{Q} -divisor of G/P .

Then, it is not enough to know that G/P is smooth to say that $(G/P, D)$ is klt, when $D \neq 0$.

- (ii) Since D is globally generated, then $(G/P, D')$ is klt for a general D' in $|D|$ (consequence of [Laz04, Lemma 9.1.9]). We can generalize this remark to spherical pairs, see [AB04, Theorem 5.3].

To prove Theorem 3.1, we use a Bott-Samelson resolution of G/P . Bott-Samelson resolution of Schubert varieties of G/B have been introduced by M. Demazure in [Dem74]. Here, we use the easy (and well-known) generalization of his work to G/P . And we choose the equivalent definition of Bott-Samelson resolutions that is now used in almost all papers on the topic.

For any simple root α , we denote by P_α the minimal parabolic subgroup containing B such that α is a simple root of P_α .

Definition 3.3. Let $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_N}$ be a reduced decomposition of w_0^P with $\alpha_1, \dots, \alpha_N$ in \mathcal{S} . We define the Bott-Samelson variety BS to be the quotient of $P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_N}$ by the right action of B^N given by,

$$(p_1, p_2, \dots, p_N) \cdot (b_1, b_2, \dots, b_N) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N b_N).$$

The map $\phi' : BS \rightarrow G/P$ that sends (p_1, p_2, \dots, p_N) to $p_1 p_2 \cdots p_N P/P$ is well-defined and birational (it is an isomorphism from the quotient of $Bs_{\alpha_1}B \times Bs_{\alpha_2}B \times \cdots \times Bs_{\alpha_N}B$ by the right action of B^N to Bw_0^P/P). (We can decompose this map by the usual map from V to the Schubert variety $\overline{Bw_0^P B/B}$ of G/B and the projection map from G/B to G/P .)

Hence, to get Z as in Theorem 3.1, we define Z to be the quotient of $P_{\alpha_1} \times \cdots \times P_{\alpha_{N-1}} \times P_{\alpha_N \cup \mathcal{I}}$ by the right action of B^{N-1} given by,

$$(p_1, \dots, p_N) \cdot (b_1, b_2, \dots, b_{N-1}) = (p_1 b_1, \dots, b_{N-1}^{-1} p_N).$$

Then, since $P_{\alpha_N \cup \mathcal{I}}/P = P_{\alpha_N}P/P \simeq P_{\alpha_N}/B$, the B -varieties Z/P and BS are isomorphic and $\phi : Z/P \rightarrow G/P$ that sends (p_1, \dots, p_N) to $p_1 \cdots p_N P/P$ is well-defined and birational.

The lines bundles and divisors Bott-Samelson varieties are well-known, so that we can describe the lines bundles of Z/P , and the divisors of Z/P and Z .

Proposition 3.4. For any $i \in \{1, \dots, N-1\}$, we define F_i to be the $B \times P$ -stable divisor of Z defined by $p_i \in B$; and we define F_N to be the $B \times P$ -stable divisor of Z defined by $p_N \in P$.

Then, we can also define E_i to be the B -stable divisor F_i/P of Z/P . Moreover, the B -stable irreducible divisors of Z/P are the E_i 's with $i \in \{1, \dots, N\}$, and the family $(E_i)_{i \in \{1, \dots, N\}}$ is a basis of the cone of effective divisors of Z/P .

First remark that the divisor $\sum_{i=1}^N E_i$ is clearly a simple normal crossing divisor. Also, since G/P is smooth and by [Kol96, VI.1, Theorem 1.5], we know that the exceptional locus of ϕ is of pure codimension one, so it is the union of the E_i 's contracted by ϕ .

Now, let λ be a character of P . It defines a line bundle $\mathcal{L}_{G/P}(\lambda)$ on G/P (where P acts on the fiber over P/P by the character λ). And by pull-back by ϕ , it defines a line bundle $\mathcal{L}_{Z/P}(\lambda)$ on Z/P .

The total space of $\mathcal{L}_{Z/P}(\lambda)$ is the quotient of $P_{\alpha_1} \times \cdots \times P_{\alpha_{N-1}} \times P_{\alpha_N \cup \mathcal{I}} \times \mathbb{C}$ by the right action of $B^{N-1} \times P$ given by,

$$(p_1, \dots, p_N, z) \cdot (b_1, \dots, b_{N-1}, p) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N p, \lambda(p)z).$$

By [Dem74, Section 2.5, Proposition 1] adapted to our notation and by induction on N , we have the following result.

Proposition 3.5. Let λ be a character of P . Then $\mathcal{L}_{G/P}(\lambda)$ is the line bundle associated to the B -stable divisor $D_\lambda := \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} \langle \lambda, \alpha^\vee \rangle D_\alpha$.

Moreover, $\phi^*(D_\lambda) = \sum_{i=1}^N \langle \lambda, \beta_i^\vee \rangle E_i$, where for any $i \in \{1, \dots, N\}$, $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$.

If $\mathcal{I} \subset \mathcal{S}$, we denote by $\mathcal{R}_{\mathcal{I}}^+$ the set of positive roots generated by simple roots of \mathcal{I} . Then we define ρ to be the half sum of positive roots, and ρ^P to be the half sum of positive roots that are not in $\mathcal{R}_{\mathcal{I}}^+$ (in particular, $\rho^B = \rho$).

It is well known that an anticanonical divisor of G/P is $D_{2\rho^P}$. Anticanonical divisors of Bott-Samelson resolutions are also well-known.

Proposition 3.6. ([Ram85, Proposition 2]) *An anticanonical divisor of Z/P is $\phi^*(D_{\rho}) + \sum_{i=1}^N E_i$.*

Corollary 3.7. *The pair $(G/P, D)$ (with $\lfloor D \rfloor = 0$ as in Theorem 3.1) is klt if and only if for any β in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$,*

$$\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_{\alpha} \varpi_{\alpha}, \beta^{\vee} \rangle > 0.$$

Proof. By Propositions 3.5 and 3.6, we get

$$\begin{aligned} K_{Z/P} - \phi^*(K_{G/P} + D) &= -\phi^*(D_{\rho}) - \sum_{i=1}^N E_i + \phi^*(D_{2\rho^P}) - \phi^*(D) \\ &= \phi^*(D_{2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_{\alpha} \varpi_{\alpha}}) - \sum_{i=1}^N E_i \\ &= \sum_{i=1}^N (\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_{\alpha} \varpi_{\alpha}, \beta_i^{\vee} \rangle - 1) E_i. \end{aligned}$$

We conclude by remarking that, since $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}$ is a reduced expression of w_0^P , the set $\{\beta_i \mid i = 1 \cdots N\}$ is $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. \square

The condition of Corollary 3.7 is always satisfied by Theorem 5.1 and the hypothesis that $\lfloor D \rfloor = 0$. Then Theorem 3.1 is proved.

4 Horospherical pairs

From the classification of horospherical G -varieties, the description of G -equivariant morphisms between horospherical G -varieties, the description of B -stable Cartier divisor of horospherical G -varieties and the description of a B -stable anticanonical divisor of horospherical G -varieties (see for example [Pas08]), we have the following result.

Proposition 4.1. *Let X be a horospherical G -variety with open G -orbit isomorphic to G/H , torus fibration over the flag variety G/P . Then, there exists a smooth toric P/H -variety Y and a G -equivariant birational morphism f from the smooth horospherical G -variety $V := G \times^P Y$ to X , such that the exceptional locus of f is of pure codimension one.*

Let D be a B -stable effective \mathbb{Q} -divisor of X such that $\lfloor D \rfloor = 0$. Write $K_V - f^(K_X + D) = -f_*^{-1}(D) + \sum_{i \in \mathcal{E}} a_i V_i$. Then, for any $i \in \mathcal{E}$, V_i is exceptional and G -stable, in particular there exists a P -stable divisor Y_i of Y such that $V_i = G \times^P Y_i$. Moreover, $a_i > -1$ for any $i \in \mathcal{E}$.*

We do not want here to recall the long description and theory of horospherical varieties. To get more details, see for example [Pas08] or [Pas15].

Proof. With the description in terms of colored fans of horospherical G -varieties and G -equivariant morphisms between them, Y can be chosen as the toric P/H -variety associated to a smooth subdivision \mathbb{F}_Y of the fan associated to the colored fan \mathbb{F}_X of X . Then we clearly have that $V := G \times^P Y$ is smooth and associated to the fan \mathbb{F}_Y considered as a colored fan without color. In particular, there exists a G -equivariant morphism from $V := G \times^P Y$ to X .

Moreover, we can choose \mathbb{F}_Y such that:

- each image of a color of \mathbb{F}_X is in an edge of \mathbb{F}_Y and,
- each cone of \mathbb{F}_Y that is not a cone of \mathbb{F}_X contains an edge that is in \mathbb{F}_Y but not in \mathbb{F}_X .

These two conditions implies that the exceptional locus of f is of pure codimension one.

Any exceptional divisor V_i of f is G -stable and of the form $G \times^P Y_i$ where Y_i is a P -stable divisor of Y .

It remains to prove that $a_i > -1$ for any $i \in \mathcal{E}$. We use that $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} a_\alpha D_\alpha$ where the X_i 's are the G -stable irreducible divisors of X and the a_α are positive integers. Similarly, with our notation, $K_V = -\sum_{i=1}^m X_i - \sum_{i \in \mathcal{E}} f_*^{-1}(X_i) - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} a_\alpha D_\alpha$. In particular, by hypothesis on D , we remark that the divisor $-K_X - D$ is strictly effective (ie, $\sum_{i=1}^m b_i X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} b_\alpha D_\alpha$, with $b_i > 0$ for any $i \in \{1, \dots, m\}$ and $b_\alpha > 0$ for any $\alpha \in \mathcal{S} \setminus \mathcal{I}$) and then, by the description of pull-backs of B -stable divisors of horospherical varieties, $f^*(-K_X - D)$ is also strictly effective. Hence, we have $a_i > -1$ for any $i \in \mathcal{E}$. \square

Theorem 4.2. *Let X be a horospherical G -variety. Let D be any B -stable \mathbb{Q} -divisor D of X such that $\lfloor D \rfloor = 0$, then (X, D) has klt singularities.*

Proof. Let f be as in Proposition 4.1 and let Z be as in Theorem 3.1. Define $V' := Z \times^P Y$ and let $\pi : V' \rightarrow V$ the natural B -equivariant morphism defined from ϕ .

We first prove that the B -equivariant morphism $f \circ \pi : V' \rightarrow X$ is a log resolution of (X, D) . By composition, it is clearly a birational morphism and its exceptional locus is the union of the inverse images $Z \times^P Y_i$ of the exceptional divisors of f and the exceptional divisors $F_i \times^P Y$ of π (the exceptional locus of π is of pure codimension one because V is smooth).

The divisor $(f \circ \pi)_*^{-1}(D) + \sum_{E \in \text{Exc}(f \circ \pi)} E$ is a B -stable divisor of V' and then has simple normal crossings. Indeed, a B -stable irreducible divisor of V' is either $F_i \times^P Y$ where F_i is one of the B -stable irreducible divisors of Z described in Proposition 3.4, or $Z \times^P Y_i$ where Y_i is a P -stable divisor of Y . (Recall that, any divisor of a smooth toric variety that is stable under the action of the torus has simple normal crossings, because such a variety is everywhere locally isomorphic to \mathbb{C}^n with the natural action of $(\mathbb{C}^*)^n$.)

Since D is B -stable, we have $D = \sum_{i=1}^m d_i X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$ where the X_i 's are the G -stable irreducible divisors of X . We denote by D_B the B -stable but not G -stable part $\sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$ of D . Then we decompose $K_{V'} - (f \circ \pi)^*(K_X + D)$ as follows:

$$(K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))) + \pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D_B)).$$

By Proposition 4.1, $K_V - f^*(K_X + D) + f_*^{-1}(D_B) = \sum_{i \in \mathcal{E}} a_i V_i$, where for any $i \in \mathcal{E}$, $a_i > -1$ and $V_i = G \times^P Y_i$ with some P -stable irreducible divisor Y_i of Y . We remark that the inverse image of V_i by π is the irreducible divisor $Z \times^P V_i$ so that $\pi^*(V_i) = Z \times^P Y_i$. Hence, $\pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D_B)) = \sum_{i \in \mathcal{E}} a_i Z \times^P Y_i$.

To compute $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))$, we use the fibrations $p : V = G \times^P Y \rightarrow G/P$ and $p' : V' = Z \times^P Y \rightarrow Z/P$, which have the same fiber. To summarize, we get the following commutative diagram.

$$\begin{array}{ccccc} V' = Z \times^P Y & \xrightarrow{\pi} & V = G \times^P Y & \xrightarrow{f} & X \\ \downarrow p' & & \downarrow p & & \\ Z/P & \xrightarrow{\phi} & G/P & & \end{array}$$

In particular, we have $K_V = p^*(K_{G/P}) + K_p$ and $K_{V'} = p'^*(K_{Z/P}) + K_{p'}$. Moreover, the relative canonical divisors $K_{p'}$ and K_p satisfy $K_{p'} = \pi^*(K_p)$.

Moreover, for any B -stable irreducible divisor D of V that is not G -stable, D is the pull-back by p of a Schubert divisor of G/P , in particular $D = p^*(p_*(D))$.

Hence, we get

$$\begin{aligned} K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) &= p'^*(K_{Z/P}) + K_{p'} - \pi^*p^*(K_{G/P}) - \pi^*(K_p) - \pi^*(f_*^{-1}(D_B)) \\ &= p'^*(K_{Z/P}) + \pi^*(K_p) - p'^*\phi^*(K_{G/P}) - \pi^*(K_p) \\ &\quad - \pi^*(p^*p_*(f_*^{-1}(D_B))) \\ &= p'^*(K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))). \end{aligned}$$

Remark that $[p_*(f_*^{-1}(D_B))] = [D_B]$, so that by Theorem 3.1, we get $K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))) = \sum_{i \in \mathcal{E}'}, a_i F_i/P$, where for any $i \in \mathcal{E}'$, we have $a_i > -1$ and F_i is a $B \times P$ -stable irreducible divisor of Z .

Hence, we have $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) = \sum_{i \in \mathcal{E}'}, a_i F_i \times^P Y$.

And finally, we have

$$K_{V'} - (f \circ \pi)^*(K_X + D) = \sum_{i \in \mathcal{E}'} a_i F_i \times^P Y + \sum_{i \in \mathcal{E}} a_i Z \times^P Y_i,$$

with, for any $i \in \mathcal{E}' \cup \mathcal{E}$, $a_i > -1$. □

5 A result on root systems

In that independent section, we prove the result that permits to deduce Theorem 3.1 from Corollary 3.7. We keep notations of section 2 and we recall that, if $\mathcal{I} \subset \mathcal{S}$, we denote by $\mathcal{R}_{\mathcal{I}}^+$ the set of positive roots generated by simple roots of \mathcal{I} , ρ denotes the half sum of positive roots, and ρ^P denotes the half sum of positive roots that are not in $\mathcal{R}_{\mathcal{I}}^+$.

Theorem 5.1. *For any (proper) parabolic subgroup P of G containing B , and for any β in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$,*

$$\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} \varpi_{\alpha}, \beta^{\vee} \rangle \geq 0. \quad (5.1.1)$$

Note that $\rho = \sum_{\alpha \in \mathcal{S}} \varpi_{\alpha}$ and that $2\rho^P = 2\rho - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma = 2 \sum_{\alpha \in \mathcal{S}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma$. Hence, equation 5.1.1 is equivalent to

$$\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle \geq 0. \quad (5.1.2)$$

Denote by $f_{\mathcal{I}}(\beta^{\vee})$ the integer $\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle$.

Remarks 5.2. (i) If $\mathcal{I} = \emptyset$ (ie if $P = B$), equations 5.1.1 and 5.1.2 are trivially satisfied.

(ii) If $\beta \in \mathcal{R}_{\mathcal{I}}^+$ then $f_{\mathcal{I}}(\beta^{\vee}) = -\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, \beta^{\vee} \rangle$ and is negative.

(iii) If $\beta^{\vee} = \beta_1^{\vee} + \beta_2^{\vee}$, with β_1 and β_2 in \mathcal{R}^+ , then $f_{\mathcal{I}}(\beta^{\vee}) = f_{\mathcal{I}}(\beta_1^{\vee}) + f_{\mathcal{I}}(\beta_2^{\vee})$. In particular, if β_1 and β_2 are not in $\mathcal{R}_{\mathcal{I}}^+$, and if equation 5.1.2 is satisfied for β_1 and β_2 , then it is also satisfied for β .

(iv) If \mathcal{I} is the disjoint union of \mathcal{I}_1 and \mathcal{I}_2 , such that for any $\alpha_1 \in \mathcal{I}_1$ and $\alpha_2 \in \mathcal{I}_2$, we have $\langle \alpha_1, \alpha_2^{\vee} \rangle = 0$, then $f_{\mathcal{I}}(\beta^{\vee}) = f_{\mathcal{I}_1}(\beta^{\vee}) + f_{\mathcal{I}_2}(\beta^{\vee})$, and $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ if and only if $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}_1}^+$ and $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}_2}^+$. It implies that, it is enough to prove Theorem 5.1 in the case where the subgraph of the Dynkin diagram with vertices in \mathcal{I} (and all possible vertices) is connected. By abuse of language, we will say that \mathcal{I} is connected.

To restrict again the cases where we have to prove Theorem 5.1, we give the following lemma.

Lemma 5.3. Denote by $\text{Supp}(\beta^\vee)$ the support of β^\vee , ie the set of simple roots $\alpha \in \mathcal{S}$ such that $\langle \varpi_\alpha, \beta^\vee \rangle \neq 0$.

For any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$,

$$f_\mathcal{I}(\beta^\vee) \geq f_{\mathcal{I} \cap \text{Supp}(\beta^\vee)}(\beta^\vee) \quad (\text{and } \beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I} \cap \text{Supp}(\beta^\vee)}^+).$$

For any positive root γ , we denote by s_γ the reflection such that, for any $\delta \in \mathcal{R}$, $s_\gamma(\delta) = \delta - \langle \delta, \gamma^\vee \rangle \gamma$. Note also that, for any $\delta \in \mathcal{R}$, $s_\gamma(\delta^\vee) = \delta^\vee - \langle \gamma, \delta^\vee \rangle \gamma^\vee$.

Proof of Lemma 5.3. First we compute easily that

$$f_\mathcal{I}(\beta^\vee) - f_{\mathcal{I} \cap \text{Supp}(\beta^\vee)}(\beta^\vee) = -\left\langle \sum_{\gamma \in \mathcal{R}_\mathcal{I}^+ \setminus \mathcal{R}_{\mathcal{I} \cap \text{Supp}(\beta^\vee)}^+} \gamma, \beta^\vee \right\rangle.$$

Let $\gamma \in \mathcal{R}_\mathcal{I}^+ \setminus \mathcal{R}_{\mathcal{I} \cap \text{Supp}(\beta^\vee)}^+$ (if it exists, if not we have nothing to prove). Then, since $s_\gamma(\beta) = \beta - \langle \beta, \gamma^\vee \rangle \gamma$ is still in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we must have $\langle \gamma, \beta^\vee \rangle \leq 0$.

We conclude that $f_\mathcal{I}(\beta^\vee) - f_{\mathcal{I} \cap \text{Supp}(\beta^\vee)}(\beta^\vee) \geq 0$. \square

To summarize, it is enough to prove Theorem 5.1 when \mathcal{I} is connected and included in the support of β^\vee . In particular, now, the result when β is a simple root is reduced to the case where $\mathcal{I} = \emptyset$ and known by Remark 5.2(i). Other small cases can be proved easily.

Proposition 5.4. If $\mathcal{I} = \{\alpha\}$, for any β in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we have $f_\mathcal{I}(\beta^\vee) \geq 0$.

Proof. We compute in that case that $f_\mathcal{I}(\beta^\vee) = \langle \varpi_\alpha - \alpha, \beta^\vee \rangle = \langle s_\alpha(\varpi_\alpha), \beta^\vee \rangle = \langle \varpi_\alpha, s_\alpha(\beta^\vee) \rangle$. But $s_\alpha(\beta^\vee)$ is a positive coroot (because $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ and $\alpha \in \mathcal{I}$), and then $f_\mathcal{I}(\beta^\vee) \geq 0$. \square

Corollary 5.5. For any \mathcal{I} and for any β in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ such that $\text{Supp}(\beta^\vee)$ is of cardinality 2, we have $f_\mathcal{I}(\beta^\vee) \geq 0$.

Proof. We can suppose that $\mathcal{I} \subset \text{Supp}(\beta^\vee)$. If $\mathcal{I} = \emptyset$ or if $\mathcal{I} = \text{Supp}(\beta^\vee)$ it is obvious. Then, the only case remained is the case where \mathcal{I} is of cardinality one, and we conclude by the proposition. \square

The strategy to prove most of cases is to use Remark 5.2(iii) and make a proof by induction on the cardinality of the support of β^\vee .

We define the type of β^\vee (or β) to be the type of the root system generated by the simple roots of the support of β^\vee .

Proposition 5.6. For any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ of type A , we have $f_\mathcal{I}(\beta^\vee) \geq 0$.

Proof. For type A_1 and A_2 , we already proved it. Let $n \geq 3$. And suppose that, for any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ of type A_{n-1} , we have $f_\mathcal{I}(\beta^\vee) \geq 0$.

Write $\beta^\vee = \alpha_1^\vee + \dots + \alpha_n^\vee$ with Bourbaki's notation ([Bou75]). We can assume that $\mathcal{I} \not\subseteq \text{Supp}(\beta^\vee)$ and that \mathcal{I} is connected. In particular α_1 or α_n is not in \mathcal{I} . By symmetry, we can suppose that $\alpha_1 \notin \mathcal{I}$.

Note now that $\beta^\vee = \alpha_1^\vee + (\alpha_2^\vee + \dots + \alpha_n^\vee)$. Then by Remark 5.2(iii), if $\alpha_2^\vee + \dots + \alpha_n^\vee$ is in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we get $f_\mathcal{I}(\beta^\vee) \geq 0$ by induction hypothesis.

And if $\alpha_2^\vee + \dots + \alpha_n^\vee$ is not in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, then it is in $\mathcal{R}_\mathcal{I}^+$; in particular, $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$. By computation, we get that

$$f_\mathcal{I}(\beta^\vee) = f_\mathcal{I}(\alpha_1^\vee) + f_\mathcal{I}(\alpha_2^\vee + \dots + \alpha_n^\vee) = (0 - (-(n-1))) + (-(n-1)) = 0.$$

(We use Remark 5.2(ii) to compute $f_\mathcal{I}(\alpha_2^\vee + \dots + \alpha_n^\vee)$.)

We have proved that, for any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ of type A_n , we have $f_\mathcal{I}(\beta^\vee) \geq 0$. \square

We prove, in a very similar way, the same result in types B , C and D .

Proposition 5.7. *For any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ of type B , we have $f_{\mathcal{I}}(\beta^{\vee}) \geq 0$.*

Proof. For type B_2 , we already proved it (Corollary 5.5). Let $n \geq 3$. And suppose that, for any \mathcal{I} , for any $2 \leq m \leq n-1$ and for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ of type B_m , we have $f_{\mathcal{I}}(\beta^{\vee}) \geq 0$.

If β^{\vee} is of type B_n , then it is one of the following coroot of B_n with $i \in \{1, \dots, n\}$ (still with Bourbaki's notation ([Bou75]):

$$\beta_i^{\vee} = \alpha_1^{\vee} + \dots + \alpha_{i-1}^{\vee} + 2\alpha_i^{\vee} + \dots + 2\alpha_{n-1}^{\vee} + \alpha_n^{\vee}.$$

We can assume that $\mathcal{I} \subsetneq \text{Supp}(\beta^{\vee})$ and that \mathcal{I} is connected. In particular α_1 or α_n is not in \mathcal{I} .

- Suppose first that $\alpha_1 \notin \mathcal{I}$.

If $i \neq 1$ then $\beta_i^{\vee} = \alpha_1^{\vee} + (\alpha_2^{\vee} + \dots + \alpha_{i-1}^{\vee} + 2\alpha_i^{\vee} + \dots + 2\alpha_{n-1}^{\vee} + \alpha_n^{\vee})$. Then by Remark 5.2(iii), if $\alpha_2^{\vee} + \dots + \alpha_{i-1}^{\vee} + 2\alpha_i^{\vee} + \dots + 2\alpha_{n-1}^{\vee} + \alpha_n^{\vee}$ is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, we get that $f_{\mathcal{I}}(\beta_i^{\vee}) \geq 0$ by induction hypothesis. Else, $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$ and we compute that

$$\begin{aligned} f_{\mathcal{I}}(\beta_i^{\vee}) &= f_{\mathcal{I}}(\alpha_1^{\vee}) + f_{\mathcal{I}}(\alpha_2^{\vee} + \dots + \alpha_{i-1}^{\vee} + 2\alpha_i^{\vee} + \dots + 2\alpha_{n-1}^{\vee} + \alpha_n^{\vee}) \\ &= (2n-3) - (2n-i-1) = i-2 \geq 0. \end{aligned}$$

If $i = 1$ then $\beta_1^{\vee} = \alpha_1^{\vee} + \beta_2^{\vee}$, and we conclude by the previous case.

- Suppose now that $\alpha_n \notin \mathcal{I}$. Then $\beta_i^{\vee} = (\alpha_1^{\vee} + \dots + \alpha_{n-1}^{\vee}) + (\alpha_i^{\vee} + \dots + \alpha_n^{\vee})$.

If $i > 1$ and $\alpha_1^{\vee} + \dots + \alpha_{n-1}^{\vee}$ is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, we get that $f_{\mathcal{I}}(\beta_i^{\vee}) \geq 0$ with Proposition 5.6 ($\alpha_1^{\vee} + \dots + \alpha_{n-1}^{\vee}$ is of type A_{n-1}) and by induction hypothesis ($\alpha_i^{\vee} + \dots + \alpha_n^{\vee}$ is of type B_{n-i+1} or A_1).

If $i = 1$, and $\alpha_1^{\vee} + \dots + \alpha_{n-1}^{\vee}$ is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, we get that $f_{\mathcal{I}}(\beta_1^{\vee}) \geq 0$ with Proposition 5.6 and using that we have just proved in the latter paragraph ($f_{\mathcal{I}}(\beta_n^{\vee}) \geq 0$).

If $\alpha_1^{\vee} + \dots + \alpha_{n-1}^{\vee}$ is not in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, then $\mathcal{I} = \{\alpha_1, \dots, \alpha_{n-1}\}$, and we compute that

$$\begin{aligned} f_{\mathcal{I}}(\beta_i^{\vee}) &= f_{\mathcal{I}}(\alpha_1^{\vee} + \dots + \alpha_{n-1}^{\vee}) + f_{\mathcal{I}}(\alpha_i^{\vee} + \dots + \alpha_{n-1}^{\vee}) + f_{\mathcal{I}}(\alpha_n^{\vee}) \\ &= -(n-1) - (n-i) + 2(n-1) = i-1 \geq 0. \end{aligned}$$

□

Proposition 5.8. *For any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ of type C , we have $f_{\mathcal{I}}(\beta^{\vee}) \geq 0$.*

Proof. For type C_2 , we already proved it (Corollary 5.5). Let $n \geq 3$. And suppose that, for any \mathcal{I} , for any $2 \leq m \leq n-1$ and for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ of type C_m , we have $f_{\mathcal{I}}(\beta^{\vee}) \geq 0$.

If β^{\vee} is of type C_n , then it is one of the following coroot of C_n with $i \in \{1, \dots, n\}$:

$$\beta_i^{\vee} = \alpha_1^{\vee} + \dots + \alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \dots + 2\alpha_n^{\vee}.$$

We can assume that $\mathcal{I} \subsetneq \text{Supp}(\beta^{\vee})$ and that \mathcal{I} is connected. In particular α_1 or α_n is not in \mathcal{I} .

- Suppose first that $\alpha_1 \notin \mathcal{I}$.

If $i \geq 2$ we can write $\beta_i^{\vee} = \alpha_1^{\vee} + (\alpha_2^{\vee} + \dots + \alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \dots + 2\alpha_n^{\vee})$. Then, if $\alpha_2^{\vee} + \dots + \alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \dots + 2\alpha_n^{\vee}$ is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, we get that $f_{\mathcal{I}}(\beta_i^{\vee}) \geq 0$ by induction hypothesis. Else, $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$ and we compute that

$$f_{\mathcal{I}}(\beta_i^{\vee}) = f_{\mathcal{I}}(\alpha_1^{\vee}) + f_{\mathcal{I}}(\alpha_2^{\vee} + \dots + \alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \dots + 2\alpha_n^{\vee}) = 2(n-1) - (2n-i-1) = i-1 > 0.$$

If $i = 1$ we can write $\beta_1^{\vee} = (\alpha_1^{\vee} + \alpha_2^{\vee}) + (\alpha_2^{\vee} + \dots + \alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \dots + 2\alpha_n^{\vee})$. If $\alpha_2^{\vee} + \dots + \alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \dots + 2\alpha_n^{\vee}$ is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, we get that $f_{\mathcal{I}}(\beta_1^{\vee}) \geq 0$ by induction hypothesis (and the result in type A_2). Else, $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$ and we compute that

$$f_{\mathcal{I}}(\beta_1^{\vee}) = f_{\mathcal{I}}(\alpha_1^{\vee}) + f_{\mathcal{I}}(\alpha_2^{\vee}) + f_{\mathcal{I}}(\alpha_2^{\vee} + 2\alpha_3^{\vee} + \dots + 2\alpha_n^{\vee}) = 2(n-1) - 1 - (2n-3) = 0.$$

- Suppose now that $\alpha_n \notin \mathcal{I}$.

If $i = n$ we can write $\beta_n^\vee = (\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee) + \alpha_n^\vee$. Then, if $\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee$ is in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we get that $f_\mathcal{I}(\beta_n^\vee) \geq 0$ with Proposition 5.6. Else, $\mathcal{I} = \{\alpha_1, \dots, \alpha_{n-1}\}$ and we compute that

$$f_\mathcal{I}(\beta_n^\vee) = f_\mathcal{I}(\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee) + f_\mathcal{I}(\alpha_n^\vee) = -(n-1) + (n-1) = 0.$$

If $i \leq n-1$ we can write $\beta_{n-1}^\vee = \beta_n^\vee + (\alpha_{i+1} + \cdots + \alpha_n^\vee)$ and $f_\mathcal{I}(\beta_i^\vee) = f_\mathcal{I}(\beta_{n-1}^\vee) + f_\mathcal{I}(\alpha_n^\vee) \geq 0$ by induction hypothesis ($\alpha_{i+1}^\vee + \cdots + \alpha_n^\vee$ is of type B_{n-i}) and using that we have just proved in the latter paragraph. □

Proposition 5.9. *For any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ of type D , we have $f_\mathcal{I}(\beta^\vee) \geq 0$.*

Proof. Let $n \geq 4$. And suppose that, for any \mathcal{I} , for any $4 \leq m \leq n-1$ and for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ of type D_m , we have $f_\mathcal{I}(\beta^\vee) \geq 0$. If β^\vee is of type D_n , then it is one of the following coroot of D_n with $i \in \{1, \dots, n-2\}$:

$$\beta_i^\vee = \alpha_1^\vee + \cdots + \alpha_i^\vee + 2\alpha_{i+1}^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee.$$

We can assume that $\mathcal{I} \subsetneq \text{Supp}(\beta^\vee)$ and that \mathcal{I} is connected. In particular α_1, α_{n-1} or α_n is not in \mathcal{I} . By symmetry, we can suppose that α_1 or α_n is not in \mathcal{I} .

- Suppose first that $\alpha_1 \notin \mathcal{I}$.

If $i \geq 2$ we can write $\beta_i^\vee = \alpha_1^\vee + (\alpha_2^\vee + \cdots + \alpha_i^\vee + 2\alpha_{i+1}^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee)$. Then, if $\alpha_2^\vee + \cdots + \alpha_i^\vee + 2\alpha_{i+1}^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee$ is in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we get that $f_\mathcal{I}(\beta_i^\vee) \geq 0$ by induction hypothesis (or, if $n = 4$, by the result in type A_3). Else, $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$ and we compute that

$$\begin{aligned} f_\mathcal{I}(\beta_i^\vee) &= f_\mathcal{I}(\alpha_1^\vee) + f_\mathcal{I}(\alpha_2^\vee + \cdots + \alpha_i^\vee + 2\alpha_{i+1}^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee) \\ &= 2(n-2) - (2n-i-3) = i-1 > 0. \end{aligned}$$

If $i = 1$ we can write $\beta_1^\vee = (\alpha_1^\vee + \alpha_2^\vee) + (\alpha_2^\vee + \cdots + \alpha_i^\vee + 2\alpha_{i+1}^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee)$. If $\alpha_2^\vee + \cdots + \alpha_i^\vee + 2\alpha_{i+1}^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee$ is in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we get that $f_\mathcal{I}(\beta_1^\vee) \geq 0$ by induction hypothesis (or, if $n = 4$, by the result in type A_3 ; and the result in type A_2). Else, $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$ and we compute that

$$\begin{aligned} f_\mathcal{I}(\beta_1^\vee) &= f_\mathcal{I}(\alpha_1^\vee) + f_\mathcal{I}(\alpha_2^\vee) + f_\mathcal{I}(\alpha_2^\vee + 2\alpha_3^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee) \\ &= 2(n-2) - 1 - (2n-5) = 0. \end{aligned}$$

- Suppose now that $\alpha_n \notin \mathcal{I}$.

We can always write $\beta_i^\vee = (\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee) + (\alpha_{i+1} + \cdots + \alpha_{n-2}^\vee + \alpha_n^\vee)$. If $\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee$ is in $\mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$, we get that $f_\mathcal{I}(\beta_i^\vee) \geq 0$ with Proposition 5.6. Else, $\mathcal{I} = \{\alpha_1, \dots, \alpha_{n-1}\}$ and we compute that

$$\begin{aligned} f_\mathcal{I}(\beta_i^\vee) &= f_\mathcal{I}(\alpha_1^\vee + \cdots + \alpha_{n-1}^\vee) + f_\mathcal{I}(\alpha_{i+1}^\vee + \cdots + \alpha_{n-2}^\vee) + f_\mathcal{I}(\alpha_n^\vee) \\ &= -(n-1) - (n-i-2) + 2(n-2) = i-1 \geq 0. \end{aligned}$$

□

It remains the four exceptional cases E_6, E_7, E_8 and F_4 (the result in type G_2 is already known by Corollary 5.5).

Proposition 5.10. *For any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_\mathcal{I}^+$ of type E , we have $f_\mathcal{I}(\beta^\vee) \geq 0$.*

Proof. Let $n \in \{6, 7, 8\}$. And suppose that, for any \mathcal{I} , for any $6 \leq m \leq n-1$ and for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ of type E_m , we have $f_{\mathcal{I}}(\beta^{\vee}) \geq 0$. Let β^{\vee} be of type E_n , we still use Bourbaki's notation. We can assume that $\mathcal{I} \subsetneq \text{Supp}(\beta^{\vee})$ and that \mathcal{I} is connected. In particular α_1, α_2 or α_n is not in \mathcal{I} .

- Case 1: $\alpha_1 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_1}, \beta^{\vee} \rangle = 1$.

Let β_2^{\vee} be the maximal coroot of $R_{\{\alpha_2, \dots, \alpha_n\}}^+$ smaller than β^{\vee} . Then, it is easy to check that β_2^{\vee} is of type D_{n-1} . In particular, by Proposition 5.9, if $\mathcal{I} \neq \{\alpha_2, \dots, \alpha_n\}$, we have $f_{\mathcal{I}}(\beta_2^{\vee}) \geq 0$.

It is also not difficult (just a little long) to check that $\beta_1^{\vee} := \beta^{\vee} - \beta_2^{\vee}$ is a coroot such that $\langle \varpi_{\alpha_1}, \beta_1^{\vee} \rangle = 1$ and of one of the following types: A_1, A_2, A_3 and A_4 , for any $n \in \{6, 7, 8\}$; A_5 and D_5 only for $n \in \{7, 8\}$; and E_6 only for $n = 8$. In particular, β_1^{\vee} is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. Then, by Propositions 5.6 and 5.9, and also by induction when $n = 8$, we have $f_{\mathcal{I}}(\beta_1^{\vee}) \geq 0$.

Hence, by Remark 5.2(iii), we only have to check the case where $\mathcal{I} = \{\alpha_2, \dots, \alpha_n\}$. For such \mathcal{I} , we compute that

$$f_{\mathcal{I}}(\beta^{\vee}) = f_{\mathcal{I}}(\alpha_1^{\vee}) + f_{\mathcal{I}}(\beta_1^{\vee} - \alpha_1^{\vee}) + f_{\mathcal{I}}(\beta_2^{\vee}) = \frac{(n-1)(n-2)}{2} - \left\langle \sum_{i=2}^n \varpi_{\alpha_i}, \beta_1^{\vee} - \alpha_1^{\vee} + \beta_2^{\vee} \right\rangle.$$

To conclude Case 1, we check that, for each β^{\vee} here, $\frac{(n-1)(n-2)}{2} - \langle \sum_{i=2}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \geq 0$: for $n = 6$, $\frac{(n-1)(n-2)}{2} = 10$ and $\langle \sum_{i=2}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \leq 10$; for $n = 7$, $\frac{(n-1)(n-2)}{2} = 15$ and $\langle \sum_{i=2}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \leq 15$ (with the hypothesis $\langle \varpi_{\alpha_1}, \beta^{\vee} \rangle = 1$); and for $n = 8$, $\frac{(n-1)(n-2)}{2} = 21$ and $\langle \sum_{i=2}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \leq 21$ (still with the hypothesis $\langle \varpi_{\alpha_1}, \beta^{\vee} \rangle = 1$).

- Case 2: $\alpha_1 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_1}, \beta^{\vee} \rangle = 2$ (and then $n \in \{7, 8\}$).

Let β_2^{\vee} be the maximal coroot of R^+ smaller than β^{\vee} such that $\langle \varpi_{\alpha_1}, \beta_2^{\vee} \rangle = 1$. In particular, by Case 1, $f_{\mathcal{I}}(\beta_2^{\vee}) \geq 0$. It is not difficult to check that $\beta_1^{\vee} := \beta^{\vee} - \beta_2^{\vee}$ is a coroot such that $\langle \varpi_{\alpha_1}, \beta_1^{\vee} \rangle = 1$ and of one of the following types: A_1 for any $n \in \{7, 8\}$; and A_i with $i \in \{2, \dots, 7\}$ only for $n = 8$. In particular, β_1^{\vee} is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. Then, by Proposition 5.6, we have $f_{\mathcal{I}}(\beta_1^{\vee}) \geq 0$.

Then, we conclude that case by Remark 5.2(iii).

- Case 3: $\alpha_2 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_2}, \beta^{\vee} \rangle = 1$.

Let β_2^{\vee} be the coroot $\alpha_1^{\vee} + \alpha_3^{\vee} + \dots + \alpha_n^{\vee}$ (of type A_{n-1}). In particular, by Proposition 5.9, if $\mathcal{I} \neq \{\alpha_1, \alpha_3, \dots, \alpha_n\}$, we have $f_{\mathcal{I}}(\beta_2^{\vee}) \geq 0$.

It is not difficult to check that $\beta_1^{\vee} := \beta^{\vee} - \beta_2^{\vee}$ is a coroot such that $\langle \varpi_{\alpha_2}, \beta_1^{\vee} \rangle = 1$ and of one of the following types: A_2, A_3 and D_4 for any $n \in \{6, 7, 8\}$; A_4 and D_5 only for $n \in \{7, 8\}$; and D_6 only for $n = 8$. In particular, β_1^{\vee} is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. Then, by Propositions 5.6 and 5.9, we have $f_{\mathcal{I}}(\beta_1^{\vee}) \geq 0$.

Hence, by Remark 5.2(iii), we only have to check the case where $\mathcal{I} = \{\alpha_1, \alpha_3, \dots, \alpha_n\}$. For such \mathcal{I} , we compute that

$$f_{\mathcal{I}}(\beta^{\vee}) = f_{\mathcal{I}}(\alpha_2^{\vee}) + f_{\mathcal{I}}(\beta_1^{\vee} - \alpha_2^{\vee}) + f_{\mathcal{I}}(\beta_2^{\vee}) = 3(n-3) - \langle \varpi_{\alpha_1} + \sum_{i=3}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle.$$

To conclude Case 3, we check that, for each β^{\vee} here, $3(n-3) - \langle \sum_{i=2}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \geq 0$: for $n = 6$, $3(n-3) = 9$ and $\langle \varpi_{\alpha_1} + \sum_{i=3}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \leq 9$; for $n = 7$, $3(n-3) = 12$ and $\langle \varpi_{\alpha_1} + \sum_{i=3}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \leq 12$ (with the hypothesis $\langle \varpi_{\alpha_2}, \beta^{\vee} \rangle = 1$); and for $n = 8$, $3(n-3) = 15$ and $\langle \varpi_{\alpha_1} + \sum_{i=3}^n \varpi_{\alpha_i}, \beta^{\vee} \rangle \leq 15$ (still with the hypothesis $\langle \varpi_{\alpha_2}, \beta^{\vee} \rangle = 1$).

- Case 4: $\alpha_2 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_2}, \beta^{\vee} \rangle = 2$.

Let β_2^\vee be the maximal coroot of R^+ smaller than β^\vee such that $\langle \varpi_{\alpha_2}, \beta_2^\vee \rangle = 1$. In particular, by Case 3, $f_{\mathcal{I}}(\beta_2^\vee) \geq 0$. It is not difficult to check that $\beta_1^\vee := \beta^\vee - \beta_2^\vee$ is a coroot such that $\langle \varpi_{\alpha_2}, \beta_1^\vee \rangle = 1$ and of one of the following types: A_1 for any $n \in \{6, 7, 8\}$; A_2, A_3 and A_4 only for $n \in \{7, 8\}$; D_4 and D_5 only for $n = 8$. In particular, β_1^\vee is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. Then, by Propositions 5.6 and 5.9, we have $f_{\mathcal{I}}(\beta_1^\vee) \geq 0$.

Then, we conclude that case by Remark 5.2(iii).

- Case 5: $\alpha_2 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_2}, \beta^\vee \rangle = 3$ (and then $n = 8$).

Let β_2^\vee be the maximal coroot of R^+ smaller than β^\vee such that $\langle \varpi_{\alpha_2}, \beta_2^\vee \rangle = 2$. In particular, by Case 4, $f_{\mathcal{I}}(\beta_2^\vee) \geq 0$. It is not difficult to check that $\beta_1^\vee := \beta^\vee - \beta_2^\vee$ is a coroot such that $\langle \varpi_{\alpha_2}, \beta_1^\vee \rangle = 1$ and of one of the following types: A_i with $i \in \{1, \dots, 6\}$. In particular, β_1^\vee is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. Then, by Proposition 5.6, we have $f_{\mathcal{I}}(\beta_1^\vee) \geq 0$.

Then, we conclude that case by Remark 5.2(iii).

- Case 6: $\alpha_n \notin \mathcal{I}$ and $\langle \varpi_{\alpha_n}, \beta^\vee \rangle = 1$.

By the symmetry of E_6 , the result in that case is already known by Case 1. Suppose now that $n \in \{7, 8\}$. If $n = 8$, we suppose that the result for E_7 is known.

Let β_2^\vee be the maximal coroot of $R_{\{\alpha_1, \dots, \alpha_{n-1}\}}^+$ smaller than β^\vee . Then, it is easy to check that β_2^\vee is of type E_{n-1} . In particular, by induction hypothesis, if $\mathcal{I} \neq \{\alpha_1, \dots, \alpha_{n-1}\}$, we have $f_{\mathcal{I}}(\beta_2^\vee) \geq 0$.

It is not difficult to check that $\beta_1^\vee := \beta^\vee - \beta_2^\vee$ is a coroot such that $\langle \varpi_{\alpha_n}, \beta_1^\vee \rangle = 1$ and of one of the following types: A_i with $i \in \{1, \dots, 6\}$ for any $n \in \{7, 8\}$; and D_7 only for $n = 8$. In particular, β_1^\vee is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$. Then, by Propositions 5.6 and 5.9, we have $f_{\mathcal{I}}(\beta_1^\vee) \geq 0$.

Hence, by Remark 5.2(iii), we only have to check the case where $\mathcal{I} = \{\alpha_1, \dots, \alpha_{n-1}\}$. For such \mathcal{I} , we compute that

$$f_{\mathcal{I}}(\beta^\vee) = f_{\mathcal{I}}(\alpha_n^\vee) + f_{\mathcal{I}}(\beta_1^\vee - \alpha_n^\vee) + f_{\mathcal{I}}(\beta_2^\vee) = A - \left\langle \sum_{i=2}^{n-1} \varpi_{\alpha_i}, \beta^\vee \right\rangle,$$

where $A = 16$ if $n = 7$ and $A = 27$ if $n = 8$. To conclude Case 6, we check that, for each β^\vee here, we have: if $n = 7$, $\langle \sum_{i=2}^{n-1} \varpi_{\alpha_i}, \beta^\vee \rangle \leq 16$; and if $n = 8$, $\langle \sum_{i=2}^{n-1} \varpi_{\alpha_i}, \beta^\vee \rangle \leq 27$.

- Case 7: $\alpha_n \notin \mathcal{I}$ and $\langle \varpi_{\alpha_n}, \beta^\vee \rangle = 2$.

Then, $n = 8$ and β^\vee is the maximal coroot of E_8 , in particular $\beta^\vee = \alpha_n^\vee + \beta_2^\vee$, where β_2^\vee is a coroot of E_8 such that $\langle \varpi_{\alpha_n}, \beta_2^\vee \rangle = 1$. We conclude that case with Case 6. □

Proposition 5.11. *For any \mathcal{I} , for any $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ of type F_4 , we have $f_{\mathcal{I}}(\beta^\vee) \geq 0$.*

Proof. Let β^\vee be of type F_4 . We can assume that $\mathcal{I} \subsetneq \text{Supp}(\beta^\vee)$ and that \mathcal{I} is connected. In particular α_1 or α_4 is not in \mathcal{I} .

- Case 1: $\alpha_1 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_1}, \beta^\vee \rangle = 1$.

Let β_2^\vee be the maximal coroot of $R_{\{\alpha_2, \alpha_3, \alpha_4\}}^+$ smaller than β^\vee . Then, it is easy to check that β_2^\vee is of type C_3 . In particular, by Proposition 5.8, if $\mathcal{I} \neq \{\alpha_2, \alpha_3, \alpha_4\}$, we have $f_{\mathcal{I}}(\beta_2^\vee) \geq 0$.

It is not difficult to check that $\beta_1^\vee := \beta^\vee - \beta_2^\vee$ is a coroot such that $\langle \varpi_{\alpha_1}, \beta_1^\vee \rangle = 1$ and of type A_1 or A_2 . In particular, β_1^\vee is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, and by Proposition 5.6, we have $f_{\mathcal{I}}(\beta_1^\vee) \geq 0$.

Hence, by Remark 5.2(iii), we only have to check the case where $\mathcal{I} = \{\alpha_2, \alpha_3, \alpha_4\}$. For such \mathcal{I} , we compute that

$$f_{\mathcal{I}}(\beta^\vee) = f_{\mathcal{I}}(\alpha_1^\vee) + f_{\mathcal{I}}(\beta_1^\vee - \alpha_1^\vee) + f_{\mathcal{I}}(\beta_2^\vee) = 6 - \left\langle \sum_{i=2}^4 \varpi_{\alpha_i}, \beta^\vee \right\rangle.$$

To conclude Case 1, we check that, for each β^\vee here, we have $\langle \sum_{i=2}^4 \varpi_{\alpha_i}, \beta^\vee \rangle \leq 6$.

- Case 2: $\alpha_1 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_1}, \beta^\vee \rangle = 2$.

Let β_2^\vee be the maximal coroot of R^+ smaller than β^\vee such that $\langle \varpi_{\alpha_1}, \beta_2^\vee \rangle = 1$. In particular, by Case 1, $f_{\mathcal{I}}(\beta_2^\vee) \geq 0$. It is not difficult to check that $\beta_1^\vee := \beta^\vee - \beta_2^\vee$ is a coroot such that $\langle \varpi_{\alpha_1}, \beta_1^\vee \rangle = 1$ and of type A_i with $i \in \{1, \dots, 4\}$. In particular, β_1^\vee is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, and by Proposition 5.6, we have $f_{\mathcal{I}}(\beta_1^\vee) \geq 0$. We conclude by Remark 5.2(iii).

- Case 3: $\alpha_4 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_4}, \beta^\vee \rangle = 1$.

Let β_2^\vee be the maximal coroot of $R_{\{\alpha_1, \alpha_2, \alpha_3\}}^+$ smaller than β^\vee . Then, it is easy to check that β_2^\vee is of type B_3 . In particular, by Proposition 5.7, if $\mathcal{I} \neq \{\alpha_1, \alpha_2, \alpha_3\}$, we have $f_{\mathcal{I}}(\beta_2^\vee) \geq 0$.

It is not difficult to check that $\beta_1^\vee := \beta^\vee - \beta_2^\vee$ is a coroot such that $\langle \varpi_{\alpha_4}, \beta_1^\vee \rangle = 1$ and of type A_1, A_2, A_3 or C_3 . In particular, β_1^\vee is in $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$, and by Propositions 5.6 and 5.8, we have $f_{\mathcal{I}}(\beta_1^\vee) \geq 0$.

Hence, by Remark 5.2(iii), we only have to check the case where $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_3\}$. For such \mathcal{I} , we compute that

$$f_{\mathcal{I}}(\beta^\vee) = f_{\mathcal{I}}(\alpha_1^\vee) + f_{\mathcal{I}}(\beta_1^\vee - \alpha_4^\vee) + f_{\mathcal{I}}(\beta_2^\vee) = 9 - \langle \sum_{i=1}^3 \varpi_{\alpha_i}, \beta^\vee \rangle.$$

To conclude Case 3, we check that, for each β^\vee here, we have $\langle \sum_{i=1}^3 \varpi_{\alpha_i}, \beta^\vee \rangle \leq 9$.

- Case 4: $\alpha_4 \notin \mathcal{I}$ and $\langle \varpi_{\alpha_4}, \beta^\vee \rangle = 2$.

Then, β^\vee is the maximal coroot of F_4 , in particular $\beta^\vee = \alpha_4^\vee + \beta_2^\vee$, where β_2^\vee is a coroot of F_4 such that $\langle \varpi_{\alpha_4}, \beta_2^\vee \rangle = 1$. We conclude that case with Case 3. □

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