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# Spectral Decay of Time and Frequency Limiting Operator.

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**Abstract**— For fixed  $c$ , the Prolate Spheroidal Wave Functions (PSWFs)  $\psi_{n,c}$  form a basis with remarkable properties for the space of band-limited functions with bandwidth  $c$ . They have been largely studied and used after the seminal work of D. Slepian, H. Landau and H. Pollack. Many of the PSWFs applications rely heavily of the behavior and the decay rate of the eigenvalues  $(\lambda_n(c))_{n \geq 0}$  of the time and frequency limiting operator, which we denote by  $\mathcal{Q}_c$ . Hence, the issue of the accurate estimation of the spectrum of this operator has attracted a considerable interest, both in numerical and theoretical studies. In this work, we give an explicit integral approximation formula for these eigenvalues. This approximation holds true starting from the plunge region where the spectrum of  $\mathcal{Q}_c$  starts to have a fast decay. As a consequence of our explicit approximation formula, we give a precise description of the super-exponential decay rate of the  $\lambda_n(c)$ . Also, we mention that the described approximation scheme provides us with fairly accurate approximations of the  $\lambda_n(c)$  with low computational load, even for very large values of the parameters  $c$  and  $n$ . Finally, we provide the reader with some numerical examples that illustrate the different results of this work.

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## 1 Introduction

For a given value  $c > 0$ , called the bandwidth, PSWFs  $(\psi_{n,c}(\cdot))_{n \geq 0}$  constitute an orthonormal basis of  $L^2([-1, 1])$ , an orthogonal system of  $L^2(\mathbf{R})$  and an orthogonal basis of the Paley-Wiener space  $B_c$ , given by  $B_c = \left\{ f \in L^2(\mathbf{R}), \text{Support } \hat{f} \subset [-c, c] \right\}$ . Here,  $\hat{f}$  denotes the Fourier transform of  $f$ . They are eigenfunctions of the compact integral operators  $\mathcal{F}_c$  and  $\mathcal{Q}_c = \frac{c}{2\pi} \mathcal{F}_c^* \mathcal{F}_c$ , defined on  $L^2([-1, 1])$  by

$$\mathcal{F}_c(f)(x) = \int_{-1}^1 e^{i c x y} f(y) dy, \quad \mathcal{Q}_c(f)(x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} f(y) dy. \quad (1)$$

Since the operator  $\mathcal{F}_c$  commutes with the Sturm-Liouville operator  $\mathcal{L}_c$ ,

$$\mathcal{L}_c(\psi) = -\frac{d}{dx} \left[ (1-x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi, \quad (2)$$

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PSWFs  $(\psi_{n,c}(\cdot))_{n \geq 0}$  are also eigenfunctions of  $\mathcal{L}_c$ . They are ordered in such a way that the corresponding eigenvalues of  $\mathcal{L}_c$ , called  $\chi_n(c)$ , are strictly increasing. Functions  $\psi_{n,c}$  are restrictions to the interval  $[-1, +1]$  of real analytic functions on the whole real line and eigenvalues  $\chi_n(c)$  are values of  $\lambda$  such that the equation  $\mathcal{L}_c \psi = \lambda \psi$  has a non zero bounded solution on the whole interval.

PSWFs have been introduced by D. Slepian, H. Landau and H. Pollak [10, 16, 17, 18] in relation with signal processing. For a detailed review on properties, numerical computations, asymptotic results and first applications of the PSWFs, the reader is referred to recent books on the subject, [6], [13].

By Plancherel identity, PSWFs are normalized so that

$$\int_{-1}^1 |\psi_{n,c}(x)|^2 dx = 1, \quad \int_{\mathbb{R}} |\psi_{n,c}(x)|^2 dx = \frac{1}{\lambda_n(c)}, \quad n \geq 0. \quad (3)$$

Here,  $(\lambda_n(c))_n$  is the infinite sequence of the eigenvalues of  $\mathcal{Q}_c$ , also arranged in the decreasing order  $1 > \lambda_0(c) > \lambda_1(c) > \dots > \lambda_n(c) > \dots$ . We call  $\mu_n(c)$  the eigenvalues of  $\mathcal{F}_c$ . They are given by

$$\mu_n(c) = i^n \sqrt{\frac{2\pi}{c} \lambda_n(c)}.$$

Also, we adopt the sign normalization of the PSWFs,

$$\psi_{n,c}(0) > 0 \quad \text{for even } n, \quad \psi'_{n,c}(0) > 0, \quad \text{for odd } n. \quad (4)$$

One of the main issues that we discuss here is the decay rate of the eigenvalues  $\lambda_n(c)$ . This decay rate plays a crucial role in most of the various concrete applications of the PSWFs. In this direction, one knows their asymptotic behaviour for  $c$  fixed, which has been given in 1964 by Widom, see [19].

$$\lambda_n(c) \sim \left( \frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1} = \lambda_n^W(c). \quad (5)$$

This gives the exact decay for  $n$  large enough, but one would like to have a more precise information in terms of uniformity of this behaviour, both in  $n$  and  $c$ . On the other hand, Landau has considered the value of the smallest integer  $n$  such that  $\lambda_n(c) \leq 1/2$  in [9]. More precisely, if we note  $c_n^*$  the unique value of  $c$  such that  $\lambda_n(c) = 1/2$ , then he proves that

$$\frac{\pi}{2}(n-1) \leq c_n^* \leq \frac{\pi}{2}(n+1) \quad \lambda_n(c_n^*) = \frac{1}{2}. \quad (6)$$

So, for  $c$  fixed, we almost know when  $\lambda_n(c)$  passes through the value  $1/2$ . Landau and Widom have also described the asymptotic behaviour, when  $c$  tends to  $\infty$ , of the distribution of the eigenvalues  $\lambda_n(c)$ .

The search for more precise estimates of the  $\lambda_n(c)$  has attracted a considerable interest, both in numerical and theoretical studies. We try here to give approximate values for  $\lambda_n(c)$  for  $c \leq c_n^*$ , with some uniformity in the quality of approximation. We rely on the exact formula for the eigenvalues  $\lambda_n(c)$ , given by integrating the following differential equation, see [5],

$$\partial_\tau \ln \lambda_n(\tau) = \frac{2|\psi_{n,\tau}(1)|^2}{\tau}. \quad (7)$$

over the interval  $(c, c_n^*)$ . This is different from the classical way to estimate the  $\lambda_n(c)$ , via equation (39), where the integration is rather done on the interval  $(0, c)$ , see for example [15, 17, 20]. Consequently, we are mainly interested in the behavior, as well as in accurate and fast schemes of approximation, of the  $\lambda_n(c)$ , given by the formula

$$\lambda_n(c) = \frac{1}{2} \exp \left( -2 \int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau \right). \quad (8)$$

We use our recent works [1, 2] to estimate the value  $\psi_{n,\tau}(1)$ . In the first paper it is proved that  $|\psi_{n,\tau}(1)| \leq 2\chi_n(\tau)^{1/4}$ , which is not sufficient to find a sharp estimate for all values  $c$ . The approximation given in the second paper leads to a second estimate of  $\psi_{n,\tau}(1)$ , valid for  $\frac{\pi n}{2} - c$  larger than some multiple of  $\ln n$ . Based on this second estimate, we define  $\widetilde{\lambda}_n(c)$  as

$$\widetilde{\lambda}_n(c) = \frac{1}{2} \exp \left( -\frac{\pi^2(n + \frac{1}{2})}{2} \int_{\Phi(\frac{2c}{\pi(n+\frac{1}{2})})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt \right). \quad (9)$$

Here  $\mathbf{E}$  is the elliptic integral of the second kind given by (13) and the function  $\Phi$  is the inverse of the function  $t \mapsto \frac{t}{\mathbf{E}(t)}$ . We prove that  $\widetilde{\lambda}_n(c)$  is comparable with  $\lambda_n(c)$  up to some power of  $n$ . This is stated in the following theorem, which is the main result of this paper.

**Theorem 1.** *There exist three constants  $\delta_1 \geq 1, \delta_2, \delta_3 \geq 0$  such that, for  $n \geq 3$  and  $c \leq \frac{\pi n}{2}$ ,*

$$\delta_1^{-1} n^{-\delta_2} \left( \frac{c}{c+1} \right)^{\delta_3} \leq \frac{\widetilde{\lambda}_n(c)}{\lambda_n(c)} \leq \delta_1 n^{\delta_2} \left( \frac{c}{c+1} \right)^{-\delta_3}. \quad (10)$$

Let us explain, roughly speaking, why Legendre elliptic integrals are involved here. The Sturm-Liouville equation  $\mathcal{L}_c \psi = \chi_n(c) \psi$  can be rewritten as

$$\psi'' - \frac{2x}{1-x^2} \psi' + \chi_n(c) \frac{1-qx^2}{1-x^2} \psi = 0,$$

with  $q = c^2/\chi_n(c)$ . Under the assumptions on  $n$  and  $c$ , the value  $q$  may be seen as a parameter. We assume that  $q < 1$  and proceed to a WKB approximation of the solution  $\psi_{n,c}$ , but not directly: the equation is transformed into its normal form  $U'' + (\chi_n + \theta)U = 0$  through the Liouville transformation given by the change of functions  $\psi = [(1-x^2)(1-qx^2)]^{-1/4} U$  and the change of variable  $x \mapsto \int_0^x \sqrt{\frac{1-qt^2}{1-t^2}} dt$ . We recognize in this change of variable the incomplete Legendre elliptic integral of the second kind, while the  $L^2$ -norm of the factor  $[(1-x^2)(1-qx^2)]^{-1/4}$  can be written in terms of the complete Legendre elliptic integral of the first kind. This has been exploited by many authors (for instance [4], [20], [11]). We refer to our recent work [2] for the approximation of  $|\psi_{n,c}(1)|^2$  in terms of Legendre elliptic integrals, which is central here and leads to the formula (9).

Let us come back to the present paper. When  $n$  tends to  $\infty$  with  $c$  fixed, we recover the asymptotic behavior given by Widom and, as a corollary, we have the following, which may be seen as a kind of quantitative Widom's Theorem.

**Corollary 1.** *Let  $m > 0$  be a positive real number and let  $M > m, \varepsilon > 0$  be given. Then there exists a constant  $A(\varepsilon, m, M)$  such that, for all  $m \leq c \leq M\sqrt{n}$  and all  $n$ , we have the inequality*

$$\lambda_n(c) \leq A(\varepsilon, m, M) e^{\varepsilon n} \left( \frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1}. \quad (11)$$

We can give an explicit constant  $A(\varepsilon, m, M)$ . Also, we will show that asymptotically,  $\widehat{\lambda}_n(c) = 2\widetilde{\lambda}_n(c)$  is equivalent to Widom's asymptotic formula (5).

The fact that we recover Widom's asymptotic behavior is already a good test of validity but we can go further numerically. In fact we recover the exact equivalent given by Widom up to the factor  $1/2$ , which justifies the approximation of  $\lambda_n(c)$  by  $\widehat{\lambda}_n(c) = 2\widetilde{\lambda}_n(c)$  instead of  $\widetilde{\lambda}_n(c)$ , at least for large values of  $n$ .

**Remark 1.** *Numerical experiments show that the approximation of  $\lambda_n(c)$  by  $\widehat{\lambda}_n(c) = 2\widetilde{\lambda}_n(c)$  is surprisingly accurate. It is striking that the values of  $|\widehat{\mu}_n(c)| = \sqrt{\frac{2\pi}{c} \widehat{\lambda}_n(c)}$  coincide with the values*

of  $|\mu_n(c)|$  that have been computed in [14], with relative errors that are less than 3%. Numerical tests indicate that this relative error bound holds true as soon as  $|\mu_n(c)| \leq 0.15$ . Moreover, the smaller the value of  $|\mu_n(c)|$ , the smaller is the corresponding relative error. For example, for  $c = 10\pi$  and  $n = 90$ , we have found that  $|\mu_n(c)| \approx 8.64288E - 57$  and  $\frac{|\widehat{\mu}_n(c) - \mu_n(c)|}{|\mu_n(c)|} \approx 7.71E - 05$ .

We try to explain the factor 2 in the expression of  $\widehat{\lambda}_n(c)$ , which is of course very small compared to the accumulated errors in the theoretical approach. Let us mention that another method to approximate the values  $\lambda_n(c)$  has been used by Osipov in [12]. The estimates given in his paper are of different nature and do not propose such a simple and accurate formula. In addition, he mainly considers values of  $n$  such that  $\frac{\pi n}{2} - c$  is smaller than some multiple of  $\ln c$ . At this moment both works may be seen complementary. But we underline the fact that numerical tests validate the accuracy of the approximant (9) even when  $c$  is close to the critical value, while our theoretical approach is not sufficient to do it.

This work is organized as follows. In Section 2, we list some estimates of the PSWFs and their associated eigenvalues  $\chi_n(c)$ . In Section 3, we prove a sharp exponential decay rate of the eigenvalues  $\lambda_n(c)$  associated with the integral operator  $\mathcal{Q}_c$ . In Section 4, we provide the reader with some numerical examples that illustrate the different results of this work.

We will systematically skip the parameter  $c$  in  $\chi_n(c)$  and  $\psi_{n,c}$ , when there is no doubt on the value of the bandwidth. We then note  $q = c^2/\chi_n$  and skip both parameters  $n$  and  $c$  when their values are obvious from the context.

## 2 Estimates of PSWFs and eigenvalues $\chi_n(c)$ .

Here we first list some classical as well as some recent results on PSWFs and their eigenvalues  $\chi_n$ , then we push forward the methods and adapt them to our study. We systematically use the same notations as in [2]. It is well known that the eigenvalues  $\chi_n$  satisfy the classical inequalities

$$n(n+1) \leq \chi_n \leq n(n+1) + c^2. \quad (12)$$

Next, we recall the Legendre elliptic integral of the first and second kind, that are given respectively, by

$$\mathbf{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} dt, \quad 0 \leq k \leq 1. \quad (13)$$

Osipov has proved in [11] that the condition  $q = \frac{c^2}{\chi_n} < 1$  is fulfilled when  $c < \frac{\pi n}{2}$ , while it is not when  $c > \frac{\pi(n+1)}{2}$ . This is part of the following statement, which gives precise lower and upper bounds of the quantity  $\sqrt{q} = \frac{c}{\sqrt{\chi_n}}$ , see [2].

**Lemma 1.** *For all  $c > 0$  and  $n \geq 2$  we have*

$$\Phi\left(\frac{2c}{\pi(n+1)}\right) < \frac{c}{\sqrt{\chi_n}} < \Phi\left(\frac{2c}{\pi n}\right), \quad (14)$$

where  $\Phi$  is the inverse of the function  $k \mapsto \frac{k}{\mathbf{E}(k)} = \Psi(k)$ ,  $0 \leq k \leq 1$ .

We refer to [2] for the proof of the previous lemma, but we add some comments. The inequalities (14) are equivalent to the fact that

$$\frac{\pi n}{2\mathbf{E}(\sqrt{q})} < \sqrt{\chi_n} < \frac{\pi(n+1)}{2\mathbf{E}(\sqrt{q})}. \quad (15)$$

The left hand side is due to Osipov [11]. Note that  $\Phi(0) = 0$  and  $\Phi(1) = 1$ . Also, we should mention that

$$0 \leq \Psi'(x) = \frac{\mathbf{E}(x) - x\mathbf{E}'(x)}{(\mathbf{E}(x))^2} = \frac{\mathbf{K}(x)}{(\mathbf{E}(x))^2}, \quad 0 \leq x < 1, \quad (16)$$

$$0 \leq \Phi'(x) = \frac{(\mathbf{E}(\Phi(x)))^2}{\mathbf{K}(\Phi(x))} \leq \frac{(\mathbf{E}(0))^2}{\mathbf{K}(0)} = \frac{\pi}{2}, \quad 0 \leq x < 1. \quad (17)$$

Hence,  $\Phi$  is an increasing function on  $[0, 1]$ . Moreover, since  $\frac{2}{\pi} \leq \frac{1}{\mathbf{E}(x)} \leq 1$ , then we have

$$\frac{2x}{\pi} \leq \Psi(x) \leq x.$$

Applying the function  $\Phi$  to the previous inequalities, one gets

$$x \leq \Phi(x) \leq \frac{\pi x}{2}, \quad 0 \leq x \leq 1. \quad (18)$$

We will use bounds for  $\psi_{n,c}$  given in [2], which have been established under the condition that  $(1-q)\sqrt{\chi_n} > \kappa \geq 4$ . We leave some flexibility for the choice of the constant  $\kappa$  and do not restrict to the choice  $\kappa = 4$  as in [2]. We will only need estimates at 1, which we give in the following lemma with a slightly different form compared to [2].

**Lemma 2.** *Let  $n \geq 3$ . We assume that the condition*

$$(1-q)\sqrt{\chi_n(c)} > \kappa \quad (19)$$

*is satisfied for some  $\kappa \geq 4$ . Then, there exists a constant  $\delta(\kappa)$  (independent of  $c$  and  $n$ ) such that one has the following bounds for  $(\psi_{n,c}(1))^2$ .*

$$\frac{\pi\sqrt{\chi_n}}{2\mathbf{K}(\sqrt{q})} (1 - \delta(\kappa)\varepsilon_n) \leq (\psi_n(1))^2 \leq \frac{\pi\sqrt{\chi_n}}{2\mathbf{K}(\sqrt{q})} (1 + \delta(\kappa)\varepsilon_n), \quad \varepsilon_n = ((1-q)\sqrt{\chi_n})^{-1}. \quad (20)$$

We refer to [2], Theorem 2, for the proof. Explicit values for the constant  $\delta(\kappa)$  can also be deduced from [2]. We can choose

$$\delta(\kappa) = \eta \left( 2 + \frac{\eta}{\kappa} \right), \quad \eta = C(\kappa) \left( \frac{\beta}{1 + (1 - \kappa^{-1}\beta)^{1/2}} + \sqrt{2}\alpha(1 + \alpha\kappa^{-1}) \right) \quad (21)$$

with  $C(\kappa)^{-1} = (1 - \kappa^{-1}\beta)^{1/2} - \sqrt{2}\alpha\kappa^{-1}(1 + \alpha\kappa^{-1})$ ,  $\alpha = 1.5$ ,  $\beta = 0.35$ .

In any case, we see that the theoretical values of  $\delta(\kappa)$  are larger than 4.73. We find approximately  $\delta(4) \approx 77.2$ ,  $\delta(12) \approx 7.6$ . Numerical tests (see Example 1 in Section 4) indicate that the numerical quantities  $\delta(\kappa)$ , for which one has equality in (21), are much smaller.

In [1], we have proved that

$$|A| = |\psi_{n,c}(1)|\chi_n(c)^{-1/4} \leq 2 \quad \text{for } c \leq \frac{\pi(n+1)}{2}. \quad (22)$$

So in particular the right hand side bound of (20) is not accurate when  $\kappa$  is small. Lemma 2 expresses the fact that, under some condition depending on a parameter  $\kappa$ , we have

$$(\psi_{n,\tau}(1))^2 \approx \frac{\pi\sqrt{\chi_n(\tau)}}{2\mathbf{K}(\sqrt{q(\tau)})} = \frac{\pi\tau}{2\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})}. \quad (23)$$

The previous formula for the approximation of the quantity  $\psi_{n,c}(1)$  still requires the approximation of  $\sqrt{q(\tau)}$ . For this last quantity, we already have the double inequalities (14). We may write

$$\sqrt{q(\tau)} \approx \sqrt{\tilde{q}(\tau)} = \Phi\left(\frac{2\tau}{\pi(n+1/2)}\right). \quad (24)$$

An error bound of the previous approximation formula is given in [2] by

$$|\sqrt{q(c)} - \sqrt{\tilde{q}(c)}| \leq \frac{c}{2\sqrt{\chi_n}\sqrt{\tilde{\chi}_n}}. \quad (25)$$

Also, in [2], we have given an explicit formula for the approximation of  $\sqrt{\chi_n(c)}$  together with its associated error,

$$\sqrt{\chi_n(c)} \approx \sqrt{\tilde{\chi}_n(c)} = \frac{c}{\Phi\left(\frac{2c}{\pi(n+1/2)}\right)}, \quad n \geq \frac{2c}{\pi}, \quad (26)$$

$$\left|\sqrt{\chi_n(c)} - \sqrt{\tilde{\chi}_n(c)}\right| \leq \frac{1}{2}. \quad (27)$$

We should mention that in [2], we have further improved the error bounds (25) and (27) in the case of large values of  $n$ . The improved asymptotic error bounds are given as follows,

$$|\sqrt{q(c)} - \sqrt{\tilde{q}(c)}| \leq \frac{c\kappa}{(1-q)\chi_n\sqrt{\tilde{\chi}_n}}, \quad |\sqrt{\tilde{\chi}_n(c)} - \sqrt{\chi_n(c)}| \leq \frac{\kappa}{(1-q)\sqrt{\chi_n}}, \quad (28)$$

for some constant  $\kappa$ . Numerical evidence indicates that in practice, the actual error of the approximation scheme (24) is much smaller than the previous theoretical error bound, see example 2 of the numerical results section.

We need to translate Condition (19) in terms of the parameters  $n, c$ , which can be done by using [Proposition 4, [2]], where the following inequality has been given. For  $n \geq 2$  and  $q < 1$ ,

$$(1-q)\sqrt{\chi_n} \geq \frac{(n - \frac{2c}{\pi}) - e^{-1}}{\log n + 5}, \quad (29)$$

A further improvement of the previous inequality is given by the following lemma.

**Lemma 3.** *Let  $n \geq 3$ ,  $q < 1$  and  $\kappa \geq 4$ . Then one of the following conditions,*

$$c \leq n - \kappa, \quad (30)$$

$$\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 9), \quad (31)$$

*implies the inequality (19), that is,*

$$(1-q)\sqrt{\chi_n(c)} > \kappa.$$

*Moreover, if we assume that  $c > \frac{n+1}{2}$ , then the condition  $\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 6)$  is sufficient.*

*Proof.* Let  $\gamma = \frac{2c}{\pi n}$ . It follows from (15) that

$$1 - \gamma < 1 - \sqrt{q} + \frac{\mathbf{E}(\sqrt{q}) - 1}{\mathbf{E}(\sqrt{q})}. \quad (32)$$

We claim that

$$\mathbf{E}(x) - 1 \leq (1-x^2) \left( \frac{1}{4} \ln\left(\frac{1}{1-x^2}\right) + \ln 2 \right). \quad (33)$$

Let us assume this and go on with the proof. It follows that

$$1 - \gamma < \frac{1 - q}{\mathbf{E}(\sqrt{q})} \left( \frac{1}{4} \ln \left( \frac{1}{1 - q} \right) + \frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} + \ln 2 \right). \quad (34)$$

We then use the elementary inequality, valid for  $0 < s < 1$ ,

$$s \ln(1/s) \leq 1/n + s \ln(n/e).$$

It implies that

$$1 - \gamma - \frac{1}{4n\mathbf{E}(\sqrt{q})} < \frac{1 - q}{\mathbf{E}(\sqrt{q})} \left( \frac{1}{4} \ln(n/e) + \frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} + \ln 2 \right).$$

We use also (15) to conclude that

$$(1 - q)\sqrt{\chi_n} \geq \frac{\pi n}{2\mathbf{E}(\sqrt{q})}(1 - q) > \kappa, \quad (35)$$

whenever

$$\frac{\pi n}{2} - c > \kappa \left( \frac{1}{4} \ln(n/e) + \frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} + \ln 2 \right) + \frac{1}{4n}.$$

This is the case, in particular, when  $\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 9)$ , using the fact that  $\frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} \leq \frac{\pi}{2}$ .

The condition  $c \geq \frac{n+1}{2}$  implies that  $q > \frac{1}{\pi}$ . Then, by using the value of  $\mathbf{E}(\sqrt{\pi^{-1}})$ , the constant 9 in (31) can be replaced by 6. It remains to prove (33). We write

$$\mathbf{E}(x) - 1 \leq (1 - x^2) \int_0^1 \frac{1}{(\sqrt{1 - x^2 t^2} + \sqrt{1 - t^2})} \frac{t dt}{\sqrt{1 - t^2}} \quad (36)$$

$$= \int_0^1 \frac{ds}{(1 - x^2 + s^2 x^2)^{\frac{1}{2}} + s}. \quad (37)$$

We cut the last integral into two parts. For the first one, from  $\sqrt{1 - x^2}$  to 1, we replace the denominator by  $2s$  and find the logarithmic term. For the second one, we replace the denominator by  $\sqrt{1 - x^2} + s$  and find  $\ln 2$ .  $\square$

We will need another inequality of the same type:

$$1 - \frac{2c}{\pi n} \leq 2(1 - q)\mathbf{K}(\sqrt{q}). \quad (38)$$

This is a consequence of (32), using the fact that  $\mathbf{E}(x) - 1 \leq (1 - x^2)\mathbf{K}(x)$ , which comes directly from (36).

### 3 Sharp decay estimates of eigenvalues $\lambda_n(c)$ .

In this section, we use some of the estimates we have given in the previous section and we prove a sharp super-exponential decay rate of the eigenvalues  $(\lambda_n(c))_n$ . We first recall that these  $\lambda_n(c)$  are governed by the following differential equation, see [5] or the more recent reference [20],

$$\partial_c \ln \lambda_n(c) = \frac{2|\psi_{n,c}(1)|^2}{c}. \quad (39)$$

As a consequence, for fixed  $n$  there exists a unique value of  $c$  for which  $\lambda_n(c) = 1/2$ . We denote this value of  $c$  by  $c_n^*$ . We know from [9] that it can be bounded below and above, namely

$$\frac{\pi}{2}(n - 1) \leq c_n^* \leq \frac{\pi}{2}(n + 1) \quad \text{with} \quad \lambda_n(c_n^*) = \frac{1}{2}. \quad (40)$$



By combining (39) and (40), one gets

$$\lambda_n(c) = \frac{1}{2} \exp \left( -2 \int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau \right). \quad (41)$$

Let us recall the following definition.

$$\widetilde{\lambda}_n(c) = \frac{1}{2} \exp \left( -\frac{\pi^2(n + \frac{1}{2})}{2} \int_{\Phi(\frac{2c}{\pi(n+\frac{1}{2})})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt \right). \quad (42)$$

Our main result is the following theorem.

**Theorem 2.** *There exist three constants  $\delta_1 \geq 1, \delta_2, \delta_3, \geq 0$  such that, for  $n \geq 3$  and  $c \leq \frac{\pi n}{2}$ ,*

$$\delta_1^{-1} n^{-\delta_2} \left( \frac{c}{c+1} \right)^{\delta_3} \leq \frac{\widetilde{\lambda}_n(c)}{\lambda_n(c)} \leq \delta_1 n^{\delta_2} \left( \frac{c}{c+1} \right)^{-\delta_3}, \quad (43)$$

The factor  $\frac{c}{c+1}$  can be replaced by 1 when  $c > 1$  and replaced by  $c$  when  $c < 1$ . We have written the formula this way to avoid to have to distinguish between the two cases,  $c \geq 1$  and  $0 < c < 1$ . It is simpler to write equivalent inequalities for logarithms, which is done in the following proposition. We keep the same notations for constants, which are of course not the same. We note  $\ln^+(x)$  the positive part of the Logarithm, that is,  $\ln^+(x) = \max(0, \ln(x))$ . The following theorem is required in the proof of the main Theorem 2.

**Theorem 3.** *There exist three non negative constants  $\delta_1, \delta_2, \delta_3$  such that, for  $n \geq 3$  and  $c \leq \frac{\pi n}{2}$ , we have*

$$\int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau = \frac{\pi^2(n + \frac{1}{2})}{4} \int_{\Phi(\frac{2c}{\pi(n+\frac{1}{2})})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt + \mathcal{E}, \quad (44)$$

with

$$|\mathcal{E}| \leq \delta_1 + \delta_2 \ln(n) + \delta_3 \ln^+(1/c). \quad (45)$$

Let us make some comments before starting the proof. At this moment the three constants are not sufficiently small and cannot be used reasonably to obtain numerical values. But they can be computed and are not that enormous. There is no hope, of course, to have found an exact formula for  $\lambda_n(c)$  and (42) gives only an approximation. But these theoretical approximation errors may be seen as a kind of theoretical validation of the quality of approximation of the  $\lambda_n(c)$ , which we test numerically in Section 4.

It has been observed by many authors, and predicted by the work of Landau and Widom [10], that for fixed  $c$  the eigenvalues  $\lambda_n(c)$  decrease first exponentially in some interval starting at  $[\frac{2c}{\pi}] + 1$  with length a multiple of  $\ln(c)$ , then super-exponentially as in the asymptotic behavior given by Widom. This is what one observes in Formula (42), but the error terms do not allow to observe the decay rate at the plunge region. In fact the tools that we use, that is, the lower and upper bounds for  $\psi_{n,\tau}(1)^2$ , are only valid for  $c_n^* - \tau$  sufficiently large in terms of  $\ln(n)$ .

We try to have small constants at each step but are certainly far from the best possible. We give an explicit bound for  $\mathcal{E}$  in (66). The following notations will be used frequently in the sequel. We define

$$I(a, b) = \int_a^b \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau. \quad (46)$$

$$\mathcal{J}(y) = \frac{\pi^2}{4} \int_{\Phi(\frac{2y}{\pi})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt \quad (47)$$

We should mention that the proofs of Theorems 2 and Theorem 3, require many steps, so we start by giving a sketch of these proofs.

**Sketch of the proof of Theorem 3.** We want to prove that

$$I(c, c_n^*) \approx (n + \frac{1}{2}) \mathcal{J} \left( \frac{c}{n + \frac{1}{2}} \right).$$

For this purpose, we use the approximation of  $\psi_{n,\tau}(1)$ , given by Formula (23). This is valid under a condition involving the parameter  $\kappa$ , and may be rewritten as  $c < c_n^\kappa$  for some  $c_n^\kappa$  that is close to  $c_n^*$  by Lemma 3. We deduce from Formula (23) that

$$I(c, c_n^\kappa) \approx \int_c^{c_n^\kappa} \frac{\pi d\tau}{2\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})}.$$

Then Lemma 1 will be interpreted as the fact that

$$\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)}) \approx \Phi \left( \frac{2\tau}{\pi(n + \frac{1}{2})} \right) \mathbf{K} \circ \Phi \left( \frac{2\tau}{\pi(n + \frac{1}{2})} \right).$$

It is then elementary to relate the new integral with the function  $\mathcal{J}$  and finally find that

$$I(c, c_n^\kappa) \approx (n + \frac{1}{2}) \mathcal{J} \left( \frac{c}{n + \frac{1}{2}} \right).$$

It remains to bound the tails of the integrals  $I(c_n^\kappa, c_n^*)$ , which we can do because the two values are sufficiently close.

Let us start the proof itself. We need a set of intermediate results that can be classified into three main steps. The first step will concern the properties of the function  $\mathcal{J}$ . In the second step, we give bounds of the tails of the integrals. Finally, in the third step, we use the results of the previous two steps and complete the proofs of Theorems 2 and 3.

**First step: Properties of  $\mathcal{J}$ .**

For an integer  $l \geq 1$ , we define

$$\mathcal{J}_l(c) = \frac{\pi}{2} \int_c^{\frac{\pi l}{2}} \frac{d\tau}{\Phi \left( \frac{2\tau}{\pi l} \right) \mathbf{K} \circ \Phi \left( \frac{2\tau}{\pi l} \right)}. \quad (48)$$

As it has been seen in the sketch, these integrals are clearly involved in the proof. We first see that they are related with  $\mathcal{J}$ .

**Lemma 4.** *We have the identity*

$$\mathcal{J}_l(c) = l\mathcal{J}(c/l). \quad (49)$$

*Proof.* We consider the substitution

$$s = \Phi \left( \frac{2\tau}{\pi l} \right), \quad \tau = \frac{\pi l}{2} \Psi(s). \quad (50)$$

We have already seen in (16) that  $\Psi'(x) = \frac{\mathbf{K}(x)}{(\mathbf{E}(x))^2}$ . Hence, we have

$$\mathcal{J}_l(c) = l \int_{\Phi(\frac{2c}{\pi l})}^1 \frac{ds}{s(\mathbf{E}(s))^2} = l\mathcal{J}(c/l).$$

□

The following proposition gives us upper and lower bounds, as well as the asymptotic behavior of  $\mathcal{J}$ .

**Proposition 1.** *For  $x \in (0, \pi/2)$ , one has the upper and lower bounds*

$$\ln^+ \left( \frac{1}{x} \right) \leq \mathcal{J}(x) \leq \frac{\pi^2}{4} \ln \left( \frac{\pi}{2x} \right). \quad (51)$$

Moreover, one can write

$$\mathcal{J}(x) = \frac{\pi^2}{4} \int_{\Phi(2x/\pi)}^1 \frac{dt}{t(\mathbf{E}(t))^2} = \ln \left( \frac{4}{ex} \right) + \mathcal{E}', \quad (52)$$

with  $|\mathcal{E}'| \leq \frac{\pi^2 x^2}{8}$ .

*Proof.* The first inequalities are an easy consequence of the bounds below and above of  $\Phi$ , given by (18). Let us prove (52). We first write, for  $0 < y < 1$ ,

$$\frac{\pi^2}{4} \int_y^1 \frac{dt}{t(\mathbf{E}(t))^2} + \ln(y) = \Delta - \int_0^y \frac{\frac{\pi^2}{4} - \mathbf{E}(t)^2}{t(\mathbf{E}(t))^2} dt = \Delta - I_1(y). \quad (53)$$

Here

$$\Delta = \int_0^1 \frac{\frac{\pi^2}{4} - \mathbf{E}(t)^2}{t(\mathbf{E}(t))^2} dt.$$

It is probably well-known that

$$\Delta = \ln \left( \frac{4}{e} \right) \quad (54)$$

but we did not find any reference. We will see it as a corollary of Widom's Theorem. The integral  $I_1(y)$  is bounded by  $\frac{\pi^2 y^2}{8}$ . This is a consequence of the elementary inequalities

$$1 \leq \mathbf{E}(s) \leq \frac{\pi}{2}, \quad \frac{\pi}{2} - \mathbf{E}(s) \leq s^2 \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}} = \frac{\pi s^2}{4}.$$

Let us now fix  $y = \Phi(2x/\pi)$ . At this point we have proved that

$$0 \leq \ln \left( \frac{x}{y} \right) - \mathcal{E}' = I_1(y) \leq \frac{\pi^2 y^2}{8}.$$

From the inequalities

$$\frac{2y}{\pi} \leq \frac{2x}{\pi} = \frac{y}{\mathbf{E}(y)} \leq \frac{2y}{\pi} \left(1 - \frac{y^2}{2}\right)^{-1} \leq \frac{2y}{\pi} (1 + y^2),$$

it follows that  $0 \leq \ln \left( \frac{x}{y} \right) + y^2$ . This concludes the proof of the proposition.  $\square$

This proposition leads to the following corollary, where we recognize the equivalent given by Widom.

**Corollary 2.** *We have the double inequality*

$$\frac{1}{2} \left( \frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1} e^{-\frac{\pi^2}{4} \frac{c^2}{n + \frac{1}{2}}} \leq \widetilde{\lambda}_n(c) \leq \frac{1}{2} \left( \frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1} e^{+\frac{\pi^2}{4} \frac{c^2}{n + \frac{1}{2}}}. \quad (55)$$

*Proof.* Just note that  $\widetilde{\lambda}_n(c) = \frac{1}{2} \exp(-(2n+1)\mathcal{J}(c/(n+1/2)))$  and use (52) with  $x = \frac{c}{n+1/2}$ .  $\square$

Let us go back to quantities  $\mathcal{J}_l$ . It is a straightforward consequence of (49) that the quantity  $\mathcal{J}_l(c)$  increases with  $l$ . The next lemma gives reverse inequalities.

**Lemma 5.** *We have the inequalities*

$$\mathcal{J}_{n+1}(c) - \frac{\pi^2}{8} \ln\left(\frac{\pi(n+1)}{2c}\right) - \frac{\pi^3}{16} \leq \mathcal{J}_{n+\frac{1}{2}}(c) \leq \mathcal{J}_n(c) - \frac{\pi^2}{8} \ln\left(\frac{\pi(n+\frac{1}{2})}{2c}\right) + \frac{\pi^3}{16}. \quad (56)$$

*Proof.* We will only prove one of the inequalities, the other one being identical. Elementary computations give

$$\mathcal{J}_{n+1}(c) - \mathcal{J}_{n+\frac{1}{2}}(c) \leq \frac{1}{2} \mathcal{J}\left(\frac{c}{n+1}\right) + \frac{\pi^2}{4} \left(n + \frac{1}{2}\right) \ln\left(\frac{\Phi\left(\frac{2c}{\pi(n+\frac{1}{2})}\right)}{\Phi\left(\frac{2c}{\pi(n+1)}\right)}\right).$$

We use (51) for the first term. The second one is bounded by

$$\frac{\pi^2}{4} \left(n + \frac{1}{2}\right) \frac{\Phi\left(\frac{2c}{\pi(n+\frac{1}{2})}\right) - \Phi\left(\frac{2c}{\pi(n+1)}\right)}{\Phi\left(\frac{2c}{\pi(n+1)}\right)} \leq \frac{\pi^3}{16}.$$

Indeed, this is a consequence of the fact that  $\Phi'(x) \leq \pi/2$  and  $\frac{x}{\Phi(x)} \leq 1$ , for  $0 < x \leq 1$ .  $\square$

**Second step: tails of the integrals.**

We fix some constant  $\kappa \geq 4$  (for instance  $\kappa = 12$ ) and we assume that  $n \geq 2\kappa + 1$ . Then, we know from Lemma 3, that the condition (19), that is,

$$(1-q)\sqrt{\chi_n} > \kappa,$$

is satisfied for  $c < \frac{n+1}{2}$ . Next, if we define

$$c_n^\kappa = \max\left(\frac{\pi n}{2} - \frac{\kappa}{4}(\ln(n) + 6), \frac{n+1}{2}\right) \quad (57)$$

then, we have the following lemma.

**Lemma 6.** *For  $n \geq 2\kappa + 1$ , we have the inequality*

$$I(c_n^\kappa, c_n^*) \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2. \quad (58)$$

*Proof.* Recall that  $|\psi_{n,c}(1)| \leq 2\chi_n^{1/4}$  and  $\sqrt{\chi_n(c)} \leq \frac{\pi}{2}(n+1)$ , see [11]. Hence, we have

$$|\psi_{n,\tau}(1)|^2 \leq 4\sqrt{\chi_n(\tau)} \leq 2\pi(n+1).$$

Consequently, one gets

$$\int_{c_n^\kappa}^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau \leq 2\pi(n+1) \ln\left(1 + \frac{\frac{\pi}{2} + \frac{\kappa}{4}(\ln(n) + 6)}{c_n^\kappa}\right).$$

We conclude by using the fact that  $c_n^\kappa \geq \frac{n+1}{2}$ .  $\square$

We conclude directly the proof of Theorem 3 in the case where  $n \geq 2\kappa + 1$  and  $c < c_n^\kappa$ . It suffices to combine the results of Proposition 1 and the previous lemma, and get the desired inequalities

$$-\frac{\pi^2}{16}(\kappa \ln(n) + 6\pi\kappa + \pi) \leq I(c, c_n^*) - (n + \frac{1}{2})\mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right) \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2 \quad (59)$$

We also conclude that Theorem 3 and Theorem 2 still hold for the finite number of missing values of  $n$ , that is,  $n \leq 2\kappa + 1$ . There is no problem to have upper bounds and lower bounds that do not depend on  $c$  for  $c < 1$ . From Corollary 2, we have a precise estimate in terms of  $c^{2n+1}$  for  $\widetilde{\lambda}_n(c)$ . The same is given for  $\lambda_n(c)$  by the following lemma.

**Lemma 7.** *Assume that  $n \geq 1$  is fixed and let  $0 < c < 1$ . Then, there exist two constants  $\delta(n), \delta'(n)$  such that*

$$\delta(n)c^{2n+1} \leq \lambda_n(c) \leq \delta'(n)c^{2n+1}. \quad (60)$$

*Proof.* We first note that  $I(1, c_n^*) \leq I(1, \frac{\pi(n+1)}{2})$ . We recall that on this interval, we have the inequality  $|\psi_{n,\tau}(1)|^2 \leq 4\frac{\pi(n+1)}{2}$ . So  $I(1, c_n^*) \leq 2\pi(n+1)\ln(\frac{\pi(n+1)}{2})$ . Inside the integral defining  $I(c, 1)$ , we use the following inequality, that may be found in [2],

$$\left| |\psi_{n,\tau}(1)| - \sqrt{n + \frac{1}{2}} \right| \leq \frac{\tau^2}{\sqrt{3(n + 1/2)}} \leq \frac{\tau^2}{2}. \quad (61)$$

So  $|I(c, 1) - (n + \frac{1}{2})\ln(\frac{1}{c})| \leq 1$ , from which we conclude.  $\square$

It remains to prove Theorem 2 and Theorem 3 when  $c > c_n^\kappa$  and  $n \geq 2\kappa + 1$ .

**Third step: Proofs of Theorems 2 and 3.**

We fix  $\kappa > 4$ . Because of the previous steps, we will only need to study the cases

$$n \geq 2\kappa + 1 \quad c < c_n^\kappa = \max\left(\frac{\pi n}{2} - \frac{\kappa}{4}(\ln(n) + 6), \frac{n+1}{2}\right).$$

In view of (44), we want to bound the quantity

$$\mathcal{E} = I(c, c_n^*) - \left(n + \frac{1}{2}\right)\mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right).$$

We have already given a bound to a first error term

$$\mathcal{E}_1 = I(c, c_n^*) - I(c, c_n^\kappa).$$

Because of (58), we know that

$$0 \leq \mathcal{E}_1 \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2. \quad (62)$$

Next, the conditions on  $\kappa$  allow us to use the double inequalities (20). Namely,

$$(\psi_{n,\tau}(1))^2 = \frac{\pi}{2\mathbf{K}(\sqrt{q})}\sqrt{\chi_n(\tau)} + \mathcal{R}(\tau), \quad |\mathcal{R}(\tau)| \leq \frac{\delta(\kappa)}{(1-q(\tau))\mathbf{K}(\sqrt{q(\tau)})}, \quad 0 \leq \tau \leq c_n^\kappa. \quad (63)$$

This leads to a second error,

$$\mathcal{E}_2 = I(c, c_n^\kappa) - \frac{\pi}{2} \int_c^{c_n^\kappa} \frac{d\tau}{\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})},$$

which is bounded as follows,

$$|\mathcal{E}_2| \leq \delta(\kappa) \int_c^{c_n^\kappa} \frac{1}{(1-q(\tau))\mathbf{K}(\sqrt{q(\tau)})} \frac{d\tau}{\tau}.$$

We then use the following lemma.

**Lemma 8.** *We have the inequality*

$$|\mathcal{E}_2| \leq 2\delta(\kappa) \left( \left(1 + \frac{\pi\kappa}{4}\right) \ln(n) + \ln^+ \left(\frac{1}{c}\right) + \frac{3\pi\kappa}{2} \right). \quad (64)$$

*Proof.* By (38), we know that

$$2(1 - q(\tau))\mathbf{K}(\sqrt{q(\tau)}) \geq 1 - \frac{2\tau}{\pi n}.$$

So we have the inequality

$$|\mathcal{E}_2| \leq 2\delta(\kappa) \int_{\frac{2c}{\pi n}}^{\frac{2c_n^\kappa}{\pi n}} \frac{ds}{(1-s)s} \leq 2\delta(\kappa) \left( \ln\left(\frac{n}{c}\right) + \ln\left(\frac{1}{1 - \frac{2c_n^\kappa}{\pi n}}\right) \right),$$

and we conclude at once.  $\square$

It remains to consider the main term, that is,

$$I_{\text{main}}(c, c_n^\kappa) = \frac{\pi}{2} \int_c^{c_n^\kappa} \frac{\sqrt{\chi_n(\tau)} d\tau}{\mathbf{K}(\sqrt{q(\tau)}) \tau} = \frac{\pi}{2} \int_c^{c_n^\kappa} \frac{d\tau}{\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})}. \quad (65)$$

We use the monotonicity properties of  $\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})$ , namely

$$\Phi\left(\frac{2\tau}{\pi(n+1)}\right) \mathbf{K} \circ \Phi\left(\frac{2\tau}{\pi(n+1)}\right) \leq \sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)}) \leq \Phi\left(\frac{2\tau}{\pi n}\right) \mathbf{K} \circ \Phi\left(\frac{2\tau}{\pi n}\right).$$

It follows that

$$\mathcal{J}_n(c) - \mathcal{J}_n(c_n^\kappa) \leq I_{\text{main}}(c, c_n^\kappa) \leq \mathcal{J}_{n+1}(c).$$

So the last error,

$$\mathcal{E}_3 = I_{\text{main}}(c, c_n^\kappa) - \mathcal{J}_{n+\frac{1}{2}}(c) = I_{\text{main}}(c, c_n^\kappa) - \left(n + \frac{1}{2}\right) \mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right),$$

satisfies the inequalities

$$\mathcal{J}_n(c) - \mathcal{J}_{n+\frac{1}{2}}(c) - \mathcal{J}_n(c_n^\kappa) \leq \mathcal{E}_3 \leq \mathcal{J}_{n+1}(c) - \mathcal{J}_{n+\frac{1}{2}}(c).$$

It remains to use (51) and (56) to conclude. We finally find that

$$|\mathcal{E}| \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2 + 2\delta(\kappa) \left( \left(1 + \frac{\pi\kappa}{4}\right) \ln(n) + \ln^+ \left(\frac{1}{c}\right) + \frac{3\pi\kappa}{2} \right) + \frac{\pi^2}{8} \ln\left(\frac{\pi(n + \frac{1}{2})}{2c}\right) + \frac{\pi^3}{16}. \quad (66)$$

So we can take the following values for  $\delta_1, \delta_2, \delta_3$ , that have been given in Theorem 3.

$$\begin{aligned} \delta_1 &= 22 + 3\pi\kappa(2 + \delta(\kappa)) \\ \delta_2 &= \frac{\pi^2}{8} + \pi\kappa + 2\delta(\kappa)\left(1 + \frac{\pi\kappa}{4}\right) \\ \delta_3 &= \frac{\pi^2}{8} + 2\delta(\kappa)\left(1 + \frac{\pi\kappa}{4}\right). \end{aligned}$$

This concludes the proofs of Theorem 3 and Theorem 2.

Note that when  $\kappa = 12$  we find  $\delta_2 \approx 200$ . We could have improved the sizes of the previous constants at each step, but not significantly. Numerical experiments indicate that in practice, these constants are much smaller.

From Theorem 3 and Corollary 2 we get the following corollary:

**Corollary 3.** *There exist three constants  $\delta_1 \geq 1, \delta_2, \delta_3, \geq 0$  such that, for  $n \geq 3$  and  $c \leq \frac{\pi n}{2}$ ,*

$$A(n, c)^{-1} \left( \frac{ec}{2(2n+1)} \right)^{2n+1} \leq \lambda_n(c) \leq A(n, c) \left( \frac{ec}{2(2n+1)} \right)^{2n+1}. \quad (67)$$

with

$$A(n, c) = \delta_1 n^{\delta_2} \left( \frac{c}{c+1} \right)^{-\delta_3} e^{+\frac{\pi^2}{4} \frac{c^2}{n}}.$$

Widom's Theorem says that  $A(n, c)$  can be replaced by a quantity that tends to 1 for  $n$  tending to  $\infty$ . We cannot give such an asymptotic behavior at this moment, but we can estimate errors for fixed  $c$  and  $n$ , which he does not. Remark that we have used the fact that  $\Delta = \ln(4/e)$ , see (54), without proving it or giving a reference. This is a consequence of the asymptotic behavior found by Widom, which cannot be valid at the same time as (67) if  $e/4$  is replaced by another constant. This implies in particular Theorem 1. It may be useful to give also the following corollaries.

**Corollary 4.** *There exist constants  $a > 0$  and  $\delta \geq 1$  such that, for  $c \geq 1$  and  $n > 1.35c$ , we have*

$$\lambda_n(c) \leq \delta e^{-an}. \quad (68)$$

*Proof.* The constant 1.35 has been chosen so that  $2 \ln(\frac{4n}{ec}) > \frac{\pi^2 c^2}{4n^2}$ , which is the case when  $n > 1.35c$ .  $\square$

One has as well a critical super-exponential decay rate given by the following lemma.

**Corollary 5.** *For any  $0 \leq a < \frac{4}{e}$ , there exists a constant  $M_a$  such that for any  $c \geq 1$ , we have*

$$\lambda_n(c) \leq e^{-2n \log(\frac{an}{c})}, \quad \forall n \geq cM_a.$$

Moreover, for any  $b > \frac{4}{e}$ , there exists a constant  $M_b$  such that for any  $c \geq 1$ , we have

$$\lambda_n(c) > e^{-2n \log(\frac{bn}{c})}, \quad \forall n \geq cM_b.$$

The above corollary is a precise answer to Boyd's question on the super-exponential decay rate of the  $\lambda_n(c)$ , see [3].

### Final discussion and comments:

We should mention that one of the problems of our method of approximation of the eigenvalues  $\lambda_n(c)$  is the fact that it cannot be good for  $(1-q)\sqrt{\chi_n}$  too close to 0, while our technique of proof starts from the writing of  $\ln(\lambda_n(c))$  as an integral from  $c$  to  $c_n^*$ . We have seen that asymptotically, for  $c$  fixed and  $n$  tending to  $\infty$ , we recover up to a factor of 1/2, the asymptotic behavior given by Widom, see Corollary 2. The asymptotics for  $n$  fixed and  $c$  tending to 0 is also well-known, see for example [20]. It may be written as

$$\lambda_n(c) \sim \left( \frac{ec}{4(n+\frac{1}{2})} \right)^{2n+1} W_n$$

with  $W_n$  that does not depend on  $c$  and tends to 1 when  $n$  tends to  $\infty$ . Because of this, we propose also the approximation of  $\lambda_n(c)$  given by

$$\widehat{\lambda}_n(c) = 2\widetilde{\lambda}_n(c) = \exp \left( -\frac{\pi^2}{2} \left( n + \frac{1}{2} \right) \int_{\Phi(\frac{2c}{\pi(n+\frac{1}{2})})}^1 \frac{dt}{t\mathbf{E}(t)^2} \right). \quad (69)$$

Note that either one of  $\widetilde{\lambda}_n(c)$  or  $\widehat{\lambda}_n(c)$  can be used to get the precise super-exponential decay rate of the  $\lambda_n(c)$ . Moreover, both formulae can be tested for the approximation of the  $\lambda_n(c)$ . Nonetheless, numerical experiments show that the approximation by  $\widehat{\lambda}_n(c)$  is surprisingly good for  $c, n$  large. For smaller values (and in particular for small values of  $(1 - q)\sqrt{\chi_n}$ ), the approximation by  $\widetilde{\lambda}_n(c)$  is better.

At this moment, we do not have a theoretical justification of this, apart from the asymptotic behavior of  $\lambda_n(c)$ . A tentative proof may start by writing  $\lambda_n(c)$  with an integral from 0 to  $c$ , instead of an integral from  $c$  to  $c_n^*$ . Unfortunately, the singularity at 0 of the integral makes estimates difficult and the idea of starting at  $c_n^*$  instead of 0 has been central here in order to benefit from the estimates on  $\psi_n(1)$ .

We do not give a formal proof but rather some heuristic arguments. Heuristically, for  $c' < c$ , we have

$$\ln\left(\frac{\lambda_n(c)}{\lambda_n(c')}\right) \approx \frac{\pi^2}{2}(n+1/2) \int_{\Phi(\frac{2c'}{\pi(n+1/2)})}^{\Phi(\frac{2c}{\pi(n+1/2)})} \frac{dt}{t\mathbf{E}(t)^2}.$$

Also, because of the asymptotic behavior of  $\lambda_n(c')$  for  $c'$  very close to 0 and  $n$  large enough, we have that

$$\ln\left(\frac{1}{\lambda_n(c')}\right) \approx \frac{\pi^2}{2}(n+1/2) \int_{\Phi(\frac{2c'}{\pi(n+1/2)})}^1 \frac{dt}{t\mathbf{E}(t)^2}.$$

As a consequence of these two approximations, we have

$$\ln\left(\frac{1}{\lambda_n(c)}\right) \approx \frac{\pi^2}{2}(n+1/2) \int_{\Phi(\frac{2c}{\pi(n+1/2)})}^1 \frac{dt}{t\mathbf{E}(t)^2}$$

as long as the approximation of the values of  $|\psi_n(1)|$  are valid. That is, as long as  $(1 - q)\sqrt{\chi_n(c)}$  is not too small. The approximation  $\lambda_n(c)$  by  $\widehat{\lambda}_n(c)$  has been tested for different values of  $n$  and  $c$  in the examples 3 and 4, below. From these simulations, we can think that the quantity  $\frac{\lambda_n(c)}{\widehat{\lambda}_n(c)}$  increases from 1 to 2 when  $n$  goes from the beginning of the plunge region to infinity.

## 4 Numerical results

In this section, we illustrate the results of the previous sections by various numerical examples.

**Example 1:** In this first example, we illustrate the fact that the actual values of the constants  $\kappa$  and  $\delta(\kappa)$ , given by (19) and (20), respectively, are much smaller than the theoretical values given in the proof of Lemma 2. We are interested in these values for  $n \geq 2c/\pi$ . For this purpose, we have considered the values of  $c = m\pi, m = 10, 20, 30, 40$ . We have used Flammer's method and computed highly accurate values of  $\chi_n(c)$  and  $\psi_{n,c}(1)$ . Then, we have computed the smallest value of  $\kappa$ , denoted by  $\kappa_c$  and ensuring the bounds (20). Also, we have computed the corresponding values  $\delta(\kappa_c)$  so that  $(\psi_{n,c}(1))^2$  is equal to its upper bound given in (20). It turns out that  $\kappa_c$ , the critical value of  $\kappa$ , is obtained for  $n$ -th eigenvalues  $\chi_n(c)$  with  $n = n_c = \lceil 2c/\pi \rceil$ . Also, by considering various consecutive values of  $n_c \leq n \leq n_c + 40$  and by computing the corresponding values of  $\kappa$  and  $\delta(\kappa)$ , we found that the  $\max \delta(\kappa)$  is of the same size as  $\kappa_c$ . Table 1 shows the values of the critical values  $\kappa_c$  and  $\delta(\kappa_c)$  for the different values of the bandwidth  $c$ . Also, we give the values of  $\max \delta(\kappa)$ .

**Example 2:** In this example, we illustrate our approximations of the quantity  $\sqrt{q}$  by  $\sqrt{\widetilde{q}}$ , given by formula (24). The accuracy of this approximation is critical for proving the exact super-exponential



$c$	$n_c$	$\kappa_c$	$\tilde{\delta}(\kappa_c)$	$\max \delta(\kappa)$ .
$10\pi$	20	0.447	0.058	0.091
$20\pi$	40	0.413	0.051	0.084
$30\pi$	60	0.394	0.047	0.080
$40\pi$	80	0.335	0.025	0.048

Table 1: Critical values of  $\kappa$ ,  $\delta(\kappa)$  and  $\max \delta(\kappa)$  for different values of  $c$ .

decay rate of the  $\lambda_n(c)$  by our formula (42). For this purpose we have considered different values of the bandwidth  $c$  and computed the previous approximations for different values of  $n$ . These approximations are then compared with highly accurate counterparts obtained by the use of Flammer's method. The obtained numerical results are given by Table 2.

$c$	$n$	$\sqrt{\tilde{q}}$	$\sqrt{q}$
10	6	0.995012670	0.99486271
	10	0.782942846	0.78302833
	15	0.585651991	0.58583492
25	16	0.99062205	0.98924622
	20	0.90491661	0.90471915
	25	0.79783057	0.79783979
50	33	0.99501269	0.99430098
	40	0.91050626	0.91045325
	50	0.80287160	0.80287326
100	64	0.99705417	0.99669712
	80	0.91330250	0.91328853
	100	0.80540660	0.80540692

Table 2: Illustrations of the approximation formula (24).

**Example 3:** In this example, we compare the explicit formula given by (69) to compute highly accurate values of  $\lambda_n(c)$ . For this purpose, we have considered the values of  $c = 10\pi, 20\pi, 30\pi$  and computed  $\lambda_n(c)$  by using the method given in [7]. Then, we have implemented formula (69) in a Maple computing software code. Figure 1 (a), (b), (c) show the graph of  $\ln(\lambda_n(c))$  versus the graph of  $\ln(\widehat{\lambda}_n(c))$ , and  $\ln(\lambda_n^W(c))$ , for the different values of  $c$  and  $n$ . Here,  $\lambda_n^W(c)$  is the Widom's asymptotic approximation of  $\lambda_n(c)$ , given by (5). Also, we have plotted in Figure 2, the graphs of the corresponding values of  $\ln\left(\frac{\widehat{\lambda}_n(c)}{\lambda_n(c)}\right)$ . These figures illustrate the surprising precision of the explicit formula (69) for computing approximate values of the  $\lambda_n(c)$  which is numerically valid whenever  $q < 1$ . In particular, the numerical results illustrated by Figure 2, indicate that at least for moderate values of  $c$ , and  $q < 1$ , the approximations of  $\lambda_n(c)$ , by either  $\widehat{\lambda}_n(c)$  or  $\widetilde{\lambda}_n(c)$  are equal to  $\lambda_n(c)$  up to a small multiplicative constant.

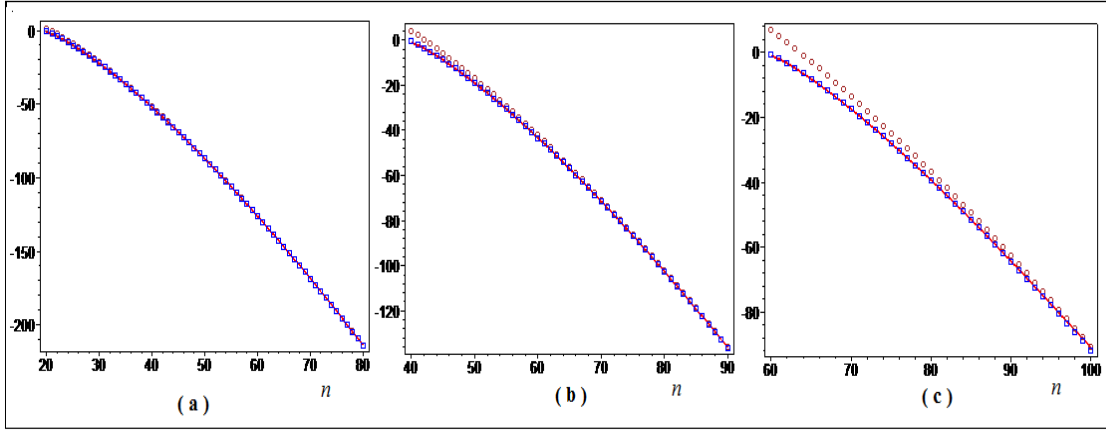


Figure 1: Graphs of  $\ln(\widehat{\lambda}_n(c))$  (boxes),  $\ln(\lambda_n^W(c))$  (circles) and  $\ln(\lambda_n(c))$  (red line) with  $c = 10\pi$  for (a),  $c = 20\pi$  for (b) and  $c = 30\pi$  for (c).

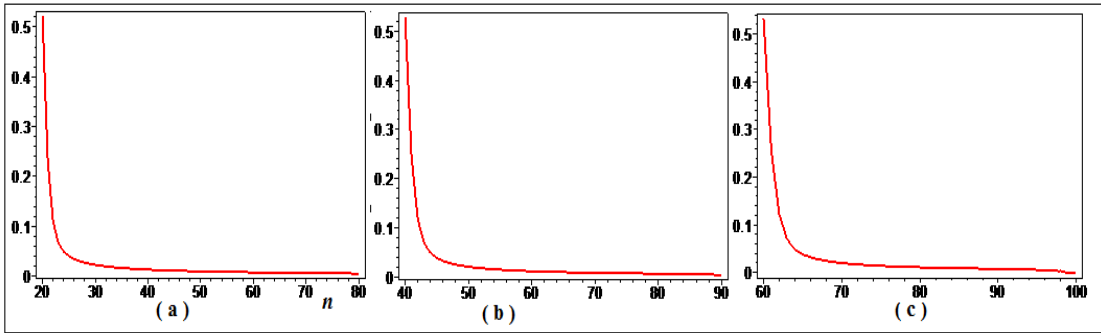


Figure 2: Graphs of  $\ln\left(\frac{\widehat{\lambda}_n(c)}{\lambda_n(c)}\right)$  with  $c = 10\pi$  for (a),  $c = 20\pi$  for (b) and  $c = 30\pi$  for (c).

**Example 4:** In this example, we illustrate the accuracy of the approximation scheme (69) in the cases where the bandwidth  $c$  has relatively large or very large values. For this purpose, we have borrowed some data given in Table 3 of [14], concerning the computation of  $|\mu_n(c)| = \sqrt{\frac{2\pi}{c}\lambda_n(c)}$ . Note that in [14] and in the present work, the roles of  $\lambda_n$  and  $\mu_n$  have been reversed. The data provided by Osipov and Rokhlin are obtained by the highly accurate numerical method for the computation of the  $\lambda_n(c)$  developed by the authors and described in [14]. These data are considered as references values and are used for comparison purpose. Table 3 gives the values of  $|\widehat{\mu}_n(c)| = \sqrt{\frac{2\pi}{c}\widehat{\lambda}_n(c)}$ , versus the corresponding references values. The numerical results of Table 3 indicate that the accuracy of formula (69) is not affected by the large values of  $c$ . Also, to check the validity condition of our explicit approximation formula of the  $\lambda_n(c)$ , for each couple  $(c, n)$ , we have provided the corresponding approximations of  $q$  and  $(1 - q)\sqrt{\chi_n}$ , given by  $\widetilde{q}$  and  $(1 - \widetilde{q})\sqrt{\widetilde{\chi}_n}$ . Note that for moderate and large values of the quantity  $(1 - q)\sqrt{\chi_n}$ , a satisfactory approximation of this latter is

given by the approximation  $(1 - \tilde{q})\sqrt{\tilde{\chi}_n}$ . In fact, from (25) and (27), we have

$$\begin{aligned} \left| (1 - q)\sqrt{\chi_n} - (1 - \tilde{q})\sqrt{\tilde{\chi}_n} \right| &\leq (1 - q) \left| \sqrt{\tilde{\chi}_n} - \sqrt{\chi_n} \right| + |q - \tilde{q}| \sqrt{\tilde{\chi}_n} \\ &\leq \frac{1}{2}(1 - \tilde{q}) + \left| \sqrt{q} - \sqrt{\tilde{q}} \right| \frac{c}{2\sqrt{\tilde{\chi}_n}} \leq \frac{3}{2} - \frac{\tilde{q}}{2}. \end{aligned}$$

$c$	$n$	$\tilde{q}$	$(1 - \tilde{q})\sqrt{\tilde{\chi}_n}$	$ \hat{\mu}_n $	$ \mu_n $
250	179	0.924218	19.707014	0.18948E-07	0.18854E-07
	184	0.903501	25.380432	0.16196E-09	0.16130E-09
	188	0.886848	30.038563	0.30609E-11	0.30500E-11
1000	659	0.981782	18.386116	0.38402E-07	0.38241E-07
	665	0.976303	23.983045	0.44139E-09	0.43991E-09
	671	0.970675	29.764638	0.42935E-11	0.42815E-11
16000	10213	0.998985	16.244476	0.56758E-07	0.56568E-07
	10222	0.998614	22.190912	0.52955E-09	0.52821E-09
	10231	0.998232	28.312611	0.42989E-11	0.42902E-11
128000	81518	0.999881	15.293549	0.42532E-07	0.42408E-07
	81529	0.999834	21.234778	0.39992E-09	0.39906E-09
	81539	0.999791	26.766672	0.51858E-11	0.51768E-11
10 <sup>6</sup>	636652	0.999986	13.738235	0.51646E-07	0.51504E-07
	636665	0.999980	19.666621	0.49076E-09	0.48980E-09
	636677	0.999975	25.260364	0.60652E-11	0.60558E-11

Table 3: Illustrations of the approximation formula (69) for large values of  $c, n$ .

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