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Justifying Dominating Options when Preferential Information is Incomplete

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Abstract. Providing convincing explanations to accompany recommendations is a key issue in decision-aiding. In the context of decisions involving multiple criteria, the problem is made very difficult because the decision model itself may involve a complex process. In this paper, we investigate the following issue: when the preferential information provided by the user is incomplete, is there a principled way to define what is a "simple" explanation for a recommended choice? We argue first that explanations may necessitate different levels of detail. Next, we show that even when a detailed explanation is necessary, it is possible to distinguish explanations of different levels of complexity. Our results rely on an original connection we establish between the "mechanics" required to compute supporting coalitions of criteria and the simplicity of the explanation.

1 Introduction

From the first expert systems to the recent recommendation systems which flourish on commercial websites, decision-aiding has been a central concern in AI. Very soon, it has become clear that providing recommendations was only part of the challenge. Indeed, *explaining* the recommended choice(s) to the decision-maker is crucial to improve the acceptance of the recommendation [9, 3, 13], but also sometimes to allow the decision-maker to justify in turn the decision against other stakeholders. What makes an explanation "convincing" is thus highly context dependent.

In a context of movie recommendation, [7] notoriously reports that a very efficient explanation is that "this recommender system has correctly predicted 80% of the time in the past". In contexts involving more critical decisions or other users, much more detailed explanations would have to be considered [11]. Of course the ultimate nature of the explanation will depend on the underlying decision model and/or on the nature of the data provided by the user. Following [7], a useful distinction to make is among *data-based* and *process-based* explanations. To put it simply, in order to explain a recommendation, a data-based approach will focus on some key data, whereas a process-based one would make explicit (part of) the steps that lead to the decision. Both aspects are considered in this paper.

We start with a collection of partial orders over the options, as provided by different *weighted* criteria (or agents). The decision model we rely on is based on the *weighted Condorcet* principle: options are compared in a pairwise fashion, and an option a is preferred to another option b when the cumulated support that a is better than b outweighs the opposite conclusion. Our aim in this paper is to provide a principled way to produce explanations to the fact that a given set of options A is *dominating* with respect to the other options, more specifically in the sense that A constitutes a *Smith set* [4].

This decision model is specified from some *preferential information* (PI) provided during interview, related to the comparison of the options on each criterion and also on the weights of criteria. Most of the time, the PI is not sufficient to uniquely specify the model. In particular, some options may be incomparable on some criteria for the decision-maker. Moreover, the elicitation process will not result in a single value of the weight vector, but rather in a set of vectors that are compatible with the PI [6]. For instance, in the context of multi-criteria decision aid (MCDA), the decision-maker provides a few learning examples that yield constraints on the weights. In social choice, instead of assigning a weight to each party, one may only know a subset of the winning coalitions (A *winning* coalition beats its complement). Then an option is said to be necessarily preferred to another one if the first option is preferred to the second for all weight vectors that are compatible with the PI, and for all orderings of the options on the criteria that are compatible with the PI [6].

Unfortunately, when the PI is incomplete, the explanation may be quite complex, even for problems of small size, because one cannot display the value of the weights to the audience as part of an explanation. Consider the following example.

Example 1. *There are 7 options* $\{a, b, c, d, e, f, g\}$ *and 4 criteria* $\{1, 2, 3, 4\}$. *The partial orderings (noted* $\succ_1, \succ_2, \succ_3, \succ_4$*) of options over the* 4 *criteria are depicted in Figure 1. The PI regarding the importance of the criteria is composed of three items:*

- *1 together with 3 are more important than 2 and 4 together;*
- *2 and 3 together are more important than criterion 1 taken alone;*
- *4 is more important than criteria 2 and 3.*

Actually, option a *is the unique dominating option. The "technical" reason is that (i)* a *dominates* e *and* f *on all criteria, (ii) coalition 1, 2, 3 is a winning coalition (preference of* a *over* b*), (iii) coalition 1,4 is a winning coalition (preference of* a *over* d*), (iv) coalition 1,3,4 is winning (preference of* a *over* g*), and (v) coalition 2,3,4 is a winning coalition (preference of* a *over* c*). But these reasons vary in terms of the effort required to understand them: (i) is trivial, and (ii), (iii) and (iv) are reinforcement of some statements of the PI. For instance, (ii) easily follows from the fact that 1 and 3 are already more important than 2 and 4. On the other hand, the underlying justification to (v) is more complex. How to deduce indeed from the PI, the statement that coalition 3,4 beats coalition 1,2?*

We will focus on MCDA but our approach can be used in social choice in a similar way. This paper advances the state of the art by characterizing minimal complete explanations to justify dominating

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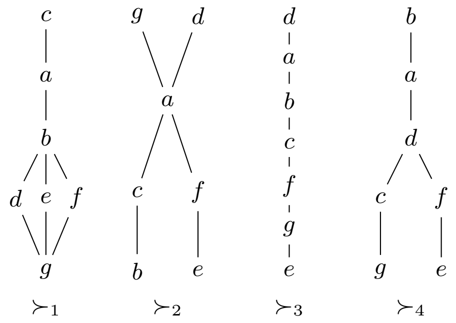

Figure 1. Partial preferences $\succ_1, \succ_2, \succ_3, \succ_4$ over the criteria 1,2,3,4.

sets in the presence of incomplete preferential information. We argue that this question calls for a process-based approach whereby comparative statements can be produced. We make precise the intuition that explanations can be of different levels of detail and complexity. Specifically, we classify explanations depending on the "operators" that were used to derive the desired statements. The major ingredient is a characterization of statements on weights that can be deduced (in terms of linear combinations) from the PI. We also show how to compute them. The remainder of this paper is as follows. Sect. 2 first details the available preferential information, the decision model, and the language used to produce explanations. We show how our approach can flexibly cater for the different degrees of accuracy that may occur within the same instance, depending whether the pairwise comparison under analysis is considered tight (Sect. 3) or obvious (Sect. 4). Sect. 5 illustrates the output as produced by our implementation. Sect. 6 discusses related work and concludes.

2 Background and basic definitions

We consider a finite set O of options and a finite set $H = \{1, \dots, m\}$ of criteria. To simplify notation, coalition $\{1, 2, 3\}$ will be noted 123.

2.1 Description of the preferential information

The decision-maker needs to provide information regarding the ranking of options O, but also regarding the relative strength of coalitions of criteria (2^H). Thus, two types of statements are considered.

Definition 1. *A preferential statement (p-statement) is of the form $[b \succ_i c]$ where $b, c \in O$ and $i \in H$, meaning that b is preferred over c on criterion i. Let $\mathcal{S}$ denote the set of all such statements.*

Definition 2. *A comparative statement (c-statement) is of the form $[I \succ J]$ where $I, J \subseteq H$ with $I \cap J = \emptyset$, meaning that the importance of the criteria in I is larger than that of the criteria in J. Let $\mathcal{V}$ denote the set of all such statements.*

It is important to remark that expressing a c-statement amounts to expressing a constraint on the feasible weight vectors attached to the criteria. Let $\mathcal{W}$ (the set of normalized weights) be the set of weights vectors $w \in [0, 1]^H$ such that $\sum_{i \in H} w_i = 1$.

We now define the operators which will make the link between the c-statements and their semantical counterpart (the weights).

Definition 3. *For a set $V \subseteq \mathcal{V}$ of c-statements, let $V^{\downarrow} := \{w \in \mathcal{W}$ s.t. $\forall [I \succ J] \in V$, $\sum_{i \in I} w_i > \sum_{i \in J} w_i\}$ be the set of weights satisfying the comparative statements V. Conversely, the set of c-statements that can be deduced from $W \subseteq \mathcal{W}$ is $W^{\uparrow} := \{[I \succ J] \in \mathcal{V}$ s.t. $\forall w \in W$, $\sum_{i \in I} w_i > \sum_{i \in J} w_i\}$. Finally, we introduce some notation:*

- *For $V \subseteq \mathcal{V}$, we set $V^{\downarrow\uparrow} = (V^{\downarrow})^{\uparrow}$ and $\mathrm{cl}(V) := V^{\downarrow\uparrow}$.*
- *For $W \subseteq \mathcal{W}$, we set $W^{\uparrow\downarrow} = (W^{\uparrow})^{\downarrow}$.*

Definition 4. *A PI is a pair $\langle S, V \rangle$ with $S \subseteq \mathcal{S}$ and $V \subseteq \mathcal{V}$.*

The information provided by the decision-maker is supposed to be "rational". Specifically, this means that the S part of the PI constitutes a *partial order* (reflexive, antisymmetric, transitive, but not necessarily complete), and that V is assumed to be consistent[4], in the sense that $V^{\downarrow} \neq \emptyset$. Note finally that the set of all linear extensions that can be obtained from S is denoted $\mathcal{S}_{\text{lin}}(S)$.

Example 2 (1 ctd.). *Given the PI of Ex. 1, $V = \{[13 \succ 24], [23 \succ 1], [4 \succ 23]\}$. We have e.g. $[c \succ_1 d] \in S$, $[b \succ_2 a] \notin S$, and $\langle 0.2, 0.1, 0.15, 0.55 \rangle \notin V^{\downarrow}$ (violation of the first constraint).*

2.2 Description of the choice problem

Definition 5. *A set $I \subseteq H$ is called a* winning coalition *(w.r.t. the PI $\langle S, W \rangle$) if $\sum_{i \in I} w_i > \frac{1}{2}$ for all $w \in V^{\downarrow}$.*

Option b is necessarily preferred to c if whatever the weight vector compatible with V, whatever the completion of S to form total orders on each criterion, the sum of the weights of criteria supporting b is larger than the sum of the weights of criteria supporting c.

Definition 6. *For $b, c \in O$, b is* necessary preferred *to c given $\langle S, V \rangle$ (noted $b \succ_{S,V} c$) if*

$$\forall w \in V^{\downarrow} \;\; \forall K \in \mathcal{S}_{\text{lin}}(S) \qquad \sum_{i \in H \,,\, [b \succ_i c] \in K} w_i > \sum_{i \in H \,,\, [c \succ_i b] \in K} w_i .$$

This corresponds to the *necessary preference relation* [6]. In qualitative decision models, this concept is similar to the dominance query for the CP nets [2]. An option $a \in O$ is called *weighted Condorcet winner w.r.t.* $\langle S, V \rangle$ (noted $\mathrm{WCW}_O(S, V)$) if for all $b \in O \setminus \{a\}$, $a \succ_{S,V} b$. When the WCW does not exist, it is usual to consider the Smith set (henceforth denoted by A). It is the smallest set of alternatives such that all elements of A beat all options outside this set. It is well defined and unique [4]. When a WCW exists, the Smith set is reduced to the WCW. Moreover, we set $O^{\star} := O \setminus A$.

There is a clear relationship between the size of A and $\langle S, V \rangle$ provided: the less informative $\langle S, V \rangle$, the more likely it is that some options cannot be compared. Specifically, the size of A will typically shrink as $\langle S, V \rangle$ gets more specified. In Example 1, e and g are incomparable because (as we shall see later) 1 and 23 are not winning coalitions given V, and e and g are incomparable on criterion 4.

This model is widely used in MCDA (note that all weights remain hidden to the user). Models not based on numerical weights also exist, but they allow less deductions to be drawn from the PI.

2.3 Description of the language for the explanation

In Example 1, we have $A = \{a\}$. When analyzing why a beats all options in $O^{\star}$, one notices that there are different situations. For options b, c, d, the preference of a over these options is not so trivial and deserves an adequate explanation. For option g, the case seems more clear, since a beats g on 134, and coalitions 13 and 34 are already winning coalitions. Now regarding options e, f, the dominance of a is clear since a is supported by unanimity of the criteria. Generalizing this example, it appears that dominated options can be partitioned

[4] In fact, many work dealing with explanations in AI address the problem of exhibiting subsets of constraints provoking an inconsistency, see *e.g.* [8].

into different classes, capturing the fact that some of them are *obviously* dominated, some are *clearly* dominated, while some others are close to a tie with some element of A. Thus, the level of detail expected by the decision maker in the produced explanation will vary.

- *unanimous* – this case occurs when an alternative b lies behind a on all criteria (technically, the option is Pareto-dominated), i.e. for all $i \in H$, $[a \succ_i b] \in S$. This requires no specific explanation.
- *large majority* – this occurs when the minimum guaranteed value of the weight of the criteria supporting a against b is larger than a threshold $\rho \in (\frac{1}{2}, 1)$ to be fixed by the designer:
$$\min_{w \in V^{\downarrow}} \sum_{i \in H,\, [a \succ_i b] \in S} w_i > \rho. \quad (1)$$
As the decision is clear-cut, the decision-maker does not need for a precise explanation.
- *weak majority* – these are the remaining cases, i.e. when the decision is not clear and a detailed explanation is required. We will focus our development mainly on this case.

The explanation process is thus as follows. For each element a in the Smith set, we denote by $O^\star_{una}[a]$, $O^\star_{large}[a]$ and $O^\star_{weak}[a]$ the set of options from $O^\star$ in the situations *unanimous*, *large majority* and *weak majority* respectively with a. The first set is easily constructed. The second will be studied at the end of the paper as we focus our analysis on the *weak majority* situation. In this case, we notice that a is a WCW of the set $O^\star_{weak}[a] \cup \{a\}$ of options (denoted by $\mathrm{WCW}_{O^\star_{weak}[a]\cup\{a\}}(S, V)$). We can thus treat each element of A separately and explain why it is a WCW of this subset of options.

3 Complete explanations for a weak majority

We turn our attention to explanations as to why $\mathrm{WCW}_{O^\star_{weak}[a]\cup\{a\}}(S, V) = \{a\}$ for some $a \in A$. We shall formally distinguish different levels of complexity required to explain c-statements in this context. This will provide a formal basis for the definition of *minimal* explanations.

3.1 Complete explanations on S and V

Following the *data-based approach* in [7], providing an explanation amounts to simplify the PI provided by the decision-maker. Accordingly, a complete explanation is a set of p-statements S' together with a set of c-statements V' such that, for any weight vector which can be deduced from V', any completion of the set of p-statements from S' yields a as a WCW. By complete, we mean that while simplifying the data, one can still prove that a is a WCW.

Definition 7. *The set of* data-based complete explanations *given* $\langle S, V\rangle$ *is:* $Ex^{\mathrm{Data}}_{S,V}(a) = \Big\{\langle S', V'\rangle \subseteq S \times V$ *s.t.* $\mathrm{WCW}_{O^\star_{weak}[a]\cup\{a\}}(S', V') = \{a\}\Big\}$.

We need the following definition to show that one can use a condition on the operator cl to prove that a is a WCW.

Definition 8. $P_S(a, b) := \{i \in H$ *s.t.* $[a \succ_i b] \in S\}$ *and* $\mathcal{V}(S) := \{[P_S(a, b) \succ H \setminus P_S(a, b)]\,,\ b \in O^\star_{weak}[a]\}$.

Lemma 1. $\mathrm{WCW}_{O^\star_{weak}[a]\cup\{a\}}(S', V') = \{a\}$ *iff* $\mathcal{V}(S') \subseteq \mathrm{cl}(V')$.

Proof : Let $b \in O^\star_{weak}[a]$. Let $L := \{[a \succ_i b]$ s.t. $[a \succ_i b] \in S'\} \cup \{[b \succ_i a]$ s.t. $[a \succ_i b] \notin S'\}$. We have $a \succ_{S',V'} b$ iff $\sum_{i\in H\,,\,[a\succ_i b]\in L} w_i > \sum_{i\in H\,,\,[a\succ_i b]\notin L} w_i$ for all $w \in V'^{\downarrow}$, iff $[P_{S'}(a, b) \succ H \setminus P_{S'}(a, b)] \in V'^{\downarrow\uparrow} = \mathrm{cl}(V')$. ■

From the previous lemma, $\langle S', V'\rangle \subseteq S \times V$ is an element of $Ex^{\mathrm{Data}}_{S,V}(a)$ iff $\mathcal{V}(S') \subseteq \mathrm{cl}(V')$.

Example 3. *Consider* 5 *criteria and four options* a, b, c, d. *Assume that* $V = \{[1 \succ 23], [34 \succ 15], [2 \succ 5]\}$ *and* $S = \{[a \succ_1 b], [a \succ_4 b], [a \succ_5 b], [a \succ_2 c], [a \succ_3 c], [a \succ_4 c], [a \succ_1 d], [a \succ_3 d], [a \succ_4 d], [b \succ_3 d]\}$. *Let* $V' = \{[1 \succ 23], [34 \succ 15]\}$ *and* $S' = S \setminus \{[b \succ_3 d]\}$. *We note that* $\langle S', V'\rangle \in Ex^{\mathrm{Data}}_{S,V}(a)$. *In the data-based approach,* $\langle S', V'\rangle$ *is the minimal complete explanation in the sense of set inclusion. However, for the decision-maker, the sole knowledge of* $\langle S', V'\rangle$ *is not sufficient to understand why* a *is a WCW, i.e. why the sets of criteria* $P_{S'}(a, b)$, $P_{S'}(a, c)$ *and* $P_{S'}(a, d)$ *appearing in* S' *form winning coalitions. In other words, the decision maker needs to understand why* $\mathcal{V}(S') = \{[145 \succ 23], [234 \succ 15], [134 \succ 15]\}$ *can be deduced from* V.

Following this example, one sees that there is a major distinction between the data-based and the process-based approaches. The first one does not allow a complete traceability from the PI to the recommendations. Hence we adopt the second one in this paper regarding the c-statements. In a process-based approach, a complete explanation is a pair composed of $S' \subseteq S$ such that $\mathcal{V}(S') \subseteq \mathrm{cl}(V)$ (proving that a is a WCW), and of an explanation noted $Ex^{\mathrm{Proc}}_V(\mathcal{V}(S'))$ of why $\mathcal{V}(S')$ results from V. Ex^{Proc}_V is not further described here; it will be done in Section 3.3.

Definition 9. *The set of* process-based complete explanations *given* $\langle S, V\rangle$ *is:* $Ex^{\mathrm{Proc}}_{S,V}(a) = \Big\{\langle S', Ex^{\mathrm{Proc}}_V(\mathcal{V}(S'))\rangle : S' \subseteq S$ *and* $\mathcal{V}(S') \subseteq \mathrm{cl}(V)\Big\}$.

In order to be able to compute S' but also to explain $\mathcal{V}(S')$, we need to give some properties of cl and characterize $\mathrm{cl}(V)$.

3.2 cl as closure

From Def. 3, $\mathrm{cl}(V)$ is the set of c-statements that can be deduced from V. A natural question is whether applying cl several times adds more c-statements. We show that this is not the case. More precisely, the operator $\mathrm{cl} : \mathcal{P}(\mathcal{V}) \to \mathcal{P}(\mathcal{V})$ is a *closure*, i.e. $V \subseteq \mathrm{cl}(V)$ for all $V \subseteq \mathcal{V}$ (extensiveness), $V \subseteq V'$ implies that $\mathrm{cl}(V) \subseteq \mathrm{cl}(V')$ for all $V, V' \subseteq \mathcal{V}$ (increasingness), and $\mathrm{cl} \circ \mathrm{cl} = \mathrm{cl}$ (idempotency), as its notation suggests.

Lemma 2. *The operator* cl *is a closure.*

Proof : The following three results are clear.

$$W \subseteq W^{\uparrow\downarrow} \text{ for all } W \subseteq \mathcal{W}. \quad (2)$$
$$V \subseteq V^{\downarrow\uparrow} \text{ for all } V \subseteq \mathcal{V}. \quad (3)$$
$$\forall V, V' \in \mathcal{V}, \quad V \subseteq V' \Rightarrow V^{\downarrow} \supseteq V'^{\downarrow}. \quad (4)$$

We now give a few useful assertions.

Assertion 1. $W^{\uparrow} = W^{\uparrow\downarrow\uparrow}$ *for all* $W \subseteq \mathcal{W}$.

Proof : We need only to prove $W^{\uparrow\downarrow\uparrow} \subseteq W^{\uparrow}$ as the opposite inclusion follows from (2). Let us consider thus $[I \succ J] \in W^{\uparrow\downarrow\uparrow}$. Hence

$$\forall w \in W^{\uparrow\downarrow} \quad \sum_{i\in I} w_i > \sum_{i\in J} w_i. \quad (5)$$

Let us fix now $w \in W$. From (2), $w \in W^{\uparrow\downarrow}$. By (5), we have $\sum_{i\in I} w_i > \sum_{i\in J} w_i$. This latter relation is satisfied for all $w \in W$. Hence $[I \succ J] \in W^{\uparrow}$. ■

Assertion 2. $V^\downarrow = V^{\downarrow\uparrow\downarrow}$ *for all* $V \subseteq \mathcal{V}$.

Proof : Similar to that of Assertion 1. ■

Extensiveness: Follows from (3).
Increasingness: Let $V \subseteq V'$ and $[I \succ J] \in V^{\downarrow\uparrow}$. Then for all $w \in V^\downarrow$, $\sum_{i\in I} w_i > \sum_{i\in J} w_i$. As $V'^\downarrow \subseteq V^\downarrow$ (by (4)), then for all $w \in V'^\downarrow$, $\sum_{i\in I} w_i > \sum_{i\in J} w_i$. Hence $[I \succ J] \in V'^{\downarrow\uparrow}$, and thus $V^{\downarrow\uparrow} \subseteq V'^{\downarrow\uparrow}$.
Idempotency: Follows from Assertions 1 and 2. ■

3.3 Explanations of c-statements

The aim of this section is to construct $Ex_V^{\mathrm{Proc}}(V')$ for $V' \subseteq \mathrm{cl}(V)$. To this end, one shall explain how any element of $\mathrm{cl}(V)$ results from V. We start with a simple example.

Example 4 (2 ctd.). *From* $[4 \succ 23] \in V$*, we can deduce a fortiori that* $[14 \succ 23] \in \mathrm{cl}(V)$ *and* $[4 \succ 3] \in \mathrm{cl}(V)$.

Monotonicity generalizes the previous example in the following way:

$$[I \succ J] \in V \Longrightarrow \forall I' \supseteq I \; \forall J' \subseteq J \quad [I' \succ J' \setminus I'] \in \mathrm{cl}(V) \quad (6)$$

Consider a more complex extension.

Example 5 (2 ctd.). $V^\downarrow$ *is composed of the weights* $w \in \mathcal{W}$ *satisfying* $w_1 + w_3 > w_2 + w_4$, $w_2 + w_3 > w_1$ *and* $w_4 > w_2 + w_3$.

New constraints can be derived by linear combinations of these constraints. For instance, the constraint $w_3 + w_4 > w_1 + w_2$ *results from the summation of constraint* $w_1 + w_3 > w_2 + w_4$ *with two times the constraints* $w_2 + w_3 > w_1$ *and* $w_4 > w_2 + w_3$.

The next proposition shows that the intuition of Example 5 holds in the general case: all c-statements that can be deduced from V results from linear combinations (with integer coefficients) of the constraints in V and of the constraints on the sign of the weights.

Proposition 1. $[I \succ J] \in \mathrm{cl}(V)$ *iff the following ILP is feasible:*

Find non-negative integers $\{\alpha_{E,F}\}_{[E\succ F]\in V}, \{\beta_i\}_{i\in H}, \gamma$ *minimizing* $\sum_{[E\succ F]\in V} \alpha_{E,F} + \sum_{i\in H} \beta_i + \gamma$ *such that*

$$\sum_{[E\succ F]\in V} \alpha_{E,F} \geq 1 \quad (7)$$

$$\beta_i + \sum_{[E\succ F]\in V,\, E\ni i} \alpha_{E,F} - \sum_{[E\succ F]\in V,\, F\ni i} \alpha_{E,F} = \begin{cases} \gamma & \text{if } i \in I \\ -\gamma & \text{if } i \in J \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

for all $i \in H$.

Proof : The normalization condition of the weights can be removed since we analyse the completion among comparative statements. Let $U := \{w \in \mathbb{R}_+^m \; : \; \forall [E \succ F] \in V \; , \; \sum_{i\in E} w_i > \sum_{i\in F} w_i\}$. Let us consider $[I \succ J] \in \mathrm{cl}(V)$. Hence for all $w \in U$, $\sum_{i\in I} w_i > \sum_{i\in J} w_i$. This means that $U' := \{w \in U \; , \; \sum_{i\in E} w_i \leq \sum_{i\in F} w_i\} = \emptyset$. Hence the linear constraints in U' are inconsistent. From Motzkin's theorem [12, pages 28-29], there exists non-negative integers $\alpha_{E,F}$, γ and β_i with at least one coefficient corresponding to the strict inequalities (i.e. at least one $\alpha_{E,F}$ non-zero – see (7)) such that the coefficients in front of each w_i in the following expression are equal to zero

$$\sum_{[E\succ F]\in V} \alpha_{E,F}\Big(\sum_{i\in E} w_i - \sum_{i\in F} w_i\Big) + \gamma\Big(\sum_{i\in J} w_i - \sum_{i\in I} w_i\Big) + \sum_{i\in H} \beta_i \, w_i.$$

Hence (8) is fulfilled for all $i \in H$. ■

The previous proposition is very important. It provides a characterization of $\mathrm{cl}(V)$. It shows precisely how constraint $[I \succ J]$ is derived from V and the sign of the weights. The values $\alpha_{E,F}$ and β_k are the coefficients that are multiplied by the constraints $\sum_{i\in E} w_i > \sum_{i\in F} w_i$ and $w_k \geq 0$ respectively. The summation yields the constraint $\gamma \times \sum_{i\in I} w_i > \gamma \times \sum_{i\in J} w_i$.

If the coefficients α, β, γ satisfy (7) and (8), multiplying these coefficients by any positive integer also verify the constraints. The use of the minimization functional in the ILP ensures that we obtain the smallest values of the coefficients and thus the simplest explanation.

Definition 10. *The* complete explanation $Ex_V^{\mathrm{Proc}}([I \succ J])$ *of the c-statement* $[I \succ J] \in \mathrm{cl}(V)$ *is* $\langle(\{\alpha_{E,F}\}_{[E\succ F]\in V}, \{\beta_i\}_{i\in H}, \gamma), [I \succ J]\rangle$.

For $V' \subseteq \mathrm{cl}(V)$, $Ex_V^{\mathrm{Proc}}(V') := \cup_{[I\succ J]\in V'} Ex_V^{\mathrm{Proc}}([I \succ J])$.

Example 6 (5 ctd.). *With* $\alpha_{13,24} = 1$, $\alpha_{23,1} = 2$, $\alpha_{4,23} = 1$, $\beta_i = 0$ *for all* $i \in H$ *and* $\gamma = 1$*, we obtain* $[34 \succ 12] \in \mathrm{cl}(V)$.

Moreover, $\mathrm{cl}(V) = \{[123 \succ 4], [124 \succ 3], [134 \succ 2], [13 \succ 24], [13 \succ 2], [14 \succ 23], [14 \succ 2], [1 \succ 2], [13 \succ 4], [14 \succ 3], [234 \succ 1], [23 \succ 1], [24 \succ 1], [34 \succ 12], [34 \succ 1], [4 \succ 1], [24 \succ 3], [34 \succ 2], [3 \succ 2], [4 \succ 23], [4 \succ 2], [4 \succ 3]\}$.

3.4 Complexity levels in explaining c-statements

In Example 6, let us consider four particular elements of $\mathrm{cl}(V)$, $[23 \succ 1]$, $[4 \succ 3]$, $[4 \succ 1]$ and $[34 \succ 12]$. The difficulty of justifying these four statements from V is not the same. Indeed, the first statement $[23 \succ 1]$ is directly contained in V so that there is no underlying complexity for the user. The second statement $[4 \succ 3]$ is directly obtained from $[4 \succ 23] \in V$ using a monotonicity argument (see Example 4). The third statement $[4 \succ 1]$ results from the summation of the two relations $w_2 + w_3 > w_1$ and $w_4 > w_2 + w_3$ of $V^\downarrow$. Lastly, as we already noticed in Example 5, the last statement $[34 \succ 12]$ is more complex to obtain. The arguments that we use to justify a statement from $\mathrm{cl}(V)$, going from the first statement to the fourth one are of increasing complexity.

It seems thus natural to decompose $\mathrm{cl}(V)$ into four nested sets. The first set $\mathrm{cl}_0(V) := V$ is the c-statements contained in the PI. The second set $\mathrm{cl}_1(V)$ is composed of the elements of $\mathrm{cl}(V)$ that can be deduced from V only using monotonicity condition (see (6)). This corresponds to the case where, in Proposition 1, all α coefficients are equal to 0, except one that is equal to 1. The third set $\mathrm{cl}_2(V)$ is composed of the elements of $\mathrm{cl}(V)$ that can be deduced from V only using summation and monotonicity conditions. This corresponds to the case when the α coefficients are either equal to 0 or 1. Finally, $\mathrm{cl}_3(V) = \mathrm{cl}(V)$. The set $\mathrm{cl}(V)$ is partitioned in the following way.

Definition 11. $\Delta_0 = \mathrm{cl}_0(V)$, $\Delta_j = \mathrm{cl}_j(V) \setminus \mathrm{cl}_{j-1}(V)$ *for* $j \in \{1, 2, 3\}$.

Example 7 (6 ctd.). *We have* $\Delta_0 = \{[13 \succ 24], [23 \succ 1], [4 \succ 23]\}$, $\Delta_1 = \{[123 \succ 4], [134 \succ 2], [13 \succ 2], [13 \succ 4], [234 \succ 1], [124 \succ 3], [14 \succ 23], [14 \succ 2], [14 \succ 3], [24 \succ 3], [34 \succ 2], [4 \succ 2], [4 \succ 3]\}$, $\Delta_2 = \{[24 \succ 1], [34 \succ 1], [4 \succ 1], [1 \succ 2], [3 \succ 2]\}$ *and* $\Delta_3 = \{[34 \succ 12]\}$.

The sets $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ are of increasing complexity. When comparing two sets $V', V'' \subseteq \mathrm{cl}(V)$, we prefer the set that has the smallest number of elements in Δ_3. In case of equality, we prefer the one that has the smallest number of elements in Δ_2. And so on. The following ordering $\rhd_{\mathcal{V}}$ depicts the complexity of understanding why a set of c-statements derives from V.

Definition 12. *For $V', V'' \subseteq \mathrm{cl}(V)$, $V' \rhd_{\mathcal{V}} V''$ iff $(|V' \cap \Delta_3|, |V' \cap \Delta_2|, |V' \cap \Delta_1|, |V' \cap \Delta_0|) \succ_{\mathrm{lex}} (|V'' \cap \Delta_3|, |V'' \cap \Delta_2|, |V'' \cap \Delta_1|, |V'' \cap \Delta_0|)$, where $\succ_{\mathrm{lex}}$ is the lexicographic ordering. $(x_1, x_2, x_3, x_4) \succ_{\mathrm{lex}} (y_1, y_2, y_3, y_4)$ if there exists $i \in \{1,2,3,4\}$ such that $x_i > y_i$ and $x_j = y_j$ for all $j \in \{1,2,3,4\}$ with $j < i$.*

We notice the elements of S and of Δ_0 are of the same complexity since they are both elements of the PI. The number of elements of p-statements is added to the number of c-statements that belong to Δ_0.

Definition 13. *Let $\langle S', Ex_V^{\mathrm{Proc}}(\mathcal{V}(S'))\rangle, \langle S'', Ex_V^{\mathrm{Proc}}(\mathcal{V}(S''))\rangle \in Ex_{S,V}^{\mathrm{Proc}}(a)$. The* complexity *of S' is $comp(S') := (|\mathcal{V}(S') \cap \Delta_3|, |\mathcal{V}(S') \cap \Delta_2|, |\mathcal{V}(S') \cap \Delta_1|, |\mathcal{V}(S') \cap \Delta_0| + |S'|)$. We define the order $\rhd$ (over $Ex_{S,V}^{\mathrm{Proc}}(a)$) by $\langle S', Ex_V^{\mathrm{Proc}}(\mathcal{V}(S'))\rangle \rhd \langle S'', Ex_V^{\mathrm{Proc}}(\mathcal{V}(S''))\rangle$ iff $comp(S') \succ_{\mathrm{lex}} comp(S'')$.*

3.5 Determination of the minimal explanations

In order to compute the minimal explanations in the sense of $\rhd$, one may proceed in three steps: (S1) determines all elements of $\mathrm{cl}_1(V)$, $\mathrm{cl}_2(V)$ and $\mathrm{cl}_3(V)$; (S2) identifies all elements of $Ex_S^{\mathrm{Proc}}(a) := \{S' \subseteq S \ : \ \mathcal{V}(S') \subseteq \mathrm{cl}(V)\}$; (S3) determines the elements S' in $Ex_S^{\mathrm{Proc}}(a)$ such that $\langle S', Ex_V^{\mathrm{Proc}}(\mathcal{V}(S'))\rangle$ is minimal in the sense of $\rhd$. To determine whether $[I \succ J] \in \mathrm{cl}(V)$, it suffices to perform an ILP (by Proposition 1). Step (S1) requires to compute all elements of $\mathrm{cl}_1(V)$, $\mathrm{cl}_2(V)$ and $\mathrm{cl}_3(V)$, which is not necessary in step (S2) and might be time consuming. Hence, we propose to perform steps (S1) and (S2) at the same time, determining the belonging to the closure only when required.

To this end, the following variables are introduced in the algorithm: $\mathit{Cl}_1, \mathit{Cl}_2, \mathit{Cl}_3$ correspond to the elements of $\mathrm{cl}_1(V)$, $\mathrm{cl}_2(V)$, $\mathrm{cl}_3(V)$ that are useful in the analysis; $\mathcal{N}\mathit{Cl}$ are c-statements that are not in $\mathrm{cl}(V)$ and ExCl contains the explanations of $\mathit{Cl}_1, \mathit{Cl}_2, \mathit{Cl}_3$. We start by initializing these variables

$$\mathit{Cl}_0 = V \ , \ \mathit{Cl}_1 \leftarrow \emptyset \ , \ \mathit{Cl}_2 \leftarrow \emptyset \ , \ \mathit{Cl}_3 \leftarrow \emptyset \ , \ \mathcal{N}\mathit{Cl} \leftarrow \emptyset \ , \ \mathit{ExCl} \leftarrow \emptyset.$$

The next algorithm checks whether a c-statements belongs to $\mathrm{cl}(V)$.

Algorithm 1. <u>**Function** $\mathrm{isInClosure}(I)$</u> *returns a boolean saying whether $[I \succ H \setminus I] \in \mathrm{cl}(V)$, and updates Cl_1, Cl_2, Cl_3, $\mathcal{N}\mathit{Cl}$, ExCl:*
- **If** $[I \succ H \setminus I] \in \mathit{Cl}_0 \cup \mathit{Cl}_1 \cup \mathit{Cl}_2 \cup \mathit{Cl}_3$ **then return true**;
- **If** $[I \succ H \setminus I] \in \mathcal{N}\mathit{Cl}$ **then return false**;
- **Launch** *the ILP of Proposition 1 on* $[I \succ H \setminus I]$;
- **If** *the ILP is infeasible* **then**
  - $\mathcal{N}\mathit{Cl} \leftarrow \mathcal{N}\mathit{Cl} \cup \{[I \succ H \setminus I]\}$; **return false**;
- $\mathit{ExCl} \leftarrow \mathit{ExCl} \cup \{\langle(\{\alpha_{E,F}\}_{[E \succ F] \in V}, \{\beta_i\}_{i \in H}, \gamma), [I \succ H \setminus I]\rangle\}$;
- **If** $\sum_{[E \succ F] \in V} \alpha_{E,F} = 1$ **then return true**;
- **If** $\alpha_{E,F} \in \{0,1\}$ *for all* $[E \succ F] \in V$ **then**
  - $\mathit{Cl}_2 \leftarrow \mathit{Cl}_2 \cup \{[I \succ H \setminus I]\}$; **return true**;
- $\mathit{Cl}_3 \leftarrow \mathit{Cl}_3 \cup \{[I \succ H \setminus I]\}$; **return true**;

The main algorithm is now described. Firstly, it computes $\mathcal{P}_S(a,b) := \{I \subseteq P_S(a,b) \text{ s.t. } [I \succ H \setminus I] \in \mathrm{cl}(V)\}$. Then steps (S2) and (S3) are performed.

Algorithm 2. <u>**Function** $\mathrm{bestExplanations}(a)$</u> *(for $a \in A$) computes the elements of $Ex_{S,V}^{\mathrm{Proc}}(a)$ that are minimal w.r.t. $\rhd$:*
- **For all** $b \in O^\star_{weak}[a]$ **do**
  - $\mathcal{P}_S(a,b) \leftarrow \emptyset$;
  - **For all** $I \subseteq P_S(a,b)$ **do**
    - **If** $\mathrm{isInClosure}(I)$ =**true then** $\mathcal{P}_S(a,b) \leftarrow \mathcal{P}_S(a,b) \cup \{I\}$;
- $Ex_S^{\mathrm{Proc}}(a) \leftarrow \Big\{\{[a \succ_i b] \ , \ i \in I_b \text{ and } b \in O^\star_{weak}[a]\}$ *for all* $I_b \in \mathcal{P}_S(a,b), b \in O^\star_{weak}[a]\Big\}$;
- $E \leftarrow \Big\{\langle S', Ex_V^{\mathrm{Proc}}(\mathcal{V}(S'))\rangle \ , \ S' \in Ex_S^{\mathrm{Proc}}(a)\Big\}$; *(where Ex_V^{Proc} is stored in ExCl)*
- **return** *the minimal elements of E w.r.t. $\rhd$;*

Note that in the first loop in Algorithm 2, when $\mathrm{isInClosure}(I)$ returns *false*, we need not explore any subset of I (they cannot belong to $\mathcal{P}_S(a,b)$). This treatment is clearly exponential in the number of criteria in the worst case. In Algo. 2, the number of calls to an ILP is at worse 2^m (where m is the number of criteria) and the total number of calls of isInClosure is at worse $|O^\star_{weak}[a]| \times 2^m$. Moreover, the cardinality of $Ex_S^{\mathrm{Proc}}(a)$ is at worse $2^{m \times |O^\star_{weak}[a]|}$. In practice, it might be much less – see Example 8 below. Moreover our approach only computes ILPs when required.

Example 8 (1 ctd.). *In the comparison of a with b, c, d, the number of ILP that are solved is 10 (instead of 16 in the worse case), and* isInClosure *is called 15 times (instead of 48 in the worse case). Moreover, $|Ex_S^{\mathrm{Proc}}(a)| = 4$ (instead of 4096 in the worse case) – see Section 5 for the details.*

4 Complete explanation for a large majority

We have seen at the beginning of the paper, that the *large majority* situation applies when condition (1) is satisfied.

Definition 14. $V_\rho^{\Updownarrow} = \Big\{I \subseteq H \text{ s.t. } \forall w \in V^{\downarrow} \ \ \sum_{i \in I} w_i > \rho\Big\}$.

This section is concerned with the identification of the coalitions in $V_\rho^{\Updownarrow}$ with the associated explanation. One first notes that if $I \in V_\rho^{\Updownarrow}$ then necessarily $[I \succ H \setminus I] \in \mathrm{cl}(V)$ as $\rho > \frac{1}{2}$.

Example 9. *Assume that $V = \{[1 \succ 2], [2 \succ 3], [3 \succ 4], [4 \succ 5]\}$, with $m = 5$. One can easily show that $w_1 + w_2 + w_3 > \frac{3}{5}$ for all $w \in V^{\downarrow}$. Hence coalition $123 \in V_\rho^{\Updownarrow}$ if ρ does not exceed $\frac{3}{5}$.*

We characterize the elements of $V_\rho^{\Updownarrow}$.

Proposition 2. *$I \in V_\rho^{\Updownarrow}$ iff the following ILP is feasible:*
Find $\{\alpha_{E,F}\}_{[E \succ F] \in V} \in \mathbb{N}_+^V, \{\beta_i\}_{i \in H} \in \mathbb{N}_+^H, \delta \in \mathbb{N}, \gamma \in \mathbb{N}_+$ minimizing $\sum_{[E \succ F] \in V} \alpha_{E,F} + \sum_{i \in H} \beta_i + \delta + \gamma$ such that

$$\sum_{[E \succ F] \in V} \alpha_{E,F} \geq 1 \tag{9}$$

$$\beta_i + \delta + \sum_{[E \succ F] \in V \ , \ E \ni i} \alpha_{E,F} - \sum_{[E \succ F] \in V \ , \ F \ni i} \alpha_{E,F} = \begin{cases} \gamma & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

for all $i \in H$, and

$$\delta \geq \gamma \, \rho \tag{11}$$

Proof (Sketch): Similar to that of Proposition 1. The normalization condition must be considered due to the presence of non-zero right hand side in (1). The inequality (11) follows from this. ■

The values $\alpha_{E,F}$, β_k and δ are the coefficients that are multiplied by the constraints $\sum_{i \in E} w_i > \sum_{i \in F} w_i$, $w_k \geq 0$ and $\sum_{i \in H} w_i = 1$ respectively.

Example 10 (Example 9 cont.). *For $\frac{1}{2} < \rho \leq \frac{3}{5}$, $123 \in V_\rho^{\Updownarrow}$ results from the coefficients $\alpha_{1,2} = 2$,,$\alpha_{2,3} = 4$, $\alpha_{3,4} = 6$, $\alpha_{4,5} = 3$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, $\delta = 3$ and $\gamma = 5$. More generally, $V_\rho^{\Updownarrow} = \{123, 124, 1234, 1235, 1245, 12345\}$.*

The generation of explanation from the coefficients in Prop. 2 can be done as for Prop. 1, based on α, β, δ and γ.

5 Output of the results

To wrap up, for all $a \in A$, $O^\star_{una}[a] = \{b \in O^\star \,,\, P_S(a,b) = H\}$, $O^\star_{large}[a] = \{b \in O^\star \setminus O^\star_{una}[a] \,,\, P_S(a,b) \in V^\Updownarrow_\rho\}$ and $O^\star_{weak}[a]$ are the remaining elements of $O^\star$. ILP is used to compute the coefficients appearing in Propositions 1 and 2. The minimal explanations for $O^\star_{weak}[a]$ are obtained thanks to Algorithm 2. As for the closure, the elements of $V^\Updownarrow_\rho$ are computed only when required, that is only for coalitions $P_S(a,b)$.

We conclude the paper by considering again the Example 1, to illustrate how our approach (implemented in JAVA) outputs the results. We recall that the Smith set is a. The explanation generated is as follows:

- a is better than e and f by *unanimity* of the criteria;
- a is better than g on a *large majority*.

By default, the system shall not give any further detail, since the case is deemed clear enough not to require any further justification. Upon request (*why?*) of the decision-maker however, the algorithm may provide the following explanation.

- In fact, the large majority is 134, and $134 \in V^\Updownarrow_\rho$, with $\rho = 0.7$ and the coefficients $\alpha_{13,24} = 1$ (for $[13 \succ 24]$), $\alpha_{4,23} = 2$ (for $[4 \succ 23]$), $\beta_3 = 2$ (for $w_3 \geq 0$), $\delta = 3$ (for $w_1+w_2+w_3+w_4 = 1$), $\gamma = 4$ (for $134 \in V^\Updownarrow_\rho$), and all other coefficients are zero.

Concerning the comparison of a with b, c and d, we apply Algorithm 2 described in Section 3.5. In particular, $Ex^{\mathrm{Proc}}_S(a) = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{[a \succ_1 b], [a \succ_3 b], [a \succ_3 c], [a \succ_4 c], [a \succ_1 d], [a \succ_4 d]\}$, $S_2 = S_1 \cup \{[a \succ_2 b]\}$, $S_3 = S_1 \cup \{[a \succ_2 c]\}$ and $S_4 = S_1 \cup \{[a \succ_2 b], [a \succ_2 c]\}$. At first sight, S_1 seems the simplest set and S_4 the most complex one. This intuition is defected. By Def. 13 and Ex. 7, $comp(S_1) = (1,0,1,7)$ as $\mathcal{V}(S_1)$ is composed of $[13 \succ 24] \in \Delta_0$ (comparison of a and b), $[34 \succ 12] \in \Delta_3$ (comparison of a and c) and $[14 \succ 23] \in \Delta_1$ (comparison of a and d), and S_1 is composed of 6 statements. Likewise, $comp(S_2) = (1,0,2,7)$, $comp(S_3) = (0,0,2,8)$, $comp(S_4) = (0,0,3,8)$. Comparing S_1 and S_3, it is apparent that simplifying over the p-statements might result in a much more complex explanation regarding the c-statements. Hence the minimal element of $Ex^{\mathrm{Proc}}_{S,V}(a)$ w.r.t. $\rhd$ is $\langle S_3, Ex^{\mathrm{Proc}}_V(\mathcal{V}(S_3))\rangle$. The latter takes the following form:

- a is better than b on the *weak majority* $13 \in \Delta_0$;
- a is better than c on the *weak majority* $234 \in \Delta_1$ such that $\alpha_{23,1} = 1$ (for $[23 \succ 1]$), $\beta_4 = 1$ (for $w_4 \geq 0$) and $\gamma = 1$;
- a is better than d on the *weak majority* $14 \in \Delta_1$ such that $\alpha_{4,23} = 1$ (for $[4 \succ 23]$), $\beta_1 = 1$ (for $w_1 \geq 0$) and $\gamma = 1$.

The previous explanation is very detailed. For a user who does not require such level of traceability (e.g. a user with a shallower understanding of the decision process), it is possible to hide the coefficients α, β and γ, by just mentioning the set statements that yield another one. We emphasize that we did not explore yet the natural language issues that occur here: the way to present and organize the same (content-wise) explanation may clearly affect the way it is perceived [3]. We leave this for further research.

6 Related work and Conclusion

In the domain of recommender systems, the issue of explanation has motivated a huge amount of studies. In their recent taxonomy proposal, [5] distinguish three distinctive features to classify generated explanations: the *reasoning model* (whether the explanation disclose, even partially, the decision model), the *recommendation paradigm* (the type of decision model), and the *information categories* which are used when the explanation is generated (more specifically, whether they use the user's model, whether they refer to the recommended item and/or to the alternative options). Our approach would be classified as follows: *white box*, *knowledge-based*, and using the *three* categories (in our case: user's rankings, and referring to both the recommended item and the dominated ones). A distinctive feature of our approach lies on the decision model used, taken together with the fact that the PI may be largely incomplete. In this context, the precise weights attached to attributes cannot be exhibited, and the challenge is to provide convincing (complete) explanations despite this constraint.

We also observe that there is at least a syntactic similarity with argumentation theories. For instance, in Definition 10, an explanation is a pair $\langle C, [I \succ J]\rangle$, where C is minimal and is the support of the explanation and $[I \succ J]$ is the conclusion. This may be seen as an argument, a pair $\langle H, h\rangle$ where h is the conclusion, H is a minimal consistent subset of the knowledge base that entails h [1]. However we emphasize again that in our context we are looking for proofs, whereas arguments support non-monotonic inferences. Under this more argumentative perspective, [10] puts forward the idea of having different levels of explanations for multi-attribute preference models.

A study of complete explanations for the same type of preference model can be found in [11]. The main difference is that completeness of the PI (for both weights and rankings) is assumed in [11]. In this context, a sentence like *"the weight of criterion 2 is 0.3"* may stand as a valid justification. Instead, this paper investigates the explanation of the importance of coalitions of criteria.

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