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Petru Mironescu. Profile decomposition and phase control for circle-valued maps in one dimension. 2015. hal-01202156v1

**HAL Id: hal-01202156**

**<https://hal.science/hal-01202156v1>**

Preprint submitted on 22 Sep 2015 (v1), last revised 3 Jan 2016 (v2)

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# Profile decomposition and phase control for circle-valued maps in one dimension

Petru Mironescu\*

September 18, 2015

**Abstract.** When  $1 < p < \infty$ , maps  $f$  in  $W^{1/p,p}((0,1); \mathbb{S}^1)$  have  $W^{1/p,p}$  phases  $\varphi$ , but the  $W^{1/p,p}$ -seminorm of  $\varphi$  is not controlled by the one of  $f$ . Lack of control is illustrated by “the kink”:  $f = e^{i\varphi}$ , where the phase  $\varphi$  moves quickly from 0 to  $2\pi$ . A similar situation occurs for maps  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , with Moebius maps playing the role of kinks. We prove that this is the only loss of control mechanism: each map  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  satisfying  $|f|_{W^{1/p,p}}^p \leq M$  can be written as

$f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\pm 1}$ , where  $M_{a_j}$  is a Moebius map vanishing at  $a_j \in \mathbb{D}$ , while the integer  $K = K(f)$

and the phase  $\psi$  are controlled by  $M$ . In particular, we have  $K \leq c_p M$  for some  $c_p$ . When  $p = 2$ , we obtain the sharp value of  $c_2$ , which is  $c_2 = 1/(4\pi^2)$ . As an application, we obtain the existence of minimal maps of degree one in  $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  with  $p \in (2 - \varepsilon, 2)$ .

**Résumé. Décomposition en profils et contrôle des phases des applications unimodulaires en dimension un.** Si  $1 < p < \infty$ , les applications  $f$  appartenant à  $W^{1/p,p}((0,1); \mathbb{S}^1)$  ont des phases  $\varphi$  dans  $W^{1/p,p}$ , mais la seminorme  $W^{1/p,p}$  de  $\varphi$  n’est pas contrôlée par celle de  $f$ . L’absence de contrôle est illustrée par “le pli”:  $f = e^{i\varphi}$ , où la phase  $\varphi$  augmente rapidement de 0 à  $2\pi$ . Pour des applications  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , le même phénomène apparaît, avec les transformations de Moebius jouant le rôle des plis. Nous prouvons que cet exemple est essentiellement le seul : toute application  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  telle que  $|f|_{W^{1/p,p}}^p \leq M$  s’écrit  $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\pm 1}$ , où  $M_{a_j}$  est une transformation de Moebius s’annulant en  $a_j \in \mathbb{D}$ , tandis que l’entier  $K = K(f)$  et la phase  $\psi$  sont contrôlés par  $M$ . En particulier, nous avons  $K \leq c_p M$  pour une constante  $c_p$ . Pour  $p = 2$ , nous obtenons la valeur optimale de  $c_2$ , qui est  $c_2 = 1/(4\pi^2)$ . Comme application, nous obtenons l’existence d’une application minimale de degré un dans  $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  avec  $p \in ]2 - \varepsilon, 2[$ .

## 1 Introduction

Let  $0 < s < 1$ ,  $1 \leq p < \infty$  and let  $f : (0,1) \rightarrow \mathbb{S}^1$  belong to the space  $W^{s,p}$ . Then  $f$  can be written as  $f = e^{i\varphi}$ , where  $\varphi \in W^{s,p}$  [4]. Once the existence of  $\varphi$  is known, a natural question is whether we can control  $|\varphi|_{W^{s,p}}$  in terms of  $|f|_{W^{s,p}}$ . For most of  $s, p$  the answer is positive. The exceptional cases are provided precisely by the spaces  $W^{1/p,p}((0,1); \mathbb{S}^1)$ , with  $1 < p < \infty$  [4]. In these spaces, lack of control is established via the following explicit example. For  $n \geq 1$ , we define  $\varphi_n$  as

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Keywords. Profile decomposition, circle-valued map, phase, Sobolev spaces

2002 AMS classification subject. 46E35

follows:

$$\varphi_n(x) := \begin{cases} 0, & \text{for } 0 < x < 1/2 \\ 2\pi n(x - 1/2), & \text{for } 1/2 < x < 1/2 + 1/n. \\ 2\pi, & \text{for } 1/2 + 1/n < x < 1 \end{cases}$$

Then  $|\varphi_n|_{W^{1/p,p}} \rightarrow \infty$  (since  $\varphi_n \rightarrow \varphi = 2\pi \chi_{(1/2,1)}$  a.e., and  $\varphi$  does not belong to  $W^{1/p,p}$ ). On the other hand, if we extend  $u_n := e^{i\varphi_n}$  with the value 1 outside  $(0,1)$  and still denote the extension  $u_n$  then, by scaling,

$$|u_n|_{W^{1/p,p}((0,1))} \leq |u_n|_{W^{1/p,p}(\mathbb{R})} = |u_1|_{W^{1/p,p}(\mathbb{R})} < \infty.$$

Thus  $|u_n|_{W^{1/p,p}((0,1))} \lesssim 1$  and  $|\varphi_n|_{W^{1/p,p}((0,1))} \rightarrow \infty$ . Finally, we invoke the fact that  $W^{1/p,p}$  phases are unique mod  $2\pi$  [4].

If one considers instead maps  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , always in the critical case  $f \in W^{1/p,p}$ ,  $1 < p < \infty$ , then a new phenomenon occurs:  $f$  has a degree  $\deg f$ , and does not have a  $W^{1/p,p}$  phase at all when  $\deg f \neq 0$  [11, Remark 10]. However, even if  $\deg f = 0$  (and thus  $f$  has a  $W^{1/p,p}$  phase  $\varphi$ ), we have a loss of control phenomenon similar to the one on  $(0,1)$ . Indeed, let  $M_a(z) := \frac{a-z}{1-\bar{a}z}$ ,  $a \in \mathbb{D}$ ,  $z \in \bar{\mathbb{D}}$ , be a Moebius transform (that we identify with its restriction to  $\mathbb{S}^1$ ,  $M_a : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ ). Let  $f_a(z) := \bar{z}M_a(z)$ , so that  $f_a$  is smooth and  $\deg f_a = 0$ . One may prove (see below) that  $|M_a|_{W^{1/p,p}} = |\text{Id}|_{W^{1/p,p}}$ , and thus  $f_a$  is bounded in  $W^{1/p,p}$ . However, if  $a \rightarrow \alpha = e^{i\xi} \in \mathbb{S}^1$ , then the smooth phase  $\varphi_a$  of  $f_a$  converges a.e. to  $\varphi(e^{i\theta}) := \begin{cases} \xi - \theta, & \text{if } \xi - \pi < \theta < \xi \\ 2\pi + \xi - \theta, & \text{if } \xi < \theta < \xi + \pi \end{cases}$ , which does not

belong to  $W^{1/p,p}$ . [Here, uniqueness of the phases and convergence hold mod  $2\pi$ .] Thus  $\varphi_a$  is not bounded as  $a \rightarrow \alpha \in \mathbb{S}^1$ . On the other hand, the plot of  $\varphi_a$  shows that  $\varphi_a$  has a ‘‘kink shape’’, and thus we have here the analog of the example on  $(0,1)$ .

There are evidences that this loss of control mechanism is the only possible one. For example, the phase of the kink is not bounded in  $W^{1/p,p}$ , but clearly is in  $W^{1,1}$  (same for  $f_a$ ). Bourgain and Brezis [3] proved that for every  $f \in W^{1/2,2}((0,1); \mathbb{S}^1)$ , we may split  $f = e^{i\psi} v$ , with  $\psi$  and  $v = e^{i\eta}$  satisfying

$$|\psi|_{W^{1/2,2}} \lesssim |f|_{W^{1/2,2}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \lesssim |f|_{W^{1/2,2}}^2. \quad (1)$$

Intuitively, one should think at  $v$  as at ‘‘the kink part of  $f$ ’’. The above result was extended by Nguyen [18] to  $1 < p < \infty$ : for every  $1 < p < \infty$  and every  $f \in W^{1/p,p}((0,1); \mathbb{S}^1)$ , we may split  $f = e^{i\psi} v$ , with  $\psi$  and  $v = e^{i\eta}$  satisfying

$$|\psi|_{W^{1/p,p}} \leq C_p |f|_{W^{1/p,p}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \leq C_p |f|_{W^{1/p,p}}^p. \quad (2)$$

Here we present another result in this direction, written for simplicity on the unit circle.

**Theorem 1.** *Let  $1 < p < \infty$  and  $M > 0$ . Then there exist constants  $c_p$  and  $F(M)$  such that: every map  $f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  satisfying  $|f|_{W^{1/p,p}}^p \leq M$  can be written as  $f = e^{i\psi} \prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$ , with  $\varepsilon_j \in \{-1, 1\}$ ,*

$$K \leq c_p M, \quad (3)$$

and

$$|\psi|_{W^{1/p,p}}^p \leq F(M). \quad (4)$$

When  $p = 2$ , we may take  $c_2 = 1/(4\pi^2)$ , and this constant is optimal.

**Corollary 1.** Let  $1 < p < \infty$  and let  $f_n, f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  be such that  $f_n \rightharpoonup f$  in  $W^{1/p,p}$ . Then, up to a subsequence, there exist  $K \in \mathbb{N}$ ,  $\varepsilon_j \in \{-1, 1\}$ ,  $a_{j_n} \in \mathbb{D}$ ,  $\alpha_j \in \mathbb{S}^1$ ,  $j = 1, \dots, K$ ,  $\psi_n \in W^{1/p,p}(\mathbb{S}^1; \mathbb{R})$ , and a constant  $C$ , such that:

- i)  $f_n = e^{i\psi_n} \prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j} f$ ;
- ii)  $a_{j_n} \rightarrow \alpha_j$  as  $n \rightarrow \infty$ ;
- iii)  $\psi_n \rightharpoonup C$  in  $W^{1/p,p}$  as  $n \rightarrow \infty$ .

The theorem and the corollary are reminiscent of profile decompositions obtained in different, often geometrical, contexts. We mention e.g. the work of Sacks and Uhlenbeck [19] on minimal 2-spheres, the analysis of Brezis and Coron [6, 7, 8] of constant mean curvature surfaces, or the one of Struwe [20] of equations involving the critical Sobolev exponent. There are also abstract approaches to bubbling as in the work of Lions [16] about concentration-compactness or the characterization of lack of compactness of critical embeddings in Gérard [12], Jaffard [15] or Bahouri, Cohen and Koch [1].

Let us comment on the connection between (2) and our theorem. First, (2) has the following version for maps on  $\mathbb{S}^1$ : we may split  $f = e^{i\psi} v$ , with  $|\psi|_{W^{1/p,p}} \leq C_p |f|_{W^{1/p,p}}$  and  $|v|_{W^{1,1}} \leq C_p |f|_{W^{1/p,p}}$ . Next, a Moebius maps satisfies  $|M_\alpha|_{W^{1,1}} = 2\pi$ , and thus

$$\left| \prod_{j=1}^K (M_{\alpha_j})^{\varepsilon_j} \right|_{W^{1,1}} \leq 2\pi K \leq 2\pi c_p M. \quad (5)$$

Estimate (5) shows that (3) is a refinement of the second part of (2). On the other hand, (4) is weaker than the first part of (2), since  $F(M)$  need not have a linear growth (and actually we do not have any control on  $F$ ). This suggests the following

**Conjecture.** Let  $1 < p < \infty$ . Then there exist constants  $c_p, d_p$  such that every  $f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  satisfying  $|f|_{W^{1/p,p}}^p \leq M$  can be decomposed as  $f = e^{i\psi} \prod_{j=1}^K (M_{\alpha_j})^{\varepsilon_j}$ , with  $\varepsilon_j \in \{-1, 1\}$ ,

$$K \leq c_p M, \quad (6)$$

and

$$|\psi|_{W^{1/p,p}}^p \leq d_p M. \quad (7)$$

In addition, when  $p = 2$ , we may take  $c_2 = 1/(4\pi^2)$ .

## 2 Proofs

We start by recalling or establishing few auxiliary results. Given  $1 \leq p < \infty$ ,  $f, f_n$  will denote maps in  $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ . When  $1 < p < \infty$ , “ $\rightharpoonup$ ” refers to weak convergence in  $W^{1/p,p}$ .

**1.** Recall that, up to a multiplicative factor  $\alpha \in \mathbb{S}^1$ , the Moebius transforms give all the conformal representations  $u : \mathbb{D} \rightarrow \mathbb{D}$ . In particular,  $M_\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a smooth orientation preserving diffeomorphism, and thus  $\deg M_\alpha = 1$ . Consequence: if  $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is continuous, then  $\deg[g \circ M_\alpha] = \deg g$ .

**2.** If  $1 \leq p < \infty$  and  $a \in \mathbb{D}$ , then  $|f \circ M_a|_{W^{1/p,p}} = |f|_{W^{1/p,p}}$ . [Here, we let  $|f|_{W^{1,1}} := \int_{\mathbb{S}^1} |f| = \int_0^{2\pi} |d[f(e^{i\theta})]/d\theta| d\theta$  and, for  $1 < p < \infty$ ,  $|f|_{W^{1/p,p}}^p := \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} |f(x) - f(y)|^p / |x - y|^2 dx dy$ .] In order to prove the desired equality when  $p = 1$ , we write  $M_\alpha(e^{i\theta}) = e^{i\varphi(\theta)}$ ,  $0 \leq \theta \leq 2\pi$ , with  $\varphi$  smooth and increasing. Then

$$|f \circ M_\alpha|_{W^{1,1}} = \int_0^{2\pi} \left| \frac{d}{d\theta} [f(e^{i\varphi(\theta)})] \right| d\theta = \int_{\varphi^{-1}(0)}^{\varphi^{-1}(2\pi)} \left| \frac{d}{d\theta} [f(e^{i\theta})] \right| d\theta = \int_0^{2\pi} \left| \frac{d}{d\theta} [f(e^{i\theta})] \right| d\theta = |f|_{W^{1,1}}.$$

When  $1 < p < \infty$ , we rely on the following identity, valid for measurable functions  $F : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow [0, \infty]$ :

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{F(M_a(x), M_a(y))}{|x - y|^2} dx dy = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{F(x, y)}{|x - y|^2} dx dy. \quad (8)$$

Proof of (8): We have  $[M_a]^{-1} = M_a$  and thus, after change of variables, (8) amounts to

$$|x - y|^2 |\dot{M}_a(x)| |\dot{M}_a(y)| = |M_a(x) - M_a(y)|^2, \quad \forall x, y \in \mathbb{S}^1. \quad (9)$$

In turn, (9) follows immediately from the straightforward equality  $|\dot{M}_a(x)| = \frac{1 - |a|^2}{|1 - \bar{a}x|^2}$ .

**3.** If  $1 \leq p < \infty$  and  $a \in \mathbb{D}$ , then  $\deg[f \circ M_a] = \deg f$ . Indeed, to start with, such  $f$  has a degree, since  $W^{1/p, p} \hookrightarrow \text{VMO}$  and VMO maps gave a degree stable with respect to BMO convergence [11]. By item **1**, the desired equality holds true for smooth  $f$ . The general case follows by density of  $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$  into  $W^{1/p, p}(\mathbb{S}^1; \mathbb{S}^1)$  [11, Lemmas A.11 and A.12] and by stability of the VMO degree.

**4.** If  $1 \leq p < \infty$  and the degree of  $f$  is  $d$ , then we may write  $f(z) = e^{i\psi(z)} z^d$ , with  $\psi \in W^{1/p, p}(\mathbb{S}^1; \mathbb{R})$ . This follows easily from the fact that maps  $f \in W^{1/p, p}((0, 1); \mathbb{S}^1)$  lift within  $W^{1/p, p}$  [4].

**5.** Let  $1 < p < \infty$ . For  $f \in W^{1/p, p}(\mathbb{S}^1; \mathbb{S}^1)$ , let  $u = u(f)$  be its harmonic extension. Set  $c'_p := \inf\{|f|_{W^{1/p, p}}^p; u(0) = 0\}$ . Clearly,  $c'_p$  is achieved, and therefore  $c'_p > 0$ .

**6.** When  $p = 2$ , we have the following straightforward calculations: if  $f = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}$ , then  $|f|_{W^{1/2, 2}}^2 = 4\pi^2 \sum_{n \in \mathbb{Z}} |n| |a_n|^2$  [10, Chapter 13], and  $\deg f = \sum_{n \in \mathbb{Z}} n |a_n|^2$  [11, eq (25)]. This leads to  $4\pi^2 |\deg f| \leq |f|_{W^{1/2, 2}}^2$ , with equality e.g. when  $f(z) := z^d$ . On the other hand, if  $u(f)(0) = 0$ , then  $a_0 = 0$  and thus

$$|f|_{W^{1/2, 2}}^2 = 4\pi^2 \sum_{n \neq 0} |n| |a_n|^2 \geq 4\pi^2 \sum_{n \neq 0} |a_n|^2 = 4\pi^2 \sum_{n \in \mathbb{Z}} |a_n|^2 = 2\pi \|f\|_{L^2}^2 = 4\pi^2.$$

Thus  $c'_2 \geq 4\pi^2$ , and the example  $f(z) := z$  shows that  $c'_2 = 4\pi^2$ .

**7.** For  $1 < p < \infty$ , there exists some constant  $c''_p$  such that  $c''_p |\deg f| \leq |f|_{W^{1/p, p}}^p$ ,  $\forall f \in W^{1/p, p}(\mathbb{S}^1, \mathbb{S}^1)$  [5, Corollary 0.5]. We let  $c''_p$  be the best constant such that this estimate holds, and set  $c_p^* := \min\{c'_p, c''_p\}$ . We also set  $c_p := 1/c_p^*$ . By item **6**, for  $p = 2$  we have  $c''_2 = c'_2 = c_2^* = 4\pi^2$ , and  $c_2 = 1/(4\pi^2)$ .

**8.** Let  $1 < p < \infty$ . Let  $\delta > 0$  and assume that  $|u(f)| \geq \delta$  in  $\mathbb{D}$ . Then there exists some  $C = C(\delta, p)$  such that

$$f = e^{i\psi}, \quad \text{with } \psi \in W^{1/p, p}(\mathbb{S}^1; \mathbb{R}) \text{ and } |\psi|_{W^{1/p, p}} \leq C |f|_{W^{1/p, p}}. \quad (10)$$

Indeed, set  $v := u/|u|$ , and write  $v = e^{i\varphi}$ , with smooth  $\varphi$ . By standard properties of the functional calculus and of trace theory, and by the lifting estimates in [4], we have  $\varphi \in W^{2/p, p}(\mathbb{D}; \mathbb{R})$ , and then  $\psi := \text{tr } \varphi \in W^{1/p, p}(\mathbb{S}^1; \mathbb{R})$  satisfies

$$|\psi|_{W^{1/p, p}} \leq C(p) |\varphi|_{W^{2/p, p}} \leq C(p) |v|_{W^{2/p, p}} \leq C(\delta, p) |u|_{W^{2/p, p}} \leq C(\delta, p) |f|_{W^{1/p, p}}.$$

**9.** Let  $1 < p < \infty$  and  $c < c'_p$ . If  $|f|_{W^{1/p, p}}^p \leq c$ , then there exists some  $\delta > 0$  such that  $|u(f)| \geq \delta$  in  $\mathbb{D}$ . Proof by contradiction: assume that  $|f_n|_{W^{1/p, p}}^p \leq c$ ,  $f_n \rightarrow g$  and  $|u(f_n)(a_n)| \leq 1/n$ . Since  $u(g \circ M_a) = [u(g)] \circ M_a$ , we may assume (by item **2**) that  $a_n = 0$ . We find that  $u(f)(0) = 0$  and  $|f|_{W^{1/p, p}}^p < c'_p$ , which is impossible.

**10.** Let  $1 < p < \infty$ . Assume that  $f_n \rightarrow f$  and  $f_n \rightarrow f$  a.e. Then  $|f_n|_{W^{1/p, p}}^p = |f|_{W^{1/p, p}}^p + |f_n \bar{f}|_{W^{1/p, p}}^p + o(1)$ . Indeed, if we set  $g_n := f_n \bar{f}$ , then this follows from the Brezis-Lieb lemma [9] and the identity

$$\overline{g_n}(x) [f_n(x) - f_n(y)] = f(x) - f(y) + \overline{g_n}(x) f(y) [g_n(x) - g_n(y)].$$

*Proof of the Theorem 1.* The proof is by complete induction on the integer part  $L := I(c_p M) = I(M/c_p^*)$  of  $c_p M$ . The case where  $L = 0$  follows from items **8** and **9**. Let  $L > 0$  and let  $M$  be such that  $I(M/c_p^*) = L$ . Assume, by contradiction, that the theorem does not hold for  $M$ . We may thus find a sequence  $(f_n)$  with the following properties:

(a)  $|f_n|_{W^{1/p,p}}^p \leq M$ .

(b) For any  $K \leq L$  and any choice of  $a_1, \dots, a_K \in \mathbb{D}$  and of signs  $\varepsilon_j = \pm 1$  such that  $\sum_{j=1}^K \varepsilon_j = \deg f_n$ , if we write  $f_n = e^{i\psi_n} \prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$ , then we have  $|\psi_n|_{W^{1/p,p}} \rightarrow \infty$ . [It is always possible to take  $K$ ,  $a_j$ ,  $\varepsilon_j$  and  $\psi_n$  as above: it suffices to let  $K := |\deg f| \leq I(M/c_p'') \leq I(M/c_p^*) = L$ ,  $\varepsilon_j := \text{sgn } \deg f$ , and  $a_j = 0$ .]

By item **8** and property (b), there exist points  $a_n \in \mathbb{D}$  such that  $u(f_n)(a_n) \rightarrow 0$ . By item **2**, we may assume in addition that  $a_n = 0$ . Thus, in addition to (a) and (b), we may assume

(c)  $f_n \rightarrow f$  and  $\underline{f}_n \rightarrow f$  a.e., for some  $f$  with  $u(f)(0) = 0$ .

Set  $g_n := f_n \bar{f}$ . By item **10** and the definition of  $c_p'$ , we have  $|f|_{W^{1/p,p}}^p \geq c_p' \geq c_p^*$ , and  $|g_n|_{W^{1/p,p}}^p = M - |f|_{W^{1/p,p}}^p + o(1)$ . Let  $N > M - |f|_{W^{1/p,p}}^p$  be such that  $I(N/c_p^*) = I((M - |f|_{W^{1/p,p}}^p)/c_p^*) \leq L - 1$ . For large  $n$ , we have  $|g_n|_{W^{1/p,p}}^p \leq N$ . By the induction hypothesis, we may write (possibly up to a subsequence)  $g_n = e^{i\eta_n} \prod_{j=1}^R (M_{b_{j_n}})^{\varepsilon_j}$ , with  $|\eta_n|_{W^{1/p,p}}^p \leq F(N)$  and  $R \leq N/c_p^*$ . On the other hand, if  $d := \deg f$ ,  $b_{j_n} := 0$  and  $\varepsilon_j := \text{sgn } d$ , then we may write  $f = e^{i\eta} \prod_{j=R+1}^{R+|d|} (M_{b_{j_n}})^{\varepsilon_j}$ , with  $\eta \in W^{1/p,p}$  (item **4**). In addition, we have  $|d| \leq |f|_{W^{1/p,p}}^p / c_p''$  (item **7**). Finally, with  $\psi_n := \eta_n + \eta$  and  $K := R + |d| \leq M/c_p^*$ , we have  $f_n = e^{i\psi_n} \prod_{j=1}^K (M_{b_{j_n}})^{\varepsilon_j}$ , and  $(\psi_n)$  is bounded in  $W^{1/p,p}$ . This contradiction completes the proof of the first part of the theorem.

Optimality of (3) when  $p = 2$  follows from the fact that, by item **6**,  $f(z) := z^d$ ,  $d > 0$ , satisfies  $|f|_{W^{1/2,2}}^2 = c_2 d$  and requires at least  $d$  Moebius maps in its decomposition.  $\square$

*Proof of Corollary 1.* By replacing  $f_n$  with  $f_n \bar{f}$ , we may assume that  $f_n \rightarrow 1$ . Up to a subsequence, we may write  $f_n = e^{i\eta_n} \prod_{j=1}^P (M_{a_{j_n}})^{\varepsilon_j}$ , with  $a_{j_n} \rightarrow \alpha_j \in \overline{\mathbb{D}}$ ,  $j = 1, \dots, P$ , and  $\eta_n \rightarrow \eta$ . With no loss of generality, we assume that  $\alpha_1, \dots, \alpha_K \in \mathbb{S}^1$  and  $\alpha_{K+1}, \dots, \alpha_P \in \mathbb{D}$ . Since (clearly)  $M_{a_{j_n}} \rightarrow \alpha_j$ ,  $j = 1, \dots, K$ , we find that  $1 = e^{i(\eta_n - C)} \prod_{j=K+1}^P (M_{a_{j_n}})^{\varepsilon_j}$  for some appropriate  $C$ . Thus, with  $\zeta_n := \eta_n - \eta$ , we have

$$f_n = e^{i(\zeta_n + C)} \prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j} \prod_{j=K+1}^P (M_{a_{j_n}} M_{\alpha_j}^{-1})^{\varepsilon_j} = e^{i\psi_n} \prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j},$$

for some  $\psi_n$  such that  $\psi_n - \zeta_n \rightarrow C$  in  $W^{1/p,p}$ , and thus  $\psi_n \rightarrow C$ .  $\square$

*Remark.* The corollary implies the ‘‘bubbling-off of circles along a sequence of graphs’’. More specifically, the behavior of weakly converging sequences of manifold-valued maps can be investigated within the theory of Cartesian currents of Giaquinta, Modica and Souček [13]; see also [14, 17] for the specific case of  $W^{1/2,2}(\mathbb{S}^1; \mathbb{S}^1)$ . When  $p = 2$ , it is possible to define (as a current) the graph  $\mathcal{G}_f$  of  $f \in W^{1/2,2}(\mathbb{S}^1; \mathbb{S}^1)$ . With the notation in the corollary, if  $p = 2$  and  $f_n \rightarrow f$ , ‘‘bubbling-off’’ reads

$$\mathcal{G}_{f_n} \rightarrow \mathcal{G}_f + \sum_{j=1}^K \varepsilon_j \delta_{\alpha_j} \times [\mathbb{S}^1] \text{ in } \mathcal{D}_1(\mathbb{S}^1 \times \mathbb{S}^1). \quad (11)$$

This can be obtained directly from (1) [17, Proposition 3.1], but also as an immediate consequence of the corollary. Details are left to the reader.

### 3 Applications

We start with an immediate consequence of Theorem 1.

**Corollary 2.** *Let  $d$  be a non negative integer and  $\delta > 0$ . Then there exist a constant  $F(d, \delta)$  such that: every map  $f \in W^{1/2,2}(\mathbb{S}^1; \mathbb{S}^1)$  satisfying  $\deg f = d$  and  $|f|_{W^{1/2,2}}^2 \leq 4\pi^2(d+1) - \delta$  can be written as  $f = e^{i\psi} \prod_{j=1}^d M_{\alpha_j}$ , with  $|\psi|_{W^{1/2,2}}^2 \leq F(d, \delta)$ .*

Corollary 2 with  $d = 1$ , as well as a weak version of the corollary when  $d \geq 2$  were obtained in [2, Theorem 4.4, Theorem 4.8]. As an application of Corollary 2, we obtain

**Theorem 2.** *There exists some  $\varepsilon > 0$  such that, for  $p \in (2 - \varepsilon, 2]$ ,*

$$m_p := \min\{|f|_{W^{1/p,p}}^p; \deg f = 1\}$$

is achieved.

*Proof.* When  $p = 2$ , it follows from item **6** that  $m_2$  is achieved by multiples of Moebius maps.

When  $1 < p < 2$ , consider a minimizing sequence for  $m_p$ . Since  $m_p \leq |\text{Id}|_{W^{1/p,p}}^p := I_p$ , we may assume that

$$|f_n|_{W^{1/p,p}}^p \leq I_p \rightarrow I_2 = 4\pi^2 \text{ as } p \rightarrow 2. \quad (12)$$

On the other hand, when  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  we have  $|f|_{H^{1/2}}^2 \leq 2^{2-p} |f|_{W^{1/p,p}}^p$ . Thus

$$|f_n|_{H^{1/2}}^2 \leq J_p := 2^{2-p} I_p \rightarrow 4\pi^2 \text{ as } p \rightarrow 2. \quad (13)$$

For  $p$  sufficiently close to 2 and fixed  $\delta > 0$ , we have  $J_p \leq 8\pi^2 - \delta$ . We next apply Corollary 2 to  $f_n$  and write  $f_n = e^{i\psi_n} M_{\alpha_n}$ , with  $|\psi_n|_{W^{1/2,2}} \leq F(1, \delta)$ . Set  $g_n := f_n \circ M_{\alpha_n}$ . By item **2**,  $(g_n)$  is a minimizing sequence for  $m_p$ . On the other hand, we have  $g_n = e^{i\varphi_n} \text{Id}$ , with  $\varphi_n := \psi_n \circ M_{\alpha_n}$  bounded in  $W^{1/2,2}(\mathbb{S}^1; \mathbb{R})$  (by (8)). Therefore, up to a subsequence  $\varphi_n \rightharpoonup \varphi$  in  $W^{1/2,2}$ , and thus  $g_n \rightharpoonup g := e^{i\varphi} \text{Id}$  in  $W^{1/2,2}$ . We find that  $\deg g = 1$ . Since  $(g_n)$  is bounded in  $W^{1/p,p}$ , we obtain that  $g_n \rightharpoonup g$  in  $W^{1/p,p}$ . By a standard argument,  $g$  achieves  $m_p$ .  $\square$

## Acknowledgements

The author was supported by the ANR project ‘‘Harmonic Analysis at its Boundaries’’, ANR-12-BS01-0013-03, and by the LABEX MILYON (ANR-10-LABX-0070) of Universit e de Lyon, within the program ‘‘Investissements d’Avenir’’ (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

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