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# Uniform approximation and explicit estimates for the prolate spheroidal wave functions.

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**Abstract**— For fixed  $c$ , Prolate Spheroidal Wave Functions (PSWFs), denoted by  $\psi_{n,c}$ , form an orthogonal basis with remarkable properties for the space of band-limited functions with bandwidth  $c$ . They have been largely studied and used after the seminal work of D. Slepian and his co-authors. In several applications, uniform estimates of the  $\psi_{n,c}$  in  $n$  and  $c$ , are needed. To progress in this direction, we push forward the uniform approximation error bounds and give an explicit approximation of their values at 1 in terms of the Legendre complete elliptic integral of the first kind. Also, we give an explicit formula for the accurate approximation the eigenvalues of the Sturm-Liouville operator associated with the PSWFs.

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## 1 Introduction

Prolate spheroidal wave functions (PSWFs) have been introduced in the sixties by D. Slepian, H. Landau and H. Pollak [15, 16, 28, 29, 30] as a fundamental tool in signal processing. One can now refer to [13] for their properties, starting from the seminal work of Slepian, Landau and Pollak. For a fixed value  $c > 0$ , called the bandwidth, PSWFs constitute an orthonormal basis of  $L^2([-1, +1])$ , an orthogonal system of  $L^2(\mathbf{R})$  and an orthogonal basis of the Paley-Wiener space  $B_c$ , given by  $B_c = \left\{ f \in L^2(\mathbf{R}), \text{Support } \widehat{f} \subset [-c, c] \right\}$ . Here,  $\widehat{f}$  denotes the Fourier transform of  $f$ . One possible definition is given by the fact that they are eigenfunctions of the compact integral operators  $\mathcal{F}_c$  and  $\mathcal{Q}_c = \frac{2c}{\pi} \mathcal{F}_c^* \mathcal{F}_c$ , defined on  $L^2([-1, 1])$  by

$$\mathcal{F}_c(f)(x) = \int_{-1}^1 e^{icxy} f(y) dy, \quad \mathcal{Q}_c(f)(x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} f(y) dy. \quad (1)$$

On the other hand, Slepian and Pollack have pointed out, and reported this property as "a lucky incident", that the operator  $\mathcal{Q}_c$  commutes with a Sturm-Liouville operator  $\mathcal{L}_c$ , which is first defined on  $C^2([-1, 1])$  by

$$\mathcal{L}_c(\psi) = -\frac{d}{dx} \left[ (1-x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi. \quad (2)$$

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So PSWFs  $(\psi_{n,c})_{n \geq 0}$  are also eigenfunctions of  $\mathcal{L}_c$ . They are ordered in such a way that the corresponding eigenvalues of  $\mathcal{L}_c$ , called  $\chi_n(c)$ , are strictly increasing. Functions  $\psi_{n,c}$  are restrictions to the interval  $[-1, +1]$  of real analytic functions on the whole real line and eigenvalues  $\chi_n(c)$  are the values of  $\lambda$  for which the equation  $\mathcal{L}_c \psi = \lambda \psi$  has a bounded solution on the whole interval.

We will use all along this paper the fact that the PSWFs are eigenfunctions of  $\mathcal{L}_c$ . Remark that the study of PSWFs as eigenfunctions of the above Sturm-Liouville problem has started a long time ago. To the best of our knowledge, C. Niven was the first, in 1880, to give a remarkably detailed theoretical, as well as computational studies of the eigenfunctions and the eigenvalues of  $\mathcal{L}_c$ , see [24]. Nowadays work on PSWFs is mainly connected with possible applications in signal processing [6, 12, 17, 18] and other scientific issues. In geophysics, for instance, they provide good approximations of the Rossby waves that constitute the planetary scale waves in the atmosphere and ocean, see [7, 20, 21, 22, 25].

One main issue concerns numerical computations of the PSWFs and related quantities, [5, 14, 32, 33]. This is a necessary step before using the bases they constitute. For  $c = 0$ , PSWFs reduce to Legendre polynomials, which are extensively used to expand functions on  $[-1, +1]$ . Nevertheless numerical evidence shows that in many cases PSWFs may be more adapted, beyond band limited signals. Accurate estimates are then needed to fix which bandwidth  $c$ , and so which specific basis, will be used to decompose the signal.

Approximation of PSWFs in terms of the Bessel function  $J_0$  (or other special functions when considering different generalizations of the PSWFs) has been developed by many authors (see for instance [8]). It is based on WKB approximation of the PSWF as eigenfunctions of the Sturm-Liouville problem that we described above. The case of (normalized) Legendre polynomials has been studied for a long time and may be summarized in the formula

$$\bar{P}_n(\cos \theta) \approx (n + 1/2)^{1/2} \left( \frac{\theta}{\sin \theta} \right)^{1/2} J_0((n + 1/2)\theta),$$

for which very precise bounds of the approximation error are known (see for instance [10]). Such formulas are sometimes called Hilb's formulas. When Legendre polynomials are replaced by  $\psi_{n,c}$ , the quantity  $n + 1/2$  is partly replaced by  $\sqrt{\chi_n(c)}$  or partly replaced by  $\psi_{n,c}(1)^2$ , since there is no simple relation between both. The change of variable linked to  $\cos \theta$  is replaced by an expression that depends on  $q = c^2/\chi_n(c)$  and involves Legendre elliptic integrals. In this work, we mainly restrict ourselves to the values of  $n, c$  such that  $q = c^2/\chi_n(c) < 1$ . Condition  $q \leq 1$  guarantees that  $\psi_{n,c}$  oscillates on the whole interval  $(-1, +1)$ , like Legendre polynomials, and gets its largest value at 1.

We briefly describe some features of our study. In a first step the equation is transformed into its Liouville normal form, so that one can make use of Olver's theorems to have precise error bounds when approximating the function  $\psi_{n,c}$  by a quantity that involves  $\chi_n(c)$  and  $\psi_{n,c}(1)$ . It should be emphasized that this formula does not appear here for the first time. Moreover the fact that one has uniform estimates when  $q$  stays far from 1 may be found in [8]. But we push forward estimates, to allow  $q$  to tend to 1 and have an explicit error of order  $[(1 - q)\sqrt{\chi_n(c)}]^{-1}$ .

Our second step consists in having an accurate estimate for  $\psi_{n,c}(1)$ , which appears as a coefficient to be fixed under the condition that the function  $\psi_{n,c}$  has a unit  $L^2([-1, 1])$ -norm. We find the explicit approximate formula,

$$\psi_{n,c}(1) \approx \chi_n(c)^{1/4} \sqrt{\frac{\pi}{2\mathbf{K}(\sqrt{q})}}$$

where  $\mathbf{K}$  is the complete Legendre elliptic integral. The relative error estimate is found of the same order  $[(1 - q)\sqrt{\chi_n(c)}]^{-1}$ .

It should be mentioned that the above estimate of  $\psi_{n,c}(1)$  plays a central role in our further study of the sharp decay of the spectrum of the operator  $\mathcal{Q}_c$ , see [4].

It turns out that our approximate expressions depend only on the values of the eigenvalues  $\chi_n(c)$ . The explicit approximation of the  $\chi_n(c)$  is also one of our concerns. There exist accurate numerical methods for computing the  $\chi_n(c)$ , but theoretical studies have their own interest. For instance the condition  $q < 1$  has been discussed by Osipov in [27] in view of replacing it by a condition that involves only  $n$  and  $c$ , not  $\chi_n(c)$ . He proved that  $q < 1$  when  $c < \frac{\pi n}{2}$ , while  $q > 1$  when  $c > \frac{\pi(n+1)}{2}$ . His study extends largely this particular comparison of  $q$  with 1. We further extend this study and state the final result in a more friendly way: we prove that there exists an explicit function  $\Phi$ , which may be written in terms of elliptic integrals, such that

$$\Phi\left(\frac{2c}{\pi(n+1)}\right) < \sqrt{q} < \Phi\left(\frac{2c}{\pi n}\right). \quad (3)$$

Moreover, we prove that, for  $q < 1$ ,

$$\sqrt{q} \approx \Phi\left(\frac{2c}{\pi(n+1/2)}\right)$$

with an error estimate of  $O(1/n^2)$ . Formula (3) is also used to interpret conditions of the type  $(1-q)\sqrt{\chi_n(c)} > \kappa$ , which appear everywhere, in terms of  $n$  and  $c$ . Roughly speaking, this is comparable with  $n - \frac{2c}{\pi} > \kappa' \log c$ . This condition is reminiscent of the description by Landau and Widom [16] of the decay of the eigenvalues of  $\mathcal{Q}_c$ : in the above range of  $n$ , the decay is super-exponential.

Let us emphasize the fact that we make a special effort towards numerical constants, in order to show that most of them remain small. Numerical experiments will be presented elsewhere.

This work is organized as follows. Section 2 is centred on the eigenvalues  $\chi_n(c)$  and formula (3). Section 3 is the main section, with accurate uniform estimates of the PSWFs for  $q < 1$ . They are given in terms of  $\chi_n(c)$  and  $\psi_{n,c}(1)$  at first, then only in terms of  $\chi_n(c)$  later on. As a consequence we give an improved error bound for the approximation of  $\chi_n(c)$  and hence of the quantity  $q$ . In Section 4, we first extend the previous techniques to get also uniform estimates of  $\psi_{n,c}$  when  $q = 1$ . Then the same kind of estimates is used to approximate  $\psi_{n,c}$  by the Legendre polynomial  $P_n$ .

Without loss of generality we assume everywhere that  $\psi_{n,c}(1) > 0$ . To alleviate notation, we systematically replace  $\psi_{n,c}$  by  $\psi_n$  and  $\chi_n(c)$  by  $\chi_n$ , the parameter  $c$  being implicit.

## 2 Bounds and estimates of the eigenvalues $\chi_n$ .

Inequalities (3), which complete previous work of Osipov, are based on properties of the equation satisfied by the PSWFs. We start by a study of this equation. For simplification, we skip the parameter  $c$  and note  $\chi_n, \psi_n$ . We also skip the parameter  $n$  for  $q = q_n(c) = c^2/\chi_n(c)$ . The equation satisfied by  $\psi_n$  is then given by

$$\frac{d}{dx} [(1-x^2)\psi'_n(x)] + \chi_n(1-qx^2)\psi_n(x) = 0, \quad x \in [-1, 1]. \quad (4)$$

Recall that the function  $\psi_n$  is smooth up to the boundary. Its  $L^2$  norm on  $[-1, 1]$  is equal to 1 and  $\psi_n(1) > 0$ .

Because of the parity of the PSWFs ( $\psi_n$  has the same parity as  $n$ ), we can restrict to the interval  $[0, 1]$ , which we do now. We then use the Liouville transformation, which transforms the equation (4) into its Liouville normal form

$$U'' + (\chi_n + \theta)U = 0. \quad (5)$$

In view of this, we first let

$$S(x) = S_{\sqrt{q}}(x) = \int_x^{\min(1, \frac{1}{\sqrt{q}})} \sqrt{\frac{1-qt^2}{1-t^2}} dt. \quad (6)$$

The function  $S$  can be written as

$$S(x) = \mathbf{E}(\sqrt{q}) - \mathbf{E}(x, \sqrt{q}),$$

where

$$\mathbf{E}(k) = \int_0^{\min(1, \frac{1}{k})} \sqrt{\frac{1-k^2t^2}{1-t^2}} dt, \quad \mathbf{E}(x, k) = \int_0^x \sqrt{\frac{1-k^2t^2}{1-t^2}} dt.$$

When  $0 \leq k \leq 1$ , we recognize from the previous equalities, the complete and incomplete elliptic integral of the second kind, respectively. Note that  $S(\cdot)$  defines a homeomorphism on the whole interval  $[0, \min(1, \frac{1}{\sqrt{q}})]$ . Liouville Transformation consists in looking for  $\psi$  under the form

$$\psi_n(x) = \varphi(x)U(S(x)), \quad \varphi(x) = (1-x^2)^{-1/4}(1-qx^2)^{-1/4}.$$

The equation satisfied by  $U$  may be written as in (5) with

$$\theta(S(x)) = \varphi(x)^{-1}(1-qx^2)^{-1} \frac{d}{dx} [(1-x^2)\varphi'(x)].$$

We have

$$\varphi'/\varphi = -\frac{1}{4}Q'/Q, \quad Q(x) = (1-x^2)(1-qx^2).$$

It follows that  $\theta \circ S$  is a rational function with poles at  $\pm 1$  and  $\pm \sqrt{\frac{1}{q}}$ , which may be written

$$\theta \circ S = \frac{1}{16}(1-qx^2)^{-1} \left[ (1-x^2) \left( \frac{Q'}{Q} \right)^2 - 4 \frac{d}{dx} \left( (1-x^2) \frac{Q'}{Q} \right) \right].$$

By computing the different derivatives appearing in the previous expression, then by writing the numerator as a polynomial in  $1-x^2$ , one can easily check that

$$\theta(S(x)) = \frac{(1-q)^2}{4(1-x^2)(1-qx^2)^3} + \frac{(1-q)^2 + 2q(3-q)(1-x^2)}{4(1-qx^2)^3}. \quad (7)$$

The following proposition shows the monotonicity of  $\theta \circ S$  for any  $q > 0$ .

**Proposition 1.** *For  $q > 0$ , the function  $\theta \circ S$  is increasing on  $[0, \min(1, 1/\sqrt{q})]$ .*

*Proof.* We recall the expression of  $\theta \circ S$  given in (7). We use the notation  $u = 1-x^2$ . Straightforward computations show that

$$\theta(S(x))' = 2x \frac{G(u)}{4u^2(1-q+qu)^4}, \quad G(u) = (1-q)^2(1-q+4qu+3qu^2) + 2q(3-q)(2qu-1+q)u^2.$$

So let us prove that  $G(u)$  is non negative when  $\max(0, 1-(1/q)) < u \leq 1$ . If  $1 \leq q \leq 3$ , we deduce from the inequality  $1-q+qu \geq 0$  that both terms are non negative. Assuming now that  $q < 1$ , by computing the minimum of  $u(2qu-1+q)$  we get the inequality  $8qu(2qu-1+q) \geq -(1-q)^2$ . Substituting the right hand side of the previous inequality in the second term of  $G$ , one gets

$$\begin{aligned} G(u) &\geq \frac{(1-q)^2}{4} (4-4q+16qu+12qu^2 - (3-q)u) \\ &\geq \frac{(1-q)^2}{4} (4-4q+u(17q-3)) \geq 0, \quad \forall 0 < q < 1, \quad 0 \leq u \leq 1. \end{aligned}$$

Finally, assuming that  $q > 3$ , direct computations show that

$$\begin{aligned} G(u) &\geq (1-q)^2(3q-3+3qu^2) - 2q(q-3)(2qu+q-1)u^2 \\ &\geq 3(q-1)^3 + (3q(1-q)^2 - 2q(q-3)(3q-1))u^2 \\ &\geq (6(q-1)^3 - 2q(q-3)(3q-1))u^2 \end{aligned}$$

which is positive.  $\square$

Let us go back to the eigenvalues  $\chi_n$ . They satisfy the classical inequalities (the left hand side has been slightly improved in [3] but we do not use this)

$$n(n+1) \leq \chi_n \leq n(n+1) + c^2.$$

On the other hand, it has been shown in [Theorem 13, [27]], that if  $n \geq 2$  and  $c^2/\chi_n < 1$ , then

$$\chi_n < \left(\frac{\pi}{2}(n+1)\right)^2.$$

As an application of the previous proposition we give new inequalities for  $\chi_n$ , which improve or complete the above bounds, and are valid for  $q \leq 1$  as well as for  $q > 1$ . We will use the fact that  $\psi_n$  has exactly  $n$  zeros in  $(-1, +1)$ . Instead of the change of variables  $S$ , we define  $\tilde{S}$  on  $(-\min(1, 1/\sqrt{q}), \min(1, 1/\sqrt{q}))$  by

$$\tilde{S}(x) = \begin{cases} \mathbf{E}(x, \sqrt{q}) & \text{for } x \geq 0 \\ -\mathbf{E}(-x, \sqrt{q}) & \text{for } x \leq 0 \end{cases}.$$

It is easily seen that the function  $\tilde{U}$ , which is such that  $\psi_n(x) = \varphi(x)\tilde{U}(\tilde{S}(x))$ , satisfies the equation

$$Y'' + \left(\chi_n + \tilde{\theta}\right)Y = 0, \tag{8}$$

with  $\tilde{\theta}$  an even function such that for  $x > 0$ , we have  $\tilde{\theta}(\tilde{S}(x)) = \theta(S(x))$ . We know from Proposition 1 that  $\tilde{\theta} \geq \chi_n + \frac{q+1}{2}$  on the interval  $(-\mathbf{E}(\sqrt{q}), +\mathbf{E}(\sqrt{q}))$ . We use Sturm comparison theorem between the equation (8) and the equation  $Y'' + \left(\chi_n + \frac{q+1}{2}\right)Y = 0$ . This allows us to say that the distance between two consecutive zeros of the equation (8) is smaller than  $\frac{\pi}{\left(\chi_n + \frac{q+1}{2}\right)^{\frac{1}{2}}}$ . On the

other hand, we know that  $\tilde{U}$ , whose zeros correspond to the ones of  $\psi_n$ , has exactly  $n$  zeros in  $(-\min(1, 1/\sqrt{q}), \min(1, 1/\sqrt{q}))$  (see [27] for  $q > 1$ ). As a consequence, we find that

$$\frac{2}{\pi}\mathbf{E}(\sqrt{q})\sqrt{\chi_n + \frac{q+1}{2}} \leq n+1. \tag{9}$$

Let  $\Phi$  denotes the inverse function of the function  $k \mapsto \frac{k}{\mathbf{E}(k)}$ , so that  $\Phi\left(\frac{k}{\mathbf{E}(k)}\right) = k$ . It is an increasing function that vanishes at 0 and takes the value 1 at 1. Then we have the following theorem, which gives a double inequality for  $\sqrt{q}$  and implies also a double inequality for  $\chi_n$ .

**Theorem 1.** *For all  $c > 0$  and  $n \geq 2$  we have*

$$\Phi\left(\frac{2c}{\pi(n+1)}\right) < \frac{c}{\sqrt{\chi_n}} < \Phi\left(\frac{2c}{\pi n}\right), \tag{10}$$

where  $\Phi$  is the inverse of the function  $k \mapsto \frac{k}{\mathbf{E}(k)}$ .

*Proof.* The left hand side comes directly from (9) and the monotonicity of  $\Phi$ , while the right hand side is a consequence of Proposition 3 in [27].  $\square$

We could as well have written a double inequality for  $\sqrt{\chi_n}$ ,

$$\frac{c}{\Phi\left(\frac{2c}{\pi n}\right)} < \sqrt{\chi_n} < \frac{c}{\Phi\left(\frac{2c}{\pi(n+1)}\right)}, \quad c > 0, \quad n \geq 2.$$

It may be rewritten as

$$c \tilde{\Phi}\left(\frac{\pi n}{2c}\right) < \sqrt{\chi_n} < c \tilde{\Phi}\left(\frac{\pi(n+1)}{2c}\right),$$

with  $\tilde{\Phi}(k) = [\Phi(\frac{1}{k})]^{-1}$ . This function is the inverse of the function  $k \mapsto k\mathbf{E}(\frac{1}{k})$  whose derivative, for any real  $k > 1$ , is given by

$$\mathbf{E}\left(\frac{1}{k}\right) + \int_0^1 \frac{t^2}{\sqrt{(1-t^2)(1-(t/k)^2)}} dt = \mathbf{E}\left(\frac{1}{k}\right) + \mathbf{K}\left(\frac{1}{k}\right) - \int_0^1 \sqrt{\frac{1-t^2}{1-(t/k)^2}} dt.$$

Note that this last term is bounded below by  $\mathbf{K}\left(\frac{1}{k}\right)$ , which in turn is bounded below by  $\frac{\pi}{2}$ . So the derivative of  $\tilde{\Phi}$  is bounded by  $\frac{2}{\pi}$  and

$$c \tilde{\Phi}\left(\frac{\pi(n+1)}{2c}\right) - c \tilde{\Phi}\left(\frac{\pi n}{2c}\right) < 1.$$

It is natural to choose the middle value for an approximate value of  $\sqrt{\chi_n}$ , that is,

$$\sqrt{\tilde{\chi}_n} = \frac{c}{\Phi\left(\frac{2c}{\pi(n+1/2)}\right)}, \quad n \geq \frac{2c}{\pi}. \quad (11)$$

We define also  $\tilde{q} = \frac{c^2}{\tilde{\chi}_n}$ . It is easy to check that we have the following approximation and relative approximation errors of  $\sqrt{\chi_n}$ :

$$\left| \sqrt{\chi_n} - \sqrt{\tilde{\chi}_n} \right| \leq \frac{1}{2}, \quad \left| \frac{\sqrt{\chi_n} - \sqrt{\tilde{\chi}_n}}{\sqrt{\chi_n}} \right| \leq \frac{1}{2\sqrt{\chi_n}} \leq \frac{1}{2n}. \quad (12)$$

As a consequence, we also have

$$\left| \sqrt{q} - \sqrt{\tilde{q}} \right| \leq \frac{c}{2\sqrt{\chi_n}\sqrt{\tilde{\chi}_n}} \leq \frac{c}{n(2n+1)}. \quad (13)$$

**Remark 1.** Formula (11) provides us with an approximation  $\tilde{\chi}_n$  of  $\chi_n$  in terms of the easily computed function  $\Phi$ . This approximation may be compared with classical numerical methods to compute  $\chi_n$ , such as Flammer's method, see [9]. Numerical experiments prove that it is a good approximation for  $n$  not too small. We will see later on that the relative error for  $\sqrt{\chi_n}$  is of order  $O(1/n^2)$  for  $q$  not very close to 1. For large values of  $n$ , our formula (11) provides us with precise values of  $\sqrt{\chi_n}$  with very low computational load compared to the classical methods.

Theorem 1 provides also new upper and lower bounds of the  $\chi_n$  which are valid for  $n < \frac{2c}{\pi} - 1$ . This is the subject of the following proposition. The left hand side of the inequality (14) below has already been stated and proved in [3].

**Proposition 2.** For  $n \geq 2$  and  $c > \frac{\pi(n+1)}{2}$ , we have the inequalities

$$\frac{\pi cn}{2} \leq \chi_n \leq 2c(n+1). \quad (14)$$

*Proof.* We first prove that for  $s > 1$ ,

$$\sqrt{\frac{\pi s}{4}} < \Phi(s) < \sqrt{s}. \quad (15)$$

Recall that the inequality  $s > 1$  is equivalent to  $k = \Phi(s) > 1$ . So it is sufficient to prove that

$$\sqrt{\frac{\pi k}{4\mathbf{E}(k)}} < k < \sqrt{\frac{k}{\mathbf{E}(k)}},$$

that is,

$$\frac{\pi}{4k} < \mathbf{E}(k) < \frac{1}{k}, \quad k > 1. \quad (16)$$

After a change of variables we have  $k\mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-t^2}{1-(t/k)^2}} dt$ . This latter decreases from 1 to  $\int_0^1 \sqrt{1-t^2} dt = \frac{\pi}{4}$ . We have proved (16) and (15). The rest of the proof of (14) is a straightforward consequence of Theorem 1, using the fact that  $\frac{2c}{\pi n} > \frac{2c}{\pi(n+1)} > 1$ .  $\square$

Before finishing this section, we use Theorem 1 to give bounds for the quantity  $(1-q)\sqrt{\chi_n}$ . This allows us to interpret conditions on  $(1-q)\sqrt{\chi_n}$  in terms of  $c$  and  $n$ . The following proposition says, roughly speaking, that one has to add a factor of  $\log n$  when passing from a condition on  $\chi_n$  to a condition on  $n$ .

**Proposition 3.** *For  $n \geq 2$  and  $q < 1$ , we have the inequalities*

$$(1-q)\sqrt{\chi_n} \geq \frac{(n - \frac{2c}{\pi}) - e^{-1}}{\log n + 5}, \quad (17)$$

$$n + 1 - \frac{2c}{\pi} \geq \frac{1}{\pi^2} (1-q)\sqrt{\chi_n} \log \left( \frac{1}{1-\sqrt{q}} \right). \quad (18)$$

*Proof.* The proof will make use of the complete Legendre elliptic integral of the first kind, which we denote by  $\mathbf{K}$ . Recall that

$$\mathbf{K}(\eta) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\eta^2 t^2)}}, \quad 0 < \eta < 1. \quad (19)$$

We will need precise estimates on the behavior of  $\mathbf{K}$ , namely

$$(1-\eta)\frac{\pi}{2} + \frac{1}{2} \log \frac{1+\eta}{1-\eta} \leq \mathbf{K}(\eta) \leq \frac{\pi}{2} + \frac{1}{2} \log \frac{1+\eta}{1-\eta}. \quad (20)$$

To prove this, we take the difference between  $\mathbf{K}(\eta)$  and the integral of  $\eta/(1-\eta^2 t^2)$ .

Let us go back to the proof of the proposition. We write  $\frac{c}{\sqrt{\chi_n}} = 1 - \frac{\delta}{n}$  and  $\frac{2c}{\pi n} = 1 - \frac{\delta^*}{n}$ . Using Theorem 1, we have

$$\Psi \left( 1 - \frac{\delta}{n} \right) \leq 1 - \frac{\delta^*}{n}$$

with  $\Psi(k) = \frac{k}{\mathbf{E}(k)}$  the inverse function of  $\Phi$ . By using the fact that  $\mathbf{E}(\cdot)$  is decreasing, one gets

$$\frac{\delta^* - \delta}{n} \leq \mathbf{E} \left( 1 - \frac{\delta}{n} \right) - 1.$$



We need to estimate of  $\mathbf{E}(\cdot) - 1$ . Writing this quantity as an integral, we get bounds in terms of elliptic integral  $\mathbf{K}$ , given by

$$(1 - k^2)(\mathbf{K}(k)/2 - 1) \leq \mathbf{E}(k) - 1 \leq (1 - k^2)\mathbf{K}(k). \quad (21)$$

By using (20), one obtain the inequalities

$$(1 - k) \left( \frac{1}{4} \log \frac{1}{1 - k} - 1 \right) \leq \mathbf{E}(k) - 1 \leq (1 - k) \left( \pi + \log \frac{2}{1 - k} \right), \quad 0 \leq k < 1. \quad (22)$$

Using this last inequality, we get

$$\delta^* \leq \delta \left( \pi + 1 + \log \frac{2n}{\delta} \right).$$

Since  $\delta \log(\frac{1}{\delta}) \leq e^{-1}$ , we finally find the inequality

$$\delta^* - e^{-1} \leq \delta (\pi + 1 + \log 2 + \log n) \leq \delta (\log n + 5),$$

from which we conclude at once.

Conversely, we change slightly the notation and write  $\frac{c}{\sqrt{\chi_n}} = 1 - \frac{\delta}{(n+1)}$  and  $\frac{2c}{\pi(n+1)} = 1 - \frac{\delta^*}{(n+1)}$ . Using Theorem 1 and the fact that  $\mathbf{E}(\cdot)$  is bounded by  $\pi/2$ , we get the inequality

$$\mathbf{E} \left( 1 - \frac{\delta}{n+1} \right) - 1 \leq \frac{\pi}{2} \frac{\delta^*}{n+1}.$$

If we use the first inequality in (22), we get the following.

$$\frac{\delta^*}{n+1} \geq \frac{2}{\pi} + \frac{2}{\pi} \frac{\delta}{n+1} \left( \frac{1}{4} \log \frac{1}{1 - \sqrt{q}} - 1 \right).$$

We obtain (18) by using the inequality  $\sqrt{\chi_n} \leq \frac{\pi}{2}(n+1)$ . □

### 3 WKB approximation of the PSWFs and corollaries.

We assume in this section that  $q = c^2/\chi_n < 1$ . We first give explicit uniform approximation for the PSWF  $\psi_n$  in terms of its value at 1, as well as in terms of the Bessel function  $J_0$  and of the associated eigenvalue  $\chi_n$ . This approximation holds under the condition that  $(1 - q)\sqrt{\chi_n}$  is large enough. We rely on properties of Sturm-Liouville equations and use the estimates given by Olver in his book [26]. The existence of such an asymptotic approximation is well-known and has been developed in a larger context, see for example [8, 19, 26]. In particular, asymptotic approximation of the PSWFs has been given in [19] for large values of the parameter  $c$  while  $n$ -th order uniform asymptotic approximations of the PSWFs are obtained in [8] as a consequence of Olver's results. In this paragraph, we go back to Olver's asymptotic approximation scheme and we give precise estimates and simple bounds of the functions that are involved in the perturbation term. As a consequence, we obtain a simple and practical expression of the approximation error of the  $\psi_n$ . Once this done, we get rid of the dependence in  $\psi_n(1)$  of these approximations. More precisely we give an approximation of  $\psi_n(1)$  in terms of  $\sqrt{\chi_n}$ ,  $n$  and  $c$ . Recall that  $\psi_n$  has  $L^2([-1, 1])$ -norm 1, which fixes the value  $\psi_n(1)$ . This approximation of  $\psi_n(1)$  is in particular a critical issue for the sharp decay rate of eigenvalues of the integral operator  $\mathcal{Q}_c$  defined in (1).

We also use the uniform approximation to improve the error bounds for our approximation of the quantities  $\chi_n$  and  $q$  via Formula (11).

### 3.1 Uniform approximation of the PSWFs knowing their value at 1.

Let us go back to the transformed equation (5), that is,

$$U''(s) + (\chi_n + \theta(s))U(s) = 0, \quad s \in [0, S(0)].$$

We use the notations of the previous section. We claim that the function  $F(\cdot) = F_q(\cdot)$ , given by

$$F(S(x)) = \frac{1}{4S^2(x)} - \theta(S(x)), \quad x \in [0, 1), \quad (23)$$

is continuous on  $[0, S(0)]$ . We postpone the proof to Lemma 2 and go on. The equation

$$U''(s) + \left(\chi_n + \frac{1}{4s^2}\right)U(s) = F(s)U(s), \quad s \in [0, S(0)]. \quad (24)$$

is a particular case of the equation considered in Olver's book, Chapter 12, Theorem 6.1. The associated homogeneous equation has the two independent solutions

$$U_1(s) = \chi_n^{1/4} \sqrt{s} J_0(\sqrt{\chi_n s}), \quad U_2(s) = \chi_n^{1/4} \sqrt{s} Y_0(\sqrt{\chi_n s}),$$

where  $J_0$  (resp.  $Y_0$ ) denotes the Bessel function of the first (resp. second) type. Hence, using the well known explicit value of the Wronskian of  $J_0, Y_0$ , the solution  $U$  may be written as

$$\begin{aligned} U(s) &= AU_1(s) + A'U_2(s) + \frac{\pi}{2\sqrt{\chi_n}} \\ &\times \int_0^s \sqrt{st\chi_n} [J_0(\sqrt{\chi_n s})Y_0(\sqrt{\chi_n t}) - J_0(\sqrt{\chi_n t})Y_0(\sqrt{\chi_n s})] F(t)U(t)dt. \end{aligned} \quad (25)$$

From now on,  $U$  is the particular solution of (25) on  $[0, S(0)]$  that we have defined in Section 2, that is,

$$U(S(x)) = ((1-x^2)(1-qx^2))^{1/4} \psi_n(x). \quad (26)$$

In the next lemma, we prove that  $S(x)/((1-x^2)(1-qx^2))^{1/4}$  goes to 1 as  $x$  goes to 1, so that

$$\lim_{s \rightarrow 0} \frac{U(s)}{\sqrt{s}} = \psi_n(1).$$

Let us prove that this behavior at 0 forces the coefficient  $A'$  to be 0. Since the function  $J_0(s)$  has the limit 1, while  $Y_0(s)$  has a singularity at 0, it is sufficient to prove that the last term in (25) is bounded by  $s$ , up to a constant. But the function inside the integral in (25) is bounded. Indeed, we have the classical inequality

$$\sup_{s \geq 0} s(J_0^2(s) + Y_0^2(s)) \leq \frac{2}{\pi}. \quad (27)$$

(see [35], p. 446-447 for instance). So we have not only proved that  $A' = 0$  but also that

$$A = \psi_n(1)\chi_n^{-1/4}.$$

We may use either  $A$  or  $\psi_n(1)$  in the next formulas, which is obviously equivalent.

Hence, (25) can be rewritten as

$$U(s) = \psi_n(1)\sqrt{s}J_0(\sqrt{\chi_n s}) + \frac{1}{\sqrt{\chi_n}} \int_0^s K_n(s, t)F(t)U(t)dt \quad (28)$$

with

$$K_n(s, t) = \frac{\pi}{2} \sqrt{st\chi_n} [J_0(\sqrt{\chi_n s})Y_0(\sqrt{\chi_n t}) - J_0(\sqrt{\chi_n t})Y_0(\sqrt{\chi_n s})].$$

The solution  $U$  given by (28) is, up to the multiplicative constant  $\psi_n(1)$ , the solution of the equation (24) that has been considered by Olver. We first give estimates on  $S(x)$  and  $\theta(S(x))$  before we use Olver's inequalities. The two corresponding lemmas contain in particular the properties that we have already used.

**Lemma 1.** *For  $0 \leq q < 1$  and  $x \in [0, 1]$ , we have*

$$-\frac{q(1-x^2)^{3/2}}{2(1-qx^2)^{1/2}} \leq S(x) - \sqrt{(1-x^2)(1-qx^2)} \leq \frac{2-q}{3}(1-x^2)^{3/2}. \quad (29)$$

Moreover, for  $x$  tending to 1 we can write

$$\sqrt{(1-x^2)(1-qx^2)}/S(x) = 1 + \left(\frac{q}{1-q} + \frac{3}{4}\right)(1-x) + o(1-x). \quad (30)$$

*Proof.* By computing the derivative of the quantity  $S(x) - \sqrt{(1-x^2)(1-qx^2)}$ , one gets

$$S(x) - \sqrt{(1-x^2)(1-qx^2)} = \int_x^1 (1-t) \sqrt{\frac{1-qt^2}{1-t^2}} dt - q \int_x^1 t \sqrt{\frac{1-t^2}{1-qt^2}} dt. \quad (31)$$

On one hand, using the fact that  $\sqrt{\frac{1-qt^2}{1-t^2}} \leq 1$ , we have

$$\int_x^1 (1-t) \sqrt{\frac{1-qt^2}{1-t^2}} dt \leq \frac{2}{3}(1-x^2)^{3/2}. \quad (32)$$

On the other hand, we have

$$\frac{(1-x^2)^{3/2}}{3} \leq \int_x^1 t \sqrt{\frac{1-t^2}{1-qt^2}} dt \leq \sqrt{\frac{(1-x^2)}{1-qx^2}} \int_x^1 t dt = \frac{(1-x^2)^{3/2}}{2\sqrt{1-qx^2}}. \quad (33)$$

Finally, by combining (31), (32) and (33), one gets (29). The computation of (30) is elementary.  $\square$

As a consequence, we get the following double inequalities

$$\left(1 - \frac{q}{2}\right) \sqrt{(1-x^2)(1-qx^2)} \leq S(x) \leq \frac{5-q}{3} \sqrt{(1-x^2)(1-qx^2)}. \quad (34)$$

The following lemma concerns the function  $F$ , which has been defined by (23).

**Lemma 2.** *For  $0 \leq q < 1$  the function  $F$  is continuous on  $[0, S(0)]$ . Moreover, we have*

$$|F(S(x))| \leq \frac{3+2q}{4} \frac{1}{(1-qx^2)^2}, \quad x \in [0, 1] \quad (35)$$

and

$$\alpha_q = (1-q) \int_0^{S(0)} |F(s)| ds \leq \frac{3+2q}{4} \mathbf{E}(\sqrt{q}) \quad (36)$$

$$\leq 1.5. \quad (37)$$

*Proof.* The function  $F$  is a priori only defined on  $[0, S(0))$  but we will prove that we can extend it at 1 by continuity. We use the notation  $F$  for its extension as well. We first consider

$$\begin{aligned} \theta(S(x)) - \frac{1}{4(1-x^2)(1-qx^2)} &= \frac{(1-q)^2 - (1-qx^2)^2}{4(1-x^2)(1-qx^2)^3} + \frac{(1-q)^2 + 2q(3-q)(1-x^2)}{4(1-qx^2)^3} \\ &= \frac{1+q(2+3x^2q-6x^2)}{4(1-qx^2)^3}, \quad 0 \leq x < 1. \end{aligned}$$

This extends to a continuous function on  $[0, 1]$ . Moreover, from the elementary inequality

$$(1 - 3qx^2)(1 - qx^2) \leq 1 + q(2 + 3x^2q - 6x^2) \leq (1 + 2q)(1 - qx^2),$$

we conclude that

$$\frac{3q - 1}{4(1 - qx^2)^2} \leq \theta(S(x)) - \frac{1}{4(1 - x^2)(1 - qx^2)} \leq \frac{1 + 2q}{4(1 - qx^2)^2}, \quad x \in [0, 1]. \quad (38)$$

Next, the extension into a continuous function at 1 of  $\frac{1}{(1-x^2)(1-qx^2)} - \frac{1}{S^2(x)}$  at 1 is an easy consequence of (30). We then use (34) and (29) to conclude that

$$-\frac{3q}{(1 - qx^2)^2} \leq \frac{1}{(1 - x^2)(1 - qx^2)} - \frac{1}{S^2(x)} \leq \frac{2}{(1 - qx^2)^2}, \quad x \in [0, 1]. \quad (39)$$

Finally, by combining (38) and (39), one gets (35).

It remains to prove (36). Using the estimate on  $F$  given by (35), together with the change of variable  $t = S(y)$ , we get

$$\int_0^{S(x)} |F(t)| dt \leq \frac{3 + 2q}{4} \int_x^1 \frac{dy}{(1 - y^2)^{1/2}(1 - qy^2)^{3/2}}.$$

Straightforward computations give the classical identity

$$\int_x^1 \frac{dy}{(1 - y^2)^{1/2}(1 - qy^2)^{3/2}} = \frac{1}{1 - q} \left( \frac{qx\sqrt{1 - x^2}}{\sqrt{1 - qx^2}} + S(x) \right). \quad (40)$$

We conclude directly for (36) by using (40) with  $x = 0$ . For the inequality (37), we use the concavity of the function  $q \mapsto \mathbf{E}(\sqrt{q})$  as well as the fact that its derivative is  $-\frac{\pi}{8}$  at 0 and prove that  $\mathbf{E}(\sqrt{q}) \leq \frac{\pi}{2}(1 - \frac{q}{4})$ , from which we conclude.  $\square$

Let us go back to the function  $U$  defined in (26) and write

$$U(s) = \psi_n(1)\sqrt{s}J_0(\sqrt{\chi_n s}) + \mathcal{E}_n(s).$$

This gives a WKB approximation of  $U$  by the first term. We now give estimates of the approximation error  $\mathcal{E}_n$ . Using Theorem 6.1 of Chapter 12 of Olver's book [26], one has

$$|\mathcal{E}_n(s)| \leq \psi_n(1)\sqrt{s} \frac{M_0(\sqrt{\chi_n s})}{E_0(\sqrt{\chi_n s})} \left[ \exp\left(\frac{\pi}{2} \int_0^s t M_0^2(ut) |F(t)| dt\right) - 1 \right]. \quad (41)$$

Here

$$E_0(x) = \begin{cases} (-Y_0(x)/J_0(x))^{1/2} & \text{if } 0 < x \leq X_0 \\ 1 & \text{if } x \geq X_0 \end{cases}, \quad M_0(x) = \begin{cases} (2|Y_0(x)|J_0(x))^{1/2} & \text{if } 0 < x \leq X_0 \\ (J_0^2(x) + Y_0^2(x))^{1/2} & \text{if } x \geq X_0 \end{cases},$$

with  $X_0$  the first zero of

$$J_0(x) + Y_0(x) = 0.$$

It follows from the classical inequality (27) that

$$tM_0^2(\sqrt{\chi_n t}) \leq \frac{2}{\pi} \frac{1}{\sqrt{\chi_n}}, \quad \chi_n^{1/4} \sqrt{s} \frac{M_0(\sqrt{\chi_n s})}{E_0(\sqrt{\chi_n s})} \leq \sqrt{\frac{2}{\pi}}. \quad (42)$$

Before stating Olver's estimates in the form that we will use later on, let us recall or fix some notations.

**Notations.** For  $0 \leq q \leq 1$  the constant  $\alpha_q$  is given by

$$\alpha_q = (1 - q) \int_0^{\mathbf{E}(\sqrt{q})} |F(s)| ds, \quad (43)$$

where  $F = F_q$  has been defined in (23).

For a positive integer  $n, c$  such that  $q = c^2/\chi_n < 1$ , the quantity  $\varepsilon_n$  is defined as

$$\varepsilon_n = \frac{1}{(1 - q)\sqrt{\chi_n}}. \quad (44)$$

The function  $\Theta$  is defined on  $(0, +\infty)$  by

$$\Theta(x) = \frac{e^x - 1}{x}, \quad x > 0. \quad (45)$$

We will be mainly interested in  $\Theta$  on the interval  $(0, 1)$ , where  $\Theta(x) \leq 1 + x \leq 2$  plays the role of a multiplicative constant.

By combining (41) and (42) one gets the estimates

$$|\mathcal{E}_n(s)| \leq \sqrt{\frac{2}{\pi}} \frac{\psi_n(1)}{\chi_n^{1/4}} \left( \exp \left( (1 - q)\varepsilon_n \int_0^s |F(t)| dt \right) - 1 \right). \quad (46)$$

The following statements provide us with an error bound for the WKB uniform approximation of the function  $U$  and consequently of the function  $\psi_n$ . They are direct corollaries of the previous inequality.

**Lemma 3.** *Let  $n, c$  be such that  $q = c^2/\chi_n < 1$ . Then the function  $U$  defined in (26) is given by*

$$U(s) = \psi_n(1)\sqrt{s}J_0(\sqrt{\chi_n}s) + \mathcal{E}_n(s),$$

with

$$\sup_{s \in (0, S(0))} |\mathcal{E}_n(s)| \leq \sqrt{\frac{2}{\pi}} \frac{\psi_n(1)}{\chi_n^{1/4}} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n).$$

**Proposition 4.** *let  $n, c$  be such that  $q = c^2/\chi_n < 1$ . Then under the previous notations, one can write*

$$\psi_n(x) = \psi_n(1) \frac{\sqrt{S(x)}J_0(\sqrt{\chi_n}S(x))}{(1 - x^2)^{1/4}(1 - qx^2)^{1/4}} + R_n(x)$$

for  $0 \leq x \leq 1$ , with

$$|R_n(x)| \leq \kappa_0 \frac{(1 - x^2)^{1/4}}{(1 - qx^2)^{3/4}} \varepsilon_n. \quad (47)$$

Here,  $\kappa_0 = \frac{5}{2} \sqrt{\frac{2}{\pi}} \frac{\psi_n(1)}{\chi_n^{1/4}} \Theta(\alpha_q \varepsilon_n)$ .

*Proof.* Lemma 3 is directly deduced from (46) and the definition of  $\Theta$ . Let us prove the proposition. Since the function  $\Theta$  is increasing, we have the estimate

$$|\mathcal{E}_n(s)| \leq \sqrt{\frac{2}{\pi}} \frac{\psi_n(1)}{\chi_n^{1/4}} (1 - q)\varepsilon_n \Theta(\alpha_q \varepsilon_n) \int_0^s |F(t)| dt.$$

So, if we use the bound given in (40) for the integral, we get the inequalities

$$|\mathcal{E}_n(S(x))| \leq \frac{2\kappa_0(1-q)\varepsilon_n}{5} \int_0^{S(x)} |F(t)| dt \leq \frac{\kappa_0\varepsilon_n}{2} \left( \frac{qx\sqrt{1-x^2}}{\sqrt{1-qx^2}} + S(x) \right) \leq \kappa_0\varepsilon_n \sqrt{\frac{1-x^2}{1-qx^2}}.$$

For the last inequality we have first used (34), then the fact that  $qx + S(x)\sqrt{\frac{1-qx^2}{1-x^2}}$  is bounded by  $qx + \frac{2}{\sqrt{1+x}}(1-qx^2)$ , then the fact that this last function is bounded by 2. We conclude for (47) by dividing by  $(1-x^2)^{1/4}(1-qx^2)^{1/4}$ .  $\square$

We end this subsection by the remark that the same kind of estimates, but with larger constants, could have been obtained directly when  $\alpha_q\varepsilon_n < 1$ , without referring to Olver's techniques. Indeed, if we go back to (28) and use (27) to see that the kernel  $K_n$  is bounded by 1, we have the inequality

$$|U(s)| \leq \psi_n(1)\sqrt{s}|J_0(\sqrt{\chi_n}s)| + \frac{1}{\sqrt{\chi_n}} \int_0^s |F(t)||U(t)| dt \leq \sqrt{\frac{2}{\pi}} \frac{\psi_n(1)}{\chi_n^{1/4}} + \alpha_q\varepsilon_n \sup_t |U(t)|. \quad (48)$$

This gives a bound above for the maximum of  $|U(t)|$  under the assumption that  $\alpha_q\varepsilon_n < 1$ , which we can use to estimate the remainder term, that is  $\mathcal{E}_n$ . We leave the details to the reader.

### 3.2 Estimates and bounds of $\psi_n(1)$ .

As we have already mentioned, the estimate of  $\psi_n(1)$ , under the adopted normalization  $\|\psi_{n,c}\|_{L^2([-1,1])} = 1$ , is a main issue. At this point, one does not know much about  $\psi_n(1)$  except for the case  $c = 0$ , for which  $\psi_{n,0}(1) = \sqrt{n + \frac{1}{2}}$ . It is accepted, but not rigorously proved, that as a function of  $c$ ,  $\psi_{n,c}(1)$  is maximum at 0 (see [32]). We proved in [3] that, for  $q \leq 2$ , one has the inequality

$$\psi_n(1) \leq \kappa_1 \chi_n^{1/4}, \quad \kappa_1 = \frac{5^{5/4}}{4}.$$

We give here an approximated value of  $\psi_n(1)$  in terms of  $\chi_n$  up to a relative error of order  $\varepsilon_n$ .

The strategy of the proof is simple. We start from the expression of  $\psi_n$  given by (4) and set

$$\tilde{\psi}_n(x) = \frac{\chi_n^{1/4} \sqrt{S(x)} J_0(\sqrt{\chi_n} S(x))}{(1-x^2)^{1/4} (1-qx^2)^{1/4}}. \quad (49)$$

We then prove that the norm of  $R_n$  is small and compute almost explicitly the  $L^2$ -norm of  $\tilde{\psi}_n$ . The conclusion comes from these two computations.

The next lemma gives bounds for the remainder  $R_n$  in the  $L^2$ -norm. Recall that  $\mathbf{K}$  denotes the complete Legendre elliptic integral of the first kind, given by (19).

**Lemma 4.** *Assume that  $n, c$  are such that  $q = c^2/\chi_n < 1$ . Then*

$$\|R_n\|_{L^2([0,1])} \leq \frac{\psi_n(1)}{\chi_n^{1/4}} \sqrt{\frac{2\mathbf{K}(\sqrt{q})}{\pi}} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n). \quad (50)$$

*Proof.* By Lemma 3, we have

$$|R_n(x)| \leq \sqrt{\frac{2}{\pi}} \frac{\psi_n(1)}{\chi_n^{1/4}} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n) (1-x^2)^{-1/4} (1-qx^2)^{-1/4}.$$

Moreover the  $L^2(0,1)$ -norm of the function  $(1-x^2)^{-1/4}(1-qx^2)^{-1/4}$  is equal to  $\sqrt{\mathbf{K}(\sqrt{q})}$ , which allows us to conclude for (50).  $\square$

In order to evaluate the  $L^2(0,1)$ -norm of  $\tilde{\psi}_n$ , we first define a constant related with Bessel functions.

**Lemma 5.** *The function  $G(x) = \frac{x^2}{2} [(J_0(x))^2 + (J_1(x))^2] - \frac{x}{\pi}$  is bounded on  $[0, \infty)$ . Moreover,*

$$\sup_{x>0} \left| \frac{x^2}{2} [(J_0(x))^2 + (J_1(x))^2] - \frac{x}{\pi} \right| = \kappa_2 = 0.17203 \dots \quad (51)$$

*Proof.* The boundedness of  $G$  comes from the fact that it has a finite limit at  $\infty$ , which is an easy consequence of the asymptotic expansion of Bessel functions. A careful study of the remainders associated with the previous asymptotic approximations of Bessel functions proves that the maximum of  $G$  is attained in the interval  $(0, 3)$ . Its monotonicity, as well as some numerical computations, are then used to get the precise value of  $\kappa_2$ . We leave the details to the reader.  $\square$

We now prove the following lemma.

**Lemma 6.** *Under the above notations, for  $0 \leq q < 1$ , we have*

$$\|\tilde{\psi}_n\|_{L^2([0,1])}^2 = \frac{\mathbf{K}(\sqrt{q})}{\pi} + \eta, \quad |\eta| \leq \kappa_2 \varepsilon_n.$$

*Proof.* Going back to the notations of Section 3. 1, that is,  $U_1(s) = \sqrt{s}J_0(s)$ , one writes

$$\tilde{\psi}_n(x) = \frac{U_1(\sqrt{\chi_n}S(x))}{(1-x^2)^{1/4}(1-qx^2)^{1/4}}, \quad x \in [0, 1].$$

If  $x(s)$  denotes the inverse function of  $S(x)$ , one has

$$\|\tilde{\psi}_n\|_{L^2([0,1])}^2 = \int_0^1 \frac{|U_1(\sqrt{\chi_n}S(x))|^2}{\sqrt{(1-x^2)(1-qx^2)}} dx = \sqrt{\chi_n} \int_0^{S(0)} \frac{s|J_0(\sqrt{\chi_n}s)|^2}{1-qs(s)^2} ds.$$

Finally, after a last change of variables one gets

$$\|\tilde{\psi}_n\|_{L^2([0,1])}^2 = \frac{1}{\sqrt{\chi_n}} \int_0^{\sqrt{\chi_n}S(0)} \theta(t)t(J_0(t))^2 dt,$$

where

$$\theta(t) = \frac{1}{1-qx^2\left(\frac{t}{\sqrt{\chi_n}}\right)}, \quad t \in [0, S(0)\sqrt{\chi_n}].$$

Since  $0 \leq q < 1$  and  $x(s)$  is decreasing and has values in  $[0, 1]$ , the function  $\theta(s)$  is smooth and decreasing on  $[0, S(0)\sqrt{\chi_n}]$ . It takes the value 1 at  $S(0)\sqrt{\chi_n}$ . To estimate the previous quantity, we proceed as follows. We first note ([2]) that

$$\int_0^x t(J_0(t))^2 dt = \frac{x^2}{2} [(J_0(x))^2 + (J_1(x))^2], \quad x > 0.$$

So, if  $G$  is defined as in Lemma 5, we have that  $xJ_0(x)^2 = G'(x) + \frac{1}{\pi}$ . Integration by parts gives the equality

$$\begin{aligned} \sqrt{\chi_n}\|\tilde{\psi}_n\|_{L^2([0,1])}^2 &= \frac{1}{\pi} \int_0^{\sqrt{\chi_n}S(0)} \theta(s)ds + \int_0^{\sqrt{\chi_n}S(0)} \theta(s)G'(s)ds \\ &= \frac{1}{\pi} \int_0^{\sqrt{\chi_n}S(0)} \theta(s)ds + G(\sqrt{\chi_n}S(0)) - \int_0^{\sqrt{\chi_n}S(0)} \theta'(s)G(s)ds. \end{aligned}$$

We use (51) to bound the second and the third term. The second one is directly bounded by  $\kappa_2$ . Since  $\theta'$  is non positive, the last term is bounded by  $\kappa_2(\theta(0) - \theta(1)) = \frac{\kappa_2}{1-q} - \kappa_2$ . Now by using the substitution  $s = \sqrt{\chi_n}S(x)$ , one gets

$$\frac{1}{\sqrt{\chi_n}} \int_0^{\sqrt{\chi_n}S(0)} \theta(s) ds = \int_0^1 \frac{dx}{\sqrt{1-qx^2}\sqrt{1-x^2}} = \mathbf{K}(\sqrt{q}).$$

By collecting everything together, one concludes that

$$\left| \|\tilde{\psi}_n\|_{L^2([0,1])}^2 - \frac{\mathbf{K}(\sqrt{q})}{\pi} \right| \leq \kappa_2 \varepsilon_n.$$

□

As a corollary, we have the following bounds for the norm of  $\tilde{\psi}_n$ . If  $\beta_q \varepsilon_n < 1$ , where

$$\beta_q = \frac{\pi \kappa_2}{\mathbf{K}(\sqrt{q})} \leq 2\kappa_2, \quad (52)$$

then we have

$$\sqrt{\frac{\mathbf{K}(\sqrt{q})}{\pi}} (1 - \beta_q \varepsilon_n)^{1/2} \leq \|\tilde{\psi}_n\|_{L^2([0,1])} \leq \sqrt{\frac{\mathbf{K}(\sqrt{q})}{\pi}} (1 + \beta_q \varepsilon_n)^{1/2}. \quad (53)$$

At this point, we see that the left hand side estimate is interesting only if  $\beta_q \varepsilon_n$  is sufficiently small. We give now a slightly stronger but also a flexible assumption that will be sufficient for the inequalities to come,

$$(1 - q)\sqrt{\chi_n} \geq 4. \quad (54)$$

Note that by using Proposition 3, we may write the above condition in terms of  $n$  and  $c$  directly, without involving  $\chi_n$ . Under Condition (54), we have

$$\alpha_q \varepsilon_n < 0.375, \quad \beta_q \varepsilon_n < 0.086, \quad (1 - \beta_q \varepsilon_n)^{1/2} > 0.96. \quad (55)$$

The following theorem provides us with an approximation of  $\psi_n(1)$ .

**Theorem 2.** *Let  $n, c$  be such that  $(1 - q)\sqrt{\chi_n} \geq 4$ . Then*

$$\chi_n^{1/4} \sqrt{\frac{\pi}{2\mathbf{K}(\sqrt{q})}} (1 - \eta \varepsilon_n) \leq \psi_n(1) \leq \chi_n^{1/4} \sqrt{\frac{\pi}{2\mathbf{K}(\sqrt{q})}} (1 + \eta' \varepsilon_n), \quad (56)$$

where we may take  $\eta = 2.75$ ,  $\eta' = 10.78$ .

*Proof.* Let  $A = \psi_{n,c}(1)\chi_n(c)^{-1/4}$  as before. By the triangular inequality, we have

$$\left| \frac{1}{\sqrt{2}} - A \|\tilde{\psi}_n\|_{L^2([0,1])} \right| = \left| \|\psi_n\|_{L^2([0,1])} - A \|\tilde{\psi}_n\|_{L^2([0,1])} \right| \leq \|R_n\|_2. \quad (57)$$

By using the previous equality and (50), one gets

$$A \left( \|\tilde{\psi}_n\|_{L^2([0,1])} - \sqrt{\frac{2\mathbf{K}(\sqrt{q})}{\pi}} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n) \right) \leq \frac{1}{\sqrt{2}} \leq A \left( \|\tilde{\psi}_n\|_{L^2([0,1])} + \sqrt{\frac{2\mathbf{K}(\sqrt{q})}{\pi}} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n) \right).$$

Moreover, from (53), we have

$$\sqrt{\frac{2\mathbf{K}(\sqrt{q})}{\pi}} \left( (1 - \beta_q \varepsilon_n)^{1/2} - \sqrt{2} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n) \right) \leq A^{-1} \leq \sqrt{\frac{2\mathbf{K}(\sqrt{q})}{\pi}} \left( (1 + \beta_q \varepsilon_n)^{1/2} + \sqrt{2} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n) \right).$$



It follows immediately that we have (56) as soon as

$$\eta \varepsilon_n \geq 1 - \frac{1}{\sqrt{1 + \beta_q \varepsilon_n} + \sqrt{2} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n)}.$$

This is in particular the case when

$$\eta > \frac{\beta_q}{2} + \sqrt{2} \alpha_q \Theta(\alpha_q \varepsilon_n). \quad (58)$$

Taking into account the estimates given in (55) and the fact that  $\alpha_q < 1.5$ , we get the stronger sufficient condition  $\eta > 2.75$ . The same method for  $\eta'$  gives

$$\eta' > \frac{\beta_q + \sqrt{2} \alpha_q \Theta(\alpha_q \varepsilon_n)}{1 - \beta_q \varepsilon_n - \sqrt{2} \alpha_q \varepsilon_n \Theta(\alpha_q \varepsilon_n)}. \quad (59)$$

Again, by taking into account the estimates given in (55), we get the stronger sufficient condition  $\eta' > 10.78$ .  $\square$

**Remark 2.** *Remark that for a fixed  $\varepsilon_n$ , the two inequalities (58) and (59) give better estimates than the numerical values given in the statement of the theorem. Moreover numerical tests show that the relative error in estimating  $A$  by  $\sqrt{\frac{\pi}{2\mathbf{K}(\sqrt{q})}}$  is much smaller than this theoretical error.*

This last theorem allows us to improve the estimate given in (3.2), at least asymptotically. By using (9), we find that

$$\psi_n(1) \leq (n+1)^{1/2} \sqrt{\frac{\pi^2}{4\mathbf{E}(\sqrt{q})\mathbf{K}(\sqrt{q})}} (1 + \eta' \varepsilon_n).$$

**Remark 3.** *This inequality does not imply the one that has been conjectured in [32] from numerical evidence, namely  $\psi_n(1) \leq \sqrt{n + \frac{1}{2}}$ . But there are many values of  $n, c$  for which it is better: by Cauchy-Schwarz Inequality we know that  $\frac{\pi^2}{4\mathbf{E}(\sqrt{q})\mathbf{K}(\sqrt{q})} < 1$  for  $q > 0$ . Moreover this quantity goes to 0 as  $q$  goes to 1.*

Next, replacing  $\psi_n(1)$  by its approximation in Proposition 4, we get the following corollary.

**Corollary 1.** *There exist constants  $C_1$  and  $C_2$  such that for all  $n, c$  for which  $(1-q)\sqrt{\chi_n(c)} \geq 4$ , we have, for  $0 \leq x \leq 1$*

$$\psi_n(x) = \sqrt{\frac{\pi}{2\mathbf{K}(\sqrt{q})}} \frac{\chi_n^{1/4} \sqrt{S(x)} J_0(\sqrt{\chi_n} S(x))}{(1-x^2)^{1/4} (1-qx^2)^{1/4}} + \tilde{R}_n(x)$$

with

$$|\tilde{R}_n(x)| \leq C_1 \varepsilon_n \sqrt{\frac{1}{K(\sqrt{q})}} \min\left(\chi_n^{1/4}, (1-x^2)^{-1/4} (1-qx^2)^{-1/4}\right). \quad (60)$$

Moreover, we have

$$\|\tilde{R}_n\|_{L^2([0,1])} \leq C_2 \varepsilon_n.$$

*Proof.* We write  $\tilde{R}_n = R_n + \left( A - \sqrt{\frac{\pi}{2K(\sqrt{q})}} \right) \tilde{\psi}_n$ . By using (47) and (54) one checks that  $|R_n(x)|$  satisfies a similar bound as the one given by (60). Moreover, since  $|J_0(s)| \leq 1$ , we know by using (34) that  $\tilde{\psi}_n(x)$  is bounded by  $\sqrt{2}\chi_n^{1/4}$ . Using also the bound of  $\sqrt{s}|J_0(s)|$  on  $\mathbb{R}_+$ , one gets

$$|\tilde{\psi}_n(x)| \leq \min \left( \sqrt{2}\chi_n^{1/4}, \sqrt{2/\pi}(1-x^2)^{-1/4}(1-qx^2)^{-1/4} \right).$$

The proof of (60) follows from the previous inequality and from (56). This later implies that the second term satisfies also a similar bound to the one given in (60). The estimates in  $L^2$  follow from (50), Lemma 6 and (60) again.  $\square$

### 3.3 An improved error bound for the eigenvalues $\chi_n$ .

In this subsection, we use the WKB approximation of the function  $U$  given by Lemma 3 to improve the bounds given for the eigenvalue  $\chi_n$ . This result is contained in the following theorem. We first recall that under the notations of Section 2, we have

$$\sqrt{\tilde{\chi}_n(c)} = c\tilde{\Phi} \left( \frac{\pi(n + \frac{1}{2})}{2c} \right). \quad (61)$$

**Theorem 3.** *There exist two constants  $\kappa, \kappa'$  such that, when  $(1-q)\sqrt{\chi_n} \geq \kappa'$ , we have*

$$|\sqrt{\tilde{\chi}_n(c)} - \sqrt{\chi_n(c)}| \leq \frac{\kappa}{(1-q)\sqrt{\chi_n}}. \quad (62)$$

Remark that this justifies the choice of the middle point  $n + \frac{1}{2}$  in the approximation of  $\sqrt{\chi_n}$  by formula (61).

*Proof.* The proof is based on the comparison of the (normalized) function  $V = \psi_n(1)^{-1}\chi_n^{1/4}U$  with the function

$$W(s) = (\sqrt{\chi_n s})^{1/2} J_0(\sqrt{\chi_n s}).$$

When  $(1-q)\sqrt{\chi_n} \geq 1$ , we know by Lemma 3 that  $V$  and  $W$  differ by  $\frac{\gamma}{(1-q)\sqrt{\chi_n}}$  on the interval  $[0, S(0)]$  (the lemma gives an explicit constant  $\gamma$ , but from this point we do not try to track constants). Moreover both functions have alternative positive and negative local extrema, with increasing absolute values (see for instance [31] p. 166 and also [3]; for the function  $V$  it is a consequence of Proposition 1). It is easy to characterize the point  $S(0)$  for the function  $V$ . Indeed, when  $n$  is odd, the function  $\psi_n$  is also odd and  $S(0)$  is the  $\frac{n+1}{2}$ -th zero of  $V$ . When  $n$  is even, then  $S(0)$  is the  $\frac{n}{2}$ -th local extremum of  $V$ . We have used the fact that zeros of  $V$  correspond to zeros of  $\psi_n$  through the change of variable given by  $S$  (except for  $s = 0$ , which we do not count), so that  $V$  has  $\frac{n+1}{2}$  zeros in  $(0, S(0)]$  when  $n$  is odd and  $\frac{n}{2}$  zeros when  $n$  is even. We claim that the two functions  $V$  and  $W$  have nearby zeros and extrema. This allows us to get an approximation of  $\sqrt{\chi_n}S(0)$ .

The proof is easier for  $n = 2m + 1$  and we first do it in this case. Let  $m_0 \approx 0.767$  be the value of the first local maximum of  $W$  and assume that  $(1-q)\sqrt{\chi_n} \geq 5\gamma$ . The maxima of  $|W|$  are all larger than  $m_0$  and smaller than  $m_\infty = \sqrt{2/\pi}$ , while, because of the fact that their difference is at most  $1/4$ , the maxima of  $|V|$  lie in the interval  $(1/2, 1)$ . It is classical to deduce from this study that the  $k$ -th zero of one of the functions lies between two consecutive extrema of the other one (see for instance [36]). We use this to conclude that  $W$  has exactly  $m$  local extrema on  $(0, S(0)]$  and that  $\sqrt{\chi_n}S(0)$  belongs to the interval  $(s_m, s_{m+1})$ . Here  $s_k$  is the value of the variable  $s$  for which  $W$  takes its  $k$ -th local extremum.

At this moment we use also the asymptotic behavior of the Bessel function  $J_0$  to write that, for  $s \geq S(0)/2$ ,

$$|(\sqrt{\chi_n s})^{1/2} J_0(\sqrt{\chi_n s}) - \sqrt{\frac{2}{\pi}} \cos(\sqrt{\chi_n s} - \frac{\pi}{4})| \leq \frac{\gamma'}{\sqrt{\chi_n}}.$$

If we set  $T(s) = \sqrt{\frac{2}{\pi}} \cos(\sqrt{\chi_n s} - \frac{\pi}{4})$ , then  $T$  and  $W$ , as well as  $T$  and  $V$ , differ from a quantity that is much smaller than their local extrema under the condition that  $(1-q)\sqrt{\chi_n} \geq \kappa'$ , with  $\kappa'$  large enough. We now assume that this is satisfied. So both  $V$  and  $W$  have their  $m+1$ -th zero between the two same consecutive extrema of  $T$ . We know that the  $m+1$ -th zero of  $J_0$  belongs to the interval  $(m\pi + \frac{\pi}{4}, (m+1)\pi + \frac{\pi}{4})$ . So  $\sqrt{\chi_n}S(0)$  belongs to the same interval.

If  $\delta = \sqrt{\chi_n}S(0) - m - \frac{3\pi}{4}$ , we have  $\delta \in (-\pi/2, +\pi/2)$ . Moreover,  $|\sin(\delta)| = \sqrt{\frac{\pi}{2}} |T(\sqrt{\chi_n}S(0))| \leq \sqrt{\frac{\pi}{2}} \frac{\gamma + \gamma'}{(1-q)\sqrt{\chi_n}}$ , using the fact that  $V$  vanishes at this point. It is elementary that this implies the inequality  $|\delta| \leq \frac{\pi\kappa}{2(1-q)\sqrt{\chi_n}}$  for  $\kappa = (\frac{\pi}{2})^{1/2}(\gamma + \gamma')$ .

We have proved that

$$|\sqrt{\chi_n} \mathbf{E}(\sqrt{q}) - \frac{\pi}{2}(n + \frac{1}{2})| \leq \frac{\pi\kappa}{2(1-q)\sqrt{\chi_n}}.$$

By dividing by  $c$ , composing with the function  $\tilde{\Phi}$ , and using the fact that its derivative is bounded by  $\frac{2}{\pi}$  and finally, by multiplying by  $c$ , one concludes that if  $(1-q)\sqrt{\chi_n} \geq \kappa'$ , then we have

$$|\sqrt{\chi_n} - \sqrt{\tilde{\chi}_n}| \leq \frac{\kappa}{(1-q)\sqrt{\chi_n}}.$$

It remains to adapt the proof to even values of  $n$ . Now  $U$  does not vanish at  $S(0)$  and the same role will be played by the largest zero  $s_0$  of  $U$ . It is elementary to see that the proof is the same as in the previous case, once we have proved that  $s_0 = S(0) - \frac{\pi}{2\sqrt{\chi_n}} + O(1/\chi_n)$ . So let us prove this last fact. It is easier to change the variable as in Section 2 and consider the first zero  $S(0) - s_0 > 0$  of the even solutions of Equation (8). By parity,  $s_0 - S(0)$  is the next zero on the left. We use Sturm comparison theorem between equation (8) and the equation  $Y'' + (\chi_n + \frac{q+1}{2})Y = 0$  to obtain that  $S(0) - s_0 \geq \frac{\pi}{2\sqrt{\chi_n + \frac{q+1}{2}}}$  as in Section 2. To have an upper bound of the quantity  $S(0) - s_0$ , we also use Sturm comparison theorem with the equation  $Y'' + (\chi_n + B)Y = 0$ , where  $B$  is an appropriate bound of  $\tilde{\theta}$ , or equivalently, where the function  $\theta \circ S$  is bounded by  $B$  between 0 and the first zero of  $\psi_n$ . Osipov has proved in [27] that the first positive zero of  $\psi_n$  lies before  $\frac{\pi}{2\sqrt{\chi_n + 1}}$ . At this point it is sufficient to consider the expression of  $\theta \circ S$  given in (7). It is bounded by  $\frac{5}{4}(1 - \frac{\pi^2}{4}\chi_n)^{-2}$ , from which we conclude.  $\square$

**Remark 4.** From (62) the approximation error caused by replacing  $\sqrt{\chi_n}$  by  $\sqrt{\tilde{\chi}_n}$ , is of the same order as the one obtained from replacing  $\psi_n$  by its WKB approximation, up to the factor  $\mathbf{K}(\sqrt{q})^{-1/2}$ . This probably explains the accuracy of numerical tests in which PSWFs are replaced by the main term of Corollary 1, this last one being computed with  $\sqrt{\tilde{\chi}_n}$  instead of  $\sqrt{\chi_n}$ .

## 4 Uniform estimates for the PSWFs at end points for $q$ .

Our previous results do not extend to the value  $q = 1$ , mainly because of the fact that the function  $F$ , which has been introduced in Section 3 is no longer continuous on the whole interval  $[0, S(0)]$ . At this moment we do not know how to deal with all values of  $q < 1$  at the same time, but we concentrate here on the case  $q = 1$ , where we can nevertheless develop uniform approximation. The underlying idea is that the previous methods are valid on the interval  $(0, 1 - \sqrt{\chi_n}^{-1})$ , while a priori

estimates allow to know the behavior of  $\psi_n$  on the missing interval. This is not only valid for  $q = 1$  but for values of  $q$  that are very close to 1, for which the change of variable related to 1 is still relevant.

In a second subsection, we give a WKB approximation of  $\psi_n$  in terms of the normalized Legendre polynomial  $\bar{P}_n$ . This may be helpful for  $c$  small and  $n$  not too large. Otherwise Legendre polynomials themselves are very well approximated in terms of  $J_0$ .

#### 4.1 Uniform estimates for the PSWFs when $c^2 \approx \chi_n$ .

We see in this paragraph that the method that we have used for  $q < 1$  holds also for  $q = 1$  and, up to some extent, when  $\chi_n$  and  $c^2$  are very close. Let us first recall that (see for example [34])

$$\partial_c(c^2 - \chi_n(c)) = 2c \int_{-1}^{+1} (1-t^2)|\psi(t)|^2 dt$$

so that, for a fixed value of the positive integer  $n$ , there is exactly one value of  $c$  for which  $q = 1$ . We go back to the equation (4) and use the change of function  $U(1-x) = \sqrt{1-x^2}\psi_n(x)$  that coincides with the previous change when  $q = 1$ . The equation for  $U$  is now

$$U''(s) + \left( \frac{1}{4s^2(1-s/2)^2} + \chi_n + \frac{\chi_n - c^2}{2s(1-s/2)} \right) U(s) = 0, \quad 0 \leq s \leq 1.$$

As before, we write this equation as

$$U''(s) + \left( \frac{1}{4s^2} + \chi_n \right) U(s) = F(s)U(s).$$

A straightforward computation leads to  $|F(s)| \leq \frac{1}{2} + |\gamma_n| + \frac{|\gamma_n|}{s}$ , where

$$\gamma_n = \frac{2(1-q)\chi_n + 1}{4}.$$

We let  $\delta_n = \frac{1}{2} + 2|\gamma_n|$  and only use the inequality  $|F(s)| \leq \frac{|\delta_n|}{s}$  to simplify expressions, even if this leads to weaker estimates. Remark in particular that, when  $\gamma_n$  vanishes, the function  $F$  is bounded and the results of the last section are directly adapted.

We do not restrict ourselves to this case but go on with the same kind of proof. For simplification, we use the same notations. We write

$$\psi_n(x) = \frac{A\chi_n^{1/4} J_0(\sqrt{\chi_n}(1-x))}{\sqrt{1+x}} + R_n(x),$$

with  $A = \sqrt{2}\psi_n(1)\chi_n^{-1/4}$ . We have

$$\sqrt{1-x^2}R_n(x) = W(1-x) = \frac{1}{\sqrt{\chi_n}} \int_0^{1-x} K_n(1-x, t)F(t)U(t) dt.$$

As in the last section the kernel  $|K_n|$  is bounded by 1. We claim that  $W$  satisfies the inequality

$$|W(s)| \leq \delta_n \chi_n^{-1/2} \begin{cases} 2A(s\sqrt{\chi_n})^{1/2} & \text{if } s \leq \chi_n^{-1/2} \\ 2A + (\log \sqrt{\chi_n}) \sup_{s \in [0,1]} |U(s)| & \text{otherwise.} \end{cases} \quad (63)$$

Indeed, we have the inequality

$$|W(s)| \leq \delta_n \chi_n^{-1/2} \int_0^s \frac{|U(t)|}{t} dt = \delta_n \chi_n^{-1/2} \int_{1-s}^1 \sqrt{\frac{1+t}{1-t}} |\psi_n(t)| dt.$$

Recall that the maximum of  $|\psi_n|$  is attained at 1. We use the last expression and the fact that  $|\psi_n(t)| \leq \frac{A}{\sqrt{2}}(\chi_n)^{1/4}$  to conclude for the first bound. For the second one, we cut the integral into two parts. From 0 to  $\chi_n^{-1/2}$ , we use the previous inequality. From  $\chi_n^{-1/2}$  to  $s$ , we just conclude directly by using the first expression.

We will not use Olver's estimates but proceed as in the proof of (48) given at the end of Section 3. 1. As a consequence of (63), we have the inequality

$$\sup_{s \in [0,1]} |U(s)| \leq \sqrt{\frac{2}{\pi}} A + \delta_n \chi_n^{-1/2} (2A + (\log \sqrt{\chi_n}) \sup_{s \in [0,1]} |U(s)|).$$

From now on we assume that  $\delta_n \chi_n^{-1/2} \log \sqrt{\chi_n} < 1/2$ , so that we conclude from the last inequality that  $\sup |U(t)| \leq 4A$ . In the sequel, we do not give explicit bounds for uniform constants  $\kappa$ . We have the following inequality, which plays the role of the estimate given by (47).

$$|R_n(x)| \leq \kappa A \delta_n \chi_n^{-1/2} \log \sqrt{\chi_n} \min(\chi_n^{1/4}, (1-x^2)^{-1/2}).$$

Moreover, it follows from the expression of  $R_n$  that

$$\|R_n\|_{L^2([0,1])} \leq \kappa A \delta_n \chi_n^{-1/2} (\log(\sqrt{\chi_n}))^{3/2}.$$

From this point, the same method as in the last section can be used. We have to find an equivalent of the  $L^2(0,1)$ -norm of the main term. This is based on the following lemma.

**Lemma 7.** *Let  $\gamma$  be the Euler constant. Then, for any real  $x > 0$ , we have*

$$\int_0^x J_0(t)^2 dt = \frac{1}{\pi} (\log(x) + \gamma + 3 \log 2) + \epsilon_n, \quad |\epsilon_n| \leq \frac{0.4}{x}. \quad (64)$$

*Proof.* The proof follows from Lemma 5 and from the following identity, given in [11]

$$\int_0^\infty \left( J_0(t)^2 - \frac{1}{1+t} \right) dt = \frac{1}{\pi} (\gamma + 3 \log 2).$$

We can write

$$\begin{aligned} \int_0^x J_0^2(t) dt &= \frac{1}{\pi} (\gamma + 3 \log 2 + \log(1+x)) - \int_x^\infty \left( J_0^2(t) - \frac{1}{\pi(t+1)} \right) dt \\ &= \frac{1}{\pi} (\gamma + 3 \log 2 + \log(x)) - \int_x^\infty \left( J_0^2(t) - \frac{1}{\pi t} \right) dt. \end{aligned}$$

If as before,  $G(x) = \frac{x^2}{2} [(J_0(x))^2 + (J_1(x))^2] - \frac{x}{\pi}$ , then a simple integration by parts gives us

$$\int_x^\infty \left( J_0^2(t) - \frac{1}{\pi t} \right) dt = \int_x^\infty \frac{1}{t} \left( t J_0^2(t) - \frac{1}{\pi} \right) dt = -\frac{G(x)}{x} + \int_x^\infty \frac{G(t)}{t^2} dt.$$

Hence, we have

$$\left| \int_x^\infty \left( J_0^2(t) - \frac{1}{\pi t} \right) dt \right| \leq \frac{2\kappa_2}{x}, \quad x > 0.$$

□

**Lemma 8.** *Let  $\tilde{\psi}(x) = \frac{\chi_n^{1/4} J_0(\sqrt{\chi_n}(1-x))}{\sqrt{1+x}}$ . Then, for  $\beta = 4 \log 2 + \gamma$ , where  $\gamma$  is the Euler constant, we have*

$$\|\tilde{\psi}\|_2^2 = \frac{\log(\sqrt{\chi_n}) + \beta}{\pi} + \epsilon_n, \quad |\epsilon_n| \leq \frac{0.8}{\sqrt{\chi_n}}.$$

*Proof.* After a change of variable, we have

$$\|\tilde{\psi}\|_2^2 = \int_0^{\sqrt{\chi_n}} \frac{|J_0(t)|^2}{1 - t/(2\sqrt{\chi_n})} dt = \int_0^{\sqrt{\chi_n}} |J_0(t)|^2 dt + \frac{1}{2\sqrt{\chi_n}} \int_0^{\sqrt{\chi_n}} \frac{t|J_0(t)|^2}{1 - t/(2\sqrt{\chi_n})} dt.$$

From (64), the first term in place is given by

$$\int_0^{\sqrt{\chi_n}} |J_0(t)|^2 dt = \frac{1}{\pi} (\log(\sqrt{\chi_n}) + \gamma + 3 \log 2) + \epsilon_n, \quad |\epsilon_n| \leq \frac{0.4}{\sqrt{\chi_n}}.$$

To estimate the second term in (8), one can proceed exactly as in the proof of Lemma 6 with now  $\theta(t) = \frac{1}{1 - t/(2\sqrt{\chi_n})}$  and find the quantity  $\frac{\log 2}{\pi} + \epsilon_n$  with  $|\epsilon_n| \leq 2\kappa_2/\sqrt{\chi_n}$ .  $\square$

This allows us to state the following result, which may be seen as an extension of the previous section in the context  $\chi_n \approx c^2$ .

**Theorem 4.** *Let  $\varepsilon > 0$  be given. There exists a constant  $\kappa$  such that, for  $\delta_n \chi_n^{-1/2} \log \sqrt{\chi_n} \leq \kappa$ , one can write*

$$\psi_n(x) = \left( \frac{\pi}{\log(\sqrt{\chi_n}) + \beta} \right)^{\frac{1}{2}} \frac{\chi_n^{1/4} J_0(\sqrt{\chi_n}(1-x))}{\sqrt{1+x}} + \tilde{R}_n(x),$$

with

$$|\tilde{R}_n(x)| \leq \varepsilon (\log(\sqrt{\chi_n}))^{-1/2} \min \left( \chi_n^{1/4}, \frac{1}{\sqrt{1-x^2}} \right), \quad \|\tilde{R}_n\|_2 \leq \varepsilon.$$

We do not give more details. Remark that we also proved that under the conditions given on  $n, c$ , we have the asymptotic value  $\left( \frac{\pi}{2 \log \sqrt{\chi_n} + \beta} \right)^{\frac{1}{2}} \chi_n^{1/4}$  for  $\psi_n(1)$ . When  $\gamma_n = \frac{2(1-q)\chi_n+1}{4} = 0$ , it is sufficient to have a condition that does not involve a logarithm, namely  $\delta_n \chi_n^{-1/2} \leq \kappa$ .

## 4.2 Uniform estimates for the PSWFs when $c^2/\chi_n \approx 0$ .

Let  $\bar{P}_n = \psi_{n,0}$  be the normalized Legendre polynomials, so that  $\|\bar{P}_n\|_{L^2[-1,1]} = 1$ . Section 3 can be used to obtain uniform estimates of  $\bar{P}_n$ . This kind of estimates for Legendre polynomials have been known for a long time, see [31]. In this case, we have  $S(x) = \arccos(x)$  and it is simpler to use  $n + 1/2$  instead of  $\chi_n$  in (24), so that uniform estimates may be written as

$$\bar{P}_n(\cos \theta) = (n + 1/2)^{1/2} \left( \frac{\theta}{\sin \theta} \right)^{1/2} J_0((n + 1/2)\theta) + O\left(\frac{1}{n}\right)$$

for  $0 \leq \theta \leq \frac{\pi}{2}$ . Precise estimates of the remainder, which improve ours, are given in [10]. In this paragraph, we use the same method as in Section 3 to approximate  $\psi_n$  by  $\bar{P}_n$  when  $c$  is close to 0. The main result is given by the next proposition. The first statement expresses the fact that  $\psi_n(1)$  is close to its value for  $c = 0$ , that is,  $\sqrt{n + 1/2}$ .

**Proposition 5.** *For all  $n \in \mathbb{N}$  and  $c \geq 0$  we have the inequalities*

$$\left| \frac{\psi_{n,c}(1)}{\sqrt{n + 1/2}} - 1 \right| \leq \frac{c^2}{\sqrt{3}(n + 1/2)}, \quad (65)$$

$$\sup_{x \in [-1,1]} |\psi_{n,c}(x) - \bar{P}_n(x)| \leq \frac{c^2}{\sqrt{3}(n + 1/2)} \left( 1 + \frac{\sqrt{3/2}}{\sqrt{n + 1/2}} \right). \quad (66)$$

*Proof.* We now rewrite the equation as follows,

$$(1 - x^2)\psi_n'' - 2x\psi_n' + n(n+1)\psi_n = (n(n+1) - \chi_n + c^2x^2)\psi_n.$$

The homogeneous equation

$$(1 - x^2)\psi_n'' - 2x\psi_n' + n(n+1)\psi_n = 0$$

has the two linearly independent solutions  $\bar{P}_n$  and  $Q_n$ , where  $Q_n$  is the Legendre function of the second kind. Recall that (see [1]), because of the normalization for  $\bar{P}_n$ , the Wronskian is given by  $W(\bar{P}_n, Q_n)(x) = \frac{n+1/2}{1-x^2}$ . The absolute value of the function  $G = n(n+1) - \chi_n + c^2x^2$  is bounded by  $c^2$ . By the method of variation of constants, we can write, for  $x > 0$ ,

$$\psi_n(x) = A\bar{P}_n(x) + \frac{1}{n+1/2} \int_x^1 L_n(x, y) \sqrt{1-y^2} G(y) \psi_n(y) dy,$$

where we have used the notation

$$L_n(x, y) = \sqrt{1-y^2} (\bar{P}_n(x)Q_n(y) - \bar{P}_n(y)Q_n(x)).$$

As in the other cases, the behavior at 1 has been used to see that there is no term in  $Q_n$ .

We have the following lemma, which is the equivalent of (27) for Bessel functions in the present context.

**Lemma 9.** *We have the inequality, valid for all  $n > 0$  and  $x < 1$ ,*

$$\sqrt{1-x^2} (\bar{P}_n(0)^{-2}\bar{P}_n(x)^2 + \bar{P}_n(0)^2Q_n(x)^2) \leq 1.$$

Let us take this lemma for granted and go on for the proof. By Cauchy-Schwarz inequality, this implies in particular that, for  $0 \leq x \leq y \leq 1$  the kernel  $|L_n|$  is bounded by 1. The consideration of the behavior of each term when  $x$  tends to 1 implies that  $B$  is equal to 0. Moreover, by using the fact that  $\psi_n$  has norm  $1/\sqrt{2}$  in  $L^2([0, 1])$  and Cauchy-Schwarz inequality, one gets the inequality

$$|\psi_n(x) - A\bar{P}_n(x)| \leq \frac{c^2}{\sqrt{2}(n+1/2)}(1-x).$$

Since the function  $\bar{P}_n$  has also  $L^2([0, 1])$ -norm  $1/\sqrt{2}$ , then we have  $|1 - A| \leq \sqrt{\frac{1}{3} \frac{c^2}{n+1/2}}$ . This gives (65). In view of (66), we have

$$\sup_{x \in [0, 1]} |\psi_n(x) - \bar{P}_n(x)| \leq \sup_{x \in [0, 1]} |\psi_n(x) - A\bar{P}_n(x)| + |A - 1| \sup_{x \in [0, 1]} |\bar{P}_n(x)|.$$

We conclude by using the previous inequalities and the fact that  $|\bar{P}_n|$  is bounded by its value at 1.  $\square$

It remains to prove Lemma 9.

*Proof of Lemma 9.* We have seen that  $\sqrt{1-x^2}|\bar{P}_n(x)|^2 \leq |\bar{P}_n(0)|^2 + \left(\frac{|\bar{P}'_n(0)|}{n+1/2}\right)^2$ . Depending on the parity, only the first term or the second term is non-zero in the right hand side. We shall assume that  $n$  is even, but the proof would be similar for odd values. This inequality is valid for all solutions of the homogeneous equation. In particular, it is valid for the particular solution  $c\bar{P}_n(x) + dQ_n(x)$ , with  $c, d \in \mathbb{R}$ . This means that for  $n$  even,  $Q_n(0) = \bar{P}'_n(0) = 0$  and

$$\sqrt{1-x^2}(c\bar{P}_n(x) + dQ_n(x))^2 \leq c^2\bar{P}_n(0)^2 + \left(\frac{d^2Q'_n(0)}{n+1/2}\right)^2.$$

Using the fact that

$$\frac{Q'_n(0)\bar{P}_n(0)}{n+1/2} = \frac{W(\bar{P}_n, Q_n)(0)}{n+1/2} = 1,$$

the previous inequality becomes

$$\sqrt{1-x^2}(c\bar{P}_n(x) + dQ_n(x))^2 \leq c^2\bar{P}_n(0)^2 + d^2\frac{1}{\bar{P}_n(0)^2}.$$

This inequality is valid for all  $c, d$ . We conclude by taking the particular choices  $c = \bar{P}_n(0)^{-2}\bar{P}_n(x)$ ,  $d = \bar{P}_n(0)^2Q_n(x)$  and dividing both sides of the inequality that we obtain by the right hand side.  $\square$

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