An effective Hamiltonian for the eigenvalue asymptotics of the Robin Laplacian with a large parameter

Konstantin Pankrashkin, Nicolas Popoff

AN EFFECTIVE HAMILTONIAN FOR THE EIGENVALUE ASYMPTOTICS OF THE ROBIN LAPLACIAN WITH A LARGE PARAMETER

KONSTANTIN PANKRASHKIN AND NICOLAS POPOFF

Abstract. We consider the Laplacian on a class of smooth domains $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, with attractive Robin boundary conditions:

$$Q^\Omega_\alpha u = -\Delta u, \quad \frac{\partial u}{\partial n} = \alpha u \text{ on } \partial\Omega, \ \alpha > 0,$$

where n is the outer unit normal, and study the asymptotics of its eigenvalues $E_j(Q^\Omega_\alpha)$ as well as some other spectral properties for α tending to $+\infty$. We work with both compact domains and non-compact ones with a suitable behavior at infinity. For domains with compact C^2 boundaries and fixed j, we show that

$$E_j(Q^\Omega_\alpha) = -\alpha^2 + \mu_j(\alpha) + \mathcal{O}(\log \alpha),$$

where $\mu_j(\alpha)$ is the j^{th} eigenvalue, as soon as it exists, of $-\Delta_S - (\nu-1)\alpha H$ with $(-\Delta_S)$ and H being respectively the positive Laplace-Beltrami operator and the mean curvature on $\partial\Omega$. Analogous results are obtained for a class of domains with non-compact boundaries. In particular, we discuss the existence of eigenvalues for non-compact domains and the existence of spectral gaps for periodic domains. We also show that the remainder estimate can be improved under stronger regularity assumptions.

The effective Hamiltonian $-\Delta_S - (\nu-1)\alpha H$ enters the framework of semi-classical Schrödinger operators on manifolds, and we provide the asymptotics of its eigenvalues for large α under various geometric assumptions. In particular, we describe several cases for which our asymptotics provides gaps between the eigenvalues of Q^Ω_α for large α.

2010 Mathematics Subject Classification: 35P15, 35J05, 49R05, 58C40.

1. Introduction

Let $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, be an open set with a sufficiently regular boundary $S := \partial\Omega$. For $\alpha \in \mathbb{R}$, denote by Q^Ω_α the operator $Q^\Omega_\alpha u = -\Delta u$ acting on the functions u defined in Ω and satisfying the Robin boundary condition

$$\frac{\partial u}{\partial n} = \alpha u \text{ on } S,$$

where n is the outer unit normal at S. More precisely, Q^Ω_α is the self-adjoint operator in $L^2(\Omega)$ associated with the quadratic form q^Ω_α defined on the domain $\mathcal{D}(q^\Omega_\alpha) = H^1(\Omega)$ by

$$q^\Omega_\alpha(u,u) = \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_S u^2 \mathrm{d}S,$$

where $\mathrm{d}S$ stands for the $(\nu-1)$-dimensional Hausdorff measure on S, which is closed and semibounded from below under suitable assumptions (e.g. if S is compact or with a suitable behavior at infinity, see below), and we denote by $E_j(Q^\Omega_\alpha)$ the j^{th} eigenvalue of Q^Ω_α below the bottom of the essential spectrum, as soon as it exists. The aim of the paper is to obtain new results on the asymptotics of the eigenvalues as α tends to $+\infty$.

The problem appears in various applications, such as reaction-diffusion processes [26] and the enhanced surface superconductivity [14], and the related questions were already discussed in various previous works. Let us present briefly the state of art for compact

domains. It was shown in [26, 27] that for piecewise smooth Lipschitz domain one has $E_1(Q^\Omega_\alpha) = -C_\Omega\alpha^2 + o(\alpha^2)$ as $\alpha \to +\infty$, where $C_\Omega \geq 1$ is a constant depending on the geometric properties of Ω. In particular, $C_\Omega = 1$ for C^1 domains, see [5,28]. More detailed asymptotic expansions for some specific non-smooth domains were considered in [18,27,32]. As for smooth domains, a more detailed result was obtained first in [9, 31] for $\nu = 2$ and then in [33] for any $\nu \geq 2$: if the domain is C^3 and $j \in \mathbb{N}$ is fixed, then

$$E_j(Q^\Omega_\alpha) = -\alpha^2 - (\nu-1)H_{\max}\alpha + \mathcal{O}(\alpha^{2/3}), \tag{1.1}$$

where $H_{\max}$ is the maximum of the mean curvature H of the boundary (the exact definition will be recalled below). Remark that this asymptotics together with isoperimetric inequalities for the mean curvature have played an important role for the so-called reverse Faber-Krahn inequality, see [11, 33]. The asymptotics (1.1) was also obtained in [8] for a class of non-compact planar domains.

Although the asymptotics (1.1) shows the influence of the geometry on first orders, it is not sufficient to distinguish the influence of the number j of the eigenvalue and to estimate the gap between eigenvalues. For $\nu = 2$, a complete asymptotic expansion of the eigenvalues of the form

$$\begin{aligned} E_j(Q^\Omega_\alpha) = -\alpha^2 - H_{\max}\alpha + (2j-1)\sqrt{\frac{\left|H''(s_0)\right|}{2}}\,\alpha^{1/2} \\ + \sum_{k=0}^{N} \gamma_{j,k}\alpha^{-\frac{j}{2}} + o(\alpha^{-\frac{N}{2}}), \quad \gamma_{j,k} \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where $j \in \mathbb{N}$ is arbitrary but fixed, was proved in [16] under the assumption that the curvature $s \mapsto H(s)$ admits a unique non-degenerated maximum at s_0 and the second derivative is taken with respect to the arc-length s. Such a hypothesis is reminiscent of several works about the first eigenvalues of the magnetic Laplacian in the semi-classical limit, see [17], which involves the localization of the eigenfunctions at the boundary and allow expansion of the associated eigenvalues. In the informal discussion [15] it was pointed out that, under the assumptions made, the leading terms in the asymptotics (1.2) appear to coincide with those obtained through the harmonic approximation of the operator $-\alpha^2 - \partial_s^2 - \alpha H$ on the boundary, see [7]. It is our principal objective to show that such a correspondence can be made rigorous in any dimension and that it does not need any particular assumptions on the geometry.

In order to describe our results, let us introduce the necessary notation and the class of domains we consider in this article. Recall that $s \mapsto n(s)$ is the Gauss map on S, i.e. $n(s)$ is the outward pointing unit normal vector at $s \in S$. Consider the shape operator L_s at $s \in S$, which is defined by $L_s := \mathrm{d}n(s) : T_sS \to T_sS$, and let $\kappa_1(s), \ldots, \kappa_{\nu-1}(s)$ be its eigenvalues, called the principal curvatures. Our results will be valid for the so-called *C^k-admissible* domains defined as follows:

Definition 1.1. *Let $k \geq 2$. A domain $\Omega \subset \mathbb{R}^\nu$ is called* C^k-admissible*, if its boundary is C^k, and, in addition, the following holds:*

(H1) *There exists $\delta > 0$ such that the map Φ defined by*

$$\Sigma := S \times (0,\delta) \ni (s,t) \mapsto \Phi(s,t) := s - tn(s) \in \Phi\big(S \times (0,\delta)\big) \tag{1.3}$$

is a diffeomorphism and its image is contained in Ω.

(H2) *The curvatures $s \mapsto \kappa_i(s)$ are in $L^\infty(S)$. Moreover, if $k \geq 3$, their gradients are also bounded.*

The assumption (H1) is quite standard if one deals with non-compact domains, and it is sometimes called the *non-overlap condition*, cf. e.g. [3]. We remark that any C^k domain with a compact boundary is C^k-admissible.

In what follows we denote by $K(s)$ the sum of the principal curvatures:

$$K(s) = \kappa_1(s) + \cdots + \kappa_{\nu-1}(s) \equiv \operatorname{tr} L_s.$$

Remark that the quantity $H := K/(\nu-1)$ is exactly the mean curvature on S. We will denote

$$K_{\max} := \sup_{s\in S} K(s).$$

Note that the operator Q_α^Ω can have a non-empty essential spectrum, and it is more convenient to work with the Rayleigh quotients instead of the eigenvalues. For this purpose, recall the min-max principle for the eigenvalues of self-adjoint operators, see e.g. [6, Sec. 4.5]: Let Q be a lower semibounded self-adjoint operator in a Hilbert space $\mathcal{H}$ and q be its quadratic form. Denote

$$E(Q) := \inf \operatorname{spec}_{\mathrm{ess}} Q \quad \text{(we use the convention } \inf\emptyset = +\infty),$$

$$E_j(Q) := \inf_{\substack{L\subset\mathcal{D}(q),\\ \dim L = j}} \sup_{\substack{u\in L,\\ u\neq 0}} \frac{q(u,u)}{\langle u,u\rangle}, \quad j\in\mathbb{N}, \tag{1.4}$$

then:

- if $E_j(Q) < E(Q)$, then $E_j(Q)$ is the j^{th} eigenvalue of Q,
- if $E_j(Q) \geq E(Q)$, then $E_k(Q) = E(Q)$ for all $k \geq j$.

Our main result gives a comparison between the quantities E_j for Q_α^Ω and for an auxiliary Schrödinger operator with a curvature-induced potential on the boundary and in which the Robin coefficient α appears as a coupling constant:

Theorem 1.2. *Let $\nu \geq 2$ and $\Omega\subset\mathbb{R}^\nu$ be a C^2-admissible domain. Furthermore, let $-\Delta_S$ denote the positive Laplace-Beltrami operator on S viewed as a self-adjoint operator in $L^2(S, dS)$. For any fixed $j\in\mathbb{N}$ one has*

$$E_j(Q_\alpha^\Omega) = -\alpha^2 + E_j(-\Delta_S - \alpha K) + \mathcal{O}(\log\alpha), \quad \alpha\to+\infty. \tag{1.5}$$

The proof is presented in Sections 4 and 5. Furthermore, in Section 6 we show that the remainder estimate can be improved under additional assumptions:

Theorem 1.3. *Let the assumptions of Theorem 1.2 hold. In addition, assume that Ω is C^3-admissible and that K reaches its maximum, then for any fixed $j\in\mathbb{N}$ one has*

$$E_j(Q_\alpha^\Omega) = -\alpha^2 + E_j(-\Delta_S - \alpha K) + \mathcal{O}(1), \quad \alpha\to+\infty. \tag{1.6}$$

The improved remainder estimate in (1.6) allows one to identify a class of domains for which the variable curvature K gives a non-trivial contribution with respect to the first-order asymptotics (1.1), see Remark 8.1 below. Let us emphasize on the fact that no assumptions are made on the behavior of K near the set $K^{-1}(K_{\max})$ for the above results. Using numerous methods available for the study of the effective Hamiltonian $-\Delta_S - \alpha K$ one can deduce more precise asymptotics under various additional hypotheses, which improves the results of preceding works or weaken the respective assumptions. In particular, as a generalization of (1.1) in Section 7 we obtain:

Corollary 1.4. *Let $\Omega\subset\mathbb{R}^\nu$ be a C^2-admissible domain, then for each fixed $j\in\mathbb{N}$ we have $E_j(Q_\alpha^\Omega) = -\alpha^2 - K_{\max}\alpha + o(\alpha)$.*

We remark that the paper [33] discussed various eigenvalue optimization problems using the quantity $K_{\max}$, hence, Corollary 1.4 extends the discussion of [33] to C^2-domains.

Remark 1.5. If the boundary S is compact, then either $\operatorname{spec}_{\rm ess} Q_\alpha^\Omega = \emptyset$ (if Ω is bounded) or $\operatorname{spec}_{\rm ess} Q_\alpha^\Omega = [0, +\infty)$ (if Ω is unbounded). In the latter case, it is standard to check that Q_α^Ω has finitely many negative eigenvalues. On the other hand, for any fixed j one has $E_j(Q_\alpha^\Omega) < 0$ if α is sufficiently large, in particular, $E_j(Q_\alpha^\Omega) < E(Q_\alpha^\Omega)$. Therefore, by the min-max principle, each $E_j(Q_\alpha^\Omega)$ is an eigenvalue of Q_α^Ω if α is sufficiently large.

The preceding observation does not hold for domains with non-compact boundaries. In particular, in [8] one can find various examples of domains Ω with curved non-compact boundaries such that the respective operators Q_α^Ω have a purely essential spectrum for any $\alpha > 0$. Nevertheless, the existence of eigenvalues can be guaranteed by an additional assumption:

$$\text{(H3):} \qquad K_\infty := \limsup_{s\to\infty} K(s) < K_{\max}, \tag{1.7}$$

which allows one to prove the following result extending several estimates of [8]:

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^\nu$ be C^k-admissible with non-compact boundary and satisfy* (1.7)*, then for any $N \in \mathbb{N}$ there exists $\alpha_N > 0$ such that for $\alpha > \alpha_N$ the operator Q_α^Ω has at least N eigenvalues below the essential spectrum. The behavior of the j^{th} eigenvalue $E_j(Q_\alpha^\Omega)$ with a fixed j is given by* (1.5) *or, if Ω is C^3, by* (1.6)*.*

The proof is given in Section 7.

More detailed asymptotic expansions for the eigenvalues can be deduced by using the toolbox of the semi-classical analysis of Schrödinger operators on manifolds, where the mean curvature acts as a potential. In Section 8 we describe the results involved by standard hypotheses on the potential K. In particular, the following hold:

Corollary 1.7. *Let $\Omega \subset \mathbb{R}^\nu$ be a C^5-admissible domain. If $\partial\Omega$ is non-compact, assume* (1.7)*. Furthermore, assume that K admits a unique global maximum at s_0 and that the Hessian of $(-K)$ at s_0 is positive-definite. Denote by μ_k its eigenvalues and set*

$$\mathcal{E} = \Big\{ \sum_{k=1}^{\nu-1} \sqrt{\frac{\mu_k}{2}}\,\big(2n_k - 1\big), n_k \in \mathbb{N} \Big\}.$$

Then for each $j \in \mathbb{N}$ there holds, as $\alpha \to +\infty$:

$$E_j(Q_\alpha^\Omega) = -\alpha^2 - K_{\max}\alpha + e_j\alpha^{1/2} + \mathcal{O}(\alpha^{1/4}),$$

where e_j is the j^{th} element of $\mathcal{E}$, counted with multiplicity. Moreover, if Ω is C^6 and if e_j is of multiplicity one, the remainder estimate can be improved to $\mathcal{O}(1)$.

Corollary 1.8. *Let $\Omega \subset \mathbb{R}^2$ be a C^{2p+3}-admissible domain with some integer $p > 1$, and assume* (1.7) *if $\partial\Omega$ is unbounded. Assume that the curvature of the boundary admits a unique global maximum at s_0, which is degenerated in the following sense:*

$$K(s) = K(s_0) - C_p(s - s_0)^{2p} + \mathcal{O}\big((s - s_0)^{2p+1}\big), \quad C_p > 0$$

where s denotes the arc length of the connected component Γ of the boundary where K is maximal, then for each $j \in \mathbb{N}$ there holds, as $\alpha \to +\infty$:

$$E_j(-\Delta_S - \alpha K) = -K_{\max}\alpha + e_j\alpha^{\frac{1}{p+1}} + \mathcal{O}\big(\alpha^{\frac{1}{2(p+1)}}\big),$$

where e_j is the j^{th} eigenvalue of the operator $-\partial_s^2 + C_p s^{2p}$ acting on $L^2(\mathbb{R})$. If $\partial\Omega$ is C^{2p+4} smooth, then the remainder can be replaced by $\mathcal{O}(1)$.

Finally, in Section 9 we consider the case when Ω is periodic with a compact elementary cell. In that case, the above main results show that the spectral bands of Q_α^Ω are determined, up to a error term, by the spectral bands of the periodic operator $-\Delta_S - \alpha K$. In

particular, we prove some sufficient conditions guaranteeing the existence of gaps in the spectrum of Q_α^Ω.

The machinery used for the proof of the main results is quite different from the previous papers on the Robin eigenvalues and is based on a detailed global analysis of the quadratic form and appears to be ideologically very close to the one for the Laplacians in thin domains, cf. [12,13,21,22]. In fact, the main results are obtained using a kind of separation of variables in tubular neighborhoods of the boundary with suitable boundary conditions at the free side, and the thickness depends on α in a special way. The reduced operator $-\Delta_S - \alpha K$ appeared already in [22] in the study of suitable Laplacians in thin neighborhoods of hypersurfaces, and the results from Section 8 provide improvements of [22, Theorem 1.1] under the respective geometric assumptions.

2. Auxiliary estimates

We remark first that, as we deal with real-valued operators only, we will work everywhere with real Hilbert spaces. Let us prove some technical estimates which will be used in the proof of the main results.

Lemma 2.1. *For $\alpha > 0$ and $\delta > 0$, denote by T^D the operator $f \mapsto -f''$ acting in $L^2(0,\delta)$ on the domain*

$$\mathcal{D}(T^D) = \big\{ f \in H^2(0,\delta) : f'(0) = -\alpha f(0),\ f(\delta) = 0 \big\}.$$

Then, as $\delta\alpha$ tends to $+\infty$, the operator T^D has a unique negative eigenvalue E^D, which satisfies

$$E^D = -\alpha^2 + \mathcal{O}(\alpha^2 e^{-\delta\alpha}). \tag{2.1}$$

Furthermore, if ψ^D is an associated normalized eigenfunction, then

$$\psi^D(0)^2 = 2\alpha + \mathcal{O}(\alpha e^{-\delta\alpha}).$$

Proof. The assertion was partially proven in Lemma A.2 of [18] by direct computations: it was shown that the operator T^D has a unique negative eigenvalue, that $E^D = -k^2$ with $k = \alpha + \mathcal{O}(\alpha e^{-\delta\alpha})$, and, finally, that $\psi^D(t) = C(e^{k(t-\delta)} - e^{-k(t-\delta)})$, where C is a normalizing constant. We have then

$$1 = \|\psi^D\|^2_{L^2(0,\delta)} = C^2\Big(\frac{e^{2\delta k} - e^{-2\delta k}}{2k} - 2\delta\Big), \quad C^2 = \frac{2ke^{-2\delta k}}{1 - 4\delta k e^{-2\delta k} - e^{-4\delta k}},$$

which gives

$$\psi^D(0)^2 = 2k\frac{(1-e^{-2\delta k})^2}{1 - 4\delta k e^{-2\delta k} - e^{-4\delta k}} = 2k + \mathcal{O}(\delta k^2 e^{-2\delta k}) = 2\alpha + \mathcal{O}(\alpha e^{-\delta\alpha}). \qquad \square$$

Lemma 2.2. *Let $\beta \geq 0$ be fixed. For $\alpha > 0$ and $\delta > 0$, denote by T^β the operator $f \mapsto -f''$ acting in $L^2(0,\delta)$ on the domain*

$$\mathcal{D}(T^\beta) = \big\{ f \in H^2(0,\delta) : f'(0) = -\alpha f(0),\ f'(\delta) = \beta f(\delta) \big\}.$$

Then, as $\delta\alpha$ tends to $+\infty$, the operator T^β has a unique negative eigenvalue E^β, which satisfies

$$E^\beta = -\alpha^2 + \mathcal{O}(\alpha^2 e^{-\delta\alpha}). \tag{2.2}$$

Furthermore, if ψ^β is an associated normalized eigenfunction, then

$$\psi^\beta(0)^2 = 2\alpha + \mathcal{O}(\alpha e^{-\delta\alpha}), \tag{2.3}$$

$$\psi^\beta(\delta)^2 = 4\alpha e^{-2\delta\alpha} + \mathcal{O}(\alpha e^{-3\delta\alpha}), \tag{2.4}$$

$$\|(\psi^\beta)'\|^2_{L^2(0,\delta)} = \alpha^2 + \mathcal{O}(\alpha^2 e^{-\delta\alpha}). \tag{2.5}$$

In addition,

$$\|f'\|^2_{L^2(0,\delta)} - \alpha f(0)^2 - \beta f(\delta)^2 \geq 0 \text{ for any } f \in H^1(0,\delta) \text{ with } \langle f, \psi^\beta\rangle_{L^2(0,\delta)} = 0. \quad (2.6)$$

Proof. Once again E^β is clearly negative, and we denote by k the positive number such that $E^\beta = -k^2$, so that

$$\psi^\beta(t) = C\bigg(\Big(1+\frac{\beta}{k}\Big)e^{k(t-\delta)} + \Big(1-\frac{\beta}{k}\Big)e^{-k(t-\delta)}\bigg),$$

with C a normalizing constant. Then the condition $\psi'(0) = -\alpha\psi(0)$ is equivalent to

$$\delta k \frac{\sinh(\delta k) - \dfrac{\beta}{k}\cosh(\delta k)}{\cosh(\delta k) - \dfrac{\beta}{k}\sinh(\delta k)} = \delta\alpha.$$

Following literally the proof of [18, Lemma A.1] treating the case $\beta = 0$, we get the existence of a unique solution as α gets large, which satisfies $k = \alpha + \mathcal{O}(\alpha e^{-\delta\alpha})$, which gives the asymptotics of $E^\beta = -k^2$. Moreover, the other eigenvalues of T^β are positive, and since the quadratic form for T^β is

$$t^\beta(f,f) = \|f'\|^2_{L^2(0,\delta)} - \alpha f(0)^2 - \beta f(\delta)^2, \quad \mathcal{D}(t^\beta) = H^1(0,\delta),$$

the assertion (2.6) follows from the spectral theorem for self-adjoint operators. Then (2.3) and (2.4) are obtained as in the the proof of Lemma 2.1. Finally, substituting this estimate into the equality $t^\beta(\psi,\psi) = E^\beta$ we obtain (2.5). □

Finally, we will need a suitable form of the Sobolev inequality, see e.g. Lemma 8 in [25]:

Lemma 2.3. *For any $0 < \ell \leq a$ and $f \in H^1(0,a)$ there holds, with $\xi \in \{0,a\}$,*

$$f(\xi)^2 \leq \ell \int_0^a f'(t)^2 dt + \frac{2}{\ell}\int_0^a f(t)^2 dt.$$

3. Reduction to the analysis near the boundary

3.1. **Dirichlet-Neumann bracketing.** The first steps of the analysis are essentially the same as in [33]. For $\delta > 0$ denote

$$\Omega_\delta := \big\{x \in \Omega : \inf_{s\in S}|x-s| < \delta\big\}, \quad \Theta_\delta := \Omega \setminus \overline{\Omega_\delta},$$

and let $q_\alpha^{\Omega,N,\delta}$ and $q_\alpha^{\Omega,D,\delta}$ be the quadratic forms given by the same expression as q_α^Ω but acting on the domains

$$\mathcal{D}(q_\alpha^{\Omega,N,\delta}) = H^1(\Omega_\delta)\oplus H^1(\Theta_\delta), \quad \mathcal{D}(q_\alpha^{\Omega,D,\delta}) = \widetilde{H}^1_0(\Omega_\delta)\oplus H^1_0(\Theta_\delta),$$
$$\widetilde{H}^1_0(\Omega_\delta) := \{f \in H^1(\Omega_\delta) : f = 0 \text{ at } \partial\Omega_\delta \setminus S\},$$

and denote by $Q_\alpha^{\Omega,N,\delta}$ and $Q_\alpha^{\Omega,D,\delta}$ the associated self-adjoint operators in $L^2(\Omega)$. The inclusions $\mathcal{D}(q_\alpha^{\Omega,D,\delta}) \subset \mathcal{D}(q_\alpha^\Omega) \subset \mathcal{D}(q_\alpha^{\Omega,N,\delta})$ and the min-max principle imply, for each $j \in \mathbb{N}$, the inequalities

$$E_j(Q_\alpha^{\Omega,N,\delta}) \leq E_j(Q_\alpha^\Omega) \leq E_j(Q_\alpha^{\Omega,D,\delta}).$$

Furthermore, $Q_\alpha^{\Omega,N,\delta} = B_\alpha^{\Omega,N,\delta}\oplus(-\Delta)^N_{\Theta_\delta}$ and $Q_\alpha^{\Omega,D,\delta} = B_\alpha^{\Omega,D,\delta}\oplus(-\Delta)^D_{\Theta_\delta}$, where $B_\alpha^{\Omega,N,\delta}$ and $B_\alpha^{\Omega,D,\delta}$ are the self-adjoint operators in $L^2(\Omega_\delta)$ associated respectively with the quadratic

forms

$$b_\alpha^{\Omega,\star,\delta}(u,u) = \int_{\Omega_\delta} |\nabla u|^2 dx - \alpha \int_S u^2 dS, \quad \star \in \{N, D\},$$
$$\mathcal{D}(b_\alpha^{\Omega,N,\delta}) = H^1(\Omega_\delta), \quad \mathcal{D}(b_\alpha^{\Omega,D,\delta}) = \widetilde{H}_0^1(\Omega_\delta),$$

and $(-\Delta)^N_{\Theta_\delta}$ and $(-\Delta)^D_{\Theta_\delta}$ denote the Neumann and the Dirichlet Laplacian in Θ_δ, respectively. As both Neumann and Dirichlet Laplacians are non-negative, we have the inequalities

$$E_j(B_\alpha^{\Omega,N,\delta}) \le E_j(Q_\alpha^\Omega) \le E_j(B_\alpha^{\Omega,D,\delta}) \text{ for all } j \text{ with } E_j(B_\alpha^{\Omega,D,\delta}) < 0. \tag{3.1}$$

The preceding inequalities are valid for any value of $\delta > 0$, but for the rest of the paper we assume that δ depends on α in a special way:

$$\text{the value of } \delta \text{ tends to 0 and the value of } \delta\alpha \text{ tends to } +\infty \text{ as } \alpha \text{ tends to } +\infty, \tag{3.2}$$

and the precise dependence will be chosen later.

3.2. **Change of variables.** In order to study the eigenvalues of the operators $B_\alpha^{\Omega,N,\delta}$ and $B_\alpha^{\Omega,D,\delta}$ we proceed first with a change of variables in Ω_δ with small δ. The computations below are very similar to those performed in [3] for a different problem.

By assumption, for $\delta > 0$ sufficiently small, the map Φ defined in (1.3) is a diffeomorphism between Σ and Ω_δ. The metric G on Σ induced by this embedding is

$$G = g \circ (I_s - tL_s)^2 + \mathrm{d}t^2, \tag{3.3}$$

where $I_s : T_sS \to T_sS$ is the identity map, and g is the metric on S induced by the embedding in $\mathbb{R}^\nu$. The associated volume form $\mathrm{d}\Sigma$ on Σ is

$$\mathrm{d}\Sigma = |\det G|^{1/2} \mathrm{d}s\,\mathrm{d}t = \varphi(s,t) |\det g|^{1/2} \mathrm{d}s\,\mathrm{d}t = \varphi\,\mathrm{d}S\,\mathrm{d}t, \tag{3.4}$$

where

$$\mathrm{d}S = |\det g|^{1/2} \mathrm{d}s$$

is the induced $(\nu-1)$-dimensional volume form on S, and the weight φ is given by

$$\varphi(s,t) := \big|\det(I_s - tL_s)\big| = 1 - t\,\mathrm{tr}\,L_s + p(s,t)t^2 \equiv 1 - K(s)t + p(s,t)t^2, \tag{3.5}$$

with p being a polynomial in t with coefficients which are bounded and continuous in s. In particular,

$$|\varphi(s,t) - 1| \le \|\partial_t \varphi\|_\infty \delta \text{ for all } (s,t) \in \Sigma. \tag{3.6}$$

Let us recall that for a function $f : S \mapsto \mathbb{R}$, the boundedness of the gradient $\nabla_s f$, as stated in Definition 1.1, is understood for the norm on the tangent spaces: $\|\nabla_s f\|_\infty = \sup_{s\in S} \|\nabla_s f(s)\|_{T_sS}$, with

$$\begin{aligned}\|\nabla_s f(s)\|^2_{T_sS} &= \sum_{\rho,\mu} g_{\rho\mu}(s) \left(\sum_k g^{\rho k}(s)\partial_k f(s)\right)\left(\sum_\ell g^{\mu\ell}(s)\partial_\ell f(s)\right) \\ &= \sum_{\rho,\mu} g^{\rho\mu}(s)\partial_\rho f(s)\partial_\mu f(s) \quad \text{with } (g^{\rho\mu}) = g^{-1}.\end{aligned}$$

For future uses, we summarize some obvious properties of φ:

Lemma 3.1. *Let Ω be a C^2-admissible domain, the for δ small, the functions L_s, K are bounded on S, and the functions $\partial_t\varphi$, $\partial_t^2\varphi$, $\partial_t\varphi^{-1/2}$, $(\partial_t\varphi^{-1/2})\varphi^{1/2}$, and $\partial_t\big((\partial_t\varphi^{-1/2})\varphi^{1/2}\big)$ are bounded on Σ. If, in addition, Ω is C^3-admissible, then $\nabla_s\partial_t(\varphi^{-1/2})$ is bounded on Σ.*

In particular, for some $C > 0$ we have

$$\Big| \sum_{\rho,\mu} g^{\rho\mu}(s)\partial_\rho\partial_t(\varphi^{-1/2})(s,t)\partial_\mu\partial_t(\varphi^{-1/2}) \Big| \le C, \quad (s,t) \in \Sigma. \tag{3.7}$$

Now consider the unitary map

$$U : L^2(\Omega_\delta) \to L^2(\Sigma, \mathrm{d}\Sigma), \quad Uf = f \circ \Phi,$$

where Φ is the map from (1.3), and the quadratic forms

$$h^\star_\alpha(f,f) = b^{\Omega,\star,\delta}_\alpha(U^{-1}f, U^{-1}f), \quad \mathcal{D}(h^\star_\alpha) = U\mathcal{D}(b^{\Omega,\star,\delta}_\alpha), \quad \star \in \{N, D\}.$$

We have then, by adopting the Einstein summation rule for indices,

$$\begin{aligned} h^N_\alpha(u,u) &= \int_\Sigma G^{jk}\partial_j u\, \partial_k u \, \mathrm{d}\Sigma - \alpha \int_S |u(s,0)|^2 \mathrm{d}S, \quad \mathcal{D}(h^N_\alpha) = H^1(\Sigma), \\ h^D_\alpha(u,u) &= \text{the restriction of } h^N_\alpha \text{ to } \mathcal{D}(h^D_\alpha) = \widetilde{H}^1_0(\Sigma) \end{aligned}$$

with

$$\widetilde{H}^1_0(\Sigma) := \big\{ u \in H^1(\Sigma) : u(\cdot,\delta) = 0 \big\}, \quad (G^{jk}) := G^{-1}.$$

Due to (3.3) we can estimate, with some $C_g > 0$, depending only on $\|L_s\|_\infty$:

$$(1 - C_g\delta)g^{-1} + \mathrm{d}t^2 \le G^{-1} \le (1 + C_g\delta)g^{-1} + \mathrm{d}t^2.$$

Therefore, we have the form inequalities

$$h^-_\alpha \le h^N_\alpha \quad \text{and} \quad h^D_\alpha \le h^+_\alpha \tag{3.8}$$

with

$$\begin{aligned} h^-_\alpha(u,u) &:= (1 - C_g\delta) \int_\Sigma g^{\rho\mu}\, \partial_\rho u\, \partial_\mu u \, \mathrm{d}\Sigma + \int_\Sigma |\partial_t u|^2 \mathrm{d}\Sigma - \alpha \int_S u(s,0)^2 \mathrm{d}S, \\ \mathcal{D}(h^-_\alpha) &= \mathcal{D}(h^N_\alpha) = H^1(\Sigma), \\ h^+_\alpha(u,u) &:= (1 + C_g\delta) \int_\Sigma g^{\rho\mu}\, \partial_\rho u\, \partial_\mu u \, \mathrm{d}\Sigma + \int_\Sigma |\partial_t u|^2 \mathrm{d}\Sigma - \alpha \int_S u(s,0)^2 \mathrm{d}S, \\ \mathcal{D}(h^+_\alpha) &= \mathcal{D}(h^D_\alpha) = \widetilde{H}^1_0(\Sigma), \end{aligned}$$

where, as usually, $(g^{\rho\mu}) = g^{-1}$. In particular, if H^-_α and H^+_α are the self-adjoint operators acting in $L^2(\Sigma, \mathrm{d}\Sigma)$ and associated with the forms h^-_α and h^+_α respectively, then it follows from (3.1) and (3.8) that

$$E_j(H^-_\alpha) \le E_j(Q^\Omega_\alpha) \le E_j(H^+_\alpha) \text{ for all } j \text{ with } E_j(H^+_\alpha) < 0. \tag{3.9}$$

4. Proof of Theorem 1.2: upper bound

Recall that the operator T^D has been defined in Lemma 2.1. We have denoted by E^D its lowest eigenvalue, and in *this section* we denote for shortness $\psi := \psi^D$ an associated normalized eigenfunction. The function ψ will be used to construct test functions for H^+_α.

4.1. **An estimate for product functions.** Recall that everywhere we assume that δ is a function of α satisfying (3.2). We have the following estimate:

Lemma 4.1. *For $v \in H^1(S)$, consider a function u defined by $u(s,t) = v(s)\psi(t)$, which belongs to $\mathcal{D}(h_\alpha^+)$. There exist positive constants c_0^+ and c_1^+ such that, as $\alpha \to +\infty$, for any $v \in H^1(S)$ there holds*

$$\begin{aligned}\frac{h_\alpha^+(u,u)}{\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)}} - E^D \\ \leq (1+c_0^+\delta)\frac{(1+c_1^+\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}s - \alpha\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)}}{\|v\|^2_{L^2(S,\mathrm{d}S)}} \\ + \mathcal{O}(1+\alpha e^{-\delta\alpha}), \quad (4.1)\end{aligned}$$

Moreover, the remainder depends only on $\|L_s\|_\infty$, $\|K\|_\infty$, $\|\partial_t\varphi\|_\infty$, and $\|\partial_t^2\varphi\|_\infty$, and it is independent of v.

Proof. Through the estimates we denote by C_j various positive constants. Using (3.6), a direct evaluation provides

$$\begin{aligned}h_\alpha^+(u,u) =&(1+C_g\delta)\int_{S\times(0,\delta)}\psi(t)^2 g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\,\varphi(s,t)\mathrm{d}S\mathrm{d}t \\ &+\int_S v(s)^2\int_0^\delta \psi'(t)^2\varphi(s,t)\mathrm{d}t\,\mathrm{d}S - \alpha\psi(0)^2\int_S v^2\mathrm{d}S \\ \leq&(1+C_1\delta)\int_S\int_0^\delta \psi(t)^2 g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\mathrm{d}t\mathrm{d}S \\ &+\int_S v(s)^2\int_0^\delta \psi'(t)^2\varphi(s,t)\mathrm{d}t\,\mathrm{d}S - \alpha\psi(0)^2\int_S v^2\mathrm{d}S \\ =&(1+C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S \\ &+\int_S v(s)^2\int_0^\delta \psi'(t)^2\varphi(s,t)\mathrm{d}t\,\mathrm{d}S - \alpha\psi(0)^2\int_S v^2\mathrm{d}S.\end{aligned} \quad (4.2)$$

Moreover, the constant C_1 depends only on $\|L_s\|_\infty$ and $\|\partial_t\varphi\|_\infty$. Using a repeated integration by parts together with the boundary conditions satisfied by ψ, we obtain for all $s \in S$:

$$\begin{aligned}&\int_0^\delta \psi'(t)^2\varphi(s,t)\mathrm{d}t \\ =&\Big[\psi(t)\psi'(t)\varphi(s,t)\Big]_{t=0}^{t=\delta} + \int_0^\delta \psi(t)\big(-\psi''(t)\big)\varphi(s,t)\mathrm{d}t - \int_0^\delta \psi(t)\psi'(t)\partial_t\varphi(s,t)\mathrm{d}t \\ =&-\psi(0)\psi'(0)\varphi(s,0) + E^D\int_0^\delta \psi(t)^2\varphi(s,t)\mathrm{d}t - \frac{1}{2}\int_0^\delta \partial_t\big(\psi(t)^2\big)\partial_t\varphi(s,t)\mathrm{d}t \\ =&\alpha\psi(0)^2 + E^D\int_0^\delta \psi(t)^2\varphi(s,t)\mathrm{d}t - \frac{1}{2}\bigg(\Big[\psi(t)^2\partial_t\varphi(s,t)\Big]_{t=0}^{t=\delta} - \int_0^\delta \psi(t)^2\partial_t^2\varphi(s,t)\mathrm{d}t\bigg) \\ =&\alpha\psi(0)^2 + E^D\int_0^\delta \psi(t)^2\varphi(s,t)\mathrm{d}t - \frac{K(s)}{2}\psi(0)^2 + \frac{1}{2}\int_0^\delta \psi(t)^2\partial_t^2\varphi(s,t)\mathrm{d}t.\end{aligned} \quad (4.3)$$

The substitution of (4.3) into (4.2) gives

$$h_\alpha^+(u,u) \le (1+C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S + E^D\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} - \frac{\psi(0)^2}{2}\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} + \frac{1}{2}\int_S\int_0^\delta v(s)^2\psi(t)^2\partial_t^2\varphi(s,t)\mathrm{d}t\,\mathrm{d}S. \quad (4.4)$$

As the functions $\partial_t^2\varphi$ and K are bounded, we estimate with the help of Lemma 2.1:

$$h_\alpha^+(u,u) - E^D\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} \le (1+C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S - \alpha\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} + \mathcal{O}(1+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)}, \quad (4.5)$$

where the $\mathcal{O}$-coefficient depends only on $\|\partial_t^2\varphi\|_\infty$ and $\|K\|_\infty$. Furthermore, due to (3.6), we have the estimate $\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} \ge (1-C_2\delta)\|v\|^2_{L^2(S,\mathrm{d}S)}$, where C_2 depends only on $\|\partial_t\varphi\|_\infty$. This gives

$$\frac{h_\alpha^+(u,u)}{\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)}} - E^D \le \frac{(1+C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S - \alpha\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} + \mathcal{O}(1+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)}}{(1-C_2\delta)\|v\|^2_{L^2(S,\mathrm{d}S)}},$$

and we deduce the lemma by choosing $c_1^+ = C_1$, and $c_0^+ > 0$ so that $(1-C_2\delta)^{-1} \le 1 + c_0^+\delta$, which is possible since δ becomes small as α tends to $+\infty$. □

4.2. **Proof of the upper bound.** Now for each j we can use the definition (1.4) by testing on the subspaces $L \subset \mathcal{D}(h_\alpha^+)$ of the form

$$L = \{u : u(s,t) = v(s)\psi(t) \text{ with } v \in \Lambda\},$$

where Λ are the j^{th} dimensional subspaces of $H^1(S)$, which is the form domain of $-\Delta_S - \alpha K$. Lemma 4.1 then implies

$$E_j(H_\alpha^+) - E^D \le (1+c_0^+\delta)E_j\big(-(1+c_1^+\delta)\Delta_S - \alpha K\big) + \mathcal{O}(1+\alpha e^{-\delta\alpha}). \quad (4.6)$$

The right-hand side can be estimated as follows:

Lemma 4.2. *For any fixed $j \in \mathbb{N}$ there holds, as $\alpha \to +\infty$,*

$$(1+c_0^+\delta)E_j\big(-(1+c_1^+\delta)\Delta_S - \alpha K\big) \le E_j(-\Delta_S - \alpha K) + \mathcal{O}(\delta\alpha),$$

where δ is a function of α satisfying (3.2)*. The constants depend only on $\|L_s\|_\infty$, $\|K\|_\infty$, $\|\partial_t\varphi\|_\infty$, $\|\partial_t^2\varphi\|_\infty$.*

Proof. We have

$$\begin{aligned}
&(1+c_0^+\delta)E_j\big(-(1+c_1^+\delta)\Delta_S-\alpha K\big)\\
&\quad=(1+c_0^+\delta)\Big(E_j\big(-(1+c_1^+\delta)\Delta_S+\alpha(K_{\max}-K)\big)-\alpha K_{\max}\Big)\\
&\quad=(1+c_0^+\delta)E_j\big(-(1+c_1^+\delta)\Delta_S+\alpha(K_{\max}-K)\big)-\alpha K_{\max}+\mathcal{O}(\delta\alpha)\\
&\quad\le(1+c_0^+\delta)E_j\big(-(1+c_1^+\delta)\Delta_S+(1+c_1^+\delta)\alpha(K_{\max}-K)\big)-\alpha K_{\max}+\mathcal{O}(\delta\alpha)\\
&\quad\le(1+C\delta)E_j(-\Delta_S+\alpha(K_{\max}-K)\big)-\alpha K_{\max}+\mathcal{O}(\delta\alpha)\\
&\quad=E_j\big(-\Delta_S+\alpha(K_{\max}-K)\big)-\alpha K_{\max}\\
&\qquad\quad+\mathcal{O}\Big(\delta E_j(-\Delta_S+\alpha(K_{\max}-K))\Big)+\mathcal{O}(\delta\alpha)\\
&\quad=E_j(-\Delta_S-\alpha K)+\mathcal{O}\Big(\delta E_j\big(-\Delta_S+\alpha(K_{\max}-K)\big)\Big)+\mathcal{O}(\delta\alpha).
\end{aligned}\tag{4.7}$$

As K is bounded, we have the rough estimate $E_j\big(-\Delta_S+\alpha(K_{\max}-K)\big)=\mathcal{O}(\alpha)$, and the remainder depends on the constants c_0^+, c_1^+ and $\|K\|_\infty$ only. □

Finally, combining Lemma 4.2 with (4.6) and Lemma 2.1, we get

$$\begin{aligned}
E_j(H_\alpha^+)&\le E^D+E_j(-\Delta_S-\alpha K)+\mathcal{O}(1+\delta\alpha+\alpha e^{-\delta\alpha})\\
&=-\alpha^2+E_j(-\Delta_S-\alpha K)+\mathcal{O}(1+\delta\alpha+\alpha^2e^{-\delta\alpha}).
\end{aligned}\tag{4.8}$$

In order to have an optimal remainder we take

$$\delta=\frac{b\log\alpha}{\alpha},\quad b\ge 2,$$

then $E_j(H_\alpha)\le E_j(H_\alpha^+)\le-\alpha^2+E_j(-\Delta_S-\alpha K)+\mathcal{O}(\log\alpha)$.

5. Proof of Theorem 1.2: lower bound

5.1. **Minoration of the quadratic form.** The operator T^β of Lemma 2.2 with $\beta=0$ will play a special role and it will be denoted by T^N. The first eigenvalue and the first normalized eigenfunction will be denoted *in this section* by E^N and ψ respectively. Recall again that δ and α obey (3.2).

We represent any function $u\in\mathcal{D}(h_\alpha^-)$ in the form

$$u(s,t)=v(s)\psi(t)+w(s,t)\tag{5.1}$$

with

$$v(s):=\int_0^\delta\psi(t)u(s,t)\mathrm{d}t,\quad v\in H^1(S).\tag{5.2}$$

Remark that the both functions $(s,t)\mapsto v(s)\psi(t)$ and w are in $\mathcal{D}(h_\alpha^-)$. The following proposition gives a lower bound on the expression $h_\alpha^-(u,u)-E^N\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)}$ in terms of this decomposition.

Proposition 5.1. *There exist positive constants c_0^- and c_1^- such that, as $\alpha\to+\infty$,*

$$\begin{aligned}
&h_\alpha^-(u,u)-E^N\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)}\\
&\quad\ge(1-c_0^-\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S-\alpha\langle v,Kv\rangle_{L^2(S,\mathrm{d}S)}-c_1^-(1+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)}\\
&\qquad+\frac{\alpha^2}{2}\int_S\int_0^\delta w(s,t)^2\mathrm{d}t\,\mathrm{d}S
\end{aligned}\tag{5.3}$$

for any $u\in\mathcal{D}(h_\alpha^-)$. The constants depend only on $\|L_s\|_\infty$, $\|\partial_t\varphi\|_\infty$ and $\|\partial_t^2\varphi\|_\infty$.

The rest of this subsection is devoted to the proof of Proposition 5.1. Using the decomposition (5.1), we clearly have

$$h_\alpha^-(u,u) = (1-C_g\delta)\int_\Sigma g^{\rho\mu}\,\partial_\rho u\,\partial_\mu u\,\mathrm{d}\Sigma + \int_\Sigma |\partial_t u|^2\mathrm{d}\Sigma - \alpha\int_S u(s,0)^2\mathrm{d}S \tag{5.4}$$

$$=: I_1 + I_2 + I_3 + I_4, \tag{5.5}$$

where we have set

$$\begin{cases} I_1 = (1-C_g\delta)\displaystyle\int_0^\delta\int_S g^{\rho\mu}\,\partial_\rho u\,\partial_\mu u\,\varphi\,\mathrm{d}S\mathrm{d}t, \\ I_2 = \displaystyle\int_S v(s)^2\int_0^\delta \psi'(t)^2\varphi(s,t)\mathrm{d}t\,\mathrm{d}S, \\ I_3 = 2\Big[\displaystyle\int_S v(s)\int_0^\delta \psi'(s)\partial_t w(s,t)\varphi(s,t)\mathrm{d}t\,\mathrm{d}S - \alpha\psi(0)\int_S v(s)w(s,0)\mathrm{d}S\Big] \\ I_4 = \displaystyle\int_S\int_0^\delta \big|\partial_t w(s,t)\big|^2\varphi(s,t)\mathrm{d}t\mathrm{d}S - \alpha\psi(0)^2\int_S v(s)^2\mathrm{d}S - \alpha\int_S w(s,0)^2\mathrm{d}S. \end{cases}$$

We estimate the four terms separately.

Lemma 5.2. *There exists $C_1 > 0$ such that, as $\alpha \to +\infty$,*

$$I_1 \geq (1-C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S. \tag{5.6}$$

Moreover, the constant C_1 depends only on $\|\partial_t\varphi\|_\infty$ and $\|L_s\|_\infty$ and is independent of u.

Proof. Following the decomposition (5.1), we get by using (3.6) a constant $C > 0$ such that

$$\begin{aligned}
&\int_0^\delta\int_S g^{\rho\mu}\,\partial_\rho u(s,t)\,\partial_\mu u(s,t)\,\varphi(s,t)\mathrm{d}S\mathrm{d}t \\
&\quad= \int_0^\delta\int_S \psi(t)^2 g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\,\varphi(s,t)\mathrm{d}S\mathrm{d}t + 2\int_0^\delta\int_S \psi(t) g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu w(s,t)\,\varphi\mathrm{d}S\mathrm{d}t \\
&\qquad+ \int_0^\delta\int_S g^{\rho\mu}\,\partial_\rho w(s,t)\,\partial_\mu w(s,t)\,\varphi\mathrm{d}S\mathrm{d}t \\
&\quad\geq (1-C\delta)\int_0^\delta\int_S \psi(t)^2 g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\mathrm{d}S\mathrm{d}t + 2\int_0^\delta\int_S \psi(t) g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu w(s,t)\,\varphi(s,t)\mathrm{d}S\mathrm{d}t \\
&\qquad+ (1-C\delta)\int_0^\delta\int_S g^{\rho\mu}\,\partial_\rho w(s,t)\,\partial_\mu w(s,t)\mathrm{d}S\mathrm{d}t \\
&\quad= (1-C\delta)\int_S g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\mathrm{d}S + 2\int_0^\delta\int_S \psi(t) g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu w(s,t)\,\varphi(s,t)\mathrm{d}S\mathrm{d}t \\
&\qquad+ (1-C\delta)\int_0^\delta\int_S g^{\rho\mu}\,\partial_\rho w(s,t)\,\partial_\mu w(s,t)\,\mathrm{d}S\mathrm{d}t,
\end{aligned} \tag{5.7}$$

where the constant C depends only on $\|\partial_t\varphi\|_\infty$. Remark that for the function w we have

$$\int_0^\delta \psi(t)w(s,t)\mathrm{d}t = 0 \text{ and, hence, } \int_0^\delta \psi(t)\partial_\rho w(s,t)\mathrm{d}t = 0, \quad s\in S. \tag{5.8}$$

We deduce:

$$\int_0^\delta \int_S \psi g^{\rho\mu}\,\partial_\rho v\,\partial_\mu w\,\varphi \mathrm{d}S\mathrm{d}t = \int_0^\delta \int_S \psi g^{\rho\mu}\,\partial_\rho v\,\partial_\mu w\,\mathrm{d}S\mathrm{d}t + \int_0^\delta \int_S \psi g^{\rho\mu}\,\partial_\rho v\,\partial_\mu w\,(\varphi-1)\mathrm{d}S\mathrm{d}t = \int_0^\delta \int_S \psi g^{\rho\mu}\,\partial_\rho v\,\partial_\mu w\,(\varphi-1)\mathrm{d}S\mathrm{d}t. \quad (5.9)$$

Using again (3.6), we estimate with the same constant C, using the Cauchy-Schwarz inequality for the metric $(g^{\rho\mu})$,

$$\begin{aligned}
&\Big|\int_0^\delta \int_S \psi(t) g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu w(s,t)\,(\varphi(s,t)-1)\mathrm{d}S\mathrm{d}t\Big| \\
&\le C\delta \int_0^\delta \int_S \Big|g^{\rho\mu}\,\psi(t)\partial_\rho v(s)\,\partial_\mu w(s,t)\Big|\mathrm{d}S\mathrm{d}t \\
&\le \frac{C\delta}{2}\int_0^\delta \int_S \psi(t)^2 g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\mathrm{d}S\mathrm{d}t + \frac{C\delta}{2}\int_0^\delta \int_S g^{\rho\mu}\,\partial_\rho w(s,t)\,\partial_\mu w(s,t)\mathrm{d}S\mathrm{d}t \quad (5.10)\\
&= \frac{C\delta}{2}\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S + \frac{C\delta}{2}\int_0^\delta \int_S g^{\rho\mu}\,\partial_\rho w(s,t)\,\partial_\mu w(s,t)\mathrm{d}S\mathrm{d}t
\end{aligned}$$

which gives

$$2\Big[\int_0^\delta \int_S \psi(t) g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu w(s,t)\,\varphi(s,t)\mathrm{d}S\mathrm{d}t\Big] \le C\delta \int_S g^{\rho\mu}\,\partial_\rho v(s)\,\partial_\mu v(s)\mathrm{d}S + C\delta\int_0^\delta \int_S g^{\rho\mu}\,\partial_\rho w(s,t)\,\partial_\mu w(s,t)\mathrm{d}S\mathrm{d}t. \quad (5.11)$$

Substituting the last inequality into (5.7) we obtain,

$$\int_0^\delta \int_S g^{\rho\mu}\,\partial_\rho u\,\partial_\mu u\,\varphi\mathrm{d}S\mathrm{d}t \ge (1-2C\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\mathrm{d}S + (1-2C\delta)\int_0^\delta \int_S g^{\rho\mu}\,\partial_\rho w\,\partial_\mu w\,\mathrm{d}S\mathrm{d}t$$

and, therefore, for sufficiently small δ:

$$(1-C_g\delta)\int_0^\delta \int_S g^{\rho\mu}\,\partial_\rho u\,\partial_\mu u\,\varphi\mathrm{d}S\mathrm{d}t \ge (1-C_g\delta)(1-2C\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\mathrm{d}S.$$

The result follows as C_g depends only on $\|L_s\|_\infty$, and C depends only on $\|\partial_t\varphi\|_\infty$. □

Lemma 5.3. *There exists $C_2 > 0$ such that, as $\alpha \to +\infty$:*

$$I_2 \ge \alpha\psi(0)^2\int_S v(s)^2\mathrm{d}S + E^N\|u-w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} - \alpha\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} - C_2(1+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)}. \quad (5.12)$$

The constant C_2 depends only on $\|K\|_\infty$, $\|\partial_t\varphi\|_\infty$ and $\|\partial_t^2\varphi\|_\infty$ and is independent of u.

Proof. As in (4.3), an integration by part leads to

$$\int_0^\delta \psi'(t)^2\varphi(s,t)\mathrm{d}t = \alpha\psi(0)^2 + E^N\int_0^\delta \psi(t)^2\varphi(s,t)\mathrm{d}t - \frac{K(s)}{2}\psi(0)^2 - \frac{\partial_t\varphi(s,\delta)}{2}\psi(\delta)^2 + \frac{1}{2}\int_0^\delta \psi(t)^2\partial_t^2\varphi(s,t)\mathrm{d}t.$$

The additional term in comparison with (4.3) comes from the fact that $\psi(\delta) \neq 0$. We deduce:

$$\begin{aligned} I_2 &= \int_S v(s)^2 \int_0^\delta \psi'(t)^2 \varphi(s,t) \mathrm{d}t \, \mathrm{d}S \\ &= \alpha\psi(0)^2 \int_S v^2 \mathrm{d}S + E^N \|u-w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} - \frac{\psi(0)^2}{2} \langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} \\ &\quad - \frac{\psi(\delta)^2}{2} \langle v, \partial_t \varphi(\cdot,\delta) v\rangle_{L^2(S,\mathrm{d}S)} + \frac{1}{2} \int_S v(s)^2 \int_0^\delta \psi(t)^2 \partial_t^2 \varphi(s,t) \mathrm{d}t \mathrm{d}S. \end{aligned} \tag{5.13}$$

Due to (2.4), $\psi(\delta)^2 = O(\alpha e^{-2\delta\alpha})$, and there exists $C > 0$ such that for α large enough one has

$$\frac{\psi(\delta)^2}{2} \langle v, \partial_t \varphi(\cdot,\delta) v\rangle_{L^2(S,\mathrm{d}S)} \leq C\alpha e^{-2\delta\alpha} \|v\|^2_{L^2(S,\mathrm{d}S)}, \tag{5.14}$$

where the constant C depends only on $\|\partial_t \varphi\|_\infty$. We also have:

$$\Big| \int_0^\delta \psi(t)^2 \partial_t^2 \varphi(s,t) \mathrm{d}t \Big| \leq \|\partial_t^2 \varphi\|_\infty. \tag{5.15}$$

Moreover, (2.3) provides $C' > 0$, depending only on $\|K\|_\infty$, such that for α large enough:

$$\frac{\psi(0)^2}{2} \langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} \leq \alpha \langle v, Kv\rangle_{L^2(S,\mathrm{d}S)} + C'\alpha e^{-\delta\alpha} \|v\|^2_{L^2(S,\mathrm{d}S)}. \tag{5.16}$$

The lemma follows by combining (5.14)–(5.16) with (5.13). □

The crossed term I_3 needs a parametric estimate:

Lemma 5.4. *There exists $C_3 > 0$ such that for any $r > 0$ for large α one has*

$$I_3 \geq 2E^N \langle u-w, w\rangle_{L^2(\Sigma,\mathrm{d}\Sigma)} - C_3 r\alpha^2 \|v\|^2_{L^2(S,\mathrm{d}S)} - C_3 \frac{1}{r} \int_S \int_0^\delta w(s,t)^2 \mathrm{d}t \mathrm{d}S.$$

Moreover, the constant C_3 depends only on $\|\partial_t \varphi\|_\infty$ and does not depend on u.

Proof. Using the integration by parts we have:

$$\begin{aligned} I_3 &= 2\int_S v(s) \int_0^\delta \psi'(s) \partial_t w(s,t) \varphi(s,t) \mathrm{d}t \, \mathrm{d}S - 2\alpha\psi(0) \int_S v(s) w(s,0) \mathrm{d}S \\ &= 2\int_S v(s) \bigg(\Big[\psi'(t) w(s,t) \varphi(s,t) \Big]_{t=0}^{t=\delta} - \int_0^\delta \psi''(t) w(s,t) \varphi(s,t) \mathrm{d}t \\ &\quad - \int_0^\delta \psi'(t) w(s,t) \partial_t \varphi(s,t) \mathrm{d}t \bigg) \mathrm{d}S - 2\alpha\psi(0) \int_S v(s) w(s,0) \mathrm{d}S \\ &= 2\alpha\psi(0) \int_S v(s) w(s,0) \varphi(s,0) \mathrm{d}S + 2E^N \int_S \int_0^\delta v(s) \psi(t) w(s,t) \varphi(s,t) \mathrm{d}t \mathrm{d}S \\ &\quad - 2\int_S \int_0^\delta v(s) \psi'(t) w(s,t) \partial_t \varphi(s,t) \mathrm{d}t \mathrm{d}S - 2\alpha\psi(0) \int_S v(s) w(s,t) \mathrm{d}S \\ &= 2E^N \langle u-w, w\rangle_{L^2(\Sigma,\mathrm{d}\Sigma)} - 2\int_S \int_0^\delta v(s) \psi'(t) w(s,t) \partial_t \varphi(s,t) \mathrm{d}t \mathrm{d}S, \end{aligned} \tag{5.17}$$

where we have used the boundary conditions $\psi'(\delta) = 0$ and $\psi'(0) = -\alpha\psi(0)$.

We estimate now, with any $r > 0$:

$$\begin{aligned}\Big|2\int_S\int_0^\delta v(s)\psi'(t)w(s,t)\partial_t\varphi(s,t)\mathrm{d}t\mathrm{d}S\Big| &\\ &\le \|\partial_t\varphi\|_\infty \int_S\int_0^\delta 2\Big|v(s)\psi'(t)w(s,t)\Big|\mathrm{d}t\mathrm{d}S\\ &\le \|\partial_t\varphi\|_\infty r\int_S\int_0^\delta \psi'(t)^2v(s)^2\mathrm{d}t\mathrm{d}S + \frac{\|\partial_t\varphi\|_\infty}{r}\int_S\int_0^\delta w(s,t)^2\mathrm{d}t\mathrm{d}S \quad (5.18)\\ &\le 2\|\partial_t\varphi\|_\infty r\alpha^2\|v\|^2_{L^2(S,\mathrm{d}S)} + \frac{\|\partial_t\varphi\|_\infty}{r}\int_S\int_0^\delta w(s,t)^2\mathrm{d}t\mathrm{d}S,\end{aligned}$$

where we have used $\|\psi'\|^2_{L^2(0,\delta)} \le 2\alpha^2$ for α large enough, see (2.5). The substitution of (5.18) into (5.17) gives the lemma by choosing $C_3 = 2\|\partial_t\varphi\|_\infty$. □

We are now able to finish the proof of Proposition 5.1. We use Lemmas 5.2–5.4 in (5.5) and deduce

$$\begin{aligned}h^-_\alpha(u,u) \ge (1-C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S - \alpha\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)}&\\ + E^N\|u-w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} + 2E^N\langle u-w, w\rangle_{L^2(\Sigma,\mathrm{d}\Sigma)} + \int_S\int_0^\delta \big|\partial_t w(s,t)\big|^2\varphi(s,t)\mathrm{d}t\mathrm{d}S&\\ - \alpha\int_S |w(s,0)|^2\mathrm{d}S - C_4(1+r\alpha^2+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)} - \frac{C_3}{r}\int_S\int_0^\delta \big|w(s,t)\big|^2\mathrm{d}t\,\mathrm{d}S,&\end{aligned} \quad (5.19)$$

where $C_4 = \max(C_2, C_3)$. We have the equality

$$\|u-w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} + 2\langle u-w, w\rangle_{L^2(\Sigma,\mathrm{d}\Sigma)} = \|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} - \|w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)}. \quad (5.20)$$

Due to (5.8), we can use (2.6), so that

$$\int_S\int_0^\delta \big|\partial_t w(s,t)\big|^2\mathrm{d}t\,\mathrm{d}S - \alpha\int_S w(s,0)^2\mathrm{d}S \ge 0,$$

and, therefore, substituting (5.20) into (5.19), we get

$$\begin{aligned}h^-_\alpha(u,u) - E^N\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} \ge (1-C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S - \alpha\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)}&\\ - C_4(1+r\alpha^2+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)} - E^N\|w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} - \frac{C_3}{r}\int_S\int_0^\delta w(s,t)^2\mathrm{d}t\,\mathrm{d}S.&\end{aligned} \quad (5.21)$$

Due to (3.6), we have

$$\|w\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} \ge (1-\|\partial_t\varphi\|_\infty\delta)\int_S\int_0^\delta w(s,t)^2\mathrm{d}t\,\mathrm{d}S.$$

We choose $r = 3C_3/\alpha^2$ in (5.21), so that the asymptotic expansion (2.2) for E^N provides a constant $C_5 > 0$ such that for α large enough:

$$\begin{aligned}h^-_\alpha(u,u) - E^N\|u\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} \ge (1-C_1\delta)\int_S g^{\rho\mu}\,\partial_\rho v\,\partial_\mu v\,\mathrm{d}S - \frac{\psi(0)^2}{2}\langle v, Kv\rangle_{L^2(S,\mathrm{d}S)}&\\ - C_5(1+\alpha e^{-\delta\alpha})\|v\|^2_{L^2(S,\mathrm{d}S)} + \frac{\alpha^2}{2}\int_S\int_0^\delta w(s,t)^2\mathrm{d}t\,\mathrm{d}S.&\end{aligned}$$

Therefore, the proof is concluded by setting $c_0^- = C_1$ and $c_1^- = C_5$. Noticing that C_4 and C_5 express with C_2 and C_3, we deduce that the constants depends only on $\|L_s\|_\infty$, $\|\partial_t\varphi\|_\infty$ and $\|\partial_t^2\varphi\|_\infty$.

5.2. **Asymptotics of E_j.** The expression on the right-hand side of (5.3) can be viewed as a lower semibounded quadratic form defined on $\mathcal{D}(h_\alpha^-) \subset L^2(\Sigma, \mathrm{d}S\mathrm{d}t)$. Denote its closure in $L^2(\Sigma, \mathrm{d}S\mathrm{d}t)$ by q, and let Q be the associated self-adjoint operator in $L^2(\Sigma, \mathrm{d}S\mathrm{d}t) \equiv L^2(S, \mathrm{d}S) \otimes L^2(0, \delta)$. It writes as

$$Q = \Big[-(1 - c_0^-\delta)\Delta_S - \alpha K - c_1^-(1 + \alpha e^{-2\delta\alpha})\Big] P + \frac{\alpha^2}{2}(1 - P),$$

where $P : L^2(\Sigma, \mathrm{d}S\mathrm{d}t) \to L^2(S, \mathrm{d}S) \otimes \psi$ is the orthogonal projector $(Pu)(s,t) := v(s)\psi(t)$ with v defined in (5.2). For each fixed j and large α we have

$$E_j\Big(-(1 - c_0^-\delta)\Delta_S - \alpha K - c_1^-(1 + \alpha e^{-2\delta\alpha})\Big) = \mathcal{O}(\alpha) < \frac{\alpha^2}{2},$$

hence,

$$E_j(Q) = E_j\Big(-(1 - c_0^-\delta)\Delta_S - \alpha K - c_1^-(1 + \alpha e^{-\delta\alpha})\Big).$$

Furthermore, using

$$\|u\|^2_{L^2(\Sigma, d\Sigma)} \leq (1 + \|\partial_t\varphi\|_\infty\delta)\|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)}$$

we have a positive constant C_φ, depending only on $\|\partial_t\varphi\|_\infty$ such that

$$\frac{h_\alpha^-(u,u)}{\|u\|^2_{L^2(\Sigma, d\Sigma)}} - E^N \geq (1 - C_\varphi\delta)\frac{q(u,u)}{\|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)}}.$$

As the identification operator $f \mapsto f$ defines an injection of $\mathcal{D}(h_\alpha^-) \subset L^2(\Sigma, d\Sigma)$ in $\mathcal{D}(q) \subset L^2(\Sigma, \mathrm{d}S\mathrm{d}t)$, it follows that

$$\begin{aligned} E_j(H_\alpha^-) &\geq (1 - C_\varphi\delta)E_j(Q) + E^N \\ &= -\alpha^2 + (1 - C_\varphi\delta)E_j\Big(-(1 - c_0^-\delta)\Delta_S - \alpha K\Big) + \mathcal{O}(1 + \alpha^2 e^{-\delta\alpha}), \quad (5.22) \end{aligned}$$

where we have used the asymptotics (2.2) for E^N. In addition, by Lemma 4.2, we get

$$(1 - C_\varphi\delta)E_j\Big(-(1 - c_0^-\delta)\Delta_S - \alpha K\Big) = E_j(-\Delta_S - \alpha K) + \mathcal{O}(\delta\alpha).$$

Hence, by substituting in (5.22),

$$E_j(H_\alpha^-) \geq -\alpha^2 + E_j(-\Delta_S - \alpha K) + \mathcal{O}(1 + \alpha^2 e^{-\delta\alpha} + \delta\alpha),$$

and the constants depend only on $\|L_s\|_\infty$, $\|K\|_\infty$, $\|\partial_t\varphi\|_\infty$ and $\|\partial_t^2\varphi\|_\infty$. Choosing

$$\delta = \frac{b\log\alpha}{\alpha}, \quad b \geq 2, \quad (5.23)$$

we arrive at the result.

6. Proof of Theorem 1.3

The main idea for improving the remainder estimate is to work in unweighted spaces from the very beginning. The weight φ is indeed C^1 with respect to the s variable now, and this allows for more precise Taylor expansions of φ in Σ, so that the comparison between the Robin Laplacian and the decoupled operator becomes more precise. We start with the following simple result:

Lemma 6.1. *Under the assumption of Theorem 1.3, for any fixed $j \in \mathbb{N}$ one has*

$$E_j(-\Delta_S - \alpha K) = -\alpha K_{\max} + \mathcal{O}(\alpha^{2/3}) \text{ as } \alpha \to +\infty.$$

Proof. Due to $(-\Delta_S) \geq 0$ we have the obvious lower bound $E_j(-\Delta_S - \alpha K) \geq -\alpha K_{\max}$. Let us prove the upper bound. For $s, s_0 \in S$, let $d(s, s_0)$ denote the geodesic distance between s and s_0. Let $s_0 \in S$ be such that $K(s_0) = K_{\max}$. As K is at least C^1, there exist $\varepsilon > 0$ and $C > 0$ such that

$$K(s) \geq K_{\max} - Cd(s, s_0) \text{ as } d(s, s_0) < \varepsilon. \tag{6.1}$$

Now let us choose j functions $f_1, \dots, f_j \in C_c^\infty(\mathbb{R}_+)$ having disjoint supports, non identically zero, and set $v_i(s) = f_i\big(r^{-1}d(s, s_0)\big)$, where $r > 0$ is small and will be chosen later. For small r, the functions v_i have pairwise disjoint supports and belong to the domain of Δ_S. In particular, they are linearly independent, and

$$\big\langle v_i, (-\Delta_S - \alpha K)v_l \big\rangle = 0 \text{ for } i \neq l.$$

On the other hand,

$$\theta_i(r) := \int_S v_i(s)^2 \mathrm{d}S = \int_S f_i\big(r^{-1}d(s, s_0)\big)^2 \mathrm{d}S = a_i r^{\nu-1} + o(r^{\nu-1}), \quad a_i > 0.$$

Using (6.1) we have

$$\begin{aligned}\big\langle v_i, (-\Delta_S - \alpha K)v_i \big\rangle = \int_S g^{\rho\mu}\partial_\rho v_i \partial_\mu v_i \mathrm{d}S - \alpha \int_S K v_i^2 \mathrm{d}S \\ \leq b_i r^{\nu-3} - \alpha K_{\max}\theta_i(r) + c_i \alpha r \theta_i(r), \quad b_i,\, c_i > 0,\end{aligned}$$

which gives

$$\frac{\big\langle v_i, (-\Delta_S - \alpha K)v_i \big\rangle}{\|v_i\|^2} \leq -\alpha K_{\max} + A_i(r^{-2} + \alpha r), \quad A_i > 0.$$

Now it is sufficient to take $r := \alpha^{-1/3}$ and to test in (1.4) on the subspace L spanned by $v_1, \dots, v_j$. □

6.1. **Toward unweighted spaces.** In order to remove the weight φ, we perform the unitary transform

$$\Theta : L^2(\Sigma, \mathrm{d}S\mathrm{d}t) \ni u \mapsto \varphi^{-1/2}u \in L^2(\Sigma, \mathrm{d}\Sigma),$$

and consider the quadratic forms $u \mapsto h_\alpha^\pm(\Theta u, \Theta u)$ defined on $\Theta^{-1}\big(\mathcal{D}(h_\alpha^\pm)\big) \subset L^2(\Sigma, \mathrm{d}S\mathrm{d}t)$. In order to reduce the analysis to decoupled operators, we prove approximation lemmas:

Lemma 6.2. *There exists $\delta_0 > 0$ and positive constants C and C' such that for all $\delta \in (0, \delta_0)$ and $u \in \Theta^{-1}\big(\mathcal{D}(h_\alpha^\pm)\big)$ one has*

$$\begin{aligned}\left| \int_\Sigma g^{\rho\mu}\partial_\rho(\varphi^{-1/2}u)\partial_\mu(\varphi^{-1/2}u)\varphi \mathrm{d}S\mathrm{d}t - \int_\Sigma g^{\rho\mu}\partial_\rho u \partial_\mu u \mathrm{d}S\mathrm{d}t \right| \\ \leq C\delta \int_\Sigma g^{\rho\mu}\partial_\rho u \partial_\mu u \mathrm{d}S\mathrm{d}t + C'\delta \|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)},\end{aligned} \tag{6.2}$$

and the constants depend only on $\|\varphi\|_\infty$ and $\|\nabla_s \partial_t(\varphi^{-1/2})\|_\infty$.

Proof. We compute

$$\begin{aligned}\int_\Sigma g^{\rho\mu}\partial_\rho(\varphi^{-1/2}u)\partial_\mu(\varphi^{-1/2}u)\varphi \mathrm{d}S\mathrm{d}t - \int_\Sigma g^{\rho\mu}\partial_\rho u \partial_\mu u \mathrm{d}S\mathrm{d}t \\ = \int_\Sigma g^{\rho\mu}\partial_\rho \varphi^{-1/2}\partial_\mu \varphi^{-1/2}\varphi u^2 \mathrm{d}S\mathrm{d}t + 2\int g^{\rho\mu}\partial_\rho \varphi^{-1/2} u \varphi^{1/2}\partial_\mu u \mathrm{d}S\mathrm{d}t.\end{aligned} \tag{6.3}$$

Due to (3.5), we have the expansion

$$\varphi(s, t)^{-1/2} = 1 + tA(s, t),$$

where A and its gradient are bounded. In particular, there exists $C_0 > 0$ with

$$\|\nabla_s \varphi^{-1/2}\|_\infty \leq C_0 \delta, \tag{6.4}$$

where C_0 is controlled by $\|\partial_t \nabla_s \varphi^{-1/2}\|_\infty$, see (3.7). We deduce that

$$\left| \int_\Sigma g^{\rho\mu} \partial_\rho \varphi^{-1/2} \partial_\mu \varphi^{-1/2} \varphi u^2 \mathrm{d}S\mathrm{d}t \right| \leq C_1 \delta^2 \|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)}, \tag{6.5}$$

where the constant C_1 is controlled by $\|\nabla_s \partial_t \varphi^{-1/2}\|_\infty$ and $\|\varphi\|_\infty$. Using the Cauchy-Schwarz inequality for the metric $(g^{\rho\mu})$, we get

$$\begin{aligned}
&\left| 2 \int_\Sigma g^{\rho\mu} \left(\partial_\rho \varphi^{-1/2} u \right) \left(\varphi^{1/2} \partial_\mu u \right) \mathrm{d}S\mathrm{d}t \right| \\
&\qquad \leq 2 \left(\int_\Sigma g^{\rho\mu} \partial_\rho \varphi^{-1/2} \partial_\mu \varphi^{-1/2} u^2 \mathrm{d}S\mathrm{d}t \int_\Sigma g^{\rho\mu} \partial_\rho u \partial_\mu u \varphi \mathrm{d}S\mathrm{d}t \right)^{1/2} \\
&\qquad \leq \delta^{-1} \int_\Sigma g^{\rho\mu} \partial_\rho \varphi^{-1/2} \partial_\mu \varphi^{-1/2} u^2 \mathrm{d}S\mathrm{d}t + \delta \int_\Sigma g^{\rho\mu} \partial_\rho u \partial_\mu u \varphi \mathrm{d}S\mathrm{d}t \\
&\qquad \leq \delta^{-1} \|\nabla_s \varphi^{-1/2}\|^2_\infty \int_\Sigma u^2 \mathrm{d}S\mathrm{d}t + \delta \int_\Sigma g^{\rho\mu} \partial_\rho u \partial_\mu u \varphi \mathrm{d}S\mathrm{d}t \\
&\qquad \leq C_1 \delta \|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)} + C_2 \delta \int_\Sigma g^{\rho\mu} \partial_\rho u \partial_\mu u \mathrm{d}S\mathrm{d}t;
\end{aligned}$$

on the last step we have used (6.4). We deduce the lemma by combining the last inequality with (6.3) and (6.5). Since C_2 is controlled by $\|\varphi\|_\infty$, we deduce that the constants depends only on $\|\varphi\|_\infty$ and $\|\nabla_s \partial_t (\varphi^{-1/2})\|_\infty$. □

Lemma 6.3. *There exists $\delta_0 > 0$ and positive constants C and β such that for all $\delta \in (0, \delta_0)$ there holds*

$$\begin{aligned}
\int_\Sigma |\partial_t(\varphi^{-1/2} u)|^2 \varphi \mathrm{d}S\mathrm{d}t \geq \int_\Sigma |\partial_t u|^2 \mathrm{d}S\mathrm{d}t - \int_S \frac{K(s)}{2} u(s,0)^2 \mathrm{d}S \\
- \beta \int_S u(s,\delta)^2 \mathrm{d}S - C\|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)} \text{ for all } u \in \Theta^{-1}\big(\mathcal{D}(h^-_\alpha)\big)
\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
\int_\Sigma \big|\partial_t(\varphi^{-1/2} u)\big|^2 \varphi \mathrm{d}S\mathrm{d}t \leq \int_\Sigma |\partial_t u|^2 \mathrm{d}S\mathrm{d}t - \int_S \frac{K(s)}{2} u(s,0)^2 \mathrm{d}S \\
+ C\|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)} \text{ for all } u \in \Theta^{-1}\big(\mathcal{D}(h^+_\alpha)\big),
\end{aligned} \tag{6.7}$$

and the constants depend on $\|\partial_t(\varphi^{-1/2})\|_\infty$, $\|\varphi^{1/2}\|_\infty$ and $\|\partial_t(\varphi^{1/2} \partial_t \varphi^{-1/2})\|_\infty$ only.

Proof. We have

$$\begin{aligned}
\int_\Sigma |\partial_t(\varphi^{-1/2} u)|^2 \varphi \mathrm{d}S\mathrm{d}t - \int_\Sigma |\partial_t u|^2 \mathrm{d}S\mathrm{d}t \\
= \int_\Sigma |\partial_t(\varphi^{-1/2})|^2 u^2 \varphi \mathrm{d}S\mathrm{d}t + 2 \int_\Sigma \partial_t(\varphi^{-1/2}) \varphi^{1/2} u \partial_t u \mathrm{d}S\mathrm{d}t,
\end{aligned} \tag{6.8}$$

and there exists $C_0 > 0$ such that

$$\left| \int_\Sigma \big|\partial_t(\varphi^{-1/2})\big|^2 u^2 \varphi \mathrm{d}S\mathrm{d}t \right| \leq C_0 \|u\|^2_{L^2(\Sigma, \mathrm{d}S\mathrm{d}t)}. \tag{6.9}$$

The second term is treated by an integration by parts:

$$2\int_\Sigma \partial_t(\varphi^{-1/2})\varphi^{1/2}u\partial_t u\mathrm{d}S\mathrm{d}t = \int_S\int_0^\delta \partial_t(\varphi^{-1/2})\varphi^{1/2}\partial_t(u^2)\mathrm{d}t\mathrm{d}S$$
$$= \int_S \Big[\partial_t(\varphi^{-1/2})\varphi^{1/2}u^2\Big]_{t=0}^{t=\delta}\mathrm{d}S - \int_\Sigma \partial_t(\partial_t(\varphi^{-1/2})\varphi^{1/2})u^2\mathrm{d}S\mathrm{d}t. \quad (6.10)$$

Due to (3.5), we have the expansion

$$\partial_t(\varphi^{-1/2})(s,t) = \frac{K(s)}{2} + tQ(s,t),$$

where Q is bounded in Σ, so that $\partial_t(\varphi^{-1/2})(s,0) = K(s)/2$, and (6.10) provides

$$\left|2\int_\Sigma \partial_t(\varphi^{-1/2})\varphi^{1/2}u\partial_t u\mathrm{d}S\mathrm{d}t + \int_S \frac{K(s)}{2}u(s,0)^2\mathrm{d}S\right| \le C_1\|u\|^2_{L^2(\Sigma,\mathrm{d}s\mathrm{d}t)} + \beta\int_S u(s,\delta)^2\mathrm{d}S,$$

where

$$\beta = \sup_{s\in S}\Big|(\varphi^{1/2}\partial_t\varphi^{-1/2})(s,\delta)\Big|.$$

By combining this with (6.8) and (6.9), we deduce the lower bound (6.6), and also the upper bound (6.7) since $u(s,\delta) = 0$ for $u \in \Theta^{-1}\big(\mathcal{D}(h_\alpha^+)\big)$. Moreover, the constant C_0 is controlled by $\|\partial_t\varphi^{-1/2}\|^2_\infty$, the constant β by $\|\partial_t\varphi^{-1/2}\varphi^{1/2}\|_\infty$ and the constant C_1 by $\|\partial_t(\partial_t\varphi^{-1/2}\varphi^{1/2})\|_\infty$. □

We deduce by combining the last two lemmas that there exist positive constants c_0 and c_1 such that for all $\delta \in (0,\delta_0)$:

$$h_\alpha^-(\Theta u,\Theta u) \ge (1-c_0\delta)\int_\Sigma g^{\rho\mu}\partial_\rho u\partial_\mu u\mathrm{d}S\mathrm{d}t + \int_\Sigma |\partial_t u|^2\mathrm{d}S\mathrm{d}t$$
$$-\int_S\left(\alpha+\frac{K}{2}\right)u(s,0)^2\mathrm{d}S - \beta\int_S u(s,\delta)^2\mathrm{d}S - c_1\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t}$$
$$\text{for } u\in\Theta^{-1}\big(\mathcal{D}(h_\alpha^-)\big), \quad (6.11)$$

and

$$h_\alpha^+(\Theta u,\Theta u) \le (1+c_0\delta)\int_\Sigma g^{\rho\mu}\partial_\rho u\partial_\mu u\mathrm{d}S\mathrm{d}t + \int_\Sigma |\partial_t u|^2\mathrm{d}S\mathrm{d}t$$
$$-\int_S\left(\alpha+\frac{K}{2}\right)u(s,0)^2\mathrm{d}S + c_1\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} \text{ for } u\in\Theta^{-1}\big(\mathcal{D}(h_\alpha^+)\big). \quad (6.12)$$

We denote by $q_\alpha^\pm$ the quadratic forms on the right-hand side of (6.11) and (6.12) respectively, defined on the form domains $\mathcal{D}(q_\alpha^\pm) := \Theta^{-1}(\mathcal{D}(h_\alpha^\pm))$. The associated self-adjoint operators, both acting on the unweighted space $L^2(\Sigma,\mathrm{d}S\mathrm{d}t)$, will be denoted by $Q_\alpha^\pm$. Due to (3.9) one has

$$E_j(Q_\alpha^-) \le E_j(Q_\alpha^\Omega) \le E_j(Q_\alpha^+) \text{ for all } j \text{ with } E_j(Q_\alpha^+) < 0. \quad (6.13)$$

6.2. **Upper bound.** Once again we estimate the quadratic form q_α^+ evaluated on the functions u that write as a product $u(s,t) = v(s)\psi(t)$, where ψ is a normalized eigenfunction of T^D associated with E^D (see Lemma 2.1) and $v \in H^1(S)$. Here we have simply $\|u\|_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} = \|v\|^2_{L^2(S,\mathrm{d}S)}$, and

$$q_\alpha^+(u,u) - E^D\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} = (1+c_0\delta)\int_S g^{\rho\mu}\partial_\rho v\partial_\mu v\mathrm{d}S - \frac{\psi(0)^2}{2}\int_S Kv^2\mathrm{d}S + c_1\|v\|^2_{L^2(S,\mathrm{d}S)}.$$

Using (2.1) we obtain

$$E_j(Q_\alpha^+) \le -\alpha^2 + E_j\big(-(1+c_0\delta)\Delta_S - \alpha K\big) + \mathcal{O}(1+\alpha e^{-\delta\alpha}). \quad (6.14)$$

To estimate the right-hand side of (6.14) we need an additional assertion:

Lemma 6.4. *For any $j \in \mathbb{N}$ there exist positive constants C, α_0 and δ_0 such that for $\delta \in (0, \delta_0)$ and $\alpha \geq \alpha_0$ the following inequalities hold:*

$$E_j\big(-(1+c_0\delta)\Delta_S - \alpha K\big) \leq E_j\big(-\Delta_S - \alpha K\big) + C\delta\alpha^{2/3}, \tag{6.15}$$

$$E_j\big(-(1-c_0\delta)\Delta_S - \alpha K\big) \geq E_j\big(-\Delta_S - \alpha K\big) - C\delta\alpha^{2/3}. \tag{6.16}$$

Proof. We only prove the upper bound, the lower bound being symmetric. We have

$$\begin{aligned} E_j\big(-(1+c_0\delta)\Delta_S - \alpha K\big) &= (1+c_0\delta)E_j\big(-\Delta_S + \frac{1}{1+c_0\delta}\alpha(K_{\max}-K)\big) - \alpha K_{\max} \\ &\leq (1+c_0\delta)E_j\big(-\Delta_S + \alpha(K_{\max}-K)\big) - \alpha K_{\max} \\ &= E_j\big(-\Delta_S - \alpha K\big) + c_0\delta E_j\big(-\Delta_S + \alpha(K_{\max}-K)\big), \end{aligned}$$

and it is sufficient to apply Lemma 6.1 to the last term. □

Now let us assume that δ is a function of α satisfying (3.2). Applying Lemma 6.4 to (6.14) we deduce, for $\alpha \to +\infty$,

$$E_j(Q_\alpha^+) \leq -\alpha^2 + E_j\big(-\Delta_S - \alpha K\big) + \mathcal{O}(\delta\alpha^{2/3} + 1 + \alpha^2 e^{-\delta\alpha}). \tag{6.17}$$

Choosing $\delta = \alpha^{-\kappa}$ with $\kappa \in [2/3, 1)$ and using (6.13) we obtain the result.

6.3. **Lower bound.** Similarly to Section 5.1, we decompose any function $u \in \mathcal{D}(q_\alpha^-)$ as

$$u(s,t) = v(s)\psi(t) + w(s,t),$$

where $\psi = \psi^\beta$ is a normalized eigenfunction of the operator T^β associated with the first eigenvalue E^β, see Lemma 2.2, and

$$v(s) = \int_0^\delta \psi(t)u(s,t)\mathrm{d}t.$$

It follows that

$$\int_0^\delta \psi(t)w(s,t)\mathrm{d}t = 0, \quad s \in S, \tag{6.18}$$

which provides

$$\int_\Sigma v(s)\psi(t)w(s,t)\mathrm{d}S\mathrm{d}t = 0 \text{ and } \int_\Sigma g^{\rho\mu}\partial_\rho v(s)\psi(t)\partial_\mu w(s,t)\mathrm{d}S\mathrm{d}t = 0. \tag{6.19}$$

A direct computation provides

$$\begin{aligned} q_\alpha^-(u,u) = {} & (1-c_0\delta)\int_S g^{\rho\mu}\partial_\rho v\partial_\mu v\mathrm{d}S + (1-c_0\delta)\int_\Sigma g^{\rho\mu}\partial_\rho w(s,t)\partial_\mu w(s,t)\mathrm{d}S\mathrm{d}t \\ & + \int_\Sigma v(s)^2\psi'(t)^2\mathrm{d}S\mathrm{d}t + 2\int_\Sigma v(s)\psi'(t)\partial_t w(s,t)\mathrm{d}S\mathrm{d}t \\ & + \int_\Sigma |\partial_t w(s,t)|^2\mathrm{d}S\mathrm{d}t - \int_S \Big(\alpha + \frac{K(s)}{2}\Big)v(s)^2\psi(0)^2\mathrm{d}S \\ & - 2\int_S \Big(\alpha + \frac{K(s)}{2}\Big)v(s)\psi(0)w(s,0)\mathrm{d}S - \int_S \Big(\alpha + \frac{K(s)}{2}\Big)w(s,0)^2\mathrm{d}S \\ & - \beta\int_S v(s)^2\psi(\delta)^2\mathrm{d}S - 2\beta\int_S v(s)\psi(\delta)w(s,\delta)\mathrm{d}S - \beta\int_S w(s,\delta)^2\mathrm{d}S \\ & - c_1\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)}, \end{aligned} \tag{6.20}$$

where we have used (6.19) to get rid of the crossed terms. We also have, using an integration by part:

$$\begin{aligned}\int_\Sigma v(s)\psi'(t)\partial_t w(s,t)\mathrm{d}S\mathrm{d}t &= \int_S v(s)\left(\Big[\psi'(t)w(s,t)\Big]_{t=0}^{t=\delta} - \int_0^\delta \psi''(t)w(s,t)\mathrm{d}t\right)\mathrm{d}S\\ &= \int_S v(s)\left(\psi'(\delta)w(s,\delta) - \psi'(0)w(s,0) + E^\beta\int_0^\delta \psi(t)w(s,t)\mathrm{d}t\right)\mathrm{d}S\\ &= \int_S \Big(\beta v(s)\psi(\delta)w(s,\delta) + \alpha v(s)\psi(0)w(s,0)\Big)\mathrm{d}S, \quad (6.21)\end{aligned}$$

where we have used (6.19) and the boundary condition for the eigenfunction ψ. Moreover, by the definition of ψ we have

$$\int_0^\delta \psi'(t)^2\mathrm{d}t - \alpha\psi(0)^2 - \beta\psi(\delta)^2 = E^\beta.$$

Inserting the last inequality and (6.21) into (6.20), we arrive at

$$\begin{aligned}q_\alpha^-(u,u) =& (1-c_0\delta)\int_S g^{\rho\mu}\partial_\rho v\partial_\mu v\mathrm{d}S + (1-c_0\delta)\int_\Sigma g^{\rho\mu}\partial_\rho w\partial_\mu w\mathrm{d}S\mathrm{d}t\\ &+ \int_\Sigma |\partial_t w|^2\mathrm{d}S\mathrm{d}t - \int_S K(s)v(s)\psi(0)w(s,0)\mathrm{d}S\\ &- \int_S \big(\alpha + \frac{K}{2}\big)w(s,0)^2\mathrm{d}S - \int_S \frac{K}{2}v(s)^2\psi(0)^2\mathrm{d}S\\ &- \beta\int_S w(s,\delta)^2\mathrm{d}S + E^\beta\|v\|^2_{L^2(S,\mathrm{d}S)} - c_1\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)}.\end{aligned} \quad (6.22)$$

Lemma 6.5. *There exist $R>0$ and $\alpha_0>0$ such that for all $\alpha\geq\alpha_0$ there holds*

$$\begin{aligned}&\int_\Sigma |\partial_t w(s,t)|^2\mathrm{d}S\mathrm{d}t - \int_S K(s)v(s)\psi(0)w(s,0)\mathrm{d}S\\ &\quad - \int_S \Big(\alpha + \frac{K(s)}{2}\Big)w(s,0)^2\mathrm{d}S - \int_S \frac{K(s)}{2}v(s)^2\psi(0)^2\mathrm{d}S - \beta\int_S w(s,\delta)^2\mathrm{d}S\\ &\qquad \geq -\frac{\alpha^2}{2}\|w\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} - \psi(0)^2\int_S \Big(\frac{K(s)}{2} + \frac{R}{\alpha}\Big)v(s)^2\mathrm{d}S \quad (6.23)\end{aligned}$$

for all $u\in\mathcal{D}(q_\alpha^-)$. Moreover, the constant R depends only on $\|K\|_\infty$.

Proof. Denote by J the term on the left-hand side of (6.23). For any $\varepsilon>0$ we have

$$\left|\int_S K(s)v(s)\psi(0)w(s,0)\mathrm{d}S\right| \leq \varepsilon\psi(0)^2\int_S v(s)^2\mathrm{d}S + \frac{1}{4\varepsilon}\int_S K(s)^2w(s,0)^2\mathrm{d}S,$$

and there holds, for sufficiently small ε,

$$\begin{aligned}J \geq& \int_\Sigma |\partial_t w(s,t)|^2\mathrm{d}S\mathrm{d}t - \big(\alpha + \frac{B}{\varepsilon}\big)\int_S w(s,0)^2\mathrm{d}S\\ &- \psi(0)^2\int_S \Big(\frac{K(s)}{2} + \varepsilon\Big)v(s)^2\mathrm{d}S - \beta\int_S w(s,\delta)^2\mathrm{d}S, \quad (6.24)\end{aligned}$$

with $B=\sup_S(K^2+|K|)$. Due to (6.18) and to the inequality (2.6) of Lemma 2.2 we have

$$\int_0^\delta \big|\partial_t w(s,t)\big|^2\mathrm{d}t - \alpha w(s,0)^2 - \beta w(s,\delta)^2 \geq 0, \quad s\in S.$$

It follows that for any $\eta \in (0,1)$ we can estimate

$$\int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t \\ \geq \eta \int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t + (1-\eta)\alpha \int_S w(s,0)^2 \mathrm{d}S + (1-\eta)\beta \int_S w(s,\delta)^2 \mathrm{d}S,$$

and the substitution into (6.24) gives

$$J \geq \eta \int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t - \Big(\eta\alpha + \frac{B}{\varepsilon}\Big) \int_S w(s,0)^2 \mathrm{d}S - \eta\beta \int_S w(s,\delta)^2 \mathrm{d}S \\ - \psi(0)^2 \int_S \Big(\frac{K(s)}{2} + \varepsilon\Big) v(s)^2 \mathrm{d}S.$$

Therefore, choosing $\varepsilon = R/\alpha$ with $R > 0$ and then using Lemma 2.3 we obtain, for any $\ell \in (0,\delta)$,

$$\begin{aligned} J &\geq \eta \int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t - \Big(\eta + \frac{B}{R}\Big)\alpha \Big(\ell \int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t + \frac{2}{\ell} \int_\Sigma |w(s,t)|^2 \mathrm{d}S\mathrm{d}t\Big) \\ &\quad - \eta\beta \left(\ell \int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t + \frac{2}{\ell} \int_\Sigma w(s,t)^2 \mathrm{d}S\mathrm{d}t\right) - \psi(0)^2 \int_S \Big(\frac{K(s)}{2} + \frac{R}{\alpha}\Big) v(s)^2 \mathrm{d}S \\ &= \Big[\eta - \ell\alpha\Big(\eta + \frac{B}{R}\Big) - \ell\beta\eta\Big] \int_\Sigma |\partial_t w(s,t)|^2 \mathrm{d}S\mathrm{d}t - \frac{2}{\ell}\Big(\eta\alpha + \frac{B\alpha}{R} + \eta\beta\Big) \int_\Sigma w(s,t)^2 \mathrm{d}S\mathrm{d}t \\ &\quad - \psi(0)^2 \int_S \Big(\frac{K(s)}{2} + \frac{R}{\alpha}\Big) v(s)^2 \mathrm{d}S. \end{aligned}$$

Choose $\ell = \rho(\alpha+\beta)^{-1}$ with $\rho \in (0,1/2)$ and $R \geq B(1-\rho)^{-1}$, then the choice

$$\eta = \frac{\ell B\alpha}{R\big(1 - \ell(\alpha+\beta)\big)} = \frac{\rho B\alpha}{(\alpha+\beta)R(1-\rho)} \in (0,1),$$

implies

$$\eta - \ell\alpha\Big(\eta + \frac{B}{R}\Big) - \ell\beta\eta = 0,$$

and

$$J \geq -\left(\frac{2B\alpha(\alpha+\beta)}{R(1-\rho)} + \frac{2B\alpha(\alpha+\beta)}{\rho R}\right) \|w\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} - \psi(0)^2 \int_S \Big(\frac{K(s)}{2} + \frac{R}{\alpha}\Big) v(s)^2 \mathrm{d}S. \tag{6.25}$$

As R can be taken arbitrary large, we may choose it in order to have

$$R > 4B\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right),$$

then there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$ we have

$$\left(\frac{2B\alpha(\alpha+\beta)}{R(1-\nu)} + \frac{2B\alpha(\alpha+\beta)}{R\nu}\right) < \frac{\alpha^2}{2},$$

which gives the result. □

Substituting the result of Lemma 6.5 into (6.22) and using the equality

$$\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} = \|v\|^2_{L^2(S,\mathrm{d}S)} + \|w\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} \tag{6.26}$$

we deduce

$$q_\alpha^-(u,u) - E^\beta \|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} \geq (1-c_0\delta)\int_S g^{\rho\mu}\partial_\rho v\partial_\mu v\mathrm{d}S$$
$$- \psi(0)^2\int_S \Big(\frac{K(s)}{2}+\frac{R}{\alpha}\Big)v(s)^2\mathrm{d}S - \Big(E^\beta+\frac{\alpha^2}{2}\Big)\|w\|_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} - c_1\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)}. \quad (6.27)$$

We choose $\delta=\alpha^{-\kappa}$, $\kappa\in[2/3,1)$, then Lemma 2.2 provides

$$E^\beta = -\alpha^2+o(1), \quad \psi(0)^2 = 2\alpha+o(1).$$

Using again (6.26) and rough estimates, we deduce, as $\alpha\to+\infty$:

$$q_\alpha^-(u,u)+\alpha^2\|u\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)} \geq (1-c_0\delta)\int_\Sigma g^{\rho\mu}\partial_\rho v\partial_\mu v\,\mathrm{d}S$$
$$-\int_\Sigma(\alpha K+2R+1+c_1)v(s)^2\mathrm{d}S+\frac{\alpha^2}{4}\|w\|^2_{L^2(\Sigma,\mathrm{d}S\mathrm{d}t)}, \quad (6.28)$$

Now we may follows closely the arguments of Subsection 5.2. The expression on the right-hand side of (6.28) represents a densely defined quadratic form in $L^2(\Sigma,\mathrm{d}S\mathrm{d}t)$, and we denote by q its closure. The associated self-adjoint operator Q writes as

$$Q=\big(-(1-c_0\delta)\Delta_S-(\alpha K+C)\big)P+\frac{\alpha^2}{4}(1-P),$$

where $C:=2R+1+c_1$ and P is the orthogonal projector, $(Pu)(s)=v(s)\psi(t)$. It follows from (6.26) and (6.28) that $E_j(Q_\alpha^-)+\alpha^2\geq E_j(Q)$, and as in Subsection 5.2 for each fixed $j\in\mathbb{N}$ we have

$$E_j(Q)=E_j\big(-(1-c_0\delta)\Delta_S-(\alpha K+C)\big) \text{ for large } \alpha.$$

Using (6.13) we arrive at $E_j(Q_\alpha^\Omega)\geq-\alpha^2+E_j\big(-(1-c_0\delta)\Delta_S-\alpha K\big)+\mathcal{O}(1)$, and, finally, we get the desired lower bound of Theorem 1.3 by using (6.16).

7. Proof of Corollaries 1.4 and 1.6

In this section, we assume that Ω is C^2-admissible. Then the function K is bounded, and an easy adaptation of [10, Proposition 1] to the non-euclidean setting gives the following:

Lemma 7.1. *For any fixed $j\in\mathbb{N}$ there holds*

$$E_j(-\Delta_S-\alpha K)=-K_{\max}\alpha+o(\alpha), \quad \alpha\to+\infty. \quad (7.1)$$

Proof of Corollary 1.4. It is sufficient to substitute the estimate (7.1) into the asymptotics (1.5) of Theorem 1.2. □

To prove Corollary 1.6 we need a rough estimate for the essential spectrum of Q_α^Ω. Recall that for a self-adjoint operator Q, we have denoted by $E(Q)$ the infimum of its essential spectrum. Then there holds:

Lemma 7.2. *Assume that $\partial\Omega$ is non-compact and denote $K_\infty:=\limsup_{s\to\infty}K(s)$, then $E(Q_\alpha^\Omega)\geq-\alpha^2-K_\infty\alpha+o(\alpha)$ for large α.*

Proof. Lets us modify a bit the construction of Subsection 3.1. By assumption, for any $K_0>K_\infty$ there exists a compact domain $S_0\subset S$ such that $K(s)\leq K_0$ for $s\in S\setminus S_0$. Set $S_1:=S\setminus\overline{S_0}$. Now let q'_α denote the quadratic form given by the same expression as q_α^Ω but acting on the domain $\mathcal{D}(q'_\alpha):=H^1(\Omega_\delta^0)\oplus H^1(\Omega_\delta^1)\oplus H^1(\Theta_\delta)$ with

$$\Omega_\delta^0:=\Phi\big(S_0,(0,\delta)\big), \quad \Omega_\delta^1:=\Phi\big(S_1,(0,\delta)\big), \quad \Theta_\delta:=\Omega\setminus\overline{\Omega_\delta^0\cup\Omega_\delta^1}$$

and δ is sufficiently small. Let Q'_α be the self-adjoint operator associated with q'_α and acting in $L^2(\Omega)$. Due to the form inequality $Q^\Omega_\alpha \geq Q'_\alpha$ we have $E(Q^\Omega_\alpha) \geq E(Q'_\alpha)$. On the other hand, one represents $Q'_\alpha = Q^0_\alpha \oplus Q^1_\alpha \oplus (-\Delta)^N_{\Theta_\delta}$, where Q^j_α, $j \in \{0,1\}$, is the self-adjoint operator in $L^2(\Omega^j_\delta)$ generated by the quadratic form

$$q^j_\alpha(u,u) = \int_{\Omega^j_\delta} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega^j_\delta \cap S} u^2 \mathrm{d}S, \quad \mathcal{D}(q^j_\alpha) = H^1(\Omega^j_\delta),$$

and $(-\Delta)^N_{\Theta_\delta}$ is the Neumann Laplacian in Θ_δ. Note that the domain Ω^0_δ is bounded, hence, the operator Q^0_α has an empty essential spectrum. It follows that

$$E(Q^\Omega_\alpha) \geq E(Q'_\alpha) = \min\big(E(Q^1_\alpha), E((-\Delta)^N_{\Theta_\delta})\big) \geq \min\big(E(Q^1_\alpha), 0\big).$$

On the other hand, the analysis of Section 5.1 can be applied to the operator Q^1_α. In particular, the choice (5.23) for δ gives

$$E_1(Q^1_\alpha) \geq -\alpha^2 + E_1(-\Delta^N_{S_1} - \alpha K) + \mathcal{O}(\log\alpha),$$

where $-\Delta^N_{S^1}$ is the Neumann realization of the positive Laplace-Beltrami operator on S_1. As $K \leq K_0$ in S_1, we have

$$E(Q^1_\alpha) \geq E_1(Q^1_\alpha) \geq -\alpha^2 - K_0\alpha + \mathcal{O}(\log\alpha)$$

and, subsequently, $E(Q^\Omega_\alpha) \geq -\alpha^2 - K_0\alpha + \mathcal{O}(\log\alpha)$. As $K_0 > K_\infty$ is arbitrary, the result follows. □

Proof of Corollary 1.6. Let $N \in \mathbb{N}$ be fixed. Due to Corollary 1.4 and Lemma 7.2 for large α we have $E_N(Q^\Omega_\alpha) < E(Q^\Omega_\alpha)$, and $E_N(Q^\Omega_\alpha)$ is the N^{th} eigenvalue of Q^Ω_α due to the min-max principle. □

8. Analysis of the reduced operator on the boundary

In this section we gather various standard estimates on the low-lying eigenvalues of $-\Delta_S - \alpha K$, depending on the hypotheses on the dimension ν and on K. In this section, for the cas of an unbounded $\partial\Omega$ we assume that the assumption (1.7) holds.

Now note that by setting $h = \alpha^{-1/2}$ and $V = -K$, the operator $-\Delta_S - \alpha K$ writes as

$$-h^{-2}\big(-h^2\Delta_S + V\big)$$

and enters naturally the framework of Schrödinger operators in the semi-classical limit $h \to 0$. The assumption (1.7) writes now

$$\liminf_{s\to\infty} V(s) > \inf_{s\in S} V(s),$$

and ensures that the asymptotics of the low-lying eigenvalues of the reduced operator can be determined by the behavior of V near its minima (that are the maxima of K), under suitable hypotheses.

Remark 8.1. Assume that the measure of the set $K^{-1}(\{K_{\max}\})$ is 0. Then the word-by-word adaptation of [1, Lemma 3.2] to the non-euclidean setting gives, for any fixed $j \in \mathbb{N}$,

$$E_j\big(-\Delta_S + \alpha(K_{\max} - K)\big) \to +\infty \text{ as } \alpha \to +\infty \tag{8.1}$$

If, in addition, Ω is C^3-admissible, then (1.6) can be decomposed as

$$E_j(Q^\Omega_\alpha) = -\alpha^2 - K_{\max}\alpha + E_j\big(-\Delta_S + \alpha(K_{\max} - K)\big) + \mathcal{O}(1),$$

and the term $E_j\big(-\Delta_S + \alpha(K_{\max} - K)\big)$ has a lower order with respect to α, see Lemma 7.1, but is large with respect to the remainder $\mathcal{O}(1)$, see (8.1), and hence provides a refinement with respect to the first order asymptotics (1.1).

The aim is now to describe more precise asymptotics on $-\Delta_S - \alpha K$, in order to see the possible gap between eigenvalues, in particular we want to compare $E_j(-\Delta_S - \alpha K) + K_{\max}\alpha$ to the remainders in Theorems 1.2 and 1.3. The most commonly studied case is when the maxima of K are non-degenerate, see [34, Theorem 5.1] or [19]:

Proposition 8.2. *Assume that Ω is C^5-admissible and, if non-compact, satisfies* (1.7). *Furthermore, assume that the function K admits a unique global maximum at $s_0 \in S$ and that the Hessian of $(-K)$ at s_0 is positive-definite. Denote by μ_k the eigenvalues of the Hessian and*

$$\mathcal{E} := \Big\{ \sum_{k=1}^{\nu-1} \sqrt{\frac{\mu_k}{2}}(2n_k - 1), \quad n_k \in \mathbb{N} \Big\}, \tag{8.2}$$

then for each fixed $j \in \mathbb{N}$ there holds:

$$E_j(-\Delta_S - \alpha K) = -K_{\max}\alpha + e_j \alpha^{1/2} + \mathcal{O}(\alpha^{1/4}) \text{ as } \alpha \to +\infty,$$

where e_j is the j^{th} element of $\mathcal{E}$, counted with multiplicity. Moreover, if Ω is C^6, and if e_j is of multiplicity one, the remainder can be replaced by $O(1)$.

By combining Proposition 8.2 with Theorem 1.2 we obtain Corollary 1.7. Remark that for $\nu = 2$ one is reduced to

$$\mathcal{E} = \Big\{ \sqrt{\frac{-K''(s_0)}{2}}(2n-1), \quad n \geq 1 \Big\}$$

and all the elements are of multiplicity one. Therefore, by combining Theorem 1.2 and Proposition 8.2, we recover the first terms of the asymptotic expansion (1.2), see [16, Theorem 1.1].

Other cases of extrema are harder to handle, due to the different notions of degeneracy for the maxima of K, and to the possible interactions with the metric near the maxima. However, in the case $\nu = 2$, we have the following:

Proposition 8.3. *Let $\nu = 2$ and Ω be C^{2p+3}-admissible with $p \geq 2$ and, if $\partial\Omega$ is non-compact, such that the assumption* (1.7) *is satisfied. Furthermore, assume that the curvature K admits a unique global maximum at s_0 with*

$$K(s) = K(s_0) - C_p(s - s_0)^{2p} + \mathcal{O}\big((s-s_0)^{2p+1}\big), \quad s \to s_0,$$

where $C_p > 0$ and s is an arc-length of the connected component Γ of the boundary at which K takes the maximal value. Then we have the following expansion

$$E_j(-\Delta_S - \alpha K) = -K_{\max}\alpha + e_j \alpha^{\frac{1}{p+1}} + \mathcal{O}(\alpha^{\frac{1}{2(p+1)}}),$$

where e_j is the j^{th} eigenvalue of the operator $-\partial_s^2 + C_p s^{2p}$ in $L^2(\mathbb{R})$. Moreover, if $\partial\Omega$ is C^{2p+4}, then the remainder can be replaced by $\mathcal{O}(1)$.

Proof. Since we are not interested in exponentially small terms, it suffices by standard arguments to reduce the analysis to a neighborhood of the minimizer in Γ, denoted by Γ_0, with Dirichlet boundary conditions at the ends, see [19]. Let $\gamma : \mathbb{R}/|\Gamma_0|\mathbb{Z} \to \Gamma_0$ be an arc-length parametrization of Γ_0. Since the parametrization is normalized and the metrics in local coordinates is $g = \|\gamma'\|$, we only have to consider $-\Delta - \alpha K$ on the interval $(s_0 - \eta, s_0 + \eta)$, with $\eta > 0$ fixed, and Dirichlet boundary condition. The following asymptotics is then a simple consequence of [29, Theorem 2.1] applied with the semi-classical parameter $h = \alpha^{-1/2}$:

$$E_j(-\Delta_S - \alpha K) = \alpha^{\frac{1}{p+1}} \left(e_j + \sum_{k \geq 1} \beta_{j,k} \alpha^{-\frac{k}{2(p+1)}} \right), \quad \beta_{j,k} \in \mathbb{R}.$$

If $\partial\Omega$ is C^{2p+4}, then the curvature is C^{2p+2}, and we have the Taylor expansion

$$K(s) = K(s_0) - C_p(s-s_0)^{2p} + C'_p(s-s_0)^{2p+1} + \mathcal{O}\Big((s-s_0\Big)^{2p+2}), \quad C'_p \in \mathbb{R}.$$

Then, by combining the simplicity of the eigenvalues $(e_j)_{j\geq 1}$, the parity of the eigenvectors of $-\partial_s^2 + C_p s^{2p}$, and the oddness of the remainder $C'_p(s-s_0)^{2p+1}$ in the asymptotic expansion of K, it is standard to show that $\beta_{j,2} = 0$ for all $j \geq 1$, see for example [7, Theorem 4.23] for the case $p = 1$. □

The combination of Proposition 8.3 with Theorem 1.2 gives Corollary 1.8.

Remark 8.4. The above statements can be adapted easily to the case where K has several maxima by using the principle that "each well creates its own series of eigenvalues".

Corollary 8.5. *Let $j \in \mathbb{N}$, and assume* one *of the two following:*

- *The hypotheses of Proposition 8.2 hold, and e_j is of multiplicity 1 in the set $\mathcal{E}$.*
- *The hypotheses of Proposition 8.3 hold.*

Then, for α large enough, $E_j(Q_\alpha^\Omega)$ is a simple eigenvalue.

When we are not in the hypotheses of Remark 8.1, few results exist on the asymptotics of the first eigenvalues. For example, we can show

Proposition 8.6. *Assume that the interior of $K^{-1}(\{K_{\max}\})$ is not empty. Then, for any fixed $j \in \mathbb{N}$,*

$$E_j(-\Delta_S - \alpha K) = -\alpha K_{\max} + \mathcal{O}(1) \text{ as } \alpha \to +\infty.$$

Proof. Denote by $\omega \subset \partial\Omega$ an open subset of the interior of $K^{-1}\{K_{\max}\}$ with a smooth boundary. Introduce $-\Delta_\omega^D$, the Laplace-Beltrami operator in ω with the Dirichlet boundary condition. This operator has compact resolvent and we denote by $E_j(-\Delta_\omega^D)$, $j \in \mathbb{N}$, its eigenvalues, and by u_j associated normalized eigenfunctions. Denote by U_j the extensions of u_j to $\partial\Omega$ by zero, then

$$\int_S g^{\rho\mu}\partial_\rho U_j \partial_\mu U_j \mathrm{d}S - \alpha\int_S K U_j^2 \mathrm{d}S = -\alpha K_{\max} + E_j(-\Delta_\omega^D).$$

As U_j are mutually orthogonal in $L^2(S, \mathrm{d}S)$, we deduce from the min-max principle that $E_j(-\Delta_S - \alpha K) \leq -\alpha K_{\max} + E_j(-\Delta_\omega^D)$, and the sought estimate follows. □

In particular, in the situation of Proposition 8.6 Theorems 1.2 and (1.3) does not provide the gap between the eigenvalues of Q_α^Ω as $\alpha \to +\infty$.

We remark that a particular case of a piecewise constant curvature was recently studied in [32], and the eigenvalue gaps appear to have finite limits.

9. Periodic case

The preceding analysis can also be applied to periodic problems. Namely, assume that there exist linearly independent vectors $a_1, \ldots, a_m$, $m \leq \nu$, such that Ω is invariant under the shifts $x \mapsto x + a_j$, $j \in \{1, \ldots, m\}$, and that the quotient (elementary cell) $\omega := \Omega/(\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_m)$ is compact, and then the quotient surface $\sigma := S/(\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_m)$ is also compact. Such a situation is covered by the Floquet theory [23]. Namely, for $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ denote by $Q_\alpha^\Omega(\theta)$ the self-adjoint operator generated by the quadratic form

$$q_\alpha^{\Omega,\theta}(u,u) := \int_\omega |\nabla u|^2 \mathrm{d}x - \alpha\int_\sigma u^2 \mathrm{d}S,$$

$$\mathcal{D}(q_\alpha^{\Omega,\theta}) = H^1_\theta(\omega) := \Big\{u \in H^1_{\mathrm{loc}}(\Omega) : u(\cdot + a_j) = e^{i\theta_j}u(\cdot), \quad j = 1, \ldots, m\Big\}.$$

It can be easily checked that the operators $Q^\Omega(\theta)$ are with compact resolvents, and it is a standard fact of the Floquet theory that for each fixed $j \in \mathbb{N}$ the so-called band function

$$\mathbb{T}^m \ni \theta \mapsto E_j(\theta, \alpha) := E_j\big(Q^\Omega_\alpha(\theta)\big)$$

is continuous, and that

$$\operatorname{spec} Q^\Omega_\alpha := \bigcup_{j\in\mathbb{N}} B_j(\alpha), \quad B_j(\alpha) := \big\{E_j(\theta,\alpha) : \theta \in \mathbb{T}^m\big\}.$$

The segment $B_j(\alpha)$ is usually called the j^{th} spectral band of Q^Ω_α.

An analogous representation of the spectrum holds for the reduced operator $-\Delta_S - \alpha K$. Namely, denote by $T_\alpha(\theta)$ the self-adjoint operator acting in $L^2(\sigma)$ associated with the quadratic form

$$t^\theta_\alpha(v,v) := \int_\sigma g^{\rho\mu}\partial_\rho v \partial_\mu v \,\mathrm{d}S - \alpha\int_\sigma Kv^2 \mathrm{d}S,$$
$$\mathcal{D}(t^\theta_\alpha) = H^1_\theta(\sigma) := \Big\{u \in H^1_{\mathrm{loc}}(S) : u(\cdot + a_j) = e^{i\theta_j}u(\cdot), \quad j = 1,\dots,m\Big\}.$$

Again, one checks that $T_\alpha(\theta)$ have compact resolvents and the band functions

$$\mathbb{T}^m \ni \theta \mapsto \varepsilon_j(\theta,\alpha) := E_j\big(T_\alpha(\theta)\big)$$

are continuous and

$$\operatorname{spec}(-\Delta_S - \alpha K) := \bigcup_{j\in\mathbb{N}} \beta_j(\alpha), \quad \beta_j(\alpha) := \big\{\varepsilon_j(\theta,\alpha) : \theta \in \mathbb{T}^m\big\},$$

and the segment $\beta_j(\alpha)$ will be called the j^{th} spectral band of $-\Delta_S - \alpha K$.

One can easily see that the proofs of Theorems 1.2 and 1.3 also work for the operators $Q^\Omega_\alpha(\theta)$, which gives the following results:

Theorem 9.1. *For any fixed $j \in \mathbb{N}$ there holds, as $\alpha \to +\infty$,*

$$E_j(\theta,\alpha) = -\alpha^2 + \varepsilon_j(\theta,\alpha) + R_j(\theta,\alpha), \quad \theta \in \mathbb{T},$$

where $R_j(\theta,\alpha) = \mathcal{O}(\log\alpha)$ if Ω is C^2 and $R_j(\theta,\alpha) = \mathcal{O}(1)$ if Ω is C^3, and the remainder estimate is uniform in $\theta \in \mathbb{T}$.

Corollary 9.2. *If $j \in \mathbb{N}$ is fixed and $\alpha \to +\infty$, then the j^{th} spectral band of $Q^\Omega_\alpha + \alpha^2$ and the j^{th} spectral band of $-\Delta_S - \alpha K$ are located in a $\mathcal{O}\big(R(\alpha)\big)$-neighborhood of each other, where $R(\alpha) = \log\alpha$ for the C^2-admissible case and $R(\alpha) = 1$ for the C^3-admissible one.*

The result of Theorem 9.1 can be used to study some spectral properties specific for periodic operators. Recall that a non-empty interval $(a,b) \subset \mathbb{R}$ is called a (spectral) gap of a self-adjoint operator A if $(a,b) \cap \operatorname{spec} A = \emptyset$ but $a, b \in \operatorname{spec} A$. The existence of spectral gaps is one of the principal questions in the spectral theory of periodic operators, cf. [2, 4, 20, 24]. In view of Theorem 9.1, the existence of sufficiently large gaps for the reduced operator $-\Delta - \alpha K$ (i.e. having the length of order α^κ with some $\kappa > 0$) implies the existence of gaps for the Robin Laplacian Q^Ω_α, and the reduced operator was studied in numerous preceding works, cf. [30, 35]. For, example the semiclassical analysis of periodic operators of the form $-h\Delta_S + V/h$ carried out in [35, Theorem 1.1] gives the following result:

Corollary 9.3. *Assume that Ω is C^∞ and periodic as described above. Furthermore, assume that the function $\sigma \ni s \mapsto K(s)$ admits a unique maximum at s_0, and that the Hessian of $(-K)$ at s_0 is positive-definite. Let μ_j be the eigenvalues of the Hessian and the numbers e_j be defined as in Proposition 8.2, then*

(1) *for each* $j \in \mathbb{N}$ *there exists* $C > 0$ *such that*

$$\operatorname{spec}(Q_\alpha^\Omega + \alpha^2 + K_{\max}\alpha) \cap \left[e_j\alpha^{1/2} - C\alpha^{2/5}, e_j\alpha^{1/2} + C\alpha^{2/5}\right] \neq \emptyset$$

for large α*, and*

(2) *for each* $C_1 > 0$ *there exist* $C_2, C_3 > 0$ *such that*

$$\operatorname{spec}(Q_\alpha^\Omega + \alpha^2 + K_{\max}\alpha) \cap \left[-C_1\alpha^{1/2}, C_1\alpha^{1/2}\right] \subset \bigcup_{e_j \leq C_3} \left[e_j\alpha^{1/2} - C_2\alpha^{2/5}, e_j\alpha^{1/2} + C_2\alpha^{2/5}\right]$$

as $\alpha \to +\infty$*.*

In particular, for any $N \in \mathbb{N}$ *there exists* $\alpha_N > 0$ *such that the operator* Q_α^Ω *has at least* N *gaps for* $\alpha > \alpha_N$*.*

The localization of the spectrum given in the preceding corollary is not expected to be optimal for periodic domains. Furthermore, it would be interesting to understand some questions related to the location of the extrema of the band functions, cf. [2]. We hope to analyze the periodic case in greater detail in subsequent works.

References

[1] Y. Belaud, B. Helffer, L. Véron: *Long-time vanishing properties of solutions of some semilinear parabolic equations.* Ann. Inst. Henri Poincaré Anal. nonlinear **18**:1 (2001) 43–68.

[2] D. Borisov, K. Pankrashkin: *Quantum waveguides with small periodic perturbations: gaps and edges of Brillouin zones.* J. Phys. A **46** (2013) 235203.

[3] G. Carron, P. Exner, D. Krejčiřík: *Topologically nontrivial quantum layers.* J. Math. Phys. **45**:2 (2004) 774–784.

[4] V. Chiadò Piat, S. A. Nazarov, K. Ruotsalainen: *Spectral gaps for water waves above a corrugated bottom.* Proc. Roy. Soc. A **469** (2013) 20120545.

[5] D. Daners, J. B Kennedy: *On the asymptotic behaviour of the eigenvalues of a Robin problem.* Differ. Integr. Eq. **23**:7/8 (2010) 659–669.

[6] E. B. Davies: *Spectral theory and differential operators.* Cambridge University Press, 1995.

[7] M. Dimassi, J. Sjöstrand: *Spectral asymptotics in the semi-classical limit*, Cambridge University Press, 1999.

[8] P. Exner, A. Minakov: *Curvature-induced bound states in Robin waveguides and their asymptotical properties.* J. Math. Phys. **55** (2014) 122101.

[9] P. Exner, A. Minakov, L. Parnovski: *Asymptotic eigenvalue estimates for a Robin problem with a large parameter.* Portugal. Math. **71**:2 (2014) 141–156.

[10] P. Freitas, D. Krejčiřík: *Instability results for the damped wave equation in unbounded domains.* J. Differential Eq. **211**:1 (2005) 168–186.

[11] P. Freitas, D. Krejčiřík: *The first Robin eigenvalue with negative boundary parameter.* Adv. Math. (to appear), preprint arXiv:1403.6666 (2014)

[12] L. Friedlander, M. Solomyak: *On the spectrum of the Dirichlet Laplacian in a narrow infinite strip.* in T. Suslina, D. Yafaev (Eds.): *Spectral theory of differential operators: M. Sh. Birman 80th anniversary collection.* (Amer. Math. Soc. Transl. Ser. 2, Vol. 225 of Advances in the Mathematical Sciences, Amer. Math. Soc., 2008) 103–116.

[13] L. Friedlander, M. Solomyak: *On the spectrum of the Dirichlet Laplacian in a narrow strip.* Israel J. Math. **170** (2009) 337–354.

[14] T. Giorgi, R. Smits: *Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity.* Z. Angew. Math. Phys. **58**:2 (2007) 224–245.

[15] B. Helffer: *Eigenvalues for the Robin Laplacian in domains with variable curvature: a semi-classical apporach.* Extended abstracts of the mini-workshop "Eigenvalue problems in surface superconductivity", Oberwolfach Reports **11**:4 (2014), to appear.

[16] B. Helffer, A. Kachmar: *Eigenvalues for the Robin Laplacian in domains with variable curvature.* Arch. Rat. Mech. Anal. (to appear), preprint arXiv:1411.2700 (2014).

[17] B. Helffer, A. Morame: *Magnetic bottles in connection with superconductivity.* J. Funct. Anal. **185**:2 (2001) 604–680.

[18] B. Helffer, K. Pankrashkin: *Tunneling between corners for Robin Laplacians.* J. London Math. Soc. **91** (2015) 225–248.
[19] B. Helffer, J. Sjöstrand: *Multiple wells in the semi-classical limit I.* Commun. PDE **9**:4 (1984) 337–408.
[20] R. Hempel, O. Post: *Spectral gaps for periodic elliptic operators with high contrast: an overview.* In H. G. W. Begehr, R. P. Gilbert, M. W. Wong (Eds.): *Progress in Analysis. Proceedings of the 3rd International ISAAC Congress Berlin 2001. Vol. 1* (World Scientific Publishing Co., Inc., River Edge, NJ, 2003) 577–587.
[21] D. Krejčiřík: *Spectrum of the Laplacian in a narrow curved strip with combined Dirichlet and Neumann boundary conditions* ESAIM: Control, Optimisation and Calculus of Variations **15** (2009) 555-568.
[22] D. Krejčiřík: *Spectrum of the Laplacian in narrow tubular neighbourhoods of hypersurfaces with combined Dirichlet and Neumann boundary conditions.* Math. Bohem. **139** (2014) 185–193.
[23] P. Kuchment: *Floquet Theory For Partial Differential Equations* (Vol. 60 of Operator Theory: Advances and Applications, Birkhäuser, 1993).
[24] A. Khrabustovskyi: *Opening up and control of spectral gaps of the Laplacian in periodic domains.* J. Math. Phys. **55** (2014) 121502.
[25] P. Kuchment: *Quantum graphs I. Some basic structures.* Waves Random Media **14** (2004) S107–S128.
[26] A. A. Lacey, J. R. Ockendon, J. Sabina: *Multidimensional reaction diffusion equations with nonlinear boundary conditions.* SIAM J. Appl. Math. **58** (1998) 1622–1647.
[27] M. Levitin, L. Parnovski: *On the principal eigenvalue of a Robin problem with a large parameter.* Math. Nachr. **281** (2008) 272–281.
[28] Y. Lou, M. Zhu: *A singularly perturbed linear eigenvalue problem in C^1 domains.* Pacific J. Math. **214**:2 (2004) 323–334.
[29] A. Martinez, M. Rouleux: *Effet tunnel entre puits dégénérés.* Commun. PDE **13**:9 (1988) 1157–1187.
[30] A. Outassourt: *Comportement semiclassique pour l'opérateur de Schrödinger à potentiel périodique.* J. Funct. Anal. **72**:1 (1987) 65–93.
[31] K. Pankrashkin: *On the asymptotics of the principal eigenvalue for a Robin problem with a large parameter in planar domains.* Nanosyst. Phys. Chem. Math. **4**:4 (2013) 474–483.
[32] K. Pankrashkin: *On the Robin eigenvalues of the Laplacian in the exterior of a convex polygon.* Nanosyst. Phys. Chem. Math. **6** (2015) 46–56.
[33] K. Pankrashkin, N. Popoff: *Mean curvature bounds and eigenvalues of Robin Laplacians.* Calc. Var. PDE. (to appear), preprint arXiv:1407.3087 (2014).
[34] B. Simon: *Semiclassical analysis of low lying eigenvalues. I. Non-degenerate minima: Asymptotic expansions.* Ann. Inst. H. Poincaré A **38**:3 (1983) 295-308.
[35] M. A. Shubin: *Semiclassical asymptotics on covering manifolds and Morse inequalities.* Geom. Funct. Anal. **6**:2 (1996) 370–409.

Laboratoire de mathématiques (UMR 8628 du CNRS), Université Paris-Sud, Bâtiment 425, 91405 Orsay Cedex, France
E-mail address: konstantin.pankrashkin@math.u-psud.fr
URL: http://www.math.u-psud.fr/~pankrash/

Institut mathématique de Bordeaux, Université Bordeaux 1, 351 cours de la libération, 33405 Talence Cedex, France
E-mail address: nicolas.popoff@math.u-bordeaux1.fr
URL: http://www.math.u-bordeaux1.fr/~npopoff/