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# Approximations in Sobolev Spaces by Prolate Spheroidal Wave Functions.

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**Abstract**— Recently, there is a growing interest in the spectral approximation by the Prolate Spheroidal Wave Functions (PSWFs)  $\psi_{n,c}$ ,  $c > 0$ . This is due to the promising new contributions of these functions in various classical as well as emerging applications from Signal Processing, Geophysics, Numerical Analysis, etc. The PSWFs form a basis with remarkable properties not only for the space of band-limited functions with bandwidth  $c$ , but also for the Sobolev space  $H^s([-1, 1])$ . The quality of the spectral approximation and the choice of the parameter  $c$  when approximating a function in  $H^s([-1, 1])$  by its truncated PSWFs series expansion, are the main issues. By considering a function  $f \in H^s([-1, 1])$  as the restriction to  $[-1, 1]$  of an almost time-limited and band-limited function, we try to give satisfactory answers to these two issues. Also, we illustrate the different results of this work by some numerical examples.

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## 1 Introduction

Let  $f$  be a function that belongs to some Sobolev space  $H^s(I)$ ,  $s > 0$ ,  $I = [-1, 1]$ . The main issue of this work concerns the speed of convergence in  $L^2(I)$  of its expansion in some PSWF basis.

Let us recall that, for a given value  $c > 0$ , called the bandwidth, PSWFs  $(\psi_{n,c})_{n \geq 0}$  constitute an orthonormal basis of  $L^2([-1, +1])$  of eigenfunctions of the two compact integral operators  $\mathcal{F}_c$  and  $\mathcal{Q}_c = \frac{c}{2\pi} \mathcal{F}_c^* \mathcal{F}_c$ , defined on  $L^2(I)$  by

$$\mathcal{F}_c(f)(x) = \int_{-1}^1 e^{icxy} f(y) dy, \quad \mathcal{Q}_c(f)(x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} f(y) dy. \quad (1)$$

PSWFs are also eigenfunctions of the Sturm-Liouville operator  $\mathcal{L}_c$ , defined by

$$\mathcal{L}_c(\psi) = -\frac{d}{dx} \left[ (1-x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi, \quad (2)$$

We call  $\chi_n(c)$  the eigenvalues of  $\mathcal{L}_c$ , and  $\lambda_n(c)$  the eigenvalues of  $\mathcal{Q}_c$ . The first ones are arranged in the increasing order, the second ones in the decreasing order  $1 > \lambda_0(c) > \lambda_1(c) > \dots > \lambda_n(c) > \dots$ .

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We finally call  $\mu_n(c)$  the eigenvalues of  $\mathcal{F}_c$ . They are given by

$$\mu_n(c) = i^n \sqrt{\frac{2\pi}{c} \lambda_n(c)}.$$

By Plancherel identity, PSWFs are normalized so that

$$\int_{-1}^1 |\psi_{n,c}(x)|^2 dx = 1, \quad \int_{\mathbb{R}} |\psi_{n,c}(x)|^2 dx = \frac{1}{\lambda_n(c)}, \quad n \geq 0. \quad (3)$$

We adopt the sign normalization of the PSWFs, given by

$$\psi_{n,c}(0) > 0 \text{ for even } n, \quad \psi'_{n,c}(0) > 0, \text{ for odd } n. \quad (4)$$

A breakthrough in the theory and the computation of the PSWFs goes back to the 1960's and is due to D. Slepian and his co-authors H. Landau and H. Pollack. For the classical and more recent developments in the area of the PSWFs, the reader is referred to the recent books on the subjects [13, 16]. This paper is a companion paper of [3] and we refer to it for further notations and references.

This question of the quality of approximation has attracted a growing interest while, at the same time, were built PSWFs based numerical schemes for solving various problems from numerical analysis, see [4, 5, 6, 12, 14, 17, 20]. In particular, in [4], the author has shown that a PSWF approximation based method outperforms in terms of spatial resolution and stability of time-step, the classical approximation methods based on Legendre or Tchebyshev polynomials. The authors of [6] were among the first to compare the quality of approximation by the PSWFs for different values of  $c$ . In particular, they have given an estimate of the decay of the PSWFs expansion coefficients of a function  $f \in H^s(I)$ , see also [4]. Recently, in [20], the author studied the speed of convergence of the expansion of such a function in a basis of PSWFs. We should mention that the methods used in the previous three references are heavily based on the use of the properties of the PSWFs as eigenfunctions of the differential operator  $\mathcal{L}_c$ , given by (2). They pose the problem of the best choice of the value of the band-width  $c > 0$ , for approximating well a given  $f \in H^s(I)$ , but their answer is mainly experimental. It has been numerically checked in [4, 20] that the smaller the value of  $s$ , the larger the value of  $c$  should be.

Our study tries to give a satisfactory answer to this important problem of the choice of the parameter  $c$ . More precisely, we show that if  $f \in H^s(I)$ , for some positive real number  $s > 0$ , then for any integer  $N \geq 1$ , we have

$$\|f - S_N f\|_{L^2(I)} \leq K(1 + c^2)^{-s/2} \|f\|_{H^s(I)} + K\sqrt{\lambda_N(c)} \|f\|_{L^2(I)}. \quad (5)$$

Here,  $S_N f = \sum_{k=0}^N \langle f, \psi_{n,c} \rangle \psi_{n,c}$  and  $K$  is a constant depending only on  $s$ . With this expression, one sees clearly how to distribute a fixed error between that part which is due to the smoothness of the function and that part which is due to the speed of convergence for the PSWFs. We also study an  $L^2(I)$ -convergence rate of the projection  $S_N f$  to  $f$ . This is done by using the decay of the eigenvalues  $(\lambda_n(c))_n$  as well as estimates of Legendre expansion coefficients of PSWFs and the decay of Fourier coefficients of PSWFs. We prove an exponential decay rate, given by

$$|\langle e^{ik\pi x}, \psi_{n,c}(x) \rangle| \leq M' e^{-an}, \quad |k| \leq n/M, \quad n \geq \max(cM, 3). \quad (6)$$

Here,  $c \geq 1$ ,  $M \geq \sqrt{2}$  and  $M', a > 0$  are two positive constants. Under these hypotheses and notations, our rate of convergence of  $S_N f$  to  $f \in H^s(I)$  is given by

$$\|f - S_N(f)\|_{L^2(I)} \leq M''(1 + N^2)^{-s/2} \|f\|_{H^s} + M' e^{-aN} \|f\|_{L^2}. \quad (7)$$

This work is organized as follows. In Section 2, we first give bounds for the moments of the PSWFs, which we use to improve estimates of the decay of the Legendre expansion coefficients of the PSWFs. In Section 3, we first consider the quality of approximation by the PSWFs in the set of almost time and band-limited functions. Then, we combine these results with those of Section 2 and give a first  $L^2(I)$ -error bound of approximating a function  $f \in H^s(I)$  by the  $N$ -th partial sum of its PSWFs series expansion. Then, we study a more elaborated error analysis of the spectral approximation by the PSWFs in the periodic Sobolev space. This is afterwards extended to the usual Sobolev space  $H^s(I)$ . These new estimates provide us with a way for the choice of the appropriate bandwidth  $c > 0$  to be used by a PSWFs based method for the approximation in a given Sobolev space  $H^s(I)$ . In Section 4, we provide the reader with some numerical examples that illustrate the different results of this work.

We will frequently skip the parameter  $c$  in  $\chi_n(c)$  and  $\psi_{n,c}$ , when there is no doubt on the value of the bandwidth. We then note  $q = c^2/\chi_n$  and skip both parameters  $n$  and  $c$  when their values are obvious from the context.

## 2 Decay estimates of the Legendre expansion coefficients

In this paragraph, we give bounds for the Legendre expansion coefficients of PSWFs, which will be used later and are of general interest. Legendre expansion coefficients of PSWFs have been the object of many studies, in particular in relation with numerical methods for their evaluation. In particular, the classical method known as Flammer's method, [7] that uses the differential operator  $\mathcal{L}_c$ , is extensively used to compute the PSWFs and their eigenvalues.

The Legendre expansion of the PSWFs is given by

$$\psi_n(x) = \sum_{k \geq 0} \beta_k^n \overline{P_k}(x). \quad (8)$$

Recall that  $\psi_n$  has the same parity as  $n$ . Hence, the previous Legendre expansion coefficients of the  $\psi_n$  satisfy  $\beta_k^n = 0$  if  $n$  and  $k$  have different parities.

It is well known that the different expansion coefficients  $(\beta_k^n)_k$  as well as the corresponding eigenvalues  $\chi_n$  are obtained by solving the following eigensystem

$$\begin{aligned} & \frac{(k+1)(k+2)}{(2k+3)\sqrt{(2k+5)(2k+1)}} c^2 \beta_{k+2}^n + \left( k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} c^2 \right) \beta_k^n \\ & + \frac{k(k-1)}{(2k-1)\sqrt{(2k+1)(2k-3)}} c^2 \beta_{k-2}^n = \chi_n(c) \beta_k^n, \quad k \geq 0. \end{aligned} \quad (9)$$

The eigenvalues  $\chi_n$  satisfy the following well known bounds

$$n(n+1) \leq \chi_n \leq n(n+1) + c^2. \quad (10)$$

In the case where  $q = c^2/\chi_n \leq 1$ , we have the following less classical bounds

$$n(n+1) + (3 - 2\sqrt{2})c^2 \leq \chi_n \leq \left( \frac{\pi}{2}(n+1) \right)^2. \quad (11)$$

The above lower bound has been given in [2] and the upper bound is given in [15]. The previous inequalities will be needed in different parts of this work.

The decay of the coefficients  $\beta_k^n$  has been the object of many studies. We give here upper bounds of the  $|\beta_k^n|$  with  $k$  varying from 0 to a certain order depending on  $n$ . Such bounds have been proposed in [6]. There was a gap in their proof, which has been filled up in the book [13] with some loss in the results. We improve these estimates that have been proposed by [6], both for the range of  $k$  and the

constant. In view of these results we also prove auxiliary properties, which may be of independent interest. We first provide bounds for the successive derivatives of  $\psi_n$  at 0. For  $n = 0$ , we have proved in [2] that, for  $q < 2$ ,

$$|\psi_n(0)|^2 + \chi_n^{-1} |\psi'_n(0)|^2 \leq 1. \quad (12)$$

Recall that either  $\psi_n(0)$  or  $\psi'_n(0)$  is zero, depending on the parity. The same is valid for higher derivatives at 0.

**Proposition 1.** *Assume that  $c^2 < \chi_n$ . Then for any integer  $k \geq 0$  satisfying  $k(k+1) \leq \chi_n$ , we have*

$$\left| \psi_n^{(k)}(0) \right| \leq (\sqrt{\chi_n})^k (|\psi_n(0)|^2 + \chi_n^{-1} |\psi'_n(0)|^2)^{1/2}. \quad (13)$$

*Proof.* Since  $\psi_{n,c}$  has same parity as  $n$ , then it is sufficient to consider even derivatives or odd derivatives depending on the parity of  $n$ . We assume that  $n$  is even and consider  $k = 2l$ . The proof is identical for odd values. By an iterative use of the identity

$$(1 - x^2)\psi_n''(x) = 2x\psi_n'(x) + (c^2x^2 - \chi_n)\psi_n(x),$$

one can easily check that the  $\psi_{n,c}^{(k)}(0) = \psi^{(k)}(0)$  are given by the recurrence relation

$$\psi^{(k+2)}(0) = (k(k+1) - \chi_n)\psi^{(k)}(0) + k(k-1)c^2\psi^{(k-2)}(0), \quad k \geq 0. \quad (14)$$

Let us show by induction that for a fixed  $n$ ,  $\psi_{n,c}^{(2l)}(0)$  has alternating signs, that is  $\psi_{n,c}^{(k)}(0)\psi_{n,c}^{(k-2)}(0) < 0$ . Indeed, with  $\psi(0) > 0$ ,  $\psi^{(2)}(0) = -\chi_n\psi(0)$  so that the induction hypothesis is fulfilled for  $k = 2$ . Multiplying both sides of (14) by  $\psi^{(k)}(0)$ , using the assumption that  $k(k+1) \leq \chi_n$  as well as the induction hypothesis, one concludes that the induction assumption holds for the order  $k$ . Consequently, we have,

$$\left| \psi^{(k+2)}(0) \right| = (\chi_n - k(k+1)) \left| \psi^{(k)}(0) \right| + k(k-1)c^2 \left| \psi^{(k-2)}(0) \right|, \quad k \geq 0. \quad (15)$$

This may be rewritten as

$$m_{k+2} = \left( 1 - \frac{k(k+1)}{\chi_n} \right) m_k + k(k-1) \frac{q}{\chi_n} m_{k-2}. \quad (16)$$

The fact that all the  $m_{2l}$  are bounded by  $m_0 = \psi(0) = m_2$  follows at once by induction. For  $n$  odd the proof follows the same lines.  $\square$

As a consequence of the previous proposition, we have the following corollary concerning the sign and the bounds of the different moments of the  $\psi_n$ .

**Corollary 1.** *Let  $c > 0$ , be a positive real number. We assume that  $q = c^2/\chi_n < 1$ . Then, for  $j(j+1) \leq \chi_n$ , all moments  $\int_{-1}^1 y^j \psi_n(y) dy$  are non negative and*

$$0 \leq \int_{-1}^1 y^j \psi_n(y) dy \leq \left( \frac{1}{q} \right)^j |\mu_n(c)|. \quad (17)$$

*Proof.* It is sufficient to consider moments of odd order when  $n$  is odd and of even order when  $n$  is even. By taking the  $j$ -th derivative at zero on both sides of  $\int_{-1}^1 e^{icxy} \psi_n(y) dy = \mu_n(c)\psi_n(x)$ , one gets

$$\int_{-1}^1 y^j \psi_n(y) dy = (-i)^j c^{-j} \mu_n(c) \psi_n^{(j)}(0). \quad (18)$$

Since  $\psi_n^{(j)}(0)$  and  $\psi_n^{(j+2)}(0)$  have opposite signs, then the previous equation implies that moments have the same sign for any positive integer  $j$  with  $j(j+1) \leq \chi_n$ . The inequality (17) follows from the previous proposition.  $\square$

We now study the positivity of  $\beta_k^n$  for small values of  $k$ . Remark that for  $c = 0$  all these coefficients vanish when  $k < n$ . The positivity of  $\beta_0^n$  (when  $n$  is even) and  $\beta_1^n$  (when  $n$  is odd) follows from the fact that

$$\beta_0^n = \frac{1}{\sqrt{2}} \int_{-1}^{+1} \psi_n(y) dy = \frac{1}{\sqrt{2}} |\mu_n(c)| \psi_n(0), \quad \beta_1^n = \sqrt{\frac{3}{2}} \int_{-1}^{+1} y \psi_n(y) dy = |\mu_n(c)| \frac{\psi_n'(0)}{c}. \quad (19)$$

**Lemma 1.** *Let  $c > 0$ , be a fixed positive real number. Then, for all positive integers  $k, n$  such that  $k(k-1) + 1.13c^2 \leq \chi_n(c)$ , we have  $\beta_k^n \geq 0$ .*

*Proof.* We recall that the  $\beta_k^n$  are given by the eigensystem (9). Let us first consider  $k = 2$  (when  $n$  is even) and  $k = 3$  (when  $n$  is odd) and compute

$$\beta_2^n = \frac{3\sqrt{5}}{2c^2} \left( \chi_n - \frac{c^2}{3} \right) \beta_0^n, \quad \beta_3^n = \frac{5\sqrt{21}}{6c^2} \left( \chi_n - 2 - \frac{3c^2}{5} \right) \beta_1^n.$$

They are positive, assuming that 2 satisfies the condition (resp. 3 satisfies the condition). For  $k \geq 2$ , taking upper bounds for the fractions as in [6], Equation (9) implies that

$$\frac{2c^2}{3\sqrt{5}} (\beta_{j+2}^n + \beta_{j-2}^n) \geq (\chi_n(c) - j(j+1) - \frac{11c^2}{21}) \beta_j^n. \quad (20)$$

The constant 1.13 has been chosen so that  $1.13 > \frac{4}{3\sqrt{5}} + \frac{11}{21}$ . Let us prove by induction that the sequence  $\beta_{2j}^n$  (resp.  $\beta_{2j+1}^n$ ) is non decreasing for  $2j \leq k$  (resp.  $2j+1 \leq k$ ) when the assumption is satisfied by  $k$ . Without loss of generality we assume now that  $n$  is even. We have  $\beta_2^n \geq \beta_0^n$ . Next, by the induction assumption on  $j$ , we know that  $\beta_{j-2}^n \leq \beta_j^n$ . We get a contradiction with inequality (20) if we assume that  $\beta_{j+2}^n < \beta_j^n$ . So  $\beta_{j+2}^n \geq \beta_j^n$ . This implies the positivity.  $\square$

The following proposition provides us with decay rate of the expansion coefficients  $\beta_k^n$  that we had in view.

**Proposition 2.** *Let  $c > 0$ , be a fixed positive real number. Then, for all positive integers  $n, k$  such that  $k(k-1) + 1.13c^2 \leq \chi_n(c)$ , we have*

$$|\beta_0^n| \leq \frac{1}{\sqrt{2}} |\mu_n(c)| \quad \text{and} \quad |\beta_k^n| \leq \sqrt{\frac{5}{4\pi}} \left( \frac{2}{\sqrt{q}} \right)^k |\mu_n(c)|. \quad (21)$$

*Proof.* The first inequality follows from (12) and (19). To prove the second inequality, we first note that the moments  $a_{jk}$  of the normalized Legendre polynomials  $\bar{P}_k$  are non-negative (see [1]). They vanish except for  $k \leq j$ , with  $k, j$  of the same parity. Moreover, for  $j = k$ , we have

$$a_{kk} = \int_{-1}^1 x^k \bar{P}_k(x) dx = \frac{\sqrt{\pi} \sqrt{k+1/2} k!}{2^k \Gamma(k + \frac{3}{2})}. \quad (22)$$

Since  $x^j = \sum_{k=0}^j a_{jk} \bar{P}_k(x)$ , the moments of the  $\psi_n$  are related to the PSWFs Legendre expansion coefficients by

$$\int_{-1}^1 x^j \psi_n(x) dx = \sum_{k=0}^j a_{jk} \beta_k^n.$$

Since by the previous lemma, we have  $\beta_k^n \geq 0$ , for any  $0 \leq k \leq j$  and since the  $a_{jk}$  are non negative, the previous equality implies that

$$\beta_j^n \leq \frac{1}{a_{jj}} \int_{-1}^1 x^j \psi_n(x) dx \leq \frac{1}{a_{jj}} \left( \frac{1}{q} \right)^{j/2} |\mu_n(c)|. \quad (23)$$

The last inequality follows from the previous corollary. On the other hand, we have

$$a_{jj} = \frac{\sqrt{\pi}\sqrt{j+1/2}j!}{2^j\Gamma(j+3/2)} = \frac{\sqrt{\pi}j!}{2^j\sqrt{j+1/2}\Gamma(j+1/2)}.$$

Moreover, it is well known that  $j^{1-s} \leq \frac{\Gamma(j+1)}{\Gamma(j+s)} \leq (j+1)^{1-s}$ . Hence, we have

$$\frac{1}{a_{jj}} \leq \frac{2^j}{\sqrt{\pi}} \sqrt{1 + \frac{1}{2j}} \leq 2^j \sqrt{\frac{5}{4\pi}}, \quad \forall j \geq 1. \quad (24)$$

By combining (23) and (24), one gets the second inequality of (21).  $\square$

**Remark 1.** *The condition  $k(k-1) + 1.13c^2 \leq \chi_n(c)$  of the previous proposition can be replaced with the following more explicit condition. Consider real numbers  $A, B > 1$  with  $A^2 + B^2 \geq A^2B^2$ . By using (11), one concludes that if  $n \geq cA$  and  $k \leq n/B$ , then the conditions for (21) are satisfied. In particular, one may take  $A = B \geq \sqrt{2}$ .*

In order to get from (21), a quantitative decay estimate for the Legendre expansion coefficients  $(\beta_k^n)_k$ , one needs some precise behaviour as well as the decay rate of the eigenvalues  $\mu_n(c)$  or  $\lambda_n(c)$ . These issues have been the subjects of many theoretical and numerical studies. To cite but a few [3, 9, 10, 19, 21]. In particular in [3], it has been shown that

$$\lambda_n(c) \leq A(n, c) \left( \frac{ec}{2(2n+1)} \right)^{2n+1}, \quad A(n, c) = \delta_1 n^{\delta_2} \left( \frac{c}{c+1} \right)^{-\delta_3} e^{+\frac{\pi^2}{4} \frac{c^2}{n}}. \quad (25)$$

As a consequence of the previous inequality, we have the following lemma borrowed from [3].

**Lemma 2.** *There exist constants  $a > 0$  and  $\delta \geq 1$  such that, for  $c \geq 1$  and  $n > 1.35c$ , we have*

$$\lambda_n(c) \leq \delta e^{-an}. \quad (26)$$

### 3 Quality of the spectral approximation by the PSWFs

In this section, we first briefly recall the quality of approximation of band-limited and almost band-limited functions by the classical PSWFs,  $\psi_n$  that are concentrated on  $[-b, b]$ , for some  $b > 0$ . These results are mainly due to Landau, Pollack and Slepian, see [11, 18]. For the sake of convenience, we also give some hints to the proofs of these results. Then, we show how to extend this study to the case of periodic and non periodic Sobolev space  $H^s([-1, 1])$ ,  $s > 0$ .

#### 3.1 Approximation of almost time and band-limited functions

In this paragraph,  $\|\cdot\|_2$  denotes the norm in  $L^2(\mathbb{R})$ . We show that the set  $\{\psi_n(x), n \geq 0\}$  is well adapted for the representation of almost time-limited and almost band-limited functions, which are defined as follows.

**Definition 1.** *Let  $T = [-a, +a]$  and  $\Omega = [-b, +b]$  be two intervals. A function  $f$ , which we assume to be normalized in such a way that  $\|f\|_2 = 1$ , is said to be  $\epsilon_T$ -concentrated in  $T$  and  $\epsilon_\Omega$ -band concentrated in  $\Omega$  if*

$$\int_{T^c} |f(t)|^2 dt \leq \epsilon_T^2, \quad \frac{1}{2\pi} \int_{\Omega^c} |\widehat{f}(\omega)|^2 d\omega \leq \epsilon_\Omega^2.$$

Up to a re-scaling of the function  $f$ , we can always assume that  $T = [-1, 1]$  and  $\Omega = [-c, +c]$ , with  $c = ab$ . Indeed, for  $f$  that is  $\epsilon_T$ -concentrated in  $T = [-a, +a]$  and  $\epsilon_\Omega$ -band concentrated in  $\Omega = [-b, +b]$ , the normalized function  $g(t) = \sqrt{a}f(at)$  is  $\epsilon_T$ -concentrated in  $[-1, +1]$  and  $\epsilon_\Omega$ -band concentrated in  $[-ab, +ab]$ .

Before stating the theorem, let us give some notations. For  $f \in L^2(\mathbf{R})$ , we consider its expansion  $f = \sum_{n \geq 0} a_n \psi_{n,c}$  in  $L^2([-1, +1])$ . Due to the normalization of the functions  $\psi_{n,c}$  given by (3), the following equality holds,

$$\int_{-1}^{+1} |f(t)|^2 dt = \sum_{n \geq 0} |a_n|^2. \quad (27)$$

We call  $S_{N,c}f$ , the  $N$ -th partial sum, defined by

$$S_{N,c}f(t) = \sum_{n < N} a_n \psi_{n,c}(t). \quad (28)$$

We write more simply  $S_N f$  when there is no ambiguity. In the next lemma, we prove that  $S_N f$  tends to  $f$  rapidly when  $f$  belongs to the space of band-limited functions. This statement is both very simple and classical, see for instance [11, 18].

**Lemma 3.** *Let  $f \in B_c$  be an  $L^2$  normalized function. Then*

$$\int_{-1}^{+1} |f - S_N f|^2 dt \leq \lambda_N(c). \quad (29)$$

*Proof.* Since the set of functions  $\psi_{n,c}$  is also an orthogonal basis of  $B_c$ , the function  $f$  may be written on  $\mathbb{R}$  as  $f = \sum_{n \geq 0} a_n \psi_{n,c}$ , with

$$\int_{\mathbb{R}} |f(t)|^2 dt = \sum_{n \geq 0} |\lambda_n(c)|^{-1} |a_n|^2. \quad (30)$$

The two expansions coincide on  $[-1, +1]$ , and, from (30) applied to  $f - S_N f$ , it follows that

$$\int_{-1}^{+1} |f - S_N f|^2 dt \leq \sup_{n \geq N} |\lambda_n(c)| \sum_{n \geq N} |\lambda_n(c)|^{-1} |a_n|^2.$$

We use the fact that the sequence  $|\lambda_n(c)|$  decreases and (30) to conclude.  $\square$

Next we define the time-limiting operator  $P_T$  and the band-limiting operator  $\Pi_\Omega$  by:

$$P_T(f)(x) = \chi_T(x) f(x), \quad \Pi_\Omega(f)(x) = \frac{1}{2\pi} \int_{\Omega} e^{ix\omega} \widehat{f}(\omega) d\omega.$$

The following proposition provides us with the quality of approximation of almost time- and band-limited functions by the PSWFs.

**Proposition 3.** *If  $f$  is an  $L^2$  normalized function that is  $\epsilon_T$ -concentrated in  $T = [-1, +1]$  and  $\epsilon_\Omega$ -band concentrated in  $\Omega = [-c, +c]$ , then for any positive integer  $N$ , we have*

$$\left( \int_{-1}^{+1} |f - S_N f|^2 dt \right)^{1/2} \leq \epsilon_\Omega + \sqrt{\lambda_N(c)} \quad (31)$$

and, as a consequence,

$$\|f - P_T S_N f\|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)}. \quad (32)$$



More generally, if  $f$  is an  $L^2$  normalized function that is  $\epsilon_T$ -concentrated in  $T = [-a, +a]$  and  $\epsilon_\Omega$ -band concentrated in  $\Omega = [-b, +b]$  then, for  $c = ab$  and for any positive integer  $N$ , we have

$$\|f - P_T S_{N,c,a} f\|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)} \quad (33)$$

where  $S_{N,c,a}$  gives the  $N$ -th partial sum for the orthonormal basis  $\frac{1}{\sqrt{a}}\psi_{n,c}(t/a)$  on  $[-a, +a]$ .

**Proof:** We first prove (31) by writing  $f$  as the sum of  $\Pi_\Omega f$  and  $g$ . Remark first that  $\int_{-1}^{+1} |g - S_N g|^2 dt \leq \|g\|_2 \leq \epsilon_\Omega$ . We then use Lemma 3 for the band-limited function  $\Pi_\Omega f$  to conclude. The rest of the proof follows at once.  $\square$

**Remark 2.** Let  $f$  be a normalized  $L^2$  function that vanishes outside  $I$  and we assume that  $f \in H^s(\mathbb{R})$ . Then  $f$  gives an example of 0-concentrated in  $I$  and  $\epsilon_c$ -band concentrated in  $[-c, +c]$ , with  $\epsilon_c \leq M_f/c^s$  and  $M_f^2 = \frac{1}{2\pi} \int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi$ .

### 3.2 Approximation by the PSWFs in Sobolev spaces

In this paragraph, we study the quality of approximation by the PSWFs in the Sobolev space  $H^s([-1, 1])$ . We provide an  $L^2([-1, 1])$ -error bound of the approximation of a function  $f \in H^s([-1, 1])$  by the  $N$ -th partial sum of its expansion in the PSWFs basis. To simplify notation we will write  $I = [-1, 1]$ .

We should mention that different spectral approximation results by the PSWFs in  $H^s(I)$  have been already given in [4, 6, 20]. It is important to mention that the error bounds of the spectral approximations given by the previous references do not indicate how to choose a “good” value of the bandwidth  $c$  to approximate a given  $f \in H^s(I)$ . By a simultaneous use of the properties of the PSWFs as eigenfunctions of the differential operator  $\mathcal{L}_c$  and the integral operator  $\mathcal{F}_c$ , we give a first answer to this question. This is the subject of the following theorem.

**Theorem 1.** Let  $c > 0$  be a positive real number. Assume that  $f \in H^s(I)$ , for some positive real number  $s > 0$ . Then for any integer  $N \geq 1$ , we have

$$\|f - S_N f\|_{L^2(I)} \leq K(1 + c^2)^{-s/2} \|f\|_{H^s(I)} + K\sqrt{\lambda_N(c)} \|f\|_{L^2(I)}. \quad (34)$$

Here, the constant  $K$  depends only on  $s$  and on the extension operator from  $H^s(I)$  to  $H^s(\mathbb{R})$ . Moreover it can be taken equal to 1 when  $f$  belongs to the space  $H_0^s(I)$ .

*Proof.* To prove (34), we first use the fact that for any real number  $s \geq 0$ , there exists a linear and continuous extension operator  $E : H^s(I) \rightarrow H^s(\mathbb{R})$ . Moreover, if  $f \in H^s(I)$  and  $F = E(f) \in H^s(\mathbb{R})$ , then there exists a constant  $K > 0$  such that

$$\|F\|_{L^2(\mathbb{R})} \leq K \|f\|_{L^2(I)}, \quad \|F\|_{H^s(\mathbb{R})} \leq K \|f\|_{H^s(I)}. \quad (35)$$

We recall that the Sobolev norm of a function  $F$  on  $\mathbb{R}$  is given by

$$\|F\|_{H^s(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi.$$

In particular, for a  $c$ -bandlimited function  $F$ , one has

$$\|F\|_{L^2(\mathbb{R})}^2 \leq (1 + c^2)^{-s} \|F\|_{H^s(\mathbb{R})}^2.$$

Next, if  $\mathcal{F}$  denotes the Fourier transform operator and if

$$\mathcal{G} = \mathcal{F}^{-1}(\widehat{F} \cdot 1_{[-c,c]}), \quad \mathcal{H} = \mathcal{F}^{-1}(\widehat{F} \cdot (1 - 1_{[-c,c]})),$$

then  $\mathcal{G}$  is  $c$ -bandlimited and  $F = \mathcal{G} + \mathcal{H}$ . Moreover, since  $\|\widehat{\mathcal{G}}\|_{L^2(\mathbb{R})} \leq \|\widehat{F}\|_{L^2(\mathbb{R})}$  and  $\|\mathcal{H}\|_{L^2(\mathbb{R})} \leq c^{-s}\|F\|_{H^s(\mathbb{R})}$ , then by using (35), one gets

$$\|\mathcal{G}\|_{L^2(\mathbb{R})} \leq K\|f\|_{L^2(I)}, \quad \|\mathcal{H}\|_{L^2(I)} \leq K(1+c^2)^{-s/2}\|f\|_{H^s(I)}. \quad (36)$$

Finally, by using the previous inequalities and the fact that  $\mathcal{G}$  is  $c$ -bandlimited, one concludes that

$$\begin{aligned} \|f - S_N f\|_{L^2(I)} &\leq \|\mathcal{G} - S_N \mathcal{G}\|_{L^2(I)} + \|\mathcal{H} - S_N \mathcal{H}\|_{L^2(I)} \\ &\leq \sqrt{\lambda_N(c)}\|\mathcal{G}\|_{L^2(\mathbb{R})} + \|\mathcal{H}\|_{L^2(I)} \\ &\leq \sqrt{\lambda_N(c)}K\|f\|_{L^2(I)} + K(1+c^2)^{-s}\|f\|_{H^s(I)}. \end{aligned}$$

This concludes the proof for general  $f$ . When  $f$  is in the subspace  $H_0^s(I)$ , one can take as extension operator the extension by 0 outside  $I$ , so that the constant  $K$  can be replaced by 1.  $\square$

In [20], the author has used a different approach for the study of the spectral approximation by the PSWFs. More precisely, by considering the weighted Sobolev space  $\widetilde{H}^r(I)$ , associated with the differential operator  $\mathcal{L}_c$  defined by

$$\widetilde{H}^r(I) = \left\{ f \in L^2(I), \|f\|_{\widetilde{H}^r(I)}^2 = \|\mathcal{L}_c^{r/2} f\|^2 = \sum_{k \geq 0} (\chi_k)^r |f_k|^2 < +\infty \right\},$$

where  $f = \sum f_k \psi_k$  is the expansion in the basis of PSWFs. Then for any  $f \in \widetilde{H}^r(I)$ , with  $r \geq 0$ , we have

$$\|f - S_N f\|_{L^2(I)} \leq (\chi_N(c))^{-r/2} \|f\|_{\widetilde{H}^r(I)} \leq N^{-r} \|f\|_{\widetilde{H}^r(I)}.$$

For more details, the reader is referred to [20].

**Remark 3.** Compared the result of Theorem 1 with Wang's result, this latter has the advantage to give an error term for all values of  $c$ , while the first term in (34) is only small for  $c$  large enough. On the other hand, Wang compares his specific Sobolev space with the classical one and finds that

$$\|f\|_{\widetilde{H}^s(I)} \leq C(1+c^2)^{s/2} \|f\|_{H^s(I)}.$$

For large values of  $N$  we clearly have  $\frac{(1+c^2)}{\chi_N} \ll (1+c^2)^{-1}$ , but it goes the other way around when  $\chi_N$  and  $1+c^2$  are comparable. So it may be useful to have both kinds of estimates in mind for numerical purpose and for the choice of the value of  $c$ .

**Remark 4.** The error bound given by Theorem 1 has the advantage to be explicitly given in terms of  $c$  and  $\lambda_n(c)$ . Nonetheless, it has a drawback that it does not imply a rate of convergence, nor even the convergence of  $S_N(f)$  to  $f$  in the usual  $L^2(I)$ -norm. To overcome this problem, we devote the remaining of this section to a more elaborated convergence analysis in the 2-periodic Sobolev space  $H_{per}^s$ , then we extend this analysis to the usual  $H^s(I)$ -space.

Next, we consider the subspace  $H_{per}^s$  of functions in  $H^s(I)$  that extend into 2-periodic functions of the same regularity. For such functions, one can also use the norm

$$\|f\|_{H_{per}^s} = \sum_{k \in \mathbb{Z}} (1 + (k\pi)^2)^s |b_k(f)|^2.$$

Here,

$$b_k(f) = \frac{1}{\sqrt{2}} \int_{-1}^{+1} f(x) e^{-i\pi k x} dx = \frac{1}{\sqrt{2}} \widehat{f}(k\pi)$$

is the coefficient of the Fourier series expansion of  $f$ . We then have the following theorem.

**Theorem 2.** Let  $c \geq 1$ , then there exist constants  $M \geq \sqrt{2}$  and  $M'$ ,  $a > 0$  such that, when  $N \geq \max(cM, 3)$  and  $f \in H_{per}^s$ ,  $s > 0$ , we have the inequality

$$\|f - S_N(f)\|_{L^2(I)} \leq M'(1 + (\pi N)^2)^{-s/2} \|f\|_{H_{per}^s} + M'e^{-aN} \|f\|_{L^2}. \quad (37)$$

*Proof.* We start with reductions of the problem, which are analogous to the ones that we have detailed above. It is sufficient to prove this separately with the constant  $M'/2$  for periodic functions  $g$  and  $h = f - g$ , where  $g$  is the projection of  $f$  onto the subspace of  $H_{per}^s$  whose Fourier coefficients  $b_k(f)$  are zero for  $|k| > N/M$ . Moreover, we have directly the inequality without a second term, since the  $L^2$  norm of  $h$  may be bounded by the first term multiplied by some constant. So, let us prove the inequality for  $g$ . This time we will prove that the inequality holds without the first term, that is,

$$\|g - S_N(g)\|_{L^2(I)} \leq \frac{M'}{2} e^{-aN} \|g\|_{L^2(I)}.$$

The next reduction consists of restricting to exponentials  $e^{ik\pi x}$ , with  $|k| \leq N/M$ . Indeed, assume that we prove the previous inequality for all of them, with a uniform bound by  $M''e^{-a'N}$ . Then, by linearity we will have

$$\|g - S_N(g)\|_{L^2(I)} \leq M''e^{-a'N} \sum |b_k(g)| \leq M''e^{-a'N} \sqrt{2[N/M] + 1} e^{-aN} \|g\|_{L^2(I)}.$$

This in turn gives up to a constant, the required form by choosing  $a < a'$ .

So we content ourselves to consider  $f(x) = e^{ik\pi x}$ , with  $|k| \leq N/M$ . Finally, since

$$\|f - S_N f\|_{L^2(I)}^2 = \sum |\langle f, \psi_n \rangle|^2,$$

then it is sufficient to have such an estimate for each  $n > N$ , and conclude by taking the sum  $\sum_{n>N} e^{-an}$ . So the proof is a consequence of the following lemma.  $\square$

The following well known identity will be needed in the sequel.

$$\int_{-1}^1 e^{i\lambda x} \overline{P_n}(x) dx = i^n \sqrt{\frac{2\pi}{\lambda}} \sqrt{n+1/2} J_{n+1/2}(\lambda), \quad (38)$$

where  $J_{n+1/2}(\cdot)$  is the Bessel function of the first type and order  $n+1/2$ .

**Lemma 4.** Let  $c \geq 1$ , then there exist constants  $M \geq \sqrt{2}$  and  $M'$ ,  $a > 0$  such that, when  $n \geq \max(cM, 3)$  and  $f(x) = e^{ik\pi x}$  with  $|k| \leq n/M$ , we have

$$|\langle f, \psi_n \rangle| \leq M'e^{-an}. \quad (39)$$

*Proof.* This scalar product can be written by using (38)

$$\begin{aligned} \langle e^{ik\pi x}, \psi_n \rangle &= \int_{-1}^1 e^{ik\pi x} \psi_n(x) dx = \sum_{m \geq 0} \beta_m^n \langle e^{ik\pi x}, \overline{P_m} \rangle = \sum_{m \geq 0} \beta_m^n \sqrt{\frac{2}{k}} \sqrt{m+1/2} J_{m+1/2}(k\pi) \\ &= \sum_{m=0}^{[n/M]} \beta_m^n \sqrt{\frac{2}{k}} \sqrt{m+1/2} J_{m+1/2}(k\pi) + \sum_{m \geq [n/M]+1} \beta_m^n \sqrt{\frac{2}{k}} \sqrt{m+1/2} J_{m+1/2}(k\pi) \\ &= I_1^n + I_2^n. \end{aligned}$$

To bound  $I_1^n$ , we first remark that the Fourier transform of  $\overline{P_n} \chi_{[-1,1]}$  is bounded by 1 and then we use remark 1 to check that (21) is satisfied whenever  $n \geq cM$  with  $M \geq 1.40$ . Hence, we have

$$\begin{aligned} |I_1^n| &\leq \sum_{m=0}^{[n/M]} |\beta_m^n| \leq \sqrt{\frac{5}{4\pi}} |\mu_n(c)| \sum_{m=0}^{[n/M]} \left( \frac{2\sqrt{\chi_n}}{c} \right)^m \\ &\leq K \left( \frac{2\sqrt{\chi_n}}{c} \right)^{[n/M]+1} |\mu_n(c)|. \end{aligned}$$

Moreover, taking into account the decay of the  $\mu_n(c)$  given by (26) and using the upper bound of  $\chi_n$ , given by (11), we conclude that

$$|I_1^n| \leq K' \left( \frac{\pi(n+1)}{c} \right)^{\frac{n}{M}+1} e^{-\delta n} \leq K'' e^{-an} \quad (40)$$

for some sufficiently small positive real number  $a$ , as soon as  $M \geq \sqrt{2}$ . To bound  $I_2^n$ , it suffices to use the fact that  $|\beta_k^n| \leq 1$  and the bound of the Bessel function given by [1],

$$|J_\alpha(x)| \leq \frac{|x|^\alpha}{2^\alpha \Gamma(\alpha+1)}, \quad \forall \alpha > -1/2, \quad \forall x \in \mathbb{R}. \quad (41)$$

One concludes that

$$\begin{aligned} |I_2^n| &\leq \sum_{m \geq n/M} \sqrt{2/k} \sqrt{m+1/2} |J_{m+1/2}(k\pi)| \leq \sum_{m \geq [n/M]+1} \sqrt{2/k} \sqrt{m+1/2} \frac{(k\pi)^{m+1/2}}{2^{m+1/2} \Gamma(m+3/2)} \\ &\leq \sqrt{\pi} \sum_{m \geq [n/M]+1} \frac{(k\pi)^m}{2^m \sqrt{m+1/2} \Gamma(m+1/2)}. \end{aligned}$$

Moreover, since  $\Gamma(m+1/2) \geq m!/\sqrt{m+1}$  and  $m! \geq (m/e)^m \sqrt{2\pi m}$ , each term is bounded by an exponential  $e^{-an}$  and we find the required estimate for  $|I_2^n|$ .  $\square$

In [6], the authors have given a different  $L^2(I)$ -convergence rate of  $S_N(f)$  to  $f$  in terms of the decay of the expansion coefficients  $a_k(f) = \int_{-1}^1 f(x) \psi_k(x) dx$ . More precisely, it has been shown in [6] that

$$|a_N(f)| \leq C \left( N^{-2/3s} \|f\|_{H^s(I)} + \left( \sqrt{\frac{c^2}{\chi_N(c)}} \right)^{\delta N} \|f\|_{L^2(I)} \right),$$

where  $C, \delta$  are independent of  $f, N$  and  $c$ .

**Remark 5.** *The previous theorem gives the rate of convergence of the truncated PSWFs series expansion of a function  $f$  from  $H_{per}^s$ . This rate of convergence will be generalized in the sequel to the usual  $H^s(I)$ -space. Note that this rate of convergence drastically improves the one given by [6]. Moreover, unlike the error bound given in [20], the decay of the error bound given by the previous theorem is still valid even when  $N$  is comparable to  $c$ . Nonetheless, in practice, Theorem 1 is useful in the sense that it provides us with a criteria for the choice of the bandwidth  $c > 0$ , that depends on the magnitude of the Sobolev exponent  $s > 0$ . The smaller  $s$ , the larger  $c$  should be and vice versa.*

**Remark 6.** *We also have a bound of the error for ordinary polynomials. Indeed, if we consider the monomial  $f(x) := x^j$ , then*

$$a_n(f) = \int_{-1}^1 y^j \psi_{n,c}(y) dy = (-i)^j c^{-j} \mu_n(c) \psi_{n,c}^{(j)}(0), \quad \text{with } i^2 = -1.$$

For fixed  $j$ , we can then use Corollary 1 to conclude that if  $c \geq 1$  and  $c^2/\chi_N < 1$ , then we have

$$\|f - S_N f\|_2^2 \leq M \sum_{k \geq N} \left( \frac{\chi_k(c)}{c^2} \right)^j |\mu_k(c)|^2 \leq M' c^{-2N} \sum_{k \geq N} k^{2j} e^{-ak}, \quad (42)$$

which also leads to an exponential decay.

As a corollary of the previous theorem and remark, we obtain the following corollary that extends the result of the previous theorem to the case of the usual Sobolev space  $H^s([-1, 1])$ .

**Corollary 2.** *Let  $c \geq 1$ , and let  $s > 0$ . There exist constants  $M \geq \sqrt{2}$  and  $M', M'_s > 0$  such that, when  $f \in H^s(I)$  and  $N \geq \max(cM, 3)$ , we have the inequality*

$$\|f - S_N(f)\|_{L^2(I)} \leq M'_s(1 + N^2)^{-s/2} \|f\|_{H^s([-1,1])} + M'e^{-aN} \|f\|_{L^2([-1,1])}. \quad (43)$$

*Proof.* We first assume that  $[s] = m$ , and  $s \notin \frac{1}{2} + \mathbf{N}$ , then there exists a polynomial  $P$ , of degree at most  $m$ , such that  $f+P \in H_{per}^s$ . Consequently, by using the previous theorem and the inequality (42), one concludes for (43). More generally, a function  $f \in H^s(I)$  can be considered as the restriction to  $I$  of a function of  $H^s(\mathbb{R})$  which may be taken to have support in  $[-2, 2]$ . So it is also the restriction to  $I$  of a periodic function of period 4. Since Lemma 4 is also valid for the exponentials  $e^{i\frac{k\pi}{2}x}$ ,  $k \in \mathbb{Z}$ , then we conclude as before.  $\square$

## 4 Numerical results

In this section, we illustrate the results of the previous sections by various numerical examples. For this purpose, we first describe a numerical method for the computation of the PSWFs series expansion coefficients of a function from the Sobolev space  $H^s(I)$ . Note that if  $f \in H_{per}^s$ ,  $s > 0$ , then its different PSWFs series expansion coefficients  $(a_n(f))_n$  can be easily approximated as follows. For a positive integer  $K$ , an approximation  $a_n^K(f)$  to  $a_n(f)$  is given by the following formula

$$a_n^K(f) = \frac{\mu_n(c)}{\sqrt{2}} \sum_{k=-K}^K b_k(f) \psi_{n,c} \left( \frac{k\pi}{c} \right) = a_n(f) + \epsilon_K, \quad (44)$$

where the  $b_k(f)$  are the Fourier coefficients of  $f$  and where  $\epsilon_K = \frac{1}{\sqrt{2}} \sum_{|k| \geq K+1} \mu_n(c) b_k(f) \psi_{n,c} \left( \frac{k\pi}{c} \right)$ . Moreover, from the well known asymptotic behavior of the  $\psi_{n,c}(x)$ , for large values of  $x$ , see for example [8], one can easily check that  $\epsilon_K = o\left(\frac{1}{((K+1)\pi)^{1+s}}\right)$ . This computational method of the  $a_n(f)$  has the advantage to work for small as well as large values of the smoothness coefficient  $s > 0$ .

Also, note that if  $f \in H^s([-1, 1])$ , where  $s > 1/2 + 2m$ ,  $m \geq 1$ , is an integer, then  $f \in C^{2m}([-1, 1])$ . Moreover since  $\psi_{n,c} \in C^\infty(\mathbb{R})$ , then the classical Gaussian quadrature method, see for example [1] gives us the following approximate value  $\tilde{a}_n(f)$  of the  $(n+1)$ -th expansion coefficient  $a_n(f) = \langle f, \psi_{n,c} \rangle$ ,

$$\tilde{a}_n(f) = \sum_{l=1}^m \omega_l f(x_l) \psi_{n,c}(x_l) = a_n(f) + \epsilon_n, \quad (45)$$

with  $|\epsilon_n| \leq \sup_{\eta \in [-1,1]} \frac{1}{b_m^2} \frac{(f \cdot \psi_{n,c})^{(2m)}(\eta)}{(2m)!}$ . Here,  $b_m$  is the highest coefficient of  $\overline{P_m}$ , and the different weights  $\omega_l$  and nodes  $x_l$ , are easily computed by the special method given in [1].

The following examples illustrate the quality of approximation in  $H^s(I)$  by the PSWFs.

**Example 1:** In this example, we show that the PSWFs outperforms the Legendre polynomials in the approximation of a class of functions from the Sobolev space  $H^s([-1, 1])$ , having significant large coefficients at some high frequency components. To fix the idea, let  $\lambda > 0$ , be a relatively large positive real number and let  $f_\lambda(x) = e^{i\lambda x}$ ,  $x \in [-1, 1]$ . The Legendre series expansion coefficients of  $f_\lambda$  are given by

$$\alpha_n(0) = \int_{-1}^1 e^{i\lambda x} \overline{P_n}(x) dx = i^n \sqrt{\frac{2\pi}{\lambda}} \sqrt{n+1/2} J_{n+1/2}(\lambda).$$

In this case, we have

$$\|f_\lambda - \sum_{n=0}^N \alpha_n(0) \overline{P_n}\|_2^2 = \frac{2\pi}{\lambda} \sum_{n \geq N+1} (n+1/2)(J_{n+1/2}(\lambda))^2. \quad (46)$$

If  $c > 0$  is a positive real number, then the corresponding PSWFs series expansion coefficients of  $f_\lambda$  are simply given as follows,

$$\alpha_n(c) = \int_{-1}^1 e^{i\lambda x} \psi_{n,c}(x) dx = \mu_n(c) \psi_{n,c}(\lambda/c).$$

Note that the analytic extension of  $\psi_{n,c}$  outside the interval  $[-1, 1]$  has been given in [18] as follows

$$\psi_n(x) = \frac{\sqrt{2\pi}}{|\mu_n(c)|} \sum_{k \geq 0} (-1)^k \beta_k^n \sqrt{k+1/2} \frac{J_{k+1/2}(cx)}{\sqrt{cx}}, \quad (47)$$

with

$$\mu_n(c) = i^n \sqrt{\frac{2\pi}{c}} \left[ \frac{\sum_{k \geq 0} (-1)^k \sqrt{k+1/2} \beta_k^n J_{k+1/2}(c)}{\sum_{k \geq 0} \beta_k^n \sqrt{k+1/2}} \right], \quad (48)$$

is the exact value of the  $n$ -th eigenvalue of the finite Fourier transform operator  $\mathcal{F}_c$ .

On the other hand, the  $L^2(I)$ -approximation error by the PSWFs is given by

$$E_N(c) = \|f - \sum_{n=0}^N \alpha_n(c) \psi_{n,c}\|_2^2 = \sum_{n \geq N+1} |\mu_n(c)|^2 \left( \psi_{n,c} \left( \frac{\lambda}{c} \right) \right)^2. \quad (49)$$

In the special case where  $c = \lambda$ , the previous error bound becomes  $E_N(\lambda) = \sum_{n \geq N+1} |\mu_n(c)|^2 (\psi_{n,c}(1))^2$ .

Since from [2], we have  $|\psi_{n,c}(1)| \leq 2\chi_n^{1/4}$ , then by using (11), one gets  $|\psi_{n,c}(1)| \leq \sqrt{2\pi(n+1)}$ . Moreover, since the super-exponential decay of the sequence  $(|\mu_n(c)|^2)_{n \geq 0}$  starts around  $N_c = [ec/4]$ , then from (46) and (49), one concludes that the PSWFs are better adapted for the approximation of the  $f_\lambda$  by its  $N$ -th order truncated PSWFs series expansion with  $c = \lambda$  and  $N = [\lambda]$ . More generally, if  $0 \leq c < \lambda$ , then  $\frac{\lambda}{c} > 1$  and the well known blow-up of the  $\psi_{n,c}(\frac{\lambda}{c})$  with  $\frac{\lambda}{c} > 1$ , implies that  $\alpha_n(c) = \mu_n(c) \psi_{n,c}(\lambda/c)$  has a lower decay than  $\alpha_n(\lambda) = \mu_n(\lambda) \psi_{n,c}(1)$ . Moreover, if  $c > \lambda$ , then the decay of the  $|\mu_n(c)|^2$  and consequently, the fast decay of the  $\alpha_n(c)$  is possible only if  $n$  lies beyond a neighbourhood of  $\frac{ec}{4} > \frac{e\lambda}{4}$ . This means that  $c = \lambda$  is the appropriate value of the bandwidth to be used to approximate the function  $f_\lambda(x) = e^{i\lambda x}$  by its first  $N$ -th truncated PSWFs series expansion, with  $N = [\lambda]$ . This explains the numerical results given in [20] concerning the approximation of the test function  $u(x) = \sin(20\pi x)$ , where the author has checked numerically that  $c = 20\pi$  is the appropriate value of the bandwidth for approximating  $u(x)$  by the PSWFs  $\psi_{n,c}$  with a given high precision and minimal number of the truncation order  $N$ . As another example, we consider the value of  $\lambda = 50$ , then we find that

$$\|f_\lambda - \sum_{n=0}^{50} \alpha_n(0) \overline{P_n}\|_2 \approx 3.087858E - 01, \quad \|f_\lambda - \sum_{n=0}^{50} \alpha_n(50) \psi_{n,50}\|_2 \approx 1.356604E - 08.$$

**Example 2:** In this example, we consider the Weierstrass function

$$W_s(x) = \sum_{k \geq 0} \frac{\cos(2^k x)}{2^{ks}}, \quad -1 \leq x \leq 1. \quad (50)$$

Table 1: Values of  $E_N(s)$  for various values of  $N$  and  $s$ .

	$s = 0.75$	$s = 1$	$s = 1.25$	$s = 1.5$	$s = 1.75$	$s = 2.0$
$N$	$E_N(s)$	$E_N(s)$	$E_N(s)$	$E_N(s)$	$E_N(s)$	$E_N(s)$
20	4.57329E-01	4.66173E-01	4.85990E-01	5.05973E-01	5.23232E-01	5.37227E-01
30	3.15869E-01	3.11677E-01	3.28241E-01	3.48562E-01	3.67260E-01	3.82963E-01
40	1.06843E-01	1.52009E-01	1.91237E-01	2.20969E-01	2.43432E-01	2.60523E-01
50	4.09844E-02	6.88472E-02	1.01827E-01	1.26518E-01	1.44809E-01	1.58520E-01
60	3.30178E-02	2.09084E-02	3.25551E-02	4.28999E-02	5.06959E-02	5.65531E-02
70	3.15097E-02	8.82446E-03	2.51157E-03	7.35725E-04	2.33066E-04	1.04137E-04
80	3.01566E-02	8.55598E-03	2.40312E-03	6.87458E-04	1.98993E-04	5.80481E-05
90	2.67972E-02	7.64167E-03	2.14661E-03	6.15062E-04	1.78461E-04	5.22848E-05
100	2.39141E-02	6.72825E-03	1.82818E-03	5.10057E-04	1.45036E-04	4.19238E-05

The choice of this function allows us to see how the approximation process works on functions that are either nowhere smooth or having small Sobolev smoothness exponent. Note that  $W_s \in H^{s-\epsilon}([-1, 1])$ ,  $\forall \epsilon < s$ ,  $s > 0$ . We have considered the value of  $c = 100$ , and computed  $W_{s,N}$ , the  $N$ -th terms truncated PSWFs series expansion of  $W_s$  with different values of  $\frac{3}{4} \leq s \leq 2$  and different values of  $20 \leq N \leq 100$ . Also, for each pair  $(s, N)$ , we have computed the corresponding approximate  $L^2$ -error bound  $E_N(s) = \left[ \frac{1}{50} \sum_{k=-50}^{50} (W_{s,N}(k/50) - W_s(k/50))^2 \right]^{1/2}$ . Table 1 lists the obtained values of  $E_N(s)$ . Note that the numerical results given by Table 1, follow what has been predicted by the theoretical results of the previous section. In fact, the  $L^2$ -errors  $\|W_s - \Pi_N W_s\|_2$  is of order  $O(N^{-s})$ , whenever  $N \geq N_c \sim \left\lceil \frac{2c}{\pi} \right\rceil + 4$ . The graphs of  $W_{3/4}(x)$  and  $W_{3/4,N}(x)$ ,  $N = 90$  are given by Figure 1.

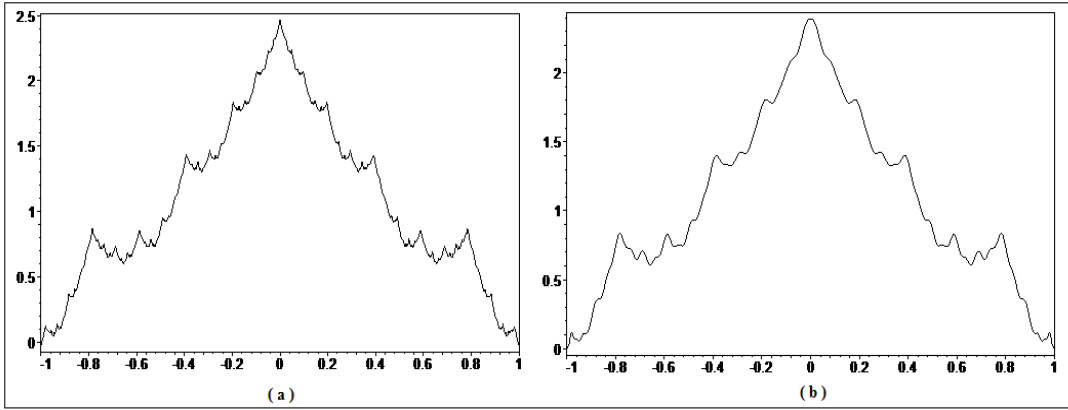


Figure 1: (a) graph of  $W_{3/4}(x)$ , (b) graph of  $W_{3/4,N}(x)$ ,  $N = 90$ .

**Example 3:** In this example, we let  $s > 0$  be any positive real number and we consider the random

function  $B_s(x)$  is given as follows.

$$B_s(x) = \sum_{k \geq 1} \frac{X_k}{k^s} \cos(k\pi x), \quad -1 \leq x \leq 1.$$

Here,  $X_k$  is a sequence of independent standard Gaussian random variables. The random process  $B_s$  behaves like a fractional Brownian motion, with Hurst parameter  $H = s - 1/2$ . It is almost surely in  $H^{s'}$  for  $s' < s - 1/2$ . For the special case  $s = 1$ , we consider the band-width  $c = 100$ , a truncation order  $N = 80$  and compute  $B_{1,N}$  the approximation of  $B_1$  by its  $N$ -th terms truncated PSWFs series expansion. The graphs of  $B_1$  and  $B_{1,N}$  are given by Figure 2.

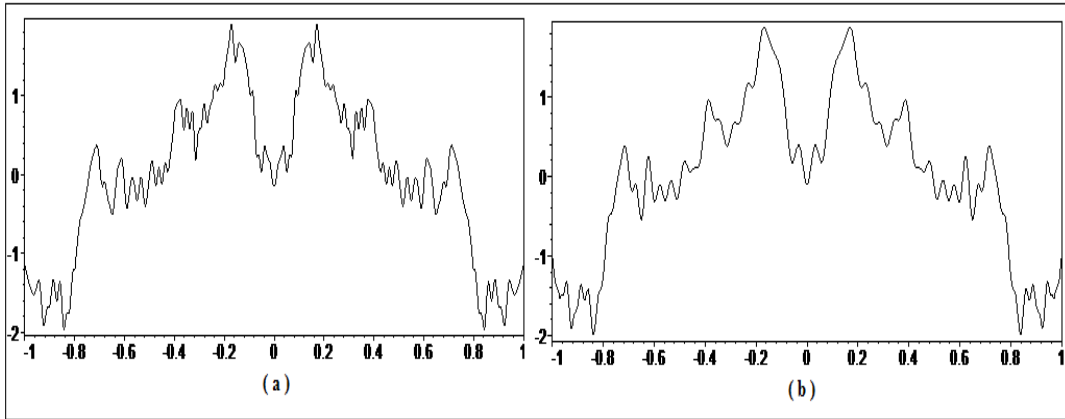


Figure 2: (a) graph of  $B_1(x)$ , (b) graph of  $B_{1,N}(x)$ ,  $N = 80$ .

**Remark 7.** From the quality of approximation in the Sobolev spaces  $H^s([-1, 1])$  given in this paper and in [4, 6, 20], one concludes that for any value of the bandwidth  $c \geq 0$ , the approximation error  $\|f - S_N f\|_2$  has the asymptotic order  $O(N^{-s})$ . Nonetheless, for a given  $f \in H^s([-1, 1])$ ,  $s > 0$  which we may assume to have a unit  $L^2$ -norm and for a given error tolerance  $\epsilon$ , the appropriate value of the bandwidth  $c \geq 0$ , corresponding to the minimum truncation order  $N$ , ensuring that  $\|f - S_N f\|_2 \leq \epsilon$ , depends on whether or not,  $f$  has some significant Fourier expansion coefficients, corresponding to large frequency components. In other words, the faster decay to zero of the Fourier coefficients of  $f$ , the smaller the value of the bandwidth should be and vice versa.

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