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Profile decomposition and phase control for circle-valued maps in one dimension

Petru Mironescu*

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Abstract. When 1 , maps <math>f in $W^{1/p,p}((0,1); \mathbb{S}^1)$ have $W^{1/p,p}$ phases φ , but the $W^{1/p,p}$ -seminorm of φ is not controlled by the one of f. Lack of control is illustrated by "the kink": $f = e^{i\varphi}$, where the phase φ moves quickly from 0 to 2π . A similar situation occurs for maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, with Moebius maps playing the role of kinks. We prove that this is the only loss of control mechanism: each map $f : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $|f|_{W^{1/p,p}}^p \leq M$ can be written as $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\pm 1}$, where M_{a_j} is a Moebius map vanishing at $a_j \in \mathbb{D}$, while the integer K = K(f) and the phase ψ are controlled by M. In particular, we have $K \leq c_p M$ for some c_p . When p = 2,

we obtain the sharp value of c_2 , which is $c_2 = 1/(4\pi^2)$. As an application, we obtain the existence of minimal maps of degree one in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ with $p \in (2-\varepsilon,2)$.

Résumé. Décomposition en profils et contrôle des phases des applications unimodulaires en dimension un. Si 1 , les applications <math>f appartenant à $W^{1/p,p}((0,1);\mathbb{S}^1)$ ont des phases φ dans $W^{1/p,p}$, mais la seminorme $W^{1/p,p}$ de φ n'est pas contrôlée par celle de f. L'absence de contrôle est illustrée par "le pli": $f = e^{i\varphi}$, où la phase φ augmente rapidement de 0 à 2π . Pour des applications $f : \mathbb{S}^1 \to \mathbb{S}^1$, le même phénomène apparaît, avec les transformations de Moebius jouant le rôle des plis. Nous prouvons que cet exemple est essentiellement le

seul : toute application $f: \mathbb{S}^1 \to \mathbb{S}^1$ telle que $|f|_{W^{1/p,p}}^p \leq M$ s'écrit $f = e^{\iota \psi} \prod_{j=1}^K (M_{a_j})^{\pm 1}$, où M_{a_j} est une transformation de Moebius s'annulant en $a_j \in \mathbb{D}$, tandis que l'entier K = K(f) et la phase ψ sont contrôlés par M. En particular, nous avons $K \leq c_p M$ pour une constante c_p . Pour p = 2, nous obtenons la valeur optimale de c_2 , qui est $c_2 = 1/(4\pi^2)$. Comme application, nous obtenons l'existence d'une application minimale de degré un dans $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ avec $p \in]2 - \varepsilon, 2[$.

1 Introduction

Let 0 < s < 1, $1 \le p < \infty$ and let $f : (0,1) \to \mathbb{S}^1$ belong to the space $W^{s,p}$. Then f can be written as $f = e^{i\varphi}$, where $\varphi \in W^{s,p}$ [4]. Once the existence of φ is known, a natural question is whether we can control $|\varphi|_{W^{s,p}}$ in terms of $|f|_{W^{s,p}}$. For most of s, p the answer is positive. The exceptional cases are provided precisely by the spaces $W^{1/p,p}((0,1);\mathbb{S}^1)$, with $1 [4]. In these spaces, lack of control is established via the following explicit example. For <math>n \ge 1$, we define φ_n as

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follows:

$$\varphi_n(x) := \begin{cases} 0, & \text{for } 0 < x < 1/2 \\ 2\pi n(x - 1/2), & \text{for } 1/2 < x < 1/2 + 1/n \\ 2\pi, & \text{for } 1/2 + 1/n < x < 1 \end{cases}$$

Then $|\varphi_n|_{W^{1/p,p}} \to \infty$ (since $\varphi_n \to \varphi = 2\pi \chi_{(1/2,1)}$ a.e., and φ does not belong to $W^{1/p,p}$). On the other hand, if we extend $u_n := e^{i\varphi_n}$ with the value 1 outside (0,1) and still denote the extension u_n then, by scaling,

 $|u_n|_{W^{1/p,p}((0,1))} \le |u_n|_{W^{1/p,p}(\mathbb{R})} = |u_1|_{W^{1/p,p}(\mathbb{R})} < \infty.$

Thus $|u_n|_{W^{1/p,p}((0,1))} \leq 1$ and $|\varphi_n|_{W^{1/p,p}((0,1))} \to \infty$. Finally, we invoke the fact that $W^{1/p,p}$ phases are unique mod 2π [4].

If one considers instead maps $f: \mathbb{S}^1 \to \mathbb{S}^1$, always in the critical case $f \in W^{1/p,p}$, 1 ,then a new phenomenon occurs: <math>f has a degree deg f, and does not have a $W^{1/p,p}$ phase at all when deg $f \neq 0$ [11, Remark 10]. However, even if deg f = 0 (and thus f has a $W^{1/p,p}$ phase φ), we have a loss of control phenomenon similar to the one on (0,1). Indeed, let $M_a(z) := \frac{a-z}{1-\overline{a}z}$, $a \in \mathbb{D}, z \in \overline{\mathbb{D}}$, be a Moebius transform (that we identify with its restriction to \mathbb{S}^1 , $M_a: \mathbb{S}^1 \to \mathbb{S}^1$). Let $f_a(z) := \overline{z} M_a(z)$, so that f_a is smooth and deg $f_a = 0$. One may prove (see below) that $|M_a|_{W^{1/p,p}} = |\mathrm{Id}|_{W^{1/p,p}}$, and thus f_a is bounded in $W^{1/p,p}$. However, if $a \to a = e^{i\xi} \in \mathbb{S}^1$, then the smooth phase φ_a of f_a converges a.e. to $\varphi(e^{i\theta}) := \begin{cases} \xi - \theta, & \text{if } \xi - \pi < \theta < \xi \\ 2\pi + \xi - \theta, & \text{if } \xi < \theta < \xi + \pi \end{cases}$, which does not

belong to $W^{1/p,p}$. [Here, uniqueness of the phases and convergence hold mod 2π .] Thus φ_a is not bounded as $a \to a \in S^1$. On the other hand, the plot of φ_a shows that φ_a has a "kink shape", and thus we have here the analog of the example on (0, 1).

There are evidences that this loss of control mechanism is the only possible one. For example, the phase of the kink is not bounded in $W^{1/p,p}$, but clearly is in $W^{1,1}$ (same for f_a). Bourgain and Brezis [3] proved that for every $f \in W^{1/2,2}((0,1);\mathbb{S}^1)$, we may split $f = e^{i\psi}v$, with ψ and $v = e^{i\eta}$ satisfying

$$|\psi|_{W^{1/2,2}} \lesssim |f|_{W^{1/2,2}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \lesssim |f|_{W^{1/2,2}}^2.$$
 (1)

Intuitively, one should think at v as at "the kink part of f". The above result was extended by Nguyen [18] to $1 : for every <math>1 and every <math>f \in W^{1/p,p}((0,1);\mathbb{S}^1)$, we may split $f = e^{i\psi}v$, with ψ and $v = e^{i\eta}$ satisfying

$$|\psi|_{W^{1/p,p}} \le C_p |f|_{W^{1/p,p}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \le C_p |f|_{W^{1/p,p}}^p.$$
(2)

Here we present another result in this direction, written for simplicity on the unit circle.

Theorem 1. Let 1 and <math>M > 0. Then there exist constants c_p and F(M) such that: every map $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $|f|_{W^{1/p,p}}^p \leq M$ can be written as $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j \in \{-1,1\}$,

$$K \le c_p M, \tag{3}$$

and

$$|\psi|_{W^{1/p,p}}^{p} \leq F(M). \tag{4}$$

When p = 2, we may take $c_2 = 1/(4\pi^2)$, and this constant is optimal.

Corollary 1. Let $1 and let <math>f_n, f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ be such that $f_n \to f$ in $W^{1/p,p}$. Then, up to a subsequence, there exist $K \in \mathbb{N}$, $\varepsilon_j \in \{-1, 1\}$, $a_{j_n} \in \mathbb{D}$, $\alpha_j \in \mathbb{S}^1$, j = 1, ..., K, $\psi_n \in W^{1/p,p}(\mathbb{S}^1; \mathbb{R})$, and a constant C, such that:

$$\begin{split} i) \ f_n &= e^{\iota \psi_n} \prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j} f; \\ ii) \ a_{j_n} &\to \alpha_j \ as \ n \to \infty; \\ iii) \ \psi_n &\to C \ in \ W^{1/p,p} \ as \ n \to \infty. \end{split}$$

The theorem and the corollary are reminiscent of profile decompositions obtained in different, often geometrical, contexts. We mention e.g. the work of Sacks and Uhlenbeck [20] on minimal 2-spheres, the analysis of Brezis and Coron [6, 7, 8] of constant mean curvature surfaces, or the one of Struwe [21] of equations involving the critical Sobolev exponent. There are also abstract approaches to bubbling as in the work of Lions [16] about concentrationcompactness or the characterization of lack of compactness of critical embeddings in Gérard [12], Jaffard [15] or Bahouri, Cohen and Koch [1].

Let us comment on the connection between (2) and our theorem. First, (2) has the following version for maps on \mathbb{S}^1 : we may split $f = e^{i\psi}v$, with $|\psi|_{W^{1/p,p}} \leq C_p |f|_{W^{1/p,p}}$ and $|v|_{W^{1,1}} \leq C_p |f|_{W^{1/p,p}}$. Next, a Moebius maps satisfies $|M_a|_{W^{1,1}} = 2\pi$, and thus

$$\left| \prod_{j=1}^{K} (M_{a_j})^{\varepsilon_j} \right|_{W^{1,1}} \le 2\pi K \le 2\pi c_p M.$$
(5)

Estimate (5) shows that (3) is a refinement of the second part of (2). On the other hand, (4) is weaker than the first part of (2), since F(M) need not have a linear growth (and actually we do not have any control on F). This suggests the following

Conjecture. Let $1 . Then there exist constants <math>c_p, d_p$ such that every $f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ satisfying $|f|_{W^{1/p,p}}^p \le M$ can be decomposed as $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j \in \{-1, 1\}$,

$$K \le c_p M, \tag{6}$$

and

$$|\psi|_{W^{1/p,p}}^{p} \le d_{p} M. \tag{7}$$

In addition, when p = 2, we may take $c_2 = 1/(4\pi^2)$.

2 Proofs

We start by recalling or establishing few auxiliary results. Given $1 \le p < \infty$, f, f_n will denote maps in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$. When $1 , "—" refers to weak convergence in <math>W^{1/p,p}$.

1. Recall that, up to a multiplicative factor $a \in \mathbb{S}^1$, the Moebius transforms give all the conformal representations $u : \mathbb{D} \to \mathbb{D}$. In particular, $M_a : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth orientation preserving diffeomorphism, and thus deg $M_a = 1$. Consequence: if $g : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous, then deg[$g \circ M_a$] = deg g.

2. If $1 \le p < \infty$ and $a \in \mathbb{D}$, then $|f \circ M_a|_{W^{1/p,p}} = |f|_{W^{1/p,p}}$. [Here, we let $|f|_{W^{1,1}} := \int_{\mathbb{S}^1} |\dot{f}| = \int_0^{2\pi} |d[f(e^{i\theta})]/d\theta | d\theta$ and, for $1 , <math>|f|_{W^{1/p,p}}^p := \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} |f(x) - f(y)|^p / |x - y|^2 dx dy$.] In order to prove the desired equality when p = 1, we write $M_a(e^{i\theta}) = e^{i\varphi(\theta)}$, $0 \le \theta \le 2\pi$, with φ smooth and increasing. Then

$$|f \circ M_{a}|_{W^{1,1}} = \int_{0}^{2\pi} \left| \frac{d}{d\theta} [f(e^{i\varphi(\theta)})] \right| d\theta = \int_{\varphi^{-1}(0)}^{\varphi^{-1}(2\pi)} \left| \frac{d}{d\theta} [f(e^{i\theta})] \right| d\theta = \int_{0}^{2\pi} \left| \frac{d}{d\theta} [f(e^{i\theta})] \right| d\theta = |f|_{W^{1,1}}.$$

When $1 , we rely on the following identity, valid for measurable functions <math>F : \mathbb{S}^1 \times \mathbb{S}^1 \to [0,\infty]$:

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{F(M_a(x), M_a(y))}{|x - y|^2} \, dx \, dy = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{F(x, y)}{|x - y|^2} \, dx \, dy. \tag{8}$$

Proof of (8): We have $[M_a]^{-1} = M_a$ and thus, after change of variables, (8) amounts to

$$|x - y|^2 |\dot{M}_a(x)| |\dot{M}_a(y)| = |M_a(x) - M_a(y)|^2, \ \forall x, y \in \mathbb{S}^1.$$
(9)

In turn, (9) follows immediately from the straightforward equality $|\dot{M}_a(x)| = \frac{1-|a|^2}{|1-\overline{a}x|^2}$.

3. If $1 \le p < \infty$ and $a \in \mathbb{D}$, then deg $[f \circ M_a)$] = deg f. Indeed, to start with, such f has a degree, since $W^{1/p,p} \hookrightarrow VMO$ and VMO maps gave a degree stable with respect to BMO convergence [11]. By item **1**, the desired equality holds true for smooth f. The general case follows by density of $C^{\infty}(\mathbb{S}^1;\mathbb{S}^1)$ into $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ [11, Lemmas A.11 and A.12] and by stability of the VMO degree.

4. If $1 \le p < \infty$ and the degree of f is d, then we may write $f(z) = e^{i\psi(z)}z^d$, with $\psi \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$. This follows easily from the fact that maps $f \in W^{1/p,p}((0,1);\mathbb{S}^1)$ lift within $W^{1/p,p}$ [4].

5. Let $1 . For <math>f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$, let u = u(f) be its harmonic extension. Set $c'_p := \inf\{|f|_{W^{1/p,p}}^p; u(0) = 0\}$. Clearly, c'_p is achieved, and therefore $c'_p > 0$.

6. When p = 2, we have the following straightforward calculations: if $f = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}$, then $|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n \in \mathbb{Z}} |n| |a_n|^2$ [10, Chapter 13], and deg $f = \sum_{n \in \mathbb{Z}} n |a_n|^2$ [11, eq (25)]. This leads to $4\pi^2 |\deg f| \le |f|^2_{W^{1/2,2}}$, with equality e.g. when $f(z) := z^d$. On the other hand, if u(f)(0) = 0, then $a_0 = 0$ and thus

$$|f|_{W^{1/2,2}}^2 = 4\pi^2 \sum_{n \neq 0} |n| \, |a_n|^2 \ge 4\pi^2 \sum_{n \neq 0} |a_n|^2 = 4\pi^2 \sum_{n \in \mathbb{Z}} |a_n|^2 = 2\pi \, \|f\|_{L^2}^2 = 4\pi^2.$$

Thus $c'_2 \ge 4\pi^2$, and the example f(z) := z shows that $c'_2 = 4\pi^2$.

7. For $1 , there exists some constant <math>c''_p$ such that $c''_p |\deg f| \le |f|^p_{W^{1/p,p}}, \forall f \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$ [5, Corollary 0.5]. We let c''_p be the best constant such that this estimate holds, and set $c^*_p := \min\{c'_p, c''_p\}$. We also set $c_p := 1/c^*_p$. By item **6**, for p = 2 we have $c''_2 = c'_2 = c^*_2 = 4\pi^2$, and $c_2 = 1/(4\pi^2)$.

8. Let $1 . Let <math>\delta > 0$ and assume that $|u(f)| \ge \delta$ in \mathbb{D} . Then there exists some $C = C(\delta, p)$ such that

$$f = e^{i\psi}, \text{ with } \psi \in W^{1/p,p}(\mathbb{S}^1; \mathbb{R}) \text{ and } |\psi|_{W^{1/p,p}} \le C |f|_{W^{1/p,p}}.$$
(10)

Indeed, set v := u/|u|, and write $v = e^{i\varphi}$, with smooth φ . By standard properties of the functional calculus and of trace theory, and by the lifting estimates in [4], we have $\varphi \in W^{2/p,p}(\mathbb{D};\mathbb{R})$, and then $\psi := \operatorname{tr} \varphi \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$ satisfies

$$|\psi|_{W^{1/p,p}} \leq C(p)|\varphi|_{W^{2/p,p}} \leq C(p)|v|_{W^{2/p,p}} \leq C(\delta,p)|u|_{W^{2/p,p}} \leq C(\delta,p)|f|_{W^{1/p,p}}.$$

9. Let $1 and <math>c < c'_p$. If $|f|^p_{W^{1/p,p}} \le c$, then there exists some $\delta > 0$ such that $|u(f)| \ge \delta$ in \mathbb{D} . Proof by contradiction: assume that $|f_n|^p_{W^{1/p,p}} \le c$, $f_n \rightharpoonup g$ and and $|u(f_n)(a_n)| \le 1/n$. Since $u(g \circ M_a) = [u(g)] \circ M_a$, we may assume (by item **2**) that $a_n = 0$. We find that u(f)(0) = 0 and $|f|^p_{W^{1/p,p}} < c'_p$, which is impossible.

10. Let $1 . Assume that <math>f_n \to f$ and $f_n \to f$ a.e. Then $|f_n|_{W^{1/p,p}}^p = |f|_{W^{1/p,p}}^p + |f_n\overline{f}|_{W^{1/p,p}}^p + o(1)$. Indeed, if we set $g_n := f_n\overline{f}$, then this follows from the Brezis-Lieb lemma [9] and the identity

$$\overline{g_n}(x)[f_n(x) - f_n(y)] = f(x) - f(y) + \overline{g_n}(x)f(y)[g_n(x) - g_n(y)].$$

Proof of the Theorem 1. The proof is by complete induction on the integer part $L := I(c_p M) = I(M/c_p^*)$ of $c_p M$. The case where L = 0 follows from items 8 and 9. Let L > 0 and let M be such that $I(M/c_p^*) = L$. Assume, by contradiction, that the theorem does not hold for M. We may thus find a sequence (f_n) with the following properties: (a) $|f_n|_{W^{1/p,p}}^p \leq M$.

(b) For any $K \leq L$ and any choice of $a_1, \ldots, a_K \in \mathbb{D}$ and of signs $\varepsilon_j = \pm 1$ such that $\sum_{j=1}^K \varepsilon_j = \deg f_n$, if we write $f_n = e^{\iota\psi_n} \prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$, then we have $|\psi_n|_{W^{1/p,p}} \to \infty$. [It is always possible to take K, a_j , ε_j and ψ_n as above: it suffices to let $K := |\deg f| \leq I(M/c_p^{\prime\prime}) \leq I(M/c_p^{\prime\prime}) = L$, $\varepsilon_j := \operatorname{sgndeg} f$, and $a_j = 0$.]

By item **8** and property (**b**), there exist points $a_n \in \mathbb{D}$ such that $u(f_n)(a_n) \to 0$. By item **2**, we may assume in addition that $a_n = 0$. Thus, in addition to (**a**) and (**b**), we may assume (**c**) $f_n \to f$ and $f_n \to f$ a.e., for some f with u(f)(0) = 0.

Set $g_n := f_n \overline{f}$. By item **10** and the definition of c'_p , we have $|f|^p_{W^{1/p,p}} \ge c'_p \ge c_p^*$, and $|g_n|^p_{W^{1/p,p}} = M - |f|^p_{W^{1/p,p}} + o(1)$. Let $N > M - |f|^p_{W^{1/p,p}}$ be such that $I(N/c_p^*) = I((M - |f|^p_{W^{1/p,p}})/c_p^*) \le L - 1$. For large n, we have $|g_n|^p_{W^{1/p,p}} \le N$. By the induction hypothesis, we may write (possibly up to a subsequence) $g_n = e^{i\eta_n} \prod_{j=1}^R (M_{b_{j_n}})^{\varepsilon_j}$, with $|\eta_n|^p_{W^{1/p,p}} \le F(N)$ and $R \le N/c_p^*$. On the other hand, if $d := \deg f$, $b_{j_n} := 0$ and $\varepsilon_j := \operatorname{sgn} d$, then we may write $f = e^{i\eta} \prod_{j=R+1}^{R+|d|} (M_{b_{j_n}})^{\varepsilon_j}$, with $\eta \in W^{1/p,p}$ (item **4**). In addition, we have $|d| \le |f|^p_{W^{1/p,p}}/c_p''$ (item **7**). Finally, with $\psi_n := \eta_n + \eta$ and $K := R + |d| \le M/c_p^*$, we have $f_n = e^{i\psi_n} \prod_{j=1}^K (M_{b_{j_n}})^{\varepsilon_j}$, and (ψ_n) is bounded in $W^{1/p,p}$. This contradiction completes the proof of the first part of the theorem.

Optimality of (3) when p = 2 follows from the fact that, by item **6**, $f(z) := z^d$, d > 0, satisfies $|f|^2_{W^{1/2,2}} = c_2 d$ and requires at least d Moebius maps in its decomposition.

Proof of Corollary 1. By replacing f_n with $f_n \overline{f}$, we may assume that $f_n \rightarrow 1$. Up to a subsequence, we may write $f_n = e^{i\eta_n} \prod_{j=1}^{P} (M_{a_{j_n}})^{\varepsilon_j}$, with $a_{j_n} \rightarrow \alpha_j \in \overline{\mathbb{D}}$, j = 1, ..., P, and $\eta_n \rightarrow \eta$. With no loss of generality, we assume that $\alpha_1, ..., \alpha_K \in \mathbb{S}^1$ and $\alpha_{K+1}, ..., \alpha_P \in \mathbb{D}$. Since (clearly) $M_{a_{j_n}} \rightarrow \alpha_j$, j = 1, ..., K, we find that $1 = e^{i(\eta - C)} \prod_{j=K+1}^{P} (M_{\alpha_j})^{\varepsilon_j}$ for some appropriate *C*. Thus, with $\zeta_n := \eta_n - \eta$, we have

$$f_n = e^{\iota(\zeta_n + C)} \prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j} \prod_{j=K+1}^P \left(M_{a_{j_n}} M_{a_j}^{-1} \right)^{\varepsilon_j} = e^{\iota\psi_n} \prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j},$$

for some ψ_n such that $\psi_n - \zeta_n \to C$ in $W^{1/p,p}$, and thus $\psi_n \to C$.

3 Applications

We start with an immediate consequence of Theorem 1.

Corollary 2. Let d be a non negative integer and $\delta > 0$. Then there exist a constant $F(d,\delta)$ such that: every map $f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying deg f = d and $|f|^2_{W^{1/2,2}} \leq 4\pi^2(d+1) - \delta$ dans be written as $f = e^{i\psi} \prod_{i=1}^d M_{a_i}$, with $|\psi|^2_{W^{1/2,2}} \leq F(d,\delta)$.

Corollary 2 with d = 1, as well as a weak version of the corollary when $d \ge 2$ were obtained in [2, Theorem 4.4, Theorem 4.8]. As an application of Corollary 2, we obtain

Theorem 2. There exists some $\varepsilon > 0$ such that, for $p \in (2 - \varepsilon, 2]$,

$$m_p := \min\{|f|_{W^{1/p,p}}^p; \deg f = 1\}$$

is achieved.

Proof. When p = 2, it follows from item **6** that m_2 is achieved by multiples of Moebius maps.

When $1 , consider a minimizing sequence for <math>m_p$. Since $m_p \le |\text{Id}|_{W^{1/p,p}}^p := I_p$, we may assume that

$$|f_n|_{W^{1/p,p}}^p \le I_p \to I_2 = 4\pi^2 \text{ as } p \to 2.$$
 (11)

On the other hand, when $f: \mathbb{S}^1 \to \mathbb{S}^1$ we have $|f|_{H^{1/2}}^2 \leq 2^{2-p} |f|_{W^{1/p,p}}^p$. Thus

$$|f_n|_{H^{1/2}}^2 \le J_p := 2^{2-p} I_p \to 4\pi^2 \text{ as } p \to 2.$$
(12)

For p sufficiently close to 2 and fixed $\delta > 0$, we have $J_p \leq 8\pi^2 - \delta$. We next apply Corollary 2 to f_n and write $f_n = e^{i\psi_n} M_{a_n}$, with $|\psi_n|_{W^{1/2,2}} \leq F(1,\delta)$. Set $g_n := f_n \circ M_{a_n}$. By item **2**, (g_n) is a minimizing sequence for m_p . On the other hand, we have $g_n = e^{i\varphi_n}$ Id, with $\varphi_n := \psi_n \circ M_{a_n}$ bounded in $W^{1/2,2}(\mathbb{S}^1;\mathbb{R})$ (by (8)). Therefore, up to a subsequence $\varphi_n \to \varphi$ in $W^{1/2,2}$, and thus $g_n \to g := e^{i\varphi}$ Id in $W^{1/2,2}$. We find that deg g = 1. Since (g_n) is bounded in $W^{1/p,p}$, we obtain that $g_n \to g$ in $W^{1/p,p}$. By a standard argument, g achieves m_p .

Corollary 1 implies the "bubbling-off of circles along a sequence of graphs", in a sense that will be specified below. A basic object within the theory of Cartesian currents of Giaquinta, Modica and Souček [13] is the one of graphs of maps, considered as currents. For smooth maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, the graph is defined (as a current) as follows. Every smooth 1-form γ on $\mathbb{S}^1 \times \mathbb{S}^1$ can be written (uniquely) as

$$\gamma(s,t) = F(s,t)\omega(s) + G(s,t)\lambda(t);$$

here, ω and λ are the 1-forms given by

$$\omega(s,t) = s_1 ds_2 - s_2 ds_1, \text{ respectively } \lambda(t) = t_1 dt_2 - t_2 dt_1, \forall s, t \in \mathbb{S}^1,$$

and F, G are smooth functions. Then (as an oriented curve on $S^1 \times S^1$) the graph \mathscr{G}_f of f acts on γ through the formula

$$\langle \mathscr{G}_f, \gamma \rangle = \int_{\mathbb{S}^1} F(s, f(s)) + \int_{\mathbb{S}^1} G(s, f(s)) f \wedge \partial_\tau f.$$
(13)

Clearly, when f is smooth formula (13) defines a current $\mathscr{G}_f \in \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1) := (\Omega^1(\mathbb{S}^1 \times \mathbb{S}^1))^*$. It was proved in [17, Section 3] (see also [14] for a higher dimensional context) that (13) can be used in order to define \mathscr{G}_f as a current when f is merely $W^{1/2,2}$. The key observation is that the integral $\int_{\mathbb{S}^1} G(s, f(s)) f \wedge \partial_{\tau} f$ can be interpreted as a duality bracket between $G(\cdot, f) \in W^{1/2,2}$ and $f \wedge \partial_{\tau} f \in (W^{1/2,2})^*$. If we set

$$\langle \mathscr{G}_{f}, \gamma \rangle := \int_{\mathbb{S}^{1}} F(s, f(s)) + \langle G(\cdot, f), f \wedge \partial_{\tau} f \rangle_{W^{1/2,2}, (W^{1/2,2})^{*}}, \forall f \in W^{1/2,2}(\mathbb{S}^{1}; \mathbb{S}^{1}),$$
(14)

then we obtain a current which coincides with the usual graph of f when f is smooth, and is continuous with respect to the strong $W^{1/2,2}$ convergence [17].

One of the aims of the theory of Cartesian currents is to describe the limiting behavior of graphs under *weak* convergence of maps. In this direction, the following result was obtained in [17, Proposition 3.1].

Proposition 1. If $f_n \to f$ in $W^{1/2,2}(\mathbb{S}^1; \mathbb{S}^1)$ then, up to a subsequence, there are finitely many points $\alpha_1, \ldots, \alpha_m \in \mathbb{S}^1$ and nonzero integers d_1, \ldots, d_m such that

$$\mathscr{G}_{f_n} \to \mathscr{G}_f + \sum_{j=1}^m d_j \delta_{a_j} \times [\mathbb{S}^1] in \, \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1).$$
(15)

Here,

$$\langle \delta_{\alpha} \times [\mathbb{S}^1], \gamma \rangle = \int_{\mathbb{S}^1} G(\alpha, t).$$

A. Pisante [19] showed me that it is still possible to define \mathscr{G}_f and to extend Proposition 1 to maps $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ with 1 .

Proposition 2. ([19]) Let $1 . It is possible to define <math>\mathscr{G}_f \in \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1)$, $\forall f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$. This definition is unique and natural, in the following sense:

- 1. \mathscr{G}_f coincides with the usual graph when f is smooth.
- 2. If $f_n \to f$ strongly in $W^{1/p,p}$, then $\mathscr{G}_{f_n} \to \mathscr{G}_f$ in $\mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1)$.

In particular, the density of $C^{\infty}(\mathbb{S}^1;\mathbb{S}^1)$ into $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ implies that \mathscr{G}_f is uniquely defined the properties 1 and 2 above.

Proof. Since we use arguments partly similar to the ones in [17, Proof of Proposition 3.1], we do not give all details. Given $G \in C^{\infty}(\mathbb{S}^1 \times \mathbb{S}^1)$, set

$$g(s) := \int_{\mathbb{S}^1} G(s,t) d\ell(t)$$

Then there exists some $h \in C^{\infty}(\mathbb{S}^1 \times \mathbb{S}^1)$, that we may choose (at least locally around some fixed G_0) to depend smoothly on G such that

$$G(s,t)\lambda(t) = g(s)\lambda(t) + d_t h(s,t).$$

Here, d_t stands for the partial differential $\partial_{\tau(t)}h(s,t)dt$; d_s is defined similarly.

When f is smooth we have

$$\langle \mathscr{G}_{f}, \gamma \rangle = \int_{\mathbb{S}^{1}} F(s, f(s)) - \int_{\mathbb{S}^{1}} d_{s} h(s, f(s)) + \int_{\mathbb{S}^{1}} g(s) f(s) \wedge \partial_{\tau} f(s).$$
(16)

Let $d := \deg f$ and write, as in item 4, $f(z) = e^{i\psi(z)}z^d$, with ψ smooth. Then (16) becomes

$$\langle \mathscr{G}_{f}, \gamma \rangle = \int_{\mathbb{S}^{1}} F(s, f(s)) - \int_{\mathbb{S}^{1}} d_{s} h(s, f(s)) + d \int_{\mathbb{S}^{1}} g(s) - \int_{\mathbb{S}^{1}} \partial_{\tau} g(s) \psi(s).$$
(17)

Clearly, formula (17) still makes sense when f is merely in $W^{1/p,p}$ (and thus ψ is merely $W^{1/p,p}$).

It is easy to see that \mathscr{G}_f defined by (17) is a current, and that its dependence on f is continuous. [The latter property comes from the fact that the degree d and the phase ψ depend continuously on f [10].]

For further use, let us note the following identity, obtained by density: if $\tilde{f} \in C^{\infty}(\mathbb{S}^1; \mathbb{S}^1)$ and $f(z) = e^{i\psi(z)}z^d$, then

$$\langle \mathcal{G}_{f\tilde{f}}, \gamma \rangle = \int_{\mathbb{S}^1} F(s, f(s)\tilde{f}(s)) - \int_{\mathbb{S}^1} d_s h(s, f(s)\tilde{f}(s)) + d \int_{\mathbb{S}^1} g(s) - \int_{\mathbb{S}^1} \partial_\tau g(s)\psi(s) + \int_{\mathbb{S}^1} g(s)\tilde{f} \wedge \partial_\tau \tilde{f}(s).$$

$$(18)$$

Proposition 3. ([19]) Let $1 . If <math>f_n \rightarrow f$ in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ then, up to a subsequence, there are finitely many points $\alpha_1, \ldots, \alpha_m \in \mathbb{S}^1$ and nonzero integers d_1, \ldots, d_m such that

$$\mathscr{G}_{f_n} \to \mathscr{G}_f + \sum_{j=1}^m d_j \delta_{a_j} \times [\mathbb{S}^1] \text{ in } \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1).$$
(19)

Sketch of proof. Let $d := \operatorname{deg} f$ and write $f(z) = e^{i\psi(z)}z^d$, with $\psi \in W^{1/p,p}$. We write f_n as in Corollary 1. Set $\tilde{f}_n := f_n \overline{e^{i\psi_n} f}$. Using (18) with f and \tilde{f} replaced by $e^{i\psi_n} f$ and \tilde{f}_n , we easily find that

$$\langle \mathscr{G}_{f_n}, \gamma \rangle = \langle \mathscr{G}_f, \gamma \rangle + \sum_{j=1}^K \int_{\mathbb{S}^1} g(s)(M_{a_j}^{\varepsilon_j}) \wedge \partial_\tau(M_{a_j}^{\varepsilon_j})(s) + o(1) \text{ as } n \to \infty.$$
(20)

A straightforward calculation shows that

$$\int_{\mathbb{S}^1} g(s)(M_{a_j}^{\varepsilon_j}) \wedge \partial_\tau(M_{a_j}^{\varepsilon_j})(s) \to 2\pi \,\varepsilon_j \, g(\alpha_j) \text{ as } j \to \infty$$

whence the conclusion of the proposition.

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