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CLTs for general branching processes related to splitting trees.

BENOIT HENRY^{1,2}

Abstract

We consider a general branching population where the lifetimes of individuals are i.i.d. with arbitrary distribution and each individual gives birth to new individuals at Poisson times independently from each others. The population counting process is a Binary Homogeneous Crump-Jagers-Mode process. Moreover, we suppose that the individuals experience mutations at Poisson rate θ under the infinite-alleles assumption. These mutations lead to a partition of the population by type. In this paper, we obtain central limit theorems for the frequency spectrum when time goes to infinity. We also obtain CLT for the population counting process. To this end, we use classical renewal theory and recent works [18, 6, 5] on this model to obtain the moments of the error. Then, we can precisely study the asymptotic behaviour of these moments thanks to Lévy processes theory. These results in conjunction with a new decomposition of the splitting tree allow us to obtain the wanted Theorems.

MSC 2000 subject classifications: Primary 60J80; secondary 92D10, 60J85, 60G51, 60K15, 60F05.

Key words and phrases. branching process – coalescent point process – splitting tree – Crump-Mode-Jagers process – linear birth-death process – allelic partition – frequency spectrum – infinite alleles model – Lévy process – scale function – Central Limit Theorem.

1 Introduction

In this work, we consider a general branching population where individuals live and reproduce independently from each other. Their lifetimes follow an arbitrary distribution \mathbb{P}_V and the births occur at Poisson times with rate b . The genealogical tree induced by this population is called a *splitting tree* [12, 11, 18] and is of main importance in the study of the model.

The population counting process N_t (giving the number of living individuals at time t) is known as binary homogeneous Crump-Mode-Jagers process. In [19], Nerman shows very general conditions for the a.s. convergence of general CMJ processes. In the supercritical case, it is known that the quantity $e^{-\alpha t} N_t$, where α is the Malthusian parameter of the population, converges almost surely. This result has been proved in [20] using the Jagers-Nerman's theory of general branching processes counted by

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random characteristics. Another proof can be found in [5], using only elementary probabilistic tools, relying on fluctuation analysis of the process.

Our purpose in this work is to investigate the behaviour of the error in the aforementioned convergence. Many works studied the second order behaviour of converging branching processes. Early works investigate the Galton-Watson case. In [14] and [15], Heyde obtained rates of convergence and get central limit Theorems in the case of supercritical Galton-Watson when the limit has finite variance. Later, in [1], Asmussen obtained the polynomial convergence rates in the general case. In our model, the particular case when the individuals never die ($\mathbb{P}_V = \delta_\infty$) means that the population counting process is a Markovian Yule process for which such study has already been made. More precisely, Athreya showed in [3], for a Markovian branching process Z with appropriate conditions, and such that $e^{-\alpha t} Z_t$ converges to some random variable W a.s., that the error

$$\frac{Z_t - e^{\alpha t} W}{\sqrt{Z_t}},$$

converge in distribution to some Gaussian random variable.

In the case of CMJ processes, very recent work of Iksanov and Meiners [16] gives sufficient conditions for the error terms in the convergence of supercritical general branching processes to be $o(t^\delta)$ in a very general background (arbitrary birth point process). Although our model is more specific, we give slightly more precise results. Indeed, we give the exact rate of convergence, $e^{\frac{\alpha}{2}t}$, and obtain a central limit Theorems for a binary homogeneous CMJ processes.

We suppose, moreover, that mutations occur on individuals and that each new mutation confers to its holder a brand new type (i.e. never seen in the population): this is the *infinitely many allele* assumption. This allows modelling the occurrence of a new type in a population (such as a new species or a new phenotype in a given species). We also suppose that every individual inherits the type of its parent. This model leads to a partition of the population by types. The frequency spectrum of the population alive at time t is defined as the sequence of number $(A(k, t))_{k \geq 1}$ where, for each k , $A(k, t)$ is the number of families of size k in the population. The famous example of Ewens sampling formula gives explicit expression for the law of the frequency spectrum [9] when the genealogy is given by the Kingman's coalescent. Other works studied similar quantities in the case of Galton-Watson branching processes (see [4] or [13]). In our model, the frequency spectrum has also been widely studied in the past [6, 7, 8, 5].

As for $e^{-\alpha t} N_t$, it is also known that the quantities $e^{-\alpha t} A(k, t)$ converge almost surely. These results can be easily obtained by conjunction of the works of [6] and [20] using the theory of general branching processes counted by random characteristics (a complete statement can be found in [8]). An alternative proof avoiding the use of the general branching processes theory can be found in [5]. In this work, we also obtain central limit Theorems for the convergence of the frequency spectrum. Moreover, we believe that our method could apply to other general branching processes counted by random characteristics, as soon as the birth point process is Poissonian. Finally, we study the Markovian case (when \mathbb{P}_V is exponential) where we can obtain more informations on the limit distribution.

To obtain these results we use a decomposition of the splitting tree at a fixed time. Then, using classical renewal theory and recent works on the moments of the frequency spectrum [5], we are able

to compute the moments of the error in the convergence of the aforementioned branching processes. The precise asymptotic behaviour of these moments is then obtained thanks to the theory of Lévy processes and their links with the splitting trees (see [18]).

The first section is devoted to the introduction of main tools used in this work. The first part recall the basic facts on splitting trees which are essentially borrowed from [18, 6, 7, 8, 5]. The second part recall some classical facts on renewal equations and the last part gives a useful Lemma on the expectation of a random integral. Section 3 is devoted to the statement of our Theorems. Theorem 3.1 is the CLT for the population counting process N_t . Theorems 3.2 and 3.4 give the corresponding results for the frequency spectrum. Finally, Theorem 3.6 concerns the error made by approximating $A(k, t)$ with $c_k N_t$, where c_k is some explicit constant. Section 4 gives the main lines of the proof of Theorem 3.1 which is finally given in Section 6. The proofs of Theorems 3.2, 3.6 and 3.4 are the subject of Sections 7, 8, and 9 respectively.

2 Model and preliminary results

This section is devoted to the statement of results which are constantly used in the sequel. The first subsection presents the model and states results on splitting trees coming from [18, 6, 7, 20, 5]. The second subsection recalls some well-known results on renewal equations. Finally, the last subsection is devoted to the statement and the proof of a Lemma for the expectation of random integrals, which is constantly used in the sequel.

2.1 Splitting trees

In this paper, we study a model of population dynamics called a splitting tree. We consider a branching tree (see Figure 1), where individuals live and reproduce independently from each other. Their lifetimes are i.i.d. following an arbitrary distribution \mathbb{P}_V . Given the lifetime of an individual, he gives birth to new individuals at Poisson times with positive constant rate b until his death independently from the other individuals. We also suppose that the population starts with a single individual called the *root*.

The finite measure $\Lambda := b\mathbb{P}_V$ is called the *lifespan measure*, and plays an important role in the study of the model.

Moreover, we assume that individuals undergo mutations at Poisson times with rate θ during their lifetimes independently from each other and from their reproduction processes. Each new mutation leads to a brand new type replacing the preceding type of the individual (*infinitely many alleles model*). Parents yield their current type to their children.

A family at given time t is a set of alive individuals carrying the same type. Our purpose is to study the distribution of the sizes of families in the population at time t .

For our study, it is easier to work with the genealogical tree of the population alive at time t . Indeed, the difference between two individuals in term of type lies on the time past since their lineages has diverged. Hence, the CPP contains the essential informations to study the allelic partition. In order to derive the law of that genealogical tree, we need to characterize the joint law of the *times of*

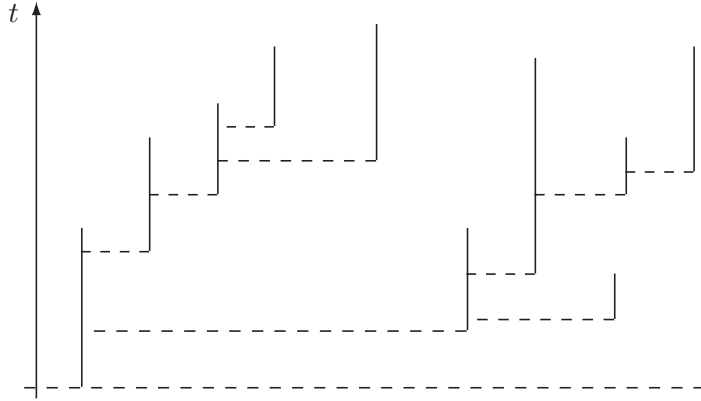


Figure 1: Graphical representation of a Splitting tree. The vertical axis represents the biological time and the horizontal axis has no biological meaning. The vertical lines represent the individuals, their lengths correspond to their lifetimes. The dashed lines denote the filiations between individuals.

coalescence between pairs of individuals in the population, which are the times since their lineages have split.

In [18], Lambert introduces a contour process Y , which codes for the tree, and hence its genealogy. Suppose we are given a tree \mathbb{T} , seen as a subset of $\mathbb{R} \times (\cup_{k \geq 0} \mathbb{N}^k)$ with some compatibility conditions (see [18]). On this object, Lambert constructs a Lebesgue measure λ and a total order relation \preceq which can be roughly summarized as follows: let x, y in \mathbb{T} , the point of birth of the lineage of x during the lifetime of the root split the tree in two connected components, then $y \preceq x$ if y belong to the same component as x but is not an ancestor of x .

Then the application,

$$\begin{aligned} \varphi : \mathbb{T} &\rightarrow [0, \lambda(\mathbb{T})], \\ x &\rightarrow \lambda(\{y \mid y \preceq x\}), \end{aligned}$$

is a bijection. Moreover, in a graphical sens (see figure 2), $\varphi(x)$ measures the length of the part of the tree which is above the lineage of x . The contour process is then defined, for all s , by,

$$Y_s^{(t)} := \Pi_{\mathbb{R}}(\varphi^{-1}(s)),$$

where $\Pi_{\mathbb{R}}$ is the projection from $\mathbb{R} \times (\cup_{k \geq 0} \mathbb{N}^k)$ to \mathbb{R} .

In a more graphical way, the contour process can be seen as the graph of an exploration process of the tree: it begins at the top of the root, decreasing with rate 1 while running back along the life of the root until it meets a birth. The contour process then jumps at the top of the life interval of the child born at this time and continues its exploration as before. If the exploration process do not encounter birth when exploring the life interval of an individual, it goes back to its parent and continues the exploration from the birth-date of the just left individual, the exploration process continues then as before (see Figure 3). It is then readily seen that the intersections of the contour process with the line of ordinate t are in one-to-one correspondence with the individuals in the tree alive at time t .

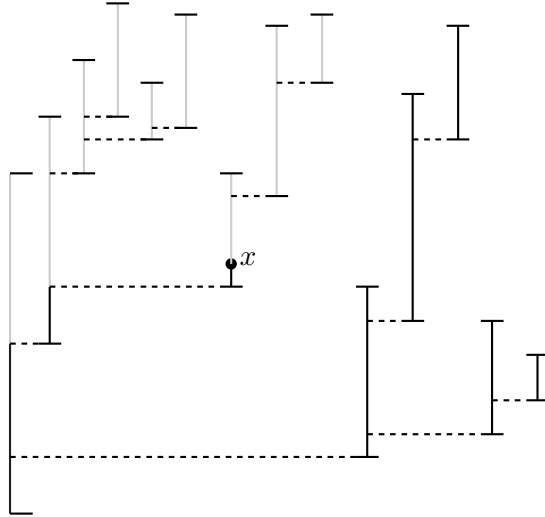


Figure 2: In gray, the set $\{y \in \mathbb{T} \mid y \preceq x\}$

In [18], Lambert shows that the contour process of the splitting tree which has been pruned from every part above t (called truncated tree above t), has the law of a spectrally positive Lévy process started at the lifespan V of the root, reflected below t and killed at 0, with Laplace exponent ψ given by,

$$\psi(x) = x - \int_{(0, \infty]} (1 - e^{-rx}) \Lambda(dr), \quad x \in \mathbb{R}_+. \quad (2.1)$$

In particular, the Laplace transform of \mathbb{P}_V can be expressed in terms of ψ ,

$$\int_{\mathbb{R}_+} e^{-\lambda v} \mathbb{P}_V(dv) = 1 + \frac{\psi(\lambda) - \lambda}{b}. \quad (2.2)$$

The largest root of ψ , denoted α , characterizes the way the population expand. In this paper, we only investigate the behavior of the population in the supercritical case, when $\alpha > 0$. In this case, the population grows exponentially fast on the survival event with rate α . In this case, it is easily seen that

$$\int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}_V(dv) = 1 - \frac{\alpha}{b}. \quad (2.3)$$

Moreover, the time of coalescence between an individual alive at time t and the next one visited by the contour, is exactly the depth of the excursion of the contour process below t between these two individuals (see Figure 3). The genealogy of the population is then described by the sequence of coalescence times, where lineages coalesce with the first deeper branch on their left (see Figure 4).

It is well known that such spectrally positive Lévy process has nice fluctuations properties (see [17]). In particular, Theorem 8.1 of [17] tells us that the law of the depth H of an excursions below t is given by,

$$\mathbb{P}(H > s) = \frac{1}{W(s)}, \quad s \in \mathbb{R}_+,$$

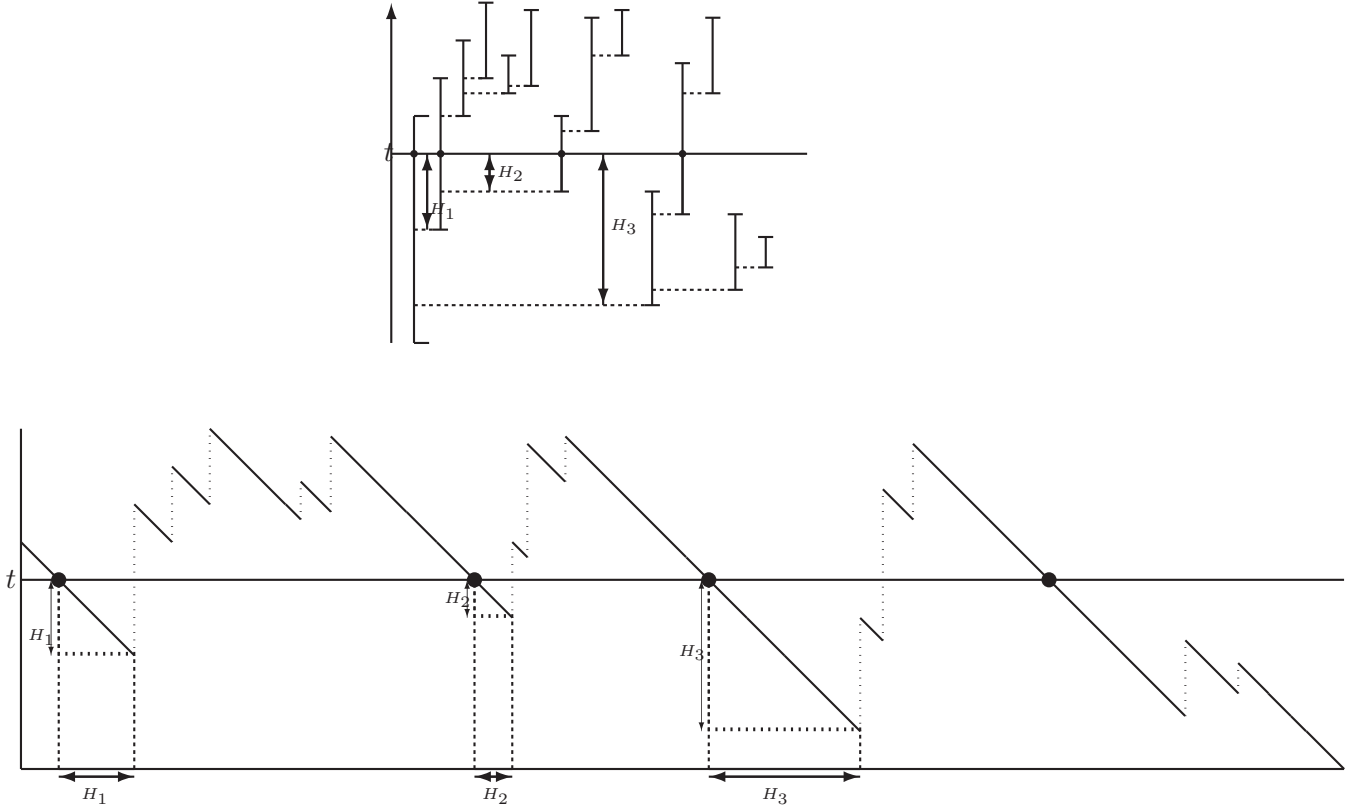


Figure 3: Construction of the contour process and link between the excursions of the contour process and the times of coalescence in the tree. There are 4 individuals alive at t , labeled from 0 to 3 according their order of exploration by the exploration process. The coalescence time between 0 and 1 is given by H_1 , etc

where W is the scale function of the Levy process characterized by its Laplace transform,

$$T_{\mathcal{L}}W(t) = \int_{(0,\infty)} e^{-rt}W(r)dr = \frac{1}{\psi(t)}, \quad t > \alpha. \quad (2.4)$$

Since, the contour process is strongly Markov, this sequence of excursion depths is i.i.d.

To summarize, given the population is still alive at time t , one can forget about the details of the splitting tree and code the genealogy by a new object called the *coalescent point process* (CPP). Its law is the law of a sequence $(H_i)_{0 \leq i \leq N_t - 1}$, where the family $(H_i)_{i \geq 1}$ is i.i.d. with the same law as H , stopped before its first value H_{N_t} greater than t , and H_0 is deterministic equal to t (see Figure 4).

Remark 2.1. Let N be a integer valued random variable. In the sequel we said that a random vector with random size $(X_i)_{1 \leq i \leq N}$ form an i.i.d. family of random variables independent of N , if and only if

$$(X_1, \dots, X_N) \stackrel{d}{=} (\tilde{X}_1, \dots, \tilde{X}_N),$$

where $(\tilde{X}_i)_{i \geq 1}$ is a sequence of i.i.d. random variables distributed as X_1 independent of N .

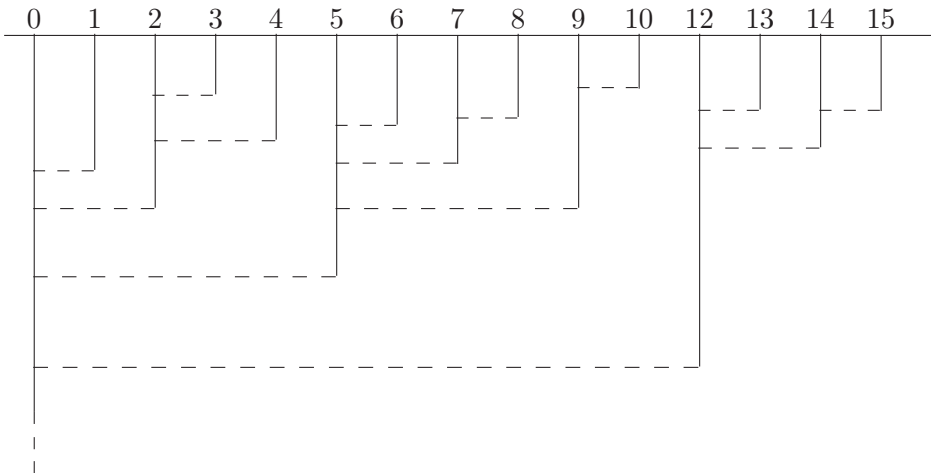


Figure 4: A coalescent point process for 16 individuals, hence 15 branches.

We denote by N_t the number of alive individuals at time t in the splitting tree, which is given in the CPP by

$$N_t = \inf\{i \geq 1 \mid H_i > t\}.$$

From the discussion above, we see that N_t is a geometric random variable conditionally on $\{N_t > 0\}$. More precisely, for a positive integer k ,

$$\mathbb{P}(N_t = k \mid N_t > 0) = \frac{1}{W(t)} \left(1 - \frac{1}{W(t)}\right)^{k-1}, \quad k \geq 1. \quad (2.5)$$

In particular,

$$\mathbb{E}[N_t \mid N_t > 0] = W(t). \quad (2.6)$$

Moreover, it can be showed (see [20]), that

$$\mathbb{E}N_t = W(t) - W \star \mathbb{P}_V(t), \quad (2.7)$$

and

$$\mathbb{P}(N_t > 0) = 1 - \frac{W \star \mathbb{P}_V(t)}{W(t)}, \quad (2.8)$$

where

$$W \star \mathbb{P}_V(t) := \int_{[0,t]} W(t-s) \mathbb{P}_V(ds).$$

Finally, since only the mutations occurring on the lineages of living individuals at time t can be observed, it follows from standard properties on Poisson random measures, that the mutation process can be defined directly on the CPP. So, let \mathcal{P} be a Poisson random measure on $(0, t) \times \mathbb{N}$ with intensity

measure $\theta\lambda \otimes C$, where C is the counting measure on \mathbb{N} , then the mutation random measure \mathcal{N} on the CPP is defined by

$$\mathcal{N}(da, di) = \mathbb{1}_{H_i > t-a} \mathbb{1}_{i < N_t} \mathcal{P}(di, da),$$

where an atom at (a, i) means that the i th branch experiences a mutation at time $t - a$. We suppose that each individual inherits the type of its parent. This rule yields a partition of the population by types. The distribution of the frequency of types in the population is called the frequency spectrum and is defined as the sequence $(A(k, t))_{k \geq 1}$ where $A(k, t)$ is the number of types carried by exactly k individuals in the alive population at time t , excluding the family holding the ancestral type of the population (i.e. individuals holding the same type as the root at time 0). This last family is called *clonal*, as the ancestral type.

In the study of the frequency spectrum, an important role is played by the clonal family. We denote by $Z_0(t)$ its size at time t .

To study this family, it is easier to consider the clonal splitting tree constructed from the original splitting tree by cutting every branches beyond mutations. This clonal splitting tree is a standard splitting tree without mutations, where individuals are killed as soon as they die or experience a mutation. The new lifespan law is therefore the minimum between an exponential random variable of parameter θ and an independent copy of V . It is straightforward by simple manipulation of Laplace transform that the Laplace exponent of the corresponding contour process is

$$\psi_\theta(x) = x - \int_{(0, \infty]} (1 - e^{-rx}) \Lambda_\theta(dr) = \frac{x\psi(x + \theta)}{x + \theta}.$$

We denote by W_θ the corresponding scale function. This leads to,

$$\mathbb{P}(Z_0(t) = k \mid Z_0(t) > 0) = \frac{1}{W_\theta(t)} \left(1 - \frac{1}{W_\theta(t)}\right)^{k-1}.$$

When $\alpha > \theta$ (resp. $\alpha = \theta$, $\alpha < \theta$), this new tree is supercritical and we talk about *clonal supercritical case* (resp. *critical*, *sub-critical case*).

Moreover, the law of Z_0 conditionally on the event $\{N_t > 0\}$ can be obtained, and is given by

$$\mathbb{P}(Z_0(t) = k \mid N_t > 0) = \frac{e^{-\theta t} W(t)}{W_\theta(t)^2} \left(1 - \frac{1}{W_\theta(t)}\right)^{k-1}, \quad \forall k \geq 1. \quad (2.9)$$

For the rest of this paper, unless otherwise stated, the notation \mathbb{P}_t refers to $\mathbb{P}(\cdot \mid N_t > 0)$ whereas \mathbb{P}_∞ refers to the probability measure conditioned on the non-extinction event (which has positive probability in the supercritical case).

Finally, we recall the asymptotic behaviour of the scale functions $W(t)$ and $W_\theta(t)$ which is widely used in the sequel,

Lemma 2.2. ([6, Thm. 3.21]) *There exist a positive constant γ such that,*

$$e^{-\alpha t} \psi'(\alpha) W(t) - 1 = \mathcal{O}(e^{-\gamma t}).$$

In the case that $\theta < \alpha$ (clonal supercritical case),

$$W_\theta(t) \underset{t \rightarrow \infty}{\sim} \frac{e^{(\alpha-\theta)t}}{\psi_\theta(\alpha-\theta)}.$$

In the case that $\theta > \alpha$ (clonal sub-critical case),

$$W_\theta(t) = \frac{\theta}{\psi(\theta)} + \mathcal{O}\left(e^{-(\theta-\alpha)t}\right).$$

In the case where $\theta = \alpha$ (clonal critical case),

$$W_\theta(t) \underset{t \rightarrow \infty}{\sim} \frac{\theta t}{\psi'(\alpha)}.$$

From this Lemma and (2.8), one can easily deduce that

$$\mathbb{P}(\text{NonEx}) = \lim_{t \rightarrow \infty} \mathbb{P}(N_t > 0) = \frac{\alpha}{b}, \quad (2.10)$$

where NonEx refer to the non-extinction event. In [5], we show that a CPP stopped at time t with scale function W can be constructed by grafting independent CPP at stopped at any fixed time $a \leq t$ on a CPP stopped at time $t - a$ with an explicit scale function different of W (see Figure 5). Moreover, we showed that the frequency spectrum can be expressed as an integral with respect to

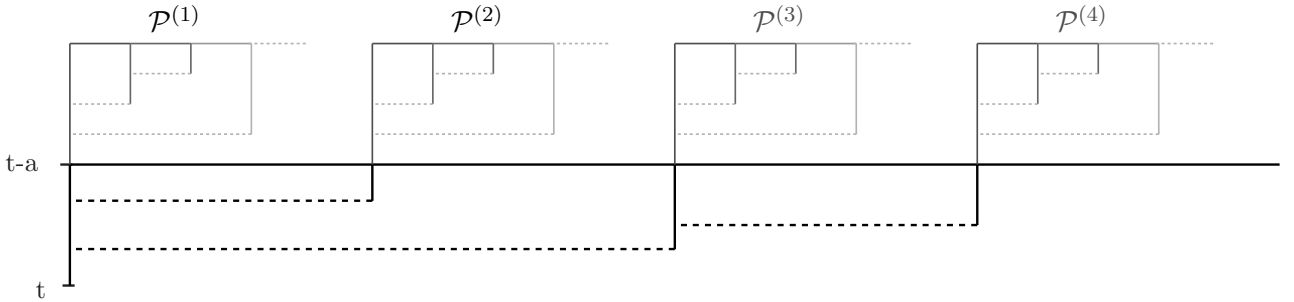


Figure 5: Adjunction of trees.

the random measure \mathcal{N} along the CPP, that is,

$$\prod_{i=1}^l A(k_i, t) = \sum_{i=1}^l \int_{[0,t] \times \mathbb{N}} \mathbf{1}_{Z_0^{(u)}(a)=k_i} \sum_{u_1:l-1=1}^{N_{t-a}^{(t)}} \prod_{\substack{j=1 \\ i \neq j}}^{l-1} A^{(u_j)}(k_j, a) \mathcal{N}(da, du), \quad (2.11)$$

where $A^{(u)}(k, a)$ (resp. $Z_0^{(u)}$) refers to the frequency spectrum (resp. clonal family) of the u th grafted sub-CPP, and $\sum_{u_1:l-1=1}^{N_{t-a}^{(t)}}$ denotes for the multi-sum

$$\sum_{u_1=1}^{N_{t-a}^{(t)}} \cdots \sum_{u_{l-1}=1}^{N_{t-a}^{(t)}}.$$

Moreover, in [5, Thm, 3.1] we show that the expectation of such integral can be computed easily when the integrand presents local independence properties with the random measure as in formula (2.11). Equation (2.11) is used later to obtain some moments estimates useful to prove our theorems. In particular, this allows to prove that (see [5]) for any positive integer k and l (see [5]),

$$\mathbb{E}_t A(k, t) = W(t) \int_0^t \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)}\right)^{k-1} ds, \quad (2.12)$$

and

$$\begin{aligned} \mathbb{E} A(k, t) A(l, t) &= 2W(t)^2 \int_0^t \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)}\right)^{k-1} ds \int_0^t \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)}\right)^{l-1} ds \\ &\quad - W(t) \int_0^t 2\theta \frac{e^{-\theta a} W(a)}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)}\right)^{l-1} \int_0^s \frac{e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(a)}\right)^{k-1} ds da \\ &\quad - W(t) \int_0^t 2\theta \frac{e^{-\theta a} W(a)}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)}\right)^{k-1} \int_0^s \frac{e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(a)}\right)^{l-1} ds da \\ &\quad + W(t) \mathbb{E} \int_0^t \theta W(a)^{-1} (\mathbb{E} [A(k, t) \mathbf{1}_{Z_0(a)=l}] + \mathbb{E} [A(l, t) \mathbf{1}_{Z_0(a)=k}]) da. \end{aligned} \quad (2.13)$$

These tools also allow us to show in [5] the next two results.

Theorem 2.3. *There exists a random variable \mathcal{E} , such that*

$$\lim_{t \rightarrow \infty} e^{-\alpha t} N_t = \frac{\mathcal{E}}{\psi'(\alpha)}, \quad \text{a.s. and in } L^2.$$

Moreover, under \mathbb{P}_∞ , \mathcal{E} is exponentially distributed with parameter one.

Theorem 2.4. *For any positive integer k ,*

$$\lim_{t \rightarrow \infty} e^{-\alpha t} A(k, t) = \frac{c_k \mathcal{E}}{\psi'(\alpha)}, \quad \text{a.s. and in } L^2,$$

where \mathcal{E} is the random variable of the Theorem 2.3 and

$$c_k = \int_0^\infty \frac{\theta e^{-\theta a}}{W_\theta(a)} \left(1 - \frac{1}{W_\theta(a)}\right)^{k-1} da. \quad (2.14)$$

2.2 A bit of renewal theory

The purpose of this part is to recall some facts on renewal equations borrowed from [10]. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a function bounded on finite intervals with support in \mathbb{R}_+ and Γ a probability measure on \mathbb{R}_+ . The equation

$$F(t) = \int_{\mathbb{R}_+} F(t-s) \Gamma(ds) + h(t),$$

called a renewal equation, is known to admit a unique solution finite on bounded interval.

Here, our interest is focused on the asymptotic behavior of F . We said that the function h is DRI (directly Riemann integrable) if for any $\delta > 0$, the quantities

$$\delta \sum_{i=0}^n \sup_{t \in [\delta i, \delta(i+1))} f(t)$$

and

$$\delta \sum_{i=0}^n \inf_{t \in [\delta i, \delta(i+1))} f(t)$$

converge as n goes to infinity respectively to some real number I_s^δ and I_i^δ , and

$$\lim_{\delta \rightarrow 0} I_s^\delta = \lim_{\delta \rightarrow 0} I_i^\delta < \infty.$$

In the sequel, we use the two following criteria for the DRI property:

Lemma 2.5. *Let h a function as defined previously. If h satisfies one of the next two conditions, then h is DRI:*

1. *h is non-negative decreasing and classically Riemann integrable on \mathbb{R}_+ ,*
2. *h is càdlàg and bounded by a DRI function.*

We can now state the next result, which is constantly used in the sequel.

Theorem 2.6. *Suppose that Γ is non-lattice, and h is DRI, then*

$$\lim_{t \rightarrow \infty} F(t) = \gamma \int_{\mathbb{R}_+} h(s) ds,$$

with

$$\gamma := \left(\int_{\mathbb{R}_+} s \Gamma(ds) \right)^{-1},$$

if the above integral is finite, and zero otherwise.

Remark 2.7. *In particular, if we suppose that Γ is a measure with mass lower than 1, and that there exists a constant $\alpha \geq 0$ such that*

$$\int_{\mathbb{R}_+} e^{\alpha t} \Gamma(dt) = 1,$$

then, one can perform the change a measure

$$\tilde{\Gamma}(dt) = e^{\alpha t} \Gamma(dt),$$

in order to apply Theorem 2.6 to a new renewal equation to obtain the asymptotic behavior of F . (See [10] for details). This method is also used in the sequel.

2.3 A lemma on the expectation of a random integral with respect to a Poisson random measure

Lemma 2.8. *Let ξ a Poisson random measure on \mathbb{R}_+ with intensity $\theta\lambda(da)$ where θ is a positive real number and λ the Lebesgue measure. Let also $(X_s^{(i)}, s \in \mathbb{R}_+)$ $_{i \geq 1}$ be an i.i.d. sequence of non-negative càdlàg random processes independent of ξ . Let also Y be a random variable independent of ξ and from the family $(X_s^{(i)}, s \in \mathbb{R}_+)$ $_{i \geq 1}$. If ξ_s denotes $\xi([0, s])$, then, for any $t \geq 0$,*

$$\mathbb{E} \int_{[0,t]} X_s^{(\xi_s)} \mathbb{1}_{Y>s} \xi(ds) = \int_0^t \mathbb{P}(Y > s) \theta \mathbb{E} X_s ds,$$

where $(X_s, s \in \mathbb{R}_+) = (X_s^{(1)}, s \in \mathbb{R}_+)$, and

$$\begin{aligned} \mathbb{E} \left[\int_{[0,t]} X_v^{(\xi_v)} \mathbb{1}_{Y>v} \xi(dv) \int_{[0,s]} X_u^{(\xi_u)} \mathbb{1}_{Y>u} \xi(du) \right] &= \int_0^t \theta \mathbb{E} [X_u^2] \mathbb{P}(Y > u) du \\ &\quad + \int_0^t \int_0^s \theta^2 \mathbb{E} X_u \mathbb{E} X_v \mathbb{P}(Y > u, Y > v) dudv. \end{aligned}$$

Proof. Since the proof the two formulas lies on the same ideas, we only give the proof of the second equation.

First of all, let $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be a positive measurable deterministic function. We recall that, for a Poisson random measure, the measures of two disjoint measurable sets are independent random variables. That is, for A, B in the Borel σ -field of \mathbb{R}_+ , $\xi(A \cap B^c)$ is independent of $\xi(B)$, which leads to

$$\mathbb{E} [\xi(A)\xi(B)] = \mathbb{E}\xi(A)\mathbb{E}\xi(B) + \text{Var}\xi(A \cap B).$$

Using the approximation of f by an increasing sequence of simple function, as in the construction of Lebesgue's integral, it follows from the Fubini-Tonelli and the monotone convergence theorem that,

$$\mathbb{E} \int_{[0,t] \times [0,s]} f(u, v) \xi(du)\xi(dv) = \int_0^t \theta f(u, u) du + \int_0^t \int_0^s \theta^2 f(u, v) dudv.$$

Since the desired relation only depends on the law of our random objects, we can assume without loss of generality that ξ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the family $(X_s^{(i)}, s \in \mathbb{R}_+)$ $_{i \geq 1}$ is defined on an other probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then, using a slight abuse of notation, we define ξ on $\Omega \times \tilde{\Omega}$ by $\xi_{(\omega, \tilde{\omega})} = \xi_\omega$, and similarly for the family X .

Then, by Fubini-Tonelli Theorem, with the notation $\xi_\omega^v = \xi_\omega([0, v])$,

$$\begin{aligned} \mathbb{E} \left[\int_{[0,t] \times [0,s]} X_v^{(\xi_v)} X_u^{(\xi_u)} \xi(ds) \right] &= \int_{\Omega \times \tilde{\Omega}} \int_{[0,t] \times [0,s]} X_v^{(\xi_v)}(\tilde{\omega}) X_u^{(\xi_u)}(\tilde{\omega}) \xi_\omega(ds) \mathbb{P} \otimes \tilde{\mathbb{P}}(d\omega, d\tilde{\omega}) \\ &= \int_{\Omega} \int_{[0,t] \times [0,s]} \int_{\tilde{\Omega}} X_v^{(\xi_v)}(\tilde{\omega}) X_u^{(\xi_u)}(\tilde{\omega}) \tilde{\mathbb{P}}(d\tilde{\omega}) \xi_\omega(ds) \mathbb{P}(d\omega). \end{aligned}$$

But since the $X^{(i)}$ are identically distributed and ξ is a simple measure (purely atomic with mass one for each atom) we deduce that, if u and v are two atoms of ξ_ω , $\xi_\omega^v = \xi_\omega^u$ if and only if $u = v$, which imply that

$$\int_{\tilde{\Omega}} X_v^{(\xi_\omega^v)}(\tilde{\omega}) X_u^{(\xi_\omega^u)}(\tilde{\omega}) \tilde{\mathbb{P}}(d\tilde{\omega}) = \begin{cases} \mathbb{E}X_u \mathbb{E}X_v, & u \neq v, \\ \mathbb{E}X_u^2, & u = v, \end{cases} \quad \xi_\omega - a.e.$$

The result follows readily, and the case with the indicator function of Y is left to the reader. \square

3 Statement of results

The a.s. convergence stated in section 2.1 suggests to study the second order properties to get central limit theorems for these convergences.

3.1 CLT for the convergence of Theorem 2.3

We recall that the Laplace distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix K is the probability distribution whose characteristic function is given, for all $\lambda \in \mathbb{R}^n$ by

$$\frac{1}{1 + \frac{1}{2} \lambda' K \lambda - i \mu' \lambda}$$

We denote this law by $\mathcal{L}(\mu, K)$. We also recall that, if G is a Gaussian random vector with mean μ and covariance matrix K and \mathcal{E} is an exponential random variable with parameter 1 independent of G , then $\sqrt{\mathcal{E}}G$ is Laplace $\mathcal{L}(\mu, K)$.

Theorem 3.1. *Under \mathbb{P}_∞ ,*

$$\lim_{t \rightarrow \infty} e^{\frac{\alpha}{2}t} (\psi'(\alpha)N_t - e^{\alpha t} \mathcal{E}) \stackrel{d}{=} \mathcal{L}(0, 2 - \psi'(\alpha)).$$

The proof of this Theorem is the subject of Section 6.

3.2 CLT for the convergence of Theorem 2.4

Theorem 3.2. *Suppose that $\theta > \alpha$ and $\int_{[0, \infty)} e^{(\theta - \alpha)v} \mathbb{P}_V(dv) > 1$. Then, we have, under \mathbb{P}_∞ ,*

$$\lim_{t \rightarrow \infty} \left(e^{\alpha \frac{t}{2}} (\psi'(\alpha)A(k, t) - e^{\alpha t} c_k \mathcal{E}) \right)_{k \in \mathbb{N}} \stackrel{d}{=} \mathcal{L}(0, K),$$

where K is some covariance matrix and the constants c_k are defined in (2.14).

The proof of this result can be found in Section 7.

Remark 3.3. *We are not able to compute explicitly the covariance matrix K in the general case due to our method of demonstration. However, all our other results give explicit formulas. In particular, the case where \mathbb{P}_V is exponential is given by the next theorem. The Yule case is also covered in the following theorem for $d = 0$ although it does not satisfy the hypothesis of Theorem 3.2.*

Theorem 3.4. *Suppose that V is exponentially distributed with parameter $d \in [0, b)$. In this case, $\alpha = b - d$. We still suppose that $\alpha < \theta$, then*

$$\lim_{t \rightarrow \infty} \left(e^{\alpha \frac{t}{2}} (\psi'(\alpha) A(k, t) - e^{\alpha t} c_k \mathcal{E}) \right)_{k \in N} \stackrel{d}{=} \mathcal{L}(0, K), \text{ w.r.t } \mathbb{P}_\infty,$$

where K is given by

$$K_{l,k} = M_{l,k} + c_k c_l \frac{\alpha}{b} \left(1 - 6 \frac{d}{\alpha} \right),$$

and

$$\begin{aligned} M_{l,k} = & 2\psi'(\alpha) \int_0^\infty \frac{\theta e^{-\theta a}}{W_\theta(a)^2} \left(\left(1 - \frac{1}{W_\theta(a)} \right)^{l-1} (\mathbb{E}_a [A(k, a)] - c_k W(a)) + \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} (\mathbb{E}_a [A(l, a)] - c_l W(a)) \right) da \\ & - \psi'(\alpha) \int_0^\infty \theta W(a)^{-1} \mathbb{E}_a [(A(k, a) - c_k N_a) \mathbb{1}_{Z_0(a)=l} + (A(l, a) - c_l N_a) \mathbb{1}_{Z_0(a)=k}], \quad (3.1) \end{aligned}$$

where W , W_θ , $\psi'(\alpha)$ are defined in the Section 2.1.

The proof of this result can be found in Section 9. Note that an explicit formula for $\mathbb{E}_t A(k, t)$ is given by (2.12). Explicit formulas for $\mathbb{E}_t [A(k, t) \mathbb{1}_{Z_0(t)=l}]$ can also be found in Proposition 4.5 of [5], and a formula for $\mathbb{E}_t [N_a \mathbb{1}_{Z_0(t)=k}]$ can be found in Proposition 4.1 of [6].

Remark 3.5. *The condition on V in Theorem 3.2 is required only to ensure controls of the moments of the considered quantities. However, although the Yule case does not satisfy this condition ($V = \infty$ p.s.) it is included in this last theorem ($d=0$). This suggests that the condition on V may not be needed.*

3.3 CLT for the error between $A(k, t)$ and $c_k N_t$

The next theorem concerns the error between $A(k, t)$ and $c_k N_t$. This case is easier to treat and we have an explicit expression of the covariance matrix of the limit.

Theorem 3.6. *Suppose that $\theta > \alpha$, then, for any multi-integer N ,*

$$\lim_{t \rightarrow \infty} \psi'(\alpha) \left(e^{\alpha \frac{t}{2}} (A(k, t) - c_k N_t) \right)_{k \in N} \stackrel{d}{=} \mathcal{L}(0, M), \text{ w.r.t. } \mathbb{P}_\infty,$$

where M is defined in relation (3.1).

The proof of this result can be found in Section 8.

Remark 3.7. *We do not know yet if the exponential random variable appearing the Gaussian mixing leading to a Laplace distribution is the same as the exponential limit of $e^{-\alpha t} A(k, t)$. However, the CLT for Markov branching processes in [2] suggest that it is, actually, the case. If, this is true in our case, it would be enough to know the correlations between the limits involved in Theorem 3.2 and 3.6 to obtain an explicit expression for the covariance matrix in Theorem 3.2.*

Remark 3.8. *Although, we only prove CLT for the frequency spectrum, we are convince that our method of proof may apply to many other general branching processes counted by random characteristics as soon as the birth point process is Poissonian. A more general result may be the subject of further works.*

4 Strategy of proof

Let $(G_n)_{n \geq 1}$ be a sequence of geometric random variables with respective parameter $\frac{1}{n}$, and $(X_i)_{i \geq 1}$ a L^2 family of i.i.d. random variables with zero mean independent of $(G_n)_{n \geq 1}$. It is easy to show that the characteristic function of

$$Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{G_n} X_i, \quad (4.1)$$

is given by

$$\mathbb{E}e^{i\lambda Z_n} = \frac{1 + o_n(1)}{1 + \lambda^2 \mathbb{E}X_1^2 + o_n(1)}, \quad (4.2)$$

from which we deduce that Z_n converges in distribution to $\mathcal{L}(0, \mathbb{E}X_1^2)$.

If we suppose that the population counting process N is a Yule Markov process, it clearly follows from the branching property that, for $s < t$,

$$N_t = \sum_{i=1}^{N_s} N_{t-s}^i, \quad (4.3)$$

where the family $(N_{t-s}^i)_{i \geq 1}$ is an i.i.d. sequence of random variables distributed as N_{t-s} and independent of N_s . Moreover, since N_s is geometrically distributed with parameter $e^{-\alpha s}$, taking the renormalized limit leads to,

$$\lim_{t \rightarrow \infty} e^{-\alpha t} N_t =: \mathcal{E} = e^{-\alpha s} \sum_{i=1}^{N_s} \mathcal{E}_i,$$

where $\mathcal{E}_1, \dots, \mathcal{E}_{N_s}$ is an i.i.d. family of exponential random variables with parameter one, and independent of N_s . Hence,

$$N_t - e^{\alpha t} \mathcal{E} = \sum_{i=1}^{N_s} \left(N_{t-s}^i - e^{\alpha(t-s)} \mathcal{E}_i \right),$$

is a geometric sum of centered i.i.d. random variables. This remark and (4.1) suggest the desired CLT in the Yule case.

However, in the general case, we need to overcome some important difficulties. First of all, equation (4.3) is wrong in general. Nevertheless, a much weaker version of (4.3) can be obtained in the general case. To make this clear, if $u < t$ are two positive real numbers, then the number of alive individuals at time t is the sum of the contributions of each subtrees $\mathbb{T}(O_i)$ induced by each alive individuals at time u (see Figure 6). Provided there are individuals alive at time u , we denote by $(O_i)_{1 \leq i \leq N_u}$ the residual lifetimes (see Figure 6) of the alive individuals at time u indexed using the canonical order on $\cup_{n \geq 0} \mathbb{N}^n$. Hence,

$$N_t = \sum_{i=1}^{N_u} N_{t-u}^i(O_i), \quad (4.4)$$

where $(N_{t-u}^i(O_i))_{i \leq N_u}$ denote the population counting processes of the subtrees $\mathbb{T}(O_i)$ induced by each individual. The notation refers to the fact that each subtree has the law of a standard splitting tree with the only difference that the lifelength of the root is given by O_i . More precisely, we

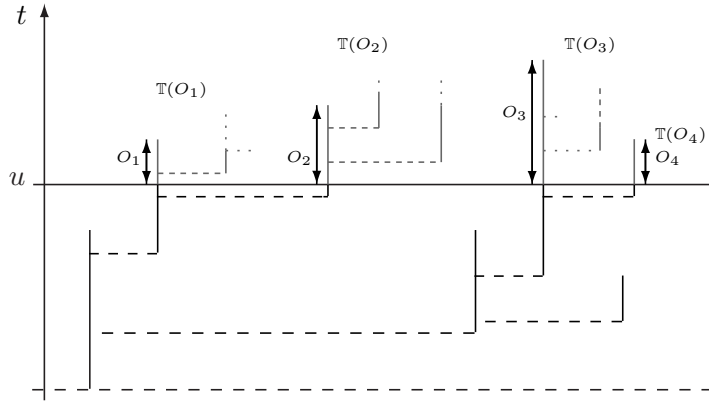


Figure 6: Residual lifetimes with subtrees associated to living individuals at time u .

define, for all $i \geq 1$ and $o \in \mathbb{R}_+$, $N_{t-u}^i(o)$ the population counting process of the splitting tree constructed from the same random objects as the i th subtree of Figure 6, where the life duration of the first individual is equal to o . Hence, from the independence properties between each individuals, $(N_{t-u}^i(o), t \geq u, o \geq 0)_{i \geq 1}$ is a family of independent processes, independent of $(O_i)_{1 \leq i \leq N_u}$, and $(N_{t-u}^i(o), t \geq u)$ has the law of the population counting process of a splitting tree but where the lifespan of the ancestor is o . Note that the lifespans of the other individuals are still distributed as V . From the discussion above, it follows that the family of processes $(N_{t-u}^i, t \geq u)_{1 \leq i \leq N_u}$ are dependent only through the residual lifetimes $(O_i)_{1 \leq i \leq N_u}$. Respectively, we define, for $1 \leq i \leq N_u$, by $(A(k, t, O_i))_{k \geq 1}$ (resp. $Z_0(t - u, O_i)$) the frequency spectrum (resp. the size of the clonal family) of the i th subtree.

Unfortunately, the computation of (4.2) does not apply to (4.4). This issue is solved by the following lemma, whose proof is very similar to the Proposition 5.5 of [18].

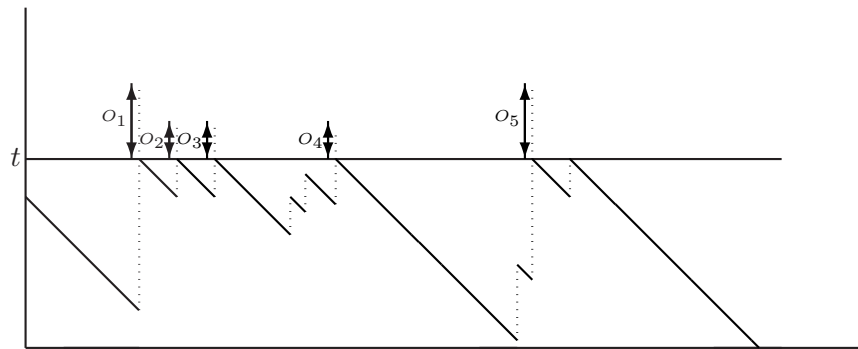


Figure 7: Reflected JCCP with overshoot over t . Independence is provided by the Markov property.

Lemma 4.1. *Let u in \mathbb{R}_+ , we denote by O_i for i an integer between 1 and N_u the residual lifetime of the i th individuals alive at time u indexed following the classical order on $\bigcup_{n \geq 0} \mathbb{N}^n$. Then under \mathbb{P}_u , the family $(O_i, i \in \llbracket 1, N_u \rrbracket)$ form a family of independent random variables, independent of N_u , and, except O_1 , having the same distribution, given by, for $2 \leq i \leq N_u$,*

$$\mathbb{P}_u(O_i \in dx) = \int_{\mathbb{R}_+} \frac{W(u-y)}{W(u)-1} b\mathbb{P}(V-y \in dx) dy. \quad (4.5)$$

Moreover, it follows that the family $(N_s(O_i), s \in \mathbb{R}_+)_{1 \leq i \leq N_u}$ is an independent family of process, i.i.d. for $i \geq 2$, and independent of N_u .

Proof. Let $(Y^{(i)})_{0 \leq i \leq N_u}$ a family of independent Lévy processes with Laplace exponent

$$\psi(x) = x - \int_{(0, \infty]} (1 - e^{-rx}) \Lambda(dr), \quad x \in \mathbb{R}_+,$$

conditioned to hit (t, ∞) before hitting 0, for $i \in \{0, \dots, N_u - 1\}$, and conditioned to hit 0 first for $i = N_u$. We also assume that,

$$Y_0^{(0)} = u \wedge V,$$

and

$$Y_0^{(i)} = u, \quad i \in \{1, \dots, N_u\}.$$

Now, denote by τ_i the exit time of the i th process out of $(0, u)$ and

$$T_n = \sum_{i=0}^{n-1} \tau_i, \quad n \in \{0, \dots, N_u + 1\}.$$

Then, the process defined, for all s , by

$$Y_s = \sum_{i=0}^{N_u} Y_{s-T_i}^{(i)} \mathbb{1}_{T_i \leq s < T_{i+1}},$$

has the law of the contour process of a splitting tree cut above t . Moreover, the quantity $Y_{\tau_i} - Y_{\tau_i-}$ is the lifetime of the i th alive individual at time t . The family of residual lifetimes $(O_i)_{1 \leq i \leq N_u}$ has then the same distribution as the sequence of the overshoots of the Y above u . Thus, the Markov property ensures us that $(O_i, i \in \llbracket 2, N_u \rrbracket)$ is an i.i.d. family of random variables. The Markov property also ensures that O_1 is independent of the other O_i 's.

It remains to derive the law of O_i . Let Y be a Lévy process with Laplace exponent ψ . We denote by τ_u^+ the time of first passage of $-Y$ above u and τ_0^- the time of first passage of $-Y$ below 0. Then, for all $i \geq 2$,

$$\mathbb{P}_u(O_i \in dx) = \mathbb{P}_0\left(-Y_{\tau_0^-} \in dx \mid \tau_0^- < \tau_u^+\right).$$

On the other hand, Theorem 8.7 of [17] gives for any measurable subsets $A \subset [0, u]$, $B \subset (0, -\infty)$,

$$\mathbb{P}_0\left(-Y_{\tau_0^-} \in B, -Y_{\tau_0^-} \in A\right) = \int_A \mathbb{P}_{-V}(B-y) \frac{W(u-y)}{W(u)} dy.$$

The result follows easily from

$$\mathbb{P}(\tau_0^- < \tau_u^+) = 1 - \frac{1}{W(u)}. \quad \square$$

Remark 4.2. *It is important to note that the law of the residual lifetimes of the individuals considered above depends on the particular time u we choose to cut the tree. That is why in the sequel, we may denote $O_i^{(u)}$ for O_i when we want to underline the dependence in time of the law of the residual lifetimes.*

In addition, as suggested by (4.2), we need to compute the expected quadratic error in the convergence of N_t ,

$$\mathbb{E} \left[(\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E})^2 \right],$$

which implies to compute $\mathbb{E}N_t\mathcal{E}$.

Although, this moment is easy to obtain in the Markovian case, the method does not extend easily to the general case. One idea is to characterize it as a solution of a renewal equation in the spirit of the theory of general CMJ processes theory.

To make this, we use the renewal structure of a splitting tree: the splitting trees can be constructed (see [18]) by grafting i.i.d. splitting tree on a branch (a tree with a single individual) of length V_\emptyset distributed as V . Therefore, there exists a family $(N_t^{(i)}, t \in \mathbb{R}_+)$ _{$i \geq 1$} of i.i.d. population counting processes, and a Poisson random measure ξ on \mathbb{R}_+ with intensity $b \bar{d}a$ such that

$$N_t = \int_{[0,t]} N_{t-u}^{(\xi_u)} \mathbb{1}_{V_\emptyset > u} \xi(du) + \mathbb{1}_{V_\emptyset > t}, \quad a.s., \quad (4.6)$$

where $\xi_u = \xi([0, u])$.

It is worth noting that any functional of the tree may have a similar decomposition (see Section 7 for $A(k, t)$).

Another difficulty comes from the fact that unlike (4.1), the quantities summed in (4.4) are time-dependent, which requires a careful analysis of the asymptotic behaviour of their moments.

The calculus and the asymptotic analysis of these moments is made in Section 6.1.1: In Lemma 6.1, we compute $\mathbb{E}N_t\mathcal{E}$, and then with Lemmas 6.2 and 6.4, we study the asymptotic behaviour of the error of order 2 and 3 respectively. Section 6.1.2 is devoted to the study of the same questions for the population counting processes of the subtrees described in Figure 6 (when the lifetime of the root is not distributed as V). Finally, Section 6.2 is devoted to the proof of Theorem 3.1. A similar plan is followed for the proof of Theorem 3.2 in Section 7, and the proof of Theorem 3.6 is much shorter and makes use of the previous moments estimates.

One of the difficulties in studying the behaviour of the moments is to get better estimates on the scale function W than those of Lemma 2.2. This is the subject of the next section.

5 Precise estimates on W using Lévy processes

Before stating and proving the result of this section, we need to recall some facts about Lévy processes. We follow the presentation of [17]. First, we recall that the law of a spectrally positive Lévy process

$(Y_t, t \in \mathbb{R}_+)$ is uniquely characterized by its Laplace exponent ψ ,

$$\psi_Y(\lambda) = \log \mathbb{E} \left[e^{-\lambda Y_1} \right], \quad \lambda \in \mathbb{R}_+,$$

which in our case take the form of (2.1):

$$\psi_Y(\lambda) = x - \int_{(0, \infty]} (1 - e^{-rx}) b\mathbb{P}_V(dr), \quad \lambda \in \mathbb{R}_+.$$

In this section, we suppose that $Y_0 = 0$. For a such Lévy process, 0 is irregular for $(0, \infty)$ and in this case the local time at the maximum $(L_t, t \in \mathbb{R})$ can be defined as

$$L_t = \sum_{i=0}^{n_t} e^i, \quad t \in \mathbb{R}_+,$$

where $(e^i)_{i \geq 0}$ is a family of i.i.d. exponential random variables with parameter 1, and

$$n_t := \text{Card}\{0 < s \leq t \mid Y_s = \sup_{u \leq s} Y_u\},$$

is the number of times Y reaches its maximum up to time t . Finally, the ascending ladder process associated to Y is defined as

$$H_t = Y_{L_t^{-1}}, \quad t \in \mathbb{R}_+,$$

where $(L_t^{-1}, t \in \mathbb{R}_+)$ is the right-inverse of L . It is easily seen that H is a subordinator whose values are the successive new maxima of Y . Conversely, in our case, the process $(\inf_{s \leq t} Y_s, t \in \mathbb{R}_+)$ is a local time at the minimum, denoted $(\hat{L}_t, t \in \mathbb{R}_+)$. The descending ladder process \hat{H} is then defined from \hat{L} as H was defined from L .

We can now state, the celebrated Wiener-Hopf factorization which allows us to connect the characteristic exponent ψ_Y of Y with the characteristic exponents of the bivariate Lévy processes $((L_t, H_t), t \in \mathbb{R}_+)$ and $((\hat{L}_t, \hat{H}_t), t \in \mathbb{R}_+)$, respectively denoted by κ and $\hat{\kappa}$. In our particular case, where Y is spectrally negative, we have

$$\begin{cases} \kappa(\alpha, \beta) = \frac{\alpha - \psi_Y(\beta)}{\phi_Y(\alpha) - \beta}, & \alpha, \beta \in \mathbb{R}_+, \\ \hat{\kappa}(\alpha, \beta) = \phi_Y(\alpha) + \beta, & \alpha, \beta \in \mathbb{R}_+, \end{cases}$$

where ϕ_Y is the right-inverse of ψ_Y . Taking $\alpha = 0$ allows us to recover the Laplace exponent ψ_H of H from which we obtain the relation,

$$\psi_Y(\lambda) = (\lambda - \phi_Y(0)) \psi_H(\lambda). \quad (5.1)$$

We have now all the notation to state and prove the main result of this section.

Proposition 5.1 (Behavior of W). *There exists a positive non-increasing càdlàg function F such that*

$$W(t) = \frac{e^{\alpha t}}{\psi'(\alpha)} - e^{\alpha t} F(t), \quad t \geq 0,$$

and

$$\lim_{t \rightarrow \infty} e^{\alpha t} F(t) = \begin{cases} \frac{1}{b\mathbb{E}V - 1} & \text{if } \mathbb{E}V < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let Y^\sharp be a spectrally negative Lévy process with Laplace exponent given by

$$\psi^\sharp(\lambda) = \lambda - \int_{\mathbb{R}_+} (1 - e^{-\lambda x}) e^{-\alpha x} b \mathbb{P}_V(dx).$$

It is known that Y^\sharp has the law of the contour process of the supercritical splitting tree with lifespan measure \mathbb{P}_V conditioned to extinction (see [18]). In this case the largest root of ψ^\sharp is zero, meaning that the process Y^\sharp does not go to infinity and that $\phi_{Y^\sharp}(0) = 0$. Elementary manipulations on Laplace transform show that the scale function W^\sharp of Y^\sharp is related to W by

$$W^\sharp(t) = e^{-\alpha t} W(t), \quad t \in \mathbb{R}_+.$$

Let H^\sharp be the ascending ladder subordinator associated to the Lévy process Y^\sharp . In the case where $\phi_{Y^\sharp}(0) = 0$, and in this case only, the scale function W^\sharp can be rewritten as (see [17] or use Laplace transform),

$$W^\sharp(t) = \int_0^\infty \mathbb{P}(H_x^\sharp \leq t) dx. \quad (5.2)$$

In other words, if we denote by U the potential measure of H^\sharp ,

$$W^\sharp(t) = U[0, t].$$

Now, it is easily seen from (5.1) that the Laplace exponent ψ_{H^\sharp} of H^\sharp takes the form,

$$\psi_{H^\sharp}(\lambda) = \psi'(\alpha) - \int_{[0, \infty]} (1 - e^{-\lambda r}) \Upsilon(dr),$$

where

$$\Upsilon(dr) = \int_{(r, \infty)} e^{-\alpha v} b \mathbb{P}_V(dv) dr = \mathbb{E}[e^{-\alpha V} \mathbf{1}_{V > r}] b dr.$$

Moreover,

$$\Upsilon(\mathbb{R}_+) = 1 - \psi'(\alpha),$$

which mean that H^\sharp is a compound Poisson process with jump rate $1 - \psi'(\alpha)$, jump distribution $J(dr) := \frac{\mathbb{E}[e^{-\alpha V} \mathbf{1}_{V > r}]}{1 - \psi'(\alpha)} dr$, and killed at rate $\psi'(\alpha)$. It is well known (or elementary by conditioning on the number of jumps at time x), that

$$\mathbb{P}_{H_x^\sharp}(dt) = e^{-\psi'(\alpha)x} \sum_{k \geq 0} e^{-(1-\psi'(\alpha))x} \frac{((1-\psi'(\alpha))x)^k}{k!} J^{*k}(dt).$$

Some calculations now lead to,

$$U(dx) = \sum_{k \geq 0} \Upsilon^{*k}(dx).$$

From this point, since Υ is a sub-probability, $U(x) := U[0, x]$ satisfies the following defective renewal equation,

$$U(x) = \int_{\mathbb{R}_+} U(x-u) \Upsilon(du) + \mathbf{1}_{\mathbb{R}_+}(x).$$

Finally, since

$$\int_{\mathbb{R}_+} e^{\alpha x} \Upsilon(dx) = 1,$$

and since, from Lemma 2.5,

$$t \rightarrow U(t, \infty),$$

is clearly a directly Riemann integrable function as a positive decreasing integrable function. Hence, as suggested in Remark 2.7,

$$e^{\alpha x} (U(\mathbb{R}_+) - U(x)) \xrightarrow{x \rightarrow \infty} \frac{1}{\alpha \mu},$$

with

$$\mu = \int_{\mathbb{R}_+} r e^{\alpha r} \Upsilon(dr) = \frac{1}{\alpha} (b\mathbb{E}V - 1),$$

if V is integrable. In the case where V is not integrable, the limit is 0.

To end the proof, note using relation (5.2) and the fact that H^\sharp is killed at rate $\psi'(\alpha)$ that,

$$W^\sharp(t) = \frac{1}{\psi'(\alpha)} - U(t, \infty). \quad \square$$

6 Proof of Theorem 3.1

We begin the proof of Theorem 3.1 by computing moments, and analysing their asymptotic behaviour. A first part is devoted to the case of a splitting tree where the lifetime of the root is distributed as V whereas a second part study the case where the lifespan of the root is arbitrary (for instance, as the subtrees described by Figure 6).

6.1 Preliminary moments estimates

This section is devoted to the calculus of the expectation of $(N_t - e^{\alpha t} \mathcal{E})^2$. We start with the simple case where the initial individual has life-length distributed as V . Secondly, we study the asymptotic behavior of these moments. In Subsection 6.1.2, we prove similar result for arbitrary initial distributions.

6.1.1 Case $V_\emptyset \stackrel{\mathcal{L}}{=} V$

We start with the computation of $\mathbb{E}N_t \mathcal{E}$.

Lemma 6.1 (Join moment of \mathcal{E} and N_t). *The function $t \rightarrow \mathbb{E}[N_t \mathcal{E}]$ is the unique solution bounded on finite intervals of the renewal equation,*

$$\begin{aligned} f(t) &= \int_{\mathbb{R}_+} f(t-u) b e^{-\alpha u} \mathbb{P}(V > u) du \\ &+ \alpha b \mathbb{E}[N.] \star \left(\int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > \cdot, V > v) dv \right) (t) \\ &+ \alpha \int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > t, V > v) dv, \end{aligned} \quad (6.1)$$

and its solution is given by

$$\left(1 + \frac{\alpha}{b} - e^{-\alpha t}\right) W(t) - (1 - e^{-\alpha t}) W \star \mathbb{P}_V(t).$$

Proof. As explained in Section 4,

$$N_t = \int_{[0,t]} N_{t-u}^{(\xi_u)} \mathbb{1}_{V_\emptyset > u} \xi(du) + \mathbb{1}_{V_\emptyset > t},$$

where ξ a Poisson point process with rate b on the real line, $(N^{(i)})_{i \geq 1}$ is a family of independent CMJ process with the same law as N and V_\emptyset is the lifespan of the root. Moreover the three objects $N^{(u)}$, ξ and V_\emptyset are independent.

It follows that, for $s > t$

$$\begin{aligned} N_t N_s &= \int_{[0,t] \times [0,s]} N_{t-u}^{(\xi_u)} N_{s-v}^{(\xi_v)} \mathbb{1}_{V_\emptyset > u} \mathbb{1}_{V_\emptyset > v} \xi(du) \xi(dv) \\ &\quad + \int_{[0,t]} N_{t-u}^{(\xi_u)} \mathbb{1}_{V_\emptyset > u} \xi(du) \mathbb{1}_{V_\emptyset > s} + \int_{[0,s]} N_{s-u}^{(\xi_u)} \mathbb{1}_{V_\emptyset > u} \xi(du) \mathbb{1}_{V_\emptyset > t} + \mathbb{1}_{V_\emptyset > t} \mathbb{1}_{V_\emptyset > s}, \end{aligned}$$

and, using Lemma 2.8,

$$\begin{aligned} \mathbb{E} N_t N_s &= \int_{[0,t]} b \mathbb{E} [N_{t-u} N_{s-u}] \mathbb{P}(V > u) du \\ &\quad + \int_{[0,t] \times [0,s]} b^2 \mathbb{E} [N_{t-u}] \mathbb{E} [N_{s-v}] \mathbb{P}(V > u, V > v) du dv \\ &\quad + \mathbb{P}(V > s) \int_{[0,t]} b \mathbb{E} [N_{t-u}] du + \int_{[0,s]} b \mathbb{E} [N_{s-u}] \mathbb{P}(V > u, V > t) du + \mathbb{P}(V > s). \end{aligned}$$

Then, thanks to the estimate $W(t) = \mathcal{O}(e^{\alpha t})$ (see Lemma 2.2 or 5.1) and the L^1 convergence of $W(s)^{-1} N_t N_s$ to $N_t \mathcal{E}$ as s goes to infinity (since, by Theorem 2.3, $\frac{N_s}{W(s)}$ converge in L^2 and using Cauchy-Schwarz inequality), we can exchange limit and integrals to obtain,

$$\begin{aligned} \lim_{s \rightarrow \infty} \mathbb{E} N_t \frac{N_s}{W(s)} &= \underbrace{\mathbb{E} N_t \mathcal{E}}_{=: f(t)} = \underbrace{\int_{[0,t]} \mathbb{E} [N_{t-u} \mathcal{E}] e^{-\alpha u} \mathbb{P}(V > u) b du}_{=: f \star G(t)} \\ &\quad + \underbrace{\int_{[0,t] \times [0,\infty)} \alpha b \mathbb{E} [N_{t-u}] e^{-\alpha v} \mathbb{P}(V > u, V > v) du dv}_{=: \zeta_1(t)} \\ &\quad + \underbrace{\int_{[0,\infty]} \alpha e^{-\alpha v} \mathbb{P}(V > v, V > t) dv}_{=: \zeta_2(t)}, \end{aligned}$$

where we used that $\lim_{t \rightarrow \infty} W(t)^{-1} \mathbb{E} N_t = \frac{\alpha}{b}$.

Now, we need to solve the last equation to obtain the last part of the lemma. To do that, we compute the Laplace transform of each part of the equation. Note that, since $W(t) = \mathcal{O}(e^{\alpha t})$, it is

easy to see that the Laplace transform of each term of (6.1) is well-defined as soon as $\lambda > \alpha$ (using Cauchy-Schwarz inequality for the first term). Now, using (2.2),

$$\begin{aligned} T_{\mathcal{L}}e^{\alpha \cdot}G(\lambda) &= b \int_{\mathbb{R}_+} e^{-\lambda t} \mathbb{P}(V > t) dt = b \int_{\mathbb{R}_+} e^{-\lambda t} \int_{(t, \infty)} \mathbb{P}_V(dv) dt \\ &= \frac{1}{\lambda} \int_{\mathbb{R}_+} (1 - e^{-\lambda v}) b \mathbb{P}_V(dv) = 1 - \frac{\psi(\lambda)}{\lambda}. \end{aligned} \quad (6.2)$$

So,

$$T_{\mathcal{L}}G(\lambda) = 1 - \frac{\psi(\lambda + \alpha)}{\lambda + \alpha}.$$

Then,

$$\begin{aligned} T_{\mathcal{L}}\zeta_1(\lambda) &= \alpha T_{\mathcal{L}}\mathbb{E}N_t(\lambda) T_{\mathcal{L}} \left(b \int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > \cdot, V > v) dv \right) (\lambda) \\ &= \left(\frac{\lambda}{\psi(\lambda)} - 1 \right) \underbrace{T_{\mathcal{L}} \left(\alpha \int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > \cdot, V > v) dv \right) (\lambda)}_{=\mathcal{L}\zeta_2(\lambda)}. \end{aligned}$$

and, using (6.2), we get

$$T_{\mathcal{L}}\zeta_2(\lambda) = \alpha \int_{\mathbb{R}_+} e^{-\lambda t} \int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > t, V > v) dv dt = \frac{1}{b} \left(\frac{\psi(\lambda + \alpha)}{\lambda} - \frac{\psi(\lambda)}{\lambda} \right).$$

Finally, we obtain,

$$T_{\mathcal{L}}f(\lambda) = T_{\mathcal{L}}f(\lambda) \left(1 - \frac{\psi(\lambda + \alpha)}{\lambda + \alpha} \right) + \left(\frac{\lambda}{\psi(\lambda)} - 1 \right) \frac{1}{b} \left(\frac{\psi(\lambda + \alpha)}{\lambda} - \frac{\psi(\lambda)}{\lambda} \right) + \frac{1}{b} \left(\frac{\psi(\lambda + \alpha)}{\lambda} - \frac{\psi(\lambda)}{\lambda} \right).$$

Hence,

$$T_{\mathcal{L}}f(\lambda) = \frac{\lambda}{b} \left(\frac{1}{\psi(\lambda)} - \frac{1}{\psi(\lambda + \alpha)} \right).$$

Finally, using (2.4) and

$$bT_{\mathcal{L}}(W \star \mathbb{P}_V)(\lambda) = \frac{(\psi(\lambda) - b + \lambda)}{\psi(\lambda)},$$

allows to inverse the Laplace transform of f and get the result. \square

Lemma 6.1 allows us to compute the expected quadratic error.

Lemma 6.2 (Quadratic error in the convergence of N_t). *Let \mathcal{E} the a.s. limit of $\psi'(\alpha)e^{-\alpha t}N_t$. Then,*

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E} (\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E})^2 = \frac{\alpha}{b} (2 - \psi'(\alpha)).$$

Proof. Let

$$\mu := \lim_{t \rightarrow \infty} e^{\alpha t} F(t),$$

where F is defined in Proposition 5.1. Since, using Proposition 5.1 and (2.3),

$$\begin{aligned} \int_{[0,t]} W(t-u) \mathbb{P}_V(du) &= \frac{e^{\alpha t}}{\psi'(\alpha)} \left(1 - \frac{\alpha}{b}\right) - \mu - \frac{e^{\alpha t}}{\psi'(\alpha)} \int_{(t,\infty)} e^{-\alpha u} \mathbb{P}_V(du) + \int_{[0,t]} \left(\mu - e^{\alpha(t-u)} F(t-u)\right) \mathbb{P}_V(du) \\ &= \frac{e^{\alpha t}}{\psi'(\alpha)} \left(1 - \frac{\alpha}{b}\right) - \mu + o(1). \end{aligned}$$

Hence, the expression of $\mathbb{E}N_t\mathcal{E}$ given by Lemma 6.1 can be rewritten, thanks to Lemmas 5.1, as

$$\mathbb{E}N_t\mathcal{E} = \frac{2\alpha e^{\alpha t}}{b\psi'(\alpha)} - \frac{\alpha}{b} \left(\frac{1}{\psi'(\alpha)} + \mu\right) + o(1), \quad (6.3)$$

Using (2.5) and (2.8) in conjunction with Proposition 5.1, we also have

$$e^{-\alpha t} \mathbb{E}N_t^2 = 2 \frac{\alpha e^{\alpha t}}{b\psi'(\alpha)^2} - \frac{2\alpha\mu}{b\psi'(\alpha)} - \frac{\alpha}{b\psi'(\alpha)} + o(1). \quad (6.4)$$

Hence, it finally follows from (6.3) and (6.4) that

$$\begin{aligned} e^{-\alpha t} \mathbb{E}(\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E})^2 &= \psi'(\alpha)^2 e^{-\alpha t} \mathbb{E}N_t^2 - 2\psi'(\alpha)\mathbb{E}N_t\mathcal{E} + \frac{2\alpha e^{\alpha t}}{b} \\ &= -2\frac{\alpha\mu}{b}\psi'(\alpha) - \frac{\alpha\psi'(\alpha)}{b} + 2\frac{\alpha}{b}(1 + \psi'(\alpha)\mu) + o(1) \\ &= \frac{\alpha}{b}(2 - \psi'(\alpha)) + o(1). \end{aligned}$$

□

It is worth noting that, using (2.8) and the method above, we have the following result.

Corollary 6.3. *We have*

$$\frac{1}{\mathbb{P}(N_t > 0)} = \frac{b}{\alpha} - \frac{b\mu\psi'(\alpha)}{\alpha} e^{-\alpha t} + o(e^{-\alpha t}), \quad (6.5)$$

which leads to

$$\mathbb{E}_t N_t \mathcal{E} = \frac{2e^{\alpha t}}{\psi'(\alpha)} - \frac{1}{\psi'(\alpha)} - 3\mu + o(1). \quad (6.6)$$

Our last estimate is the boundedness of the third moments.

Lemma 6.4 (Boundedness of the third moment). *The third moment of the error is asymptotically bounded, that is*

$$\mathbb{E} \left[\left| e^{-\alpha t} (\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E}) \right|^3 \right] = \mathcal{O}(1).$$

Proof. We define for all $t \geq 0$, N_t^∞ as the number of individuals alive at time t which have an infinite descent. According to Proposition 6.1 of [5], N^∞ is a Yule process under \mathbb{P}_∞ .

We have

$$\mathbb{E} \left[\left| \frac{\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E}}{e^{\frac{\alpha}{2}t}} \right|^3 \right] \leq 8\mathbb{E} \left[\left| \frac{\psi'(\alpha)N_t - N_t^\infty}{e^{\frac{\alpha}{2}t}} \right|^3 \right] + 8\mathbb{E} \left[\left| \frac{N_t^\infty - e^{\alpha t}\mathcal{E}}{e^{\frac{\alpha}{2}t}} \right|^3 \right].$$

Now, we know according to the proof of Theorem 6.2 of [5] (and this is easy to prove using the decomposition of Figure 6) that N^∞ can be decomposed as

$$N_t^\infty = \sum_{i=1}^{N_t} B_i^{(t)},$$

where $(B_i^{(t)})_{i \geq 1}$ is a family of independent Bernoulli random variables, which is i.i.d. for $i \geq 2$, under \mathbb{P}_t . Hence,

$$\mathbb{E}_t \left[\left| \frac{\psi'(\alpha)N_t - N_t^\infty}{e^{\frac{\alpha}{2}t}} \right|^3 \right] \leq e^{-\frac{3}{2}\alpha t} \mathbb{E}_t \left[\left(\sum_{i=1}^{N_t} (\psi'(\alpha) - B_i^{(t)}) \right)^4 \right]^{\frac{3}{4}}.$$

Since, it is known from the proof of Theorem 6.2 of [5] that

$$\mathbb{E}B_2^{(t)} = \psi'(\alpha) + \mathcal{O}(e^{-\alpha t}),$$

it is straightforward that

$$\mathbb{E}_t \left[\left| \frac{\psi'(\alpha)N_t - N_t^\infty}{e^{\frac{\alpha}{2}t}} \right|^3 \right]$$

is bounded.

On the other hand, we know that a Yule process is a time-changed Poisson process (see for instance [2], Theorem III.11.2), that is, if P_t is a Poisson process independent of \mathcal{E} under \mathbb{P}_∞ ,

$$\mathbb{E} \left[\left| \frac{N_t^\infty - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2}t}} \right|^3 \right] = \mathbb{E}_\infty \left[\left| \frac{P_{\mathcal{E}(e^{\alpha t}-1)} - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2}t}} \right|^3 \right] \mathbb{P}(\text{NonEx}).$$

Now, using Hölder inequality, it remains to bound

$$\mathbb{E}_\infty \left[\left(\frac{P_{\mathcal{E}(e^{\alpha t}-1)} - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2}t}} \right)^4 \right] = e^{-2\alpha t} \int_{\mathbb{R}_+} \mathbb{E}_\infty \left[(P_{x(e^{\alpha t}-1)} - e^{\alpha t} x)^4 \right] e^{-x} dx.$$

Finally, for a Poissonian random variable X with parameter ν , straightforward computations give that $\mathbb{E}[(X - \nu)^4] = 3\nu^2 + \nu$, which allows us to end the proof. \square

6.1.2 Case with arbitrary initial distribution \mathbb{P}_{V_0}

In order to study the behavior of the sub-splitting trees involved in the decomposition described in Figure 6, we investigate the behaviour of a splitting tree where the ancestor lifelength is not distributed as V , but follows an arbitrary distribution. Let Ξ be a random variable in $(0, \infty]$, giving to the life-length of the ancestor and by $N(\Xi)$ the associated population counting process.

Using the decomposition of $N(\Xi)$ over the lifespan of the ancestor, as described in Section 4, we have

$$N_t(\Xi) = \int_{\mathbb{R}_+} N_{t-u}^{(\xi_u)} \mathbf{1}_{\Xi > u} \xi(du) + \mathbf{1}_{\Xi > t}, \quad (6.7)$$

where $(N^i)_{i \geq 1}$ is a family of i.i.d. CMJ processes with the same law as N independent of Ξ and ξ , as described in section 4. Let, for all $i \geq 1$, \mathcal{E}_i be

$$\mathcal{E}_i := \lim_{t \rightarrow \infty} \psi'(\alpha) e^{-\alpha t} N_t^i, \quad a.s., \quad (6.8)$$

and, let $\mathcal{E}(\Xi)$ be the random variable defined by

$$\mathcal{E}(\Xi) := \int_{[0, \infty]} \mathcal{E}_{(\xi_u)} e^{-\alpha u} \mathbf{1}_{\Xi > u} \xi(du). \quad (6.9)$$

Lemma 6.5 (First moment). *The first moment is asymptotically bounded, that is*

$$\mathbb{E}(\psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi)) = \mathcal{O}(1),$$

uniformly with respect to the random variable Ξ .

Proof. Using Lemma 2.8, (6.7) and (6.9) with have

$$\mathbb{E}(\psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi)) = \int_{[0, t]} \left(\psi'(\alpha) \mathbb{E} N_{t-u} - e^{\alpha(t-u)} \mathbb{E} \mathcal{E} \right) e^{-\alpha u} \mathbb{P}(\Xi > u) bdu,$$

which leads using (2.7) and (2.10) to

$$\mathbb{E}(\psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi)) = \int_{[0, t]} \underbrace{\left(\psi'(\alpha) W(t-u) - \psi'(\alpha) W \star \mathbb{P}_V(t-u) - \frac{\alpha}{b} e^{\alpha(t-u)} \right)}_{=: I_{t-u}} e^{-\alpha u} \mathbb{P}(\Xi > u) bdu. \quad (6.10)$$

We get using Proposition 5.1 and (2.3),

$$\begin{aligned} I_s &= e^{\alpha s} - \psi'(\alpha) e^{\alpha s} F(s) - e^{\alpha s} \left(1 - \frac{\alpha}{b} \right) \\ &\quad + \psi'(\alpha) \int_{[0, s]} e^{\alpha(s-v)} F(s-v) \mathbb{P}_V(dv) + e^{\alpha s} \int_{(s, \infty)} e^{-\alpha v} \mathbb{P}_V(dv) - \frac{\alpha}{b} e^{\alpha s} \\ &= e^{\alpha s} \int_{(s, \infty)} e^{-\alpha v} \mathbb{P}_V(dv) + o(1). \end{aligned}$$

Hence, $(I_s)_{s \geq 0}$ is bounded. The result, now, follows from (6.10). \square

Lemma 6.6 (L^2 convergence in the general case). $\psi'(\alpha) e^{-\alpha t} N_t(\Xi)$ converge a.s. and in L^2 to $\mathcal{E}(\Xi)$, and

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E}(\psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi))^2 = \frac{\alpha}{b} (2 - \psi'(\alpha)) \int_{\mathbb{R}_+} e^{-\alpha s} \mathbb{P}(\Xi > s) bds,$$

where the convergence is uniform with respect to Ξ in $(0, \infty]$. In the particular case when Ξ follows the distribution of $O_2^{(\beta t)}$ given by (4.5), we have, for $0 < \beta < \frac{1}{2}$,

$$\lim_{t \rightarrow \infty} e^{\alpha t} \mathbb{E} \left(e^{-\alpha t} \psi'(\alpha) N_t(O_2^{(\beta t)}) - \mathcal{E}(O_2^{(\beta t)}) \right)^2 = (2 - \psi'(\alpha)) \psi'(\alpha).$$

Proof. From (6.7) and (6.9), we have

$$\left(e^{-\alpha t} \psi'(\alpha) N_t(\Xi) - \mathcal{E}(\Xi) \right)^2 = \left[\int_{\mathbb{R}_+} \left(e^{-\alpha(t-u)} \psi'(\alpha) N_{t-u}^{(\xi_u)} - \mathcal{E}(u) \right) e^{-\alpha u} \mathbf{1}_{\Xi > u} \xi(du) + e^{-\alpha t} \mathbf{1}_{\Xi > t} \right]^2 \quad (6.11)$$

and, using Lemma 2.8,

$$\begin{aligned} & \mathbb{E} \left(\psi'(\alpha) e^{-\alpha t} N_t(\Xi) - \mathcal{E}(\Xi) \right)^2 \\ = & \mathbb{E} \left(\int_{\mathbb{R}_+} \left(\psi'(\alpha) e^{-\alpha(t-u)} N_{t-u}^{(\xi_u)} - \mathcal{E}(u) \right) e^{-\alpha u} \mathbf{1}_{\Xi > u} \xi(du) \right)^2 \\ & + e^{-2\alpha t} \mathbb{P}(\Xi > t) + 2e^{-\alpha t} \mathbb{E} \mathbf{1}_{\Xi > t} \int_{\mathbb{R}_+} \left(\psi'(\alpha) e^{-\alpha(t-u)} N_{t-u}^{(\xi_u)} - \mathcal{E}(u) \right) e^{-\alpha u} \mathbf{1}_{\Xi > u} \xi(du), \\ = & \int_{\mathbb{R}_+} \mathbb{E} \left[\left(\psi'(\alpha) e^{-\alpha(t-u)} N_{t-u}^{(\xi_u)} - \mathcal{E}(u) \right)^2 \right] e^{-2\alpha u} \mathbb{P}(\Xi > u) \, bdu \\ & + \int_{\mathbb{R}_+} \mathbb{E} \left(\psi'(\alpha) e^{-\alpha(t-u)} N_{t-u}^{(\xi_u)} - \mathcal{E}(u) \right) \mathbb{E} \left(\psi'(\alpha) e^{-\alpha(t-v)} N_{t-v}^{(\xi_v)} - \mathcal{E}(v) \right) e^{-\alpha(u+v)} \mathbb{P}(\Xi > u, \Xi > v) \, bdu \, dv \\ & + e^{-2\alpha t} \mathbb{P}(\Xi > t) + 2e^{-\alpha t} \int_{\mathbb{R}_+} \mathbb{E} \left(\psi'(\alpha) e^{-\alpha(t-u)} N_{t-u}^{(\xi_u)} - \mathcal{E}(u) \right) e^{-\alpha u} \mathbb{P}(\Xi > u, \Xi > t) \, bdu. \end{aligned}$$

Moreover, since,

$$\psi'(\alpha) \mathbb{E} e^{-\alpha t} N_t - \mathcal{E} = \mathcal{O}(e^{-\alpha t}),$$

this leads, using Lemma 6.5, to

$$\lim_{t \rightarrow \infty} e^{\alpha t} \mathbb{E} \left(e^{-\alpha t} \psi'(\alpha) N_t(\Xi) - \mathcal{E}(\Xi) \right)^2 = \frac{\alpha}{b} (2 - \psi'(\alpha)) \int_{\mathbb{R}_+} e^{-\alpha u} \mathbb{P}(\Xi > u) \, bdu.$$

Now, we have from (4.5) and Lemma 2.2,

$$\lim_{u \rightarrow \infty} \mathbb{P}_u(O_2 > s) = \lim_{u \rightarrow \infty} \int_{\mathbb{R}_+} \frac{W(u-y)}{W(u)-1} \mathbb{P}(V > s+y) \, bdy = \int_{\mathbb{R}_+} e^{-\alpha y} \mathbb{P}(V > s+y) \, bdy.$$

It follows then from Lebesgue theorem that,

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}_+} e^{-\alpha s} \mathbb{P}_{\beta t}(O_2 > s) \, bds = \frac{b\psi'(\alpha)}{\alpha}. \quad \square$$

Lemma 6.7 (Boundedness in the general case.). *The error of order 3 in asymptotically bounded, that is*

$$e^{-\frac{3}{2}\alpha t} \mathbb{E} \left| \psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi) \right|^3 = \mathcal{O}(1),$$

uniformly w.r.t. Ξ .

Proof. Rewriting $N(\Xi)$ and $\mathcal{E}(\Xi)$ as in the proof of Lemma 6.6, we see that,

$$\begin{aligned} e^{-\frac{3}{2}t} \mathbb{E} \left| \psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi) \right|^3 &= e^{-\frac{3}{2}t} \mathbb{E} \left[\left| \int_{[0,t]} \left(\psi'(\alpha) N_{t-u}^{(\xi_u)} - e^{\alpha(t-u)} \mathcal{E}(u) \right) \mathbf{1}_{\Xi > u} \xi(du) + \psi'(\alpha) \mathbf{1}_{\Xi > t} \right|^3 \right] \\ &\leq 8 \mathbb{E} \left| \int_{[0,t]} e^{-\frac{3}{2}(t-u)} \left(\psi'(\alpha) N_{t-u}^{(\xi_u)} - e^{\alpha(t-u)} \mathcal{E}(u) \right) e^{-\frac{1}{2}u} \mathbf{1}_{\Xi > u} \xi(du) \right|^3 + 8 \psi'(\alpha) e^{-\frac{1}{2}t} \mathbb{P}(\Xi > t)^3 \end{aligned}$$

We denote by I the first term of the r.h.s. of the last inequality, leading to

$$\begin{aligned}
I &\leq 8\mathbb{E} \int_{[0,t]^3} \prod_{i=1}^3 \left| e^{-\frac{1}{2}(t-s_i)} \left(\psi'(\alpha) N_{t-s_i}^{(\xi_{s_i})} - e^{\alpha(t-s_i)} \mathcal{E}_{(s_i)} \right) \right| e^{-\frac{1}{2}s_i} \mathbf{1}_{\Xi > s_i} \xi(ds_1) \xi(ds_2) \xi(ds_3) \\
&\leq 8\mathbb{E} \int_{[0,t]^3} \sum_{j=1}^3 \left| e^{-\frac{1}{2}(t-s_j)} \left(\psi'(\alpha) N_{t-s_j}^{(\xi_{s_j})} - e^{\alpha(t-s_j)} \mathcal{E}_{(s_j)} \right) \right|^3 \prod_{i=1}^3 e^{-\frac{1}{2}s_i} \mathbf{1}_{\Xi > s_i} \xi(ds_1) \xi(ds_2) \xi(ds_3) \\
&\leq 24\mathbb{E} \int_{[0,t]} \left| e^{-\frac{1}{2}(t-u)} \left(\psi'(\alpha) N_{t-u}^{(\xi_u)} - e^{\alpha(t-u)} \mathcal{E}_{(u)} \right) \right|^3 e^{-\frac{1}{2}u} \mathbf{1}_{\Xi > u} \xi(du) \left(\int_{[0,t]} e^{-\frac{1}{2}u} \xi(du) \right)^2 \\
&\leq 24\mathbb{E} \int_{[0,t]} \left| e^{-\frac{1}{2}(t-u)} \left(\psi'(\alpha) N_{t-u}^{(\xi_u)} - e^{\alpha(t-u)} \mathcal{E}_{(u)} \right) \right|^3 e^{-\frac{1}{2}u} \mathbf{1}_{\Xi > u} \mu(du),
\end{aligned}$$

with

$$\mu(du) = \left(\int_{[0,t]} e^{-\frac{1}{2}s} \xi(ds) \right)^2 \xi(du).$$

Now, since μ is independent from the family $(N^{(i)})$ and $(\mathcal{E}_{(i)})$, an easy adaptation of the proof of Lemma 2.8, leads to

$$\begin{aligned}
e^{-\frac{3}{2}t} \mathbb{E} \left| \psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi) \right|^3 &\leq 24\mathbb{E} \int_{[0,t]} \mathbb{E} \left[\left| e^{-\frac{1}{2}(t-u)} \left(\psi'(\alpha) N_{t-u} - e^{\alpha(t-u)} \mathcal{E} \right) \right|^3 \right] e^{-\frac{1}{2}u} \mathbf{1}_{\Xi > u} \mu(du) \\
&\quad + 8\psi'(\alpha) e^{-\frac{1}{2}t} \mathbb{P}(\Xi > t)
\end{aligned}$$

Using Lemma 6.4 to bound

$$\mathbb{E} \left| e^{-\frac{3}{2}(t-u)} \left(N_{t-u} - e^{\alpha(t-u)} \mathcal{E} \right) \right|^3,$$

in the previous expression, finally leads to

$$e^{-\frac{3}{2}t} \mathbb{E} \left| \psi'(\alpha) N_t(\Xi) - e^{\alpha t} \mathcal{E}(\Xi) \right|^3 \leq \mathcal{C} \left(\mathbb{E} \left(\int_{\mathbb{R}_+} e^{-\frac{1}{2}u} \xi(du) \right)^3 + 1 \right),$$

for some real positive constant \mathcal{C} . □

6.2 Proof of Theorem 3.1

We fix a positive real number u . From this point, we recall the decomposition of the splitting tree as described in Section 4 (see also Figure 6). We also recall that, for all i in $\{1, \dots, N_u\}$, the process $(N_s^i(O_i), s \in \mathbb{R}_+)$ is the population counting process of the (sub-)splitting tree $\mathbb{T}(O_i)$.

As explained in Section 4, it follows from the construction of the splitting tree, that, for all i in $\{1, \dots, N_u\}$, there exists an i.i.d. family of processes $(N^{i,j})_{j \geq 1}$ with the same law as $(N_t, t \in \mathbb{R}_+)$, and an i.i.d. family $(\xi^{(i)})_{1 \leq i \leq N_u}$ of random measure with same law as ξ , such that

$$N_t^i(O_i) = \int_{[0,t]} N_{t-u}^{i,j} \mathbf{1}_{O_i > u} \xi^{(i)}(du) + \mathbf{1}_{O_i > t}, \quad \forall t \in \mathbb{R}_+, \quad \forall i \in \{1, \dots, N_u\}. \quad (6.12)$$

As in (6.9), we define, for all i in $\{1, \dots, N_u\}$,

$$\mathcal{E}(O_i) := \int_{[0,t]} \mathcal{E}_{i,\xi_u^{(i)}} e^{-\alpha u} \mathbf{1}_{O_i > u} \xi^{(i)}(du), \quad (6.13)$$

where $\mathcal{E}_{i,j} := \lim_{t \rightarrow \infty} \psi'(\alpha) e^{-\alpha t} N_t^{i,j}$.

Hence, it follows from Lemma 6.6, that $e^{-\alpha t} N_t^i(O_i)$ converges to $\mathcal{E}(O_i)$ in L^2 .

Note also that, from Lemma 4.1, the family $(N_t^i(O_i), t \in \mathbb{R}_+)_{2 \leq i \leq N_u}$ is i.i.d. and independent from N_u , as well as the family $(\mathcal{E}(O_i))_{2 \leq i \leq N_u}$ (in the sens of Remark 2.1).

Lemma 6.8 (Decomposition of \mathcal{E}). *We have the following decomposition of \mathcal{E} ,*

$$\mathcal{E} = e^{-\alpha u} \sum_{i=1}^{N_u} \mathcal{E}_i(O_i), \quad a.s.$$

Moreover, under \mathbb{P}_u , the random variables $(\mathcal{E}_i(O_i))_{i \geq 1}$ (defined by (6.13)) are independent, independent of N_u , and identically distributed for $i \geq 2$.

Proof. Step 1: Decomposition of \mathcal{E} .

For all t in \mathbb{R}_+ , we denote by N_t^∞ the number of individuals alive at time t which have an infinite descent. For all i , we define, for all $t \geq 0$, $N_t^\infty(O_i)$ from $\mathbb{T}(O_i)$ as N_t^∞ was defined from the whole tree. Now, it is easily seen that

$$N_t^\infty = \sum_{i=1}^{N_u} N_{t-u}^\infty(O_i).$$

Hence, if $e^{-\alpha t} N_t^\infty(O_i)$ converges a.s. to $\mathcal{E}(O_i)$, then

$$\lim_{t \rightarrow \infty} e^{-\alpha t} N_t^\infty = \lim_{t \rightarrow \infty} e^{-\alpha u} \sum_{i=1}^{N_u} e^{-\alpha(t-u)} N_{t-u}^\infty(O_i) = e^{-\alpha u} \sum_{i=1}^{N_u} \mathcal{E}(O_i).$$

So, it just remains to prove the a.s. convergence to get the desired result.

Step 2: a.s. convergence of $N^\infty(O_i)$ to $\mathcal{E}(O_i)$.

For this step, we fix $i \in \{1, \dots, N_u\}$.

In the same spirit as (6.12) (see also Section 4), it follows from the construction of the splitting tree $\mathbb{T}(O_i)$, that there exists, an i.i.d. sequence of processes $(N_s^{j,\infty}, s \in \mathbb{R}_+)_{j \geq 1}$ with the same law as $(N_t^\infty, t \in \mathbb{R}_+)$, such that

$$N_t^\infty(O_i) = \int_{[0,t]} N_{t-u}^{\xi_u^{(i)},\infty} \mathbf{1}_{O_i > u} \xi^{(i)}(du) + \mathbf{1}_{O_i = \infty}, \quad \forall t \geq 0.$$

Now, it follows from Theorem 6.2 of [5], that for all j ,

$$\lim_{t \rightarrow \infty} e^{-\alpha t} N_t^{j,\infty} = \mathcal{E}_{i,j}, \quad a.s.,$$

where $\mathcal{E}_{i,j}$ was defined in the begin of this section. Let

$$\mathcal{C}_j := \sup_{t \in \mathbb{R}_+} e^{-\alpha t} N_t^{j,\infty}, \quad \forall j \geq 1,$$

and

$$\mathcal{C} := \sup_{t \in \mathbb{R}_+} e^{-\alpha t} N_t^\infty.$$

Then, the family $(\mathcal{C}_j)_{j \geq 1}$ is i.i.d., since the processes $(N^{j, \infty})_{j \geq 1}$ are i.i.d, with the same law as \mathcal{C} . Hence,

$$\int_{[0, t]} e^{-\alpha(t-u)} N_{t-u}^{\xi_u^{(i)}, \infty} e^{-\alpha u} \mathbf{1}_{O_i > u} \xi^{(i)}(du) \leq \int_{[0, t]} \mathcal{C}_{\xi_u^{(i)}} e^{-\alpha u} \mathbf{1}_{O_i > u} \xi^{(i)}(du). \quad (6.14)$$

It is easily seen that $\mathbb{E}[\mathcal{C}] = \mathbb{P}(\text{NonEx}) \mathbb{E}_\infty[\mathcal{C}]$. Now, since, from Proposition 6.1 of [5], N_t^∞ is a Yule process under \mathbb{P}_∞ (and hence $e^{-\alpha t} N_t^\infty$ is a martingale), Doob's inequalities entails that the random variable \mathcal{C} is integrable. Hence, the right hand side of the (6.14) is a.s. finite, and we can apply Lebesgue Theorem to get

$$\lim_{t \rightarrow \infty} e^{-\alpha t} N_t^\infty(O_i) = \int_{[0, t]} \mathcal{E}_{i, \xi_u^{(i)}} e^{-\alpha u} \mathbf{1}_{O_i > u} \Gamma(du) = \mathcal{E}(O_i), \quad a.s.,$$

where the last equality is just the definition of $\mathcal{E}(O_i)$. □

We have now all the tools needed to prove the central limit theorem for N_t .

Proof of Theorem 3.1. Let $u < t$, two positive real numbers. From Lemma 6.8 and section 4, we have

$$N_t = \sum_{i=1}^{N_u} N_{t-u}^{(i)}(O_i)$$

and

$$e^{\alpha t} \mathcal{E} = \sum_{i=1}^{N_u} e^{\alpha(t-u)} \mathcal{E}_i(O_i).$$

Then,

$$\frac{\psi'(\alpha) N_t - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2} t}} = \sum_{i=1}^{N_u} \frac{\psi'(\alpha) N_{t-u}^{(i)}(O_i) - e^{\alpha(t-u)} \mathcal{E}_i(O_i)}{e^{\frac{\alpha}{2}(t-u)} e^{\frac{\alpha}{2} u}}. \quad (6.15)$$

Using Lemma 4.1, we know that, under \mathbb{P}_u , $(N_{t-u}^{(i)}(O_i), t > u)_{1 \leq i \leq N_u}$ are independent processes, i.i.d. for $i \geq 2$ and independent of N_u . Let us denote by φ and $\tilde{\varphi}$ the characteristic functions

$$\varphi(\lambda) := \mathbb{E} \left[\exp \left(i \lambda \left(\frac{\psi'(\alpha) N_{t-u}^2(O_2) - e^{\alpha(t-u)} \mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-u)}} \right) \right) \right], \quad \lambda \in \mathbb{R}$$

and

$$\tilde{\varphi}(\lambda) := \mathbb{E} \left[\exp \left(i \lambda \left(\frac{\psi'(\alpha) N_{t-u}^1(O_1) - e^{\alpha(t-u)} \mathcal{E}_1(O_1)}{e^{\frac{\alpha}{2}(t-u)}} \right) \right) \right], \quad \lambda \in \mathbb{R}.$$

It follows from (6.15) and Lemma 4.1 that,

$$\mathbb{E}_u \left[\exp \left(i \lambda \frac{\psi'(\alpha) N_t - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2} t}} \right) \right] = \frac{\tilde{\varphi} \left(\frac{\lambda}{e^{\frac{\alpha}{2} u}} \right)}{\varphi \left(\frac{\lambda}{e^{\frac{\alpha}{2} u}} \right)} \mathbb{E}_u \left[\varphi \left(\frac{\lambda}{e^{\frac{\alpha}{2} u}} \right)^{N_u} \right]$$

Since N_u is geometric with parameter $W(u)^{-1}$ under \mathbb{P}_u ,

$$\mathbb{E}_u \left[\exp \left(i\lambda \frac{\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E}}{e^{\frac{\alpha}{2}t}} \right) \right] = \frac{\tilde{\varphi} \left(\frac{\lambda}{e^{\frac{\alpha}{2}u}} \right)}{\varphi \left(\frac{\lambda}{e^{\frac{\alpha}{2}u}} \right)} \frac{W(u)^{-1}\varphi \left(\frac{\lambda}{e^{\frac{\alpha}{2}u}} \right)}{1 - (1 - W(u)^{-1})\varphi \left(\frac{\lambda}{e^{\frac{\alpha}{2}u}} \right)}$$

Using Taylor formula for φ , we obtain,

$$\mathbb{E}_u \left[\exp \left(i\lambda \frac{\psi'(\alpha)N_t - e^{\alpha t}\mathcal{E}}{e^{\frac{\alpha}{2}t}} \right) \right] = \tilde{\varphi} \left(\frac{\lambda}{e^{\frac{\alpha}{2}u}} \right) \frac{1}{D(\lambda, t, u)}$$

where,

$$D(\lambda, t, u) = W(u)$$

$$\begin{aligned} & - (W(u) - 1) \left(1 + i\lambda \mathbb{E} \left[\frac{\psi'(\alpha)N_{t-u}^i(O_2) - e^{\alpha(t-u)}\mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-u)}e^{\frac{\alpha}{2}u}} \right] \right. \\ & \quad \left. - \frac{\lambda^2}{2} \mathbb{E} \left[\left(\frac{\psi'(\alpha)N_{t-u}^i(O_2) - e^{\alpha(t-u)}\mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-u)}e^{\frac{\alpha}{2}u}} \right)^2 \right] + R(\lambda, t, u) \right) \\ & = 1 - i\lambda \frac{W(u) - 1}{e^{\frac{\alpha}{2}u}} \mathbb{E} \left[\frac{\psi'(\alpha)N_{t-u}^i(O_2) - e^{\alpha(t-u)}\mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-u)}} \right] \\ & \quad + \frac{\lambda^2}{2} \frac{W(u) - 1}{e^{\alpha u}} \mathbb{E} \left[\left(\frac{\psi'(\alpha)N_{t-u}^i(O_2) - e^{\alpha(t-u)}\mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-u)}} \right)^2 \right] \\ & \quad - (W(u) - 1)R(\lambda, t, u), \end{aligned}$$

with, for all $\epsilon > 0$ and all λ in $(-\epsilon, \epsilon)$,

$$|R(\lambda, t, u)| \leq \sup_{\lambda \in (-\epsilon, \epsilon)} \left| \frac{\partial^3}{\partial \lambda^3} \varphi(\lambda) \right| \leq \mathbb{E} \left[\left| \left(\frac{\psi'(\alpha)N_{t-u}^i(O_2) - e^{\alpha(t-u)}\mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-u)}} \right) \right|^3 \right] \frac{\epsilon^3 e^{-\frac{3}{2}\alpha u}}{6} \leq C\epsilon^3 e^{-\frac{3}{2}u}, \quad (6.16)$$

for some real positive constant C obtained using Lemma 6.7.

From this point, we set $u = \beta t$ with $0 < \beta < \frac{1}{2}$. It follows then from the Lemmas 6.6 and 4.1, that

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\beta t} \left[\left(\frac{\psi'(\alpha)N_{t-\beta t}^i(O_2) - e^{\alpha(t-\beta t)}\mathcal{E}_2(O_2)}{e^{\frac{\alpha}{2}(t-\beta t)}} \right)^2 \right] = \psi'(\alpha) (2 - \psi'(\alpha)). \quad (6.17)$$

Moreover, we have from Lemma 6.5, and since $\beta < \frac{1}{2}$,

$$\lim_{t \rightarrow \infty} W(\beta t) e^{-\frac{\alpha}{2}t} \mathbb{E} [\psi'(\alpha)N_t^i(O_2) - e^{\alpha t}\mathcal{E}_2(O_2)] = 0. \quad (6.18)$$

Finally, the relations (6.16), (6.17) and (6.18) lead to

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\beta t} \left[\exp \left(i\lambda \frac{N_t - e^{\alpha t}\mathcal{E}}{e^{\frac{\alpha}{2}t}} \right) \right] = \frac{1}{1 + \frac{\lambda^2}{2} (2 - \psi'(\alpha))}.$$

To conclude, note that,

$$\begin{aligned} \left| \mathbb{E}_{\beta t} \left[\exp \left(i\lambda \frac{N_t - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2} t}} \right) \right] - \mathbb{E}_{\infty} \left[\exp \left(i\lambda \frac{N_t - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2} t}} \right) \right] \right| &= \left| \mathbb{E} \left[e^{i\lambda \frac{\psi'(\alpha) N_t - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2} t}}} \left(\frac{\mathbb{1}_{N_{\beta t} > 0}}{\mathbb{P}(N_{\beta t} > 0)} - \frac{\mathbb{1}_{\text{NonEx}}}{\mathbb{P}(\text{NonEx})} \right) \right] \right| \\ &\leq \mathbb{E} \left[\left| \frac{\mathbb{1}_{N_{\beta t} > 0}}{\mathbb{P}(N_{\beta t} > 0)} - \frac{\mathbb{1}_{\text{NonEx}}}{\mathbb{P}(\text{NonEx})} \right| \right] \end{aligned}$$

goes to 0 as t goes to infinity. This ends the proof of Theorem 3.1. \square

7 Proof of Theorem 3.2

The proof of this theorem follows the same structure as in the preceding proof. It begins by some estimate on moments.

7.1 Preliminary moments estimates

We start by computing the moment in the case of a standard splitting tree.

7.1.1 Case $V_{\emptyset} \stackrel{\mathcal{L}}{=} V$

One of the main difficulties to extend the preceding proof to the frequency spectrum is to get estimates on

$$\mathbb{E} \left[(\psi'(\alpha) A(k, t) - e^{\alpha t} c_k \mathcal{E})^n \right], \text{ for } n = 2 \text{ or } 3.$$

We first study the renewal equation satisfied by $\mathbb{E} A(k, t) \mathcal{E}$ similarly as in Lemma 6.1.

Lemma 7.1 (Joint moment of \mathcal{E} and $A(k, t)$). *$\mathbb{E} [A(k, t) \mathcal{E}]$ is the unique solution bounded on finite intervals of the renewal equation,*

$$\begin{aligned} f(t) &= \int_{\mathbb{R}_+} f(t-u) b e^{-\alpha u} \mathbb{P}(V > u) du \\ &\quad + \alpha \mathbb{E} [A(k, \cdot)] \star b \left(\int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > \cdot, V > v) dv \right) (t) \\ &\quad + \alpha \mathbb{E} [\mathcal{E} X_t], \end{aligned} \tag{7.1}$$

with X_t the number of families of size k alive at time t whose original mutation has taken place during the lifetime of the ancestor individual.

Proof. We recall that $A(k, t)$ is the number of non-ancestral families of size k at time t . Similarly as for N_t , $A(k, t)$ can be obtained as the sum of the contributions of all the trees grafted on the lifetime of the ancestor individual in addition to the mutations which take place on the ancestral branch, that is,

$$A(k, t) = \int_{[0, t]} A(k, t-u, \xi_u) \mathbb{1}_{V_{\emptyset} > u} \xi(du) + X_t,$$

where $(A(k, t, i), t \in \mathbb{R}_+)_{i \geq 1}$ is a family of independent processes having the same law as $A(k, t)$. Now, taking the product $A(k, t)N_s$ and using the same arguments as in the proof of lemma 6.1 to take the limit in s leads to the result. In particular, the last term is obtained using that

$$\lim_{s \rightarrow \infty} \mathbb{E} \left[X_t \frac{N_s}{W(s)} \right] = \mathbb{E} [X_t \mathcal{E}].$$

□

The result of Lemma 7.1 is quite disappointing since the presence of the mysterious process X_t prevents any explicit resolution of equation (7.1). However, one may note that equation (7.1) is quite similar to equation (6.1) driving $\mathbb{E}N_t \mathcal{E}$, so if the contribution of X_t in the renewal structure of the process is small enough, one can expect the same asymptotic behaviour for $\mathbb{E}A(k, t) \mathcal{E}$ as for $\mathbb{E}N_t \mathcal{E}$. Moreover, we clearly have on X_t the following a.s. estimate,

$$X_t \leq \int_{[0, t]} \mathbb{1}_{Z_0^{(i)}(t-u) > 0} \mathbb{1}_{V > u} \xi(du), \quad (7.2)$$

where $Z_0^{(i)}$ denote for the ancestral families on the i th trees grafted on the ancestral branch. Hence, if we take $\theta > \alpha$ and we suppose $V < \infty$ a.s., one can expect that X_t decreases very fast. These are the ideas the following Lemma is based on. Moreover, as it is seen in the proof of the following lemma, the hypothesis $V < \infty$ a.s. can be weakened.

Lemma 7.2. *Under the hypothesis of Theorem 3.2, for all $k \geq 1$, there exists a constant $\gamma_k \in \mathbb{R}$ such that,*

$$\lim_{t \rightarrow \infty} \mathbb{E}N_t \mathcal{E} c_k - \mathbb{E}A(k, t) \mathcal{E} = \gamma_k. \quad (7.3)$$

Proof. Combining equations (6.1) and (7.1), we get that,

$$\begin{aligned} \mathbb{E}N_t \mathcal{E} c_k - \mathbb{E}A(k, t) \mathcal{E} &= \int_{\mathbb{R}_+} (\mathbb{E}N_{t-u} \mathcal{E} c_k - \mathbb{E}A(k, t-u) \mathcal{E}) b e^{-\alpha u} \mathbb{P}(V > u) du \\ &+ \underbrace{\alpha b (c_k \mathbb{E}N_t - \mathbb{E}[A(k, \cdot)]) \star \left(\int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > \cdot, V > v) dv \right)}_{:= \xi_1^{(k)}(t)}(t) \\ &+ \underbrace{c_k \mathbb{P}(V > t) - \alpha \mathbb{E}[X_t \mathcal{E}]}_{:= \xi_2^{(k)}(t)}, \end{aligned}$$

which is also a renewal equation. On one hand, using equations (2.6) and (2.12) imply that

$$\mathbb{E}_t [c_k N_t - A(k, t)] = W(t) \int_t^\infty \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)} \right)^{k-1} ds,$$

which leads using Lemma 2.2, to

$$\begin{aligned}
\xi_1(t) &= \alpha \int_{\mathbb{R}_+} (c_k \mathbb{E} N_{t-u} - \mathbb{E}[A(k, t-u)]) \int_{\mathbb{R}_+} e^{-\alpha v} \mathbb{P}(V > u, V > v) dv du \\
&\leq \mathcal{C} \int_{[0,t]} e^{(\alpha-\theta)t-u} \mathbb{P}(V > u) du \int_{[0,\infty)} e^{-\alpha u} du \\
&\leq \frac{\mathcal{C}}{\alpha} e^{-(\theta-\alpha)t} \int_0^t e^{(\theta-\alpha)u} \mathbb{P}(V > u) du,
\end{aligned} \tag{7.4}$$

for some positive real constant \mathcal{C} .

The derivative of the r.h.s. of (7.4) is given by

$$\frac{\mathcal{C}}{\alpha} e^{-(\theta-\alpha)t} \left(e^{(\theta-\alpha)t} \mathbb{P}(V > t) - (\alpha - \theta) \int_0^t e^{(\theta-\alpha)u} \mathbb{P}(V > u) du \right), \quad t > 0, \tag{7.5}$$

which is equal to

$$\frac{\mathcal{C}}{\alpha} e^{-(\theta-\alpha)t} \left(1 - \int_{[0,t]} e^{(\theta-\alpha)s} \mathbb{P}_V(ds) \right), \quad t > 0,$$

using Stieljes integration by parts. Now, since,

$$\int_{[0,\infty)} e^{(\theta-\alpha)s} \mathbb{P}_V(ds) > 1,$$

this shows that the right hand side of (7.4) is decreasing for t large enough. Moreover, it is straightforward to show that the r.h.s. of (7.4) is also integrable. This implies that $\xi_1^{(k)}$ is DRI from the same Lemma. On the other hand, it follows from (7.2) that

$$X_t \mathcal{E} \leq \mathcal{E} \int_{[0,t]} \mathbb{1}_{Z_0^{(u)}(t-u) > 0} \mathbb{1}_{V > t} \xi(du). \tag{7.6}$$

Then, we obtain using Cauchy-Schwarz inequality, that

$$\mathbb{E}[X_t \mathcal{E}] \leq \sqrt{\frac{2\alpha}{b}} \mathbb{E} \left[\left(\int_{[0,t]} \mathbb{1}_{Z_0^{(u)}(t-u) > 0} \mathbb{1}_{V > t} \xi(du) \right)^2 \right]^{1/2}.$$

It follows that we need to investigate the behavior of

$$\mathbb{E} \left[\left(\int_{(0,t)} \mathbb{1}_{Z_0^{(u)}(t-u) > 0} \mathbb{1}_{V > t} \xi(du) \right)^2 \right],$$

which is equal to

$$\int_0^t \mathbb{P}(Z_0(t-u) > 0) \mathbb{P}(V > t) b du + \int_{[0,t]^2} \mathbb{P}(Z_0(t-v) > 0) \mathbb{P}(Z_0(t-u) > 0) \mathbb{P}(V > u, V > v) b^2 du dv,$$

using Lemma 2.8. Then, since, from (2.9) and Lemma 2.2,

$$\mathbb{P}_{t-u}(Z_0(t-u) > 0) = \frac{e^{-\theta(t-u)} W(t-u)}{W_\theta(t-u)} = \mathcal{O}(e^{-(\theta-\alpha)(t-u)}),$$

it follows, using that the right hand side of (7.4) is DRI and Lemma 2.5, that $\xi_2^{(k)}$ is DRI. Finally, it comes from Theorem 2.6, that

$$\lim_{t \rightarrow \infty} \mathbb{E} N_t \mathcal{E} c_k - \mathbb{E} A(k, t) \mathcal{E} = \frac{\alpha}{\psi'(\alpha)} \int_{\mathbb{R}_+} \xi_1^{(k)}(s) + \xi_2^{(k)}(s) ds. \quad (7.7)$$

□

Using the preceding lemma, we can now get the quadratic error in the convergence of the frequency spectrum.

Lemma 7.3 (Quadratic error for the convergence of $A(k, t)$). *Let k and l two positive integers. Then under the hypothesis of Theorem 3.2, there exists a family of real numbers $(a_{k,l})_{l,k \geq 1}$ such that,*

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E}_t [(\psi'(\alpha) A(k, t) - e^{\alpha t} \mathcal{E} c_k) (\psi'(\alpha) A(l, t) - e^{\alpha t} \mathcal{E} c_l)] = a_{k,l},$$

where the sequence $(c_k)_{k \geq 1}$ is defined by (2.14).

Proof. Now, noting

$$c_k(t) := \int_0^t \frac{\theta e^{-\theta a}}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} da, \quad (7.8)$$

we have, from (2.13),

$$\psi'(\alpha)^2 \mathbb{E}_t [A(k, t) A(l, t)] = 2W(t)^2 c_l(t) c_k(t) + W(t) R + o(e^{-\alpha t}), \quad (7.9)$$

with

$$\begin{aligned} R := & -\psi'(\alpha) \int_0^\infty 2\theta \frac{e^{-\theta a} W(a)}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{l-1} \int_0^a \frac{e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} ds da \\ & -\psi'(\alpha) \int_0^\infty 2\theta \frac{e^{-\theta a} W(a)}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} \int_0^a \frac{e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{l-1} ds da \\ & + \psi'(\alpha) \int_0^\infty \theta W(a)^{-1} (\mathbb{E}_t [A(k, t) \mathbf{1}_{Z_0(a)=l}] + \mathbb{E}_t [A(l, t) \mathbf{1}_{Z_0(a)=k}]) da, \end{aligned}$$

Now, using (6.5), we have

$$\mathbb{E}_t \mathcal{E}^2 - 2 = -2\mu \psi'(\alpha) e^{-\alpha t} + o(e^{-\alpha t}),$$

which leads to

$$\begin{aligned} & \mathbb{E}_t [(e^{-\alpha t} \psi'(\alpha) A(k, t) - \mathcal{E} c_k) (e^{-\alpha t} \psi'(\alpha) A(l, t) - \mathcal{E} c_l)] \\ & = \mathbb{E}_t [e^{-2\alpha t} \psi'(\alpha)^2 A(k, t) A(l, t)] - c_l \mathbb{E}_t [e^{-\alpha t} \psi'(\alpha) A(k, t) \mathcal{E}] - c_k \mathbb{E}_t [e^{-\alpha t} \psi'(\alpha) A(l, t) \mathcal{E}] \\ & \quad + 2c_k c_l - 2c_k c_l \mu \psi'(\alpha) e^{-\alpha t} + o(e^{-\alpha t}), \\ & = 2(c_k(t) - c_k)(c_l(t) - c_l) - 2\mu \psi'(\alpha) c_k c_l e^{-\alpha t} + R e^{-\alpha t} \\ & \quad - (2c_k(t) c_l + 2c_l(t) c_k - 2c_k c_l \psi'(\alpha) e^{-\alpha t} \mathbb{E}_t N_t \mathcal{E}) \\ & \quad + \psi'(\alpha) c_l e^{-\alpha t} \mathbb{E}_t [(c_k N_t - A(k, t)) \mathcal{E}] + \psi'(\alpha) c_k e^{-\alpha t} \mathbb{E}_t [(c_l N_t - A(l, t)) \mathcal{E}], \end{aligned}$$

Since, by Lemma 2.2

$$c_k(t) = c_k + \mathcal{O}(e^{-\theta t}) = c_k + o(e^{-\alpha t}),$$

the result follows combining (7.7) (6.6), and Lemma 7.2 that,

$$\begin{aligned} & e^{\alpha t} \mathbb{E}_t \left[(e^{-\alpha t} \psi'(\alpha) A(k, t) - \mathcal{E} c_k) (e^{-\alpha t} \psi'(\alpha) A(l, t) - \mathcal{E} c_l) \right] \\ &= \psi'(\alpha) (c_k \gamma_l + c_l \gamma_k) + c_k c_l (2e^{\alpha t} - 2\psi'(\alpha) \mathbb{E}_t N_t \mathcal{E}) + R - 2\mu \psi'(\alpha) c_k c_l + o(1) \\ &= \psi'(\alpha) (c_k \gamma_l + c_l \gamma_k) + c_k c_l \left(\frac{1}{\psi'(\alpha)} + 3\mu \right) + R - 2\mu \psi'(\alpha) c_k c_l + o(1). \end{aligned}$$

The result follows readily. □

Lemma 7.4 (Boundedness of the third moment). *Let k_1, k_2, k_3 three positive integers, then*

$$\mathbb{E} \left[\prod_{i=1}^3 \left| e^{-\frac{\alpha}{2} t} (\psi'(\alpha) A(k_i, t) - e^{\alpha t} \mathcal{E} c_{k_i}) \right| \right] = \mathcal{O}(1).$$

Proof. We have,

$$\mathbb{E} \left[\left| \prod_{i=1}^3 \frac{(\psi'(\alpha) A(k_i, t) - e^{\alpha t} \mathcal{E} c_{k_i})}{e^{\frac{\alpha}{2} t}} \right| \right] \leq \prod_{i=1}^3 \left(\mathbb{E} \left[\left| \frac{(\psi'(\alpha) A(k_i, t) - e^{\alpha t} \mathcal{E} c_{k_i})}{e^{\frac{\alpha}{2} t}} \right|^3 \right] \right)^{\frac{1}{3}}.$$

Hence, we only have to prove the Lemma for $k_1 = k_2 = k_3 = k$. Hence,

$$\begin{aligned} \mathbb{E} \left[\left| \frac{(\psi'(\alpha) A(k, t) - e^{\alpha t} \mathcal{E} c_k)}{e^{\frac{\alpha}{2} t}} \right|^3 \right] &\leq 8 \mathbb{E} \left[\left| \frac{\psi'(\alpha) A(k, t) - c_k N_t}{e^{\frac{\alpha}{2} t}} \right|^3 \right] + 8c_k \mathbb{E} \left[\left| \frac{\psi'(\alpha) N_t - N_t^\infty}{e^{\frac{\alpha}{2} t}} \right|^3 \right] \\ &\quad + 8c_k \mathbb{E} \left[\left| \frac{N_t^\infty - e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2} t}} \right|^3 \right]. \end{aligned}$$

The last two terms have been treated in the proof of Lemma 6.4, and the boundedness of

$$\mathbb{E} \left[\left| \frac{\psi'(\alpha) A(k, t) - c_k N_t}{e^{\frac{\alpha}{2} t}} \right|^3 \right],$$

follows from the following Lemma 7.5 and Hölder's inequality. □

Lemma 7.5. *For all $k \geq 1$,*

$$\mathbb{E} \left[\left(\frac{A(k, t) - c_k N_t}{e^{-\frac{\alpha}{2} t}} \right)^4 \right],$$

is bounded.

Due to technicality, the proof of this lemma is postponed to the end in appendix.

7.1.2 Arbitrary initial distribution case

The following Lemmas are the counter part of Lemmas 6.5, 6.6, and 6.7. They play the same role as in the proof of Theorem 3.1. In the sequel, we denote by $(A(k, t, \Xi))_{k \geq 1}$, the frequency spectrum of the splitting tree where the lifetime of the ancestral individual is Ξ , in the same manner as for $N_t(\Xi)$ in the previous section.

Lemma 7.6 (*L^2 convergence in the general case*). *Consider the general frequency spectrum $(A(k, t, \Xi))_{k \geq 1}$, then, for all k , $\psi'(\alpha)e^{-\alpha t}A(k, t, \Xi)$ converge to $\mathcal{E}(\Xi)$ (see 6.9) in L^2 as t goes to infinity and*

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E} \left[(\psi'(\alpha)A(k, t, \Xi) - e^{\alpha t} \mathcal{E}(\Xi)c_k) (\psi'(\alpha)A(l, t, \Xi) - e^{\alpha t} \mathcal{E}(\Xi)c_l) \right] = a_{k,l} \int_{\mathbb{R}_+} e^{-\alpha u} \mathbb{P}(\Xi > u) b du,$$

where the convergence is uniform w.r.t. the random variable Ξ . In the case where Ξ is distributed as $O_2^{(\beta t)}$, for $0 < \beta < \frac{1}{2}$ (see section 4), we get

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E} \left[\left(\psi'(\alpha)A(k, t, O_2^{(\beta t)}) - e^{\alpha t} \mathcal{E}(O_2^{(\beta t)})c_k \right) \left(\psi'(\alpha)A(l, t, O_2^{(\beta t)}) - e^{\alpha t} \mathcal{E}(O_2^{(\beta t)})c_l \right) \right] = \frac{b\psi'(\alpha)}{\alpha} a_{k,l}.$$

Lemma 7.7 (First moment). *The first moments are asymptotically bounded, that is, for all $k \geq 1$,*

$$\mathbb{E} (\psi'(\alpha)A(k, t)(\Xi) - e^{\alpha t} c_k \mathcal{E}(\Xi)) \leq \mathcal{O}(1),$$

uniformly with respect to the random variable Ξ .

Lemma 7.8 (Boundedness in the general case.). *Let k_1, k_2, k_3 three positive integers, then*

$$\mathbb{E} \left[\left| \prod_{i=1}^3 \frac{(\psi'(\alpha)A(k_i, t) - e^{\alpha t} \mathcal{E} c_{k_i})}{e^{\frac{\alpha}{2}t}} \right| \right] = \mathcal{O}(1),$$

uniformly with respect to the random variable Ξ .

We do not detail the proofs of these results since they are direct adaptations of the proofs of Lemmas 6.5, 6.6 and 6.7.

7.2 Proof of the result

The following result is based on the fact that, in the clonal sub-critical case, the lifetime of a family is expected to be small. It follows that, in the decomposition of Figure 6, one can expect that all the family of size k live in different subtrees as soon as $t \gg u$. This is the point of the following lemma.

Lemma 7.9. *Suppose that $\alpha < \theta$. If we denote by $\Gamma_{u,t}$ the event,*

$$\Gamma_{u,t} = \{ \text{"there is no family in the population at time } t \text{ which is older than } u \},$$

then, for all β in $(0, 1 - \frac{\alpha}{\theta})$, we have

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\beta t}(\Gamma_{\beta t, t}) = 1.$$

Proof. The proof of this Lemma, as the calculation of the moments of $A(k, t)$ relies on the representation of the genealogy of the living population at time t as a coalescent point process [5]. Moreover, we denote by $\tilde{N}_u^{(t)}$ the number of living individuals at time u who have alive descent at time t . In [5], we showed that, under \mathbb{P}_t , $\tilde{N}_u^{(t)}$ is geometrically distributed with parameter $\frac{W(t-u)}{W(t)}$.

Now, $\mathbb{1}_{\Gamma_{u,t}}$ can be rewritten as

$$\mathbb{1}_{\Gamma_{u,t}} = \prod_{i=1}^{\tilde{N}_u^{(t)}} \mathbb{1}_{\{Z_0^i(t-u)=0\}},$$

where $Z_0^i(t-u)$ denotes the number of individuals descending from the i th individual alive at time u having alive descent at time t and carrying the same type as their ancestor at time u . Moreover, from Proposition 4.3 of [5], we know that that under \mathbb{P}_t , the family $Z_0^{(i)}(t-u)$ is an i.i.d. family of random variables distributed as $Z_0(t-u)$ under \mathbb{P}_{t-u} , and $\tilde{N}_u^{(t)}$ is independent of $Z_0^{(i)}(t-u)$ (still under \mathbb{P}_t).

Then,

$$\mathbb{P}_t(\Gamma_{t,u}) = \mathbb{E}_t \left[\mathbb{P}_{t-u}(Z_0(t-u) = 0)^{\tilde{N}_u^{(t)}} \right] = \frac{\mathbb{P}_{t-u}(Z_0(t-u) = 0) \frac{W(t-u)}{W(t)}}{1 - \mathbb{P}_{t-u}(Z_0(t-u) = 0) \left(1 - \frac{W(t-u)}{W(t)}\right)}.$$

Using (2.9), some calculus leads to,

$$\mathbb{P}_t(\Gamma_{t,u}) = 1 - \frac{1}{1 + \frac{W_\theta(t-u)}{e^{-\theta(t-u)}W(t)} \left(1 - \frac{e^{-\theta(t-u)}W(t-u)}{W_\theta(t-u)}\right)}.$$

Now, since,

$$\mathbb{P}_t(\Gamma_{t,u}) = \mathbb{P}_u(\Gamma_{t,u}) \frac{\mathbb{P}(N_u > 0)}{\mathbb{P}(N_t > 0)} + \frac{\mathbb{P}(\Gamma_{t,u}, N_t = 0, N_u > 0)}{\mathbb{P}(N_t > 0)},$$

taking $u = \beta t$, we obtain, using Lemma 2.2 and

$$\mathbb{P}(N_t = 0, N_{\beta t} > 0) = \mathbb{P}(N_{\beta t} > 0) - \mathbb{P}(N_t > 0) \xrightarrow[t \rightarrow \infty]{} 0,$$

the desired result. \square

Proof of Theorem 3.2. Fix $0 < u < t$. Note that the event $\Gamma_{u,t}$ of Lemma 7.9 can be rewritten as

$$\mathbb{1}_{\Gamma_{u,t}} = \prod_{i=1}^{N_u} \mathbb{1}_{\{Z_0^i(t-u, O_i)=0\}}, \quad (7.10)$$

where $Z_0^i(t-u, O_i)$ denote the number of individuals alive at time t carrying the same type as the i th alive individual at time u , that is the ancestral family of the splitting constructed from the residual lifetime of the i th individual (see Section 4).

Let K be a multi-integer, we denote by $\mathcal{L}^{(K)}$ (resp. $A(K, t)$) the random vector $(\mathcal{L}^{k_1}, \dots, \mathcal{L}^{k_N})$ (resp. $(A(k_1, t), \dots, A(k_N, t))$) with

$$\mathcal{L}_t^{k_i} = \frac{\psi'(\alpha)A(k, t) - c_k e^{\alpha t} \mathcal{E}}{e^{\frac{\alpha}{2}t}}.$$

On the event $\Gamma_{u,t}$, we have a.s.,

$$A(k_l, t) = \sum_{i=1}^{N_u} A^{(i)}(k_l, t - u, O_i), \quad \forall l = 1, \dots, N,$$

where the family $(A^{(i)}(k_l, t - u, O_i))_{i \geq 1}$ stand for the frequency spectrum for each subtree, which are independent from Lemma 4.1 (see also Section 4 and Figure 6). Hence, using Lemma 6.8,

$$\mathcal{L}_t^{k_l} = \sum_{i=1}^{N_u} \frac{\psi'(\alpha) A^{(i)}(k_l, t - u, O_i) - e^{\alpha(t-u)} \mathcal{E}_i(O_i) c_{k_l}}{e^{\frac{\alpha}{2}u} e^{\frac{\alpha}{2}(t-u)}}.$$

By Lemma 4.1, that the family $(A^{(i)}(k_l, t - u, O_i))_{2 \leq i \leq N_u}$ is i.i.d. under \mathbb{P}_u .

In the sequel, we denote, for all l and $i \geq 1$,

$$\tilde{A}^{(i)}(k_l, t - u, O_i) = \frac{\psi'(\alpha) A^{(i)}(k_l, t - u, O_i) - e^{\alpha(t-u)} \mathcal{E}_i(O_i) c_{k_l}}{e^{\frac{\alpha}{2}(t-u)}}.$$

As in the proof of Theorem 3.1, let

$$\begin{aligned} \varphi_K(\xi) &:= \mathbb{E} \left[\exp \left(i < \tilde{A}(K, t - u, O_2), \xi > \right) \mathbb{1}_{Z_0^2(t-u, O_2)=0} \right], \\ \tilde{\varphi}_K(\xi) &:= \mathbb{E} \left[\exp \left(i < \tilde{A}(K, t - u, O_1), \xi > \right) \mathbb{1}_{Z_0^1(t-u, O_1)=0} \right]. \end{aligned}$$

From this point, following closely the proof of Theorem 3.1, with β in $(0, \frac{1}{2} \wedge (1 - \frac{\alpha}{\theta}))$, the only difficulty is to handle the indicator function $\mathbb{1}_{Z_0(t-u, O_i) > 0}$ in the Taylor development of φ_K . We show how it can be done for one of the second order terms, and leave the rest of the details to the reader.

It follows from Hölder's inequality that

$$\begin{aligned} &\mathbb{E} \left[\left(\frac{\psi'(\alpha) A^{(i)}(k_l, (1-\beta)t, O_i) - e^{\alpha((1-\beta)t)} \mathcal{E}_i(O_i) c_{k_l}}{e^{\frac{\alpha}{2}((1-\beta)t)}} \right)^2 \mathbb{1}_{Z_0^2((1-\beta)t, O_2) > 0} \right] \\ &\leq \mathbb{E} \left[\left(\frac{\psi'(\alpha) A^{(i)}(k_l, (1-\beta)t, O_i) - e^{\alpha(1-\beta)t} \mathcal{E}_i(O_i) c_{k_l}}{e^{\frac{\alpha}{2}(1-\beta)t}} \right)^3 \right]^{\frac{2}{3}} \mathbb{P}(Z_0^2((1-\beta)t, O_2) > 0)^{\frac{1}{3}}, \quad (7.11) \end{aligned}$$

from which it follows, using Lemma 7.8, that the l.h.s. of this last inequality is $\mathcal{O}(\mathbb{P}(Z_0^2(t-u, O_2) > 0)^{\frac{1}{3}})$. Now, using (7.10) and Lemma 7.9, it is easily seen that

$$\lim_{t \rightarrow \infty} \mathbb{P}(Z_0^2((1-\beta)t, O_2) > 0) = 0.$$

Finally, using Lemma 7.3, we get

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\left(\frac{\psi'(\alpha) A^{(i)}(k_l, t - u, O_i) - e^{\alpha(t-u)} \mathcal{E}_i(O_i) c_{k_l}}{e^{\frac{\alpha}{2}(t-u)}} \right)^2 \mathbb{1}_{Z_0^2(t-u, O_2)=0} \right] = \frac{b\psi'(\alpha) a_{k,k}}{\alpha}.$$

These allow us to conclude that

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\beta t} \left[e^{i \langle \mathcal{L}_t^{(K)}, \xi \rangle} \mathbb{1}_{\Gamma_t} \right] = \frac{1}{1 + \sum_{i,j=1}^N \mathcal{M}_{i,j} \xi_i \xi_j},$$

where $K_{i,j}$ is given by

$$\mathcal{M}_{i,j} := \frac{b\psi'(\alpha)}{\alpha} a_{K_i, K_j},$$

with K is the multi-integer (k_1, \dots, k_N) , and the $a_{l,k}$ s are defined in Lemma 7.3.

To end the proof, note that,

$$\left| \mathbb{E}_{\infty} \left[e^{i \langle \mathcal{L}_t^{(K)}, \xi \rangle} \right] - \mathbb{E}_{\beta t} \left[e^{i \langle \mathcal{L}_t^{(K)}, \xi \rangle} \mathbb{1}_{\Gamma_{\beta t, t}} \right] \right| \leq \mathbb{E} \left[\left| \frac{\mathbb{1}_{\text{NonEx}}}{\mathbb{P}(\text{NonEx})} - \frac{\mathbb{1}_{N_{\beta t} > 0} \mathbb{1}_{\Gamma_{\beta t, t}}}{\mathbb{P}(N_{\beta t} > 0)} \right| \right] \xrightarrow{t \rightarrow \infty} 0,$$

thanks to Lemma 7.9. □

8 Proof of Theorem 3.6

Since all the ideas of the proof of this theorem have been developed in the last two section, we do not detail all the proof. The only step which needs clarification is the computation of the covariance matrix of the Laplace limit law \mathcal{M} . According to the proof of Theorem 3.2, it is given by

$$\begin{aligned} \mathcal{M}_{i,j} := & \lim_{t \rightarrow \infty} \frac{W(\beta t)}{e^{\alpha \beta t}} \mathbb{E} \left[\left(\frac{\psi'(\alpha) A^{(i)}(k_i, (1-\beta)t, O_i) - \psi'(\alpha) c_{k_i} N_{(1-\beta)t}}{e^{\frac{\alpha}{2}((1-\beta)t)}} \right) \right. \\ & \left. \times \left(\frac{\psi'(\alpha) A^{(i)}(k_j, (1-\beta)t, O_i) - c_{k_j} N_{(1-\beta)t}}{e^{\frac{\alpha}{2}((1-\beta)t)}} \right) \mathbb{1}_{Z_0^2((1-\beta)t, O_2) > 0} \right], \end{aligned}$$

which is equal, thanks to (7.11) and an easy adaptation of Lemma 6.6, to

$$\mathcal{M}_{i,j} = \lim_{t \rightarrow \infty} \frac{b\psi'(\alpha)}{\alpha} \frac{W(\beta t)}{e^{\alpha \beta t}} e^{\alpha t} \mathbb{E} \left[(e^{-\alpha t} A(k_i, t) - c_{k_i} e^{-\alpha t} N_t) (e^{-\alpha t} A(k_j, t) - c_{k_j} e^{-\alpha t} N_t) \right].$$

So it remains to get the limit of

$$e^{\alpha t} \mathbb{E} \left[(e^{-\alpha t} \psi'(\alpha) A(k, t) - \psi'(\alpha) c_k e^{-\alpha t} N_t) (e^{-\alpha t} \psi'(\alpha) A(l, t) - c_l e^{-\alpha t} \psi'(\alpha) N_t) \right],$$

as t goes to infinity. We recall that using the calculus made in the proof of Theorem 6.3 of [5], we have

$$\mathbb{E}_t A(k, t) N_t = 2W(t)^2 c_k(t) - 2W(t) \int_{[0,t]} \theta \mathbb{P}_a(Z_0(a) = k) da + W(t) \int_{[0,t]} \theta W(a)^{-1} \mathbb{E}_a [N_a \mathbb{1}_{Z_0(a)=k}] da. \quad (8.1)$$

Moreover, (7.9) entails

$$\psi'(\alpha)^2 \mathbb{E}_t A(k, t) A(l, t) = 2W(t)^2 c_k(t) c_l(t) + RW(t) + o(e^{-\alpha t}),$$

with

$$\begin{aligned}
R := & -\psi'(\alpha) \int_0^\infty 2\theta W(a)^{-1} \mathbb{P}_a(Z_0(a) = k) \mathbb{E}_a[A(l, a)] da \\
& + \psi'(\alpha) \int_0^\infty 2\theta W(a)^{-1} \mathbb{P}_a(Z_0(a) = l) \mathbb{E}_a[A(k, a)] da \\
& + \psi'(\alpha) \int_0^\infty \theta W(a)^{-1} (\mathbb{E}_t[A(k, t) \mathbb{1}_{Z_0(a)=l}] + \mathbb{E}_t[A(l, t) \mathbb{1}_{Z_0(a)=k}]) da.
\end{aligned}$$

These identities allow us to obtain

$$\begin{aligned}
\mathbb{E}_t[(A(k, t) - c_k N_t)(A(l, t) - c_l N_t)] &= 2W(t)^2 c_k(t) c_l(t) + e^{-\alpha t} R + o(e^{-\alpha t}), \\
-2c_l c_k(t) W(t)^2 + 2c_l W(t) \int_{[0, t]} \theta \mathbb{P}_a(Z_0(a) = k) da - c_l W(t) \int_{[0, t]} \theta W(a)^{-1} \mathbb{E}_a[N_a \mathbb{1}_{Z_0(a)=k}] da \\
-2c_k c_l(t) W(t)^2 + 2c_l W(t) \int_{[0, t]} \theta \mathbb{P}_a(Z_0(a) = l) da - c_k W(t) \int_{[0, t]} \theta W(a)^{-1} \mathbb{E}_a[N_a \mathbb{1}_{Z_0(a)=l}] da \\
+ c_k c_l W(t)^2 \left(2 - \frac{1}{W(t)}\right) \\
= 2W(t)^2 (c_k(t) - c_l)(c_l(t) - c_k) + e^{-\alpha t} \frac{R}{\psi'(\alpha)} + o(e^{-\alpha t}), \\
+ 2c_l W(t) \int_{[0, t]} \theta \mathbb{P}_a(Z_0(a) = k) da - c_l W(t) \int_{[0, t]} \theta W(a)^{-1} \mathbb{E}_a[N_a \mathbb{1}_{Z_0(a)=k}] da \\
+ 2c_l W(t) \int_{[0, t]} \theta \mathbb{P}_a(Z_0(a) = l) da - c_k W(t) \int_{[0, t]} \theta W(a)^{-1} \mathbb{E}_a[N_a \mathbb{1}_{Z_0(a)=l}] da \\
- c_k c_l W(t).
\end{aligned}$$

Taking the limit as t goes to infinity leads to

$$\begin{aligned}
M_{k,l} := \lim_{t \rightarrow \infty} \psi'(\alpha)^2 e^{-\alpha t} \mathbb{E}_t[(A(k, t) - c_k N_t)(A(l, t) - c_l N_t)] &= R \\
+ 2\psi'(\alpha) c_l \int_{[0, \infty]} \theta \mathbb{P}_a(Z_0(a) = k) da - \psi'(\alpha) c_l \int_{[0, \infty]} \theta W(a)^{-1} \mathbb{E}_a[N_a \mathbb{1}_{Z_0(a)=k}] da \\
+ 2\psi'(\alpha) c_l \int_{[0, \infty]} \theta \mathbb{P}_a(Z_0(a) = l) da - \psi'(\alpha) c_k \int_{[0, \infty]} \theta W(a)^{-1} \mathbb{E}_a[N_a \mathbb{1}_{Z_0(a)=l}] da \\
- \psi'(\alpha) c_k c_l. \tag{8.2}
\end{aligned}$$

Finally, since $\mathbb{P}(N_t > 0) \sim \frac{\alpha}{\theta}$,

$$\mathcal{M}_{i,j} = M_{k_i, k_j}.$$

9 Markovian cases

Theorem 3.1 for the Markovian case is already well known (see [2]), however the allelic partition for such model has not been studied. We can get more information on the unknown covariance matrix K in the case where the life duration distribution is exponential. Our study also cover the case

$\mathbb{P}_V = \delta_\infty$ (Yule case), although it does not fit the conditions required by the Theorem 3.2. The reason comes from our method of calculation for $\mathbb{E}[A(k, t)\mathcal{E}]$. Let us consider the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, where \mathcal{F}_t is the σ -field generated by the tree truncated above t and the restriction of the mutation measure on $[0, t)$.

Then N_t is Markovian with respect to \mathcal{F}_t and for all positive real numbers $t \leq s$,

$$\mathbb{E}[A(k, t)N_s | \mathcal{F}_t] = A(k, t)N_t \mathbb{E}[N_{s-t}].$$

So that,

$$\mathbb{E}[A(k, t)N_s] = \mathbb{E}[A(k, t)N_t] (W(s-t) - \mathbb{P}_V \star W(s-t)).$$

By making a renormalization by $e^{-\alpha s}$ and taking the limit as s goes to infinity, we get,

$$\mathbb{E}[A(k, t)\mathcal{E}] = \psi'(\alpha)e^{-\alpha t} \mathbb{E}[A(k, t)N_t],$$

since, in the Markovian case, it is known from [8] that

$$\frac{\alpha}{b} = \psi'(\alpha).$$

Suppose first that $d > 0$. It follows that,

$$\begin{aligned} \mathbb{E}[(\psi'(\alpha)A(k, t) - e^{\alpha t}c_k\mathcal{E})(\psi'(\alpha)A(l, t) - e^{\alpha t}c_l\mathcal{E})] &= \psi'(\alpha)^2 \mathbb{E}_t[A(k, t)A(l, t)] \mathbb{P}(N_t > 0) \\ &\quad - c_k\psi'(\alpha)^2 \mathbb{E}_t[A(l, t)N_t] \mathbb{P}(N_t > 0) - c_l\psi'(\alpha)^2 \mathbb{E}_t[A(k, t)N_t] \mathbb{P}(N_t > 0) \\ &\quad + 2\psi'(\alpha)e^{2\alpha t}c_kc_l \end{aligned}$$

By (6.5),

$$\mathbb{P}(N_t > 0) = \psi'(\alpha) + \psi'(\alpha)^2 \mu e^{-\alpha t} + o(e^{-\alpha t}),$$

so

$$\begin{aligned} &\mathbb{E}[(\psi'(\alpha)A(k, t) - e^{\alpha t}c_k\mathcal{E})(\psi'(\alpha)A(l, t) - e^{\alpha t}c_l\mathcal{E})] \\ &= \mathbb{P}(N_t > 0) \psi'(\alpha)^2 \mathbb{E}_t[(A(k, t) - c_kN_t)(A(l, t) - c_lN_t)] + c_kc_l\psi'(\alpha)(2e^{2\alpha t} - \psi'(\alpha)\mathbb{E}_t[N_t^2]) \mathbb{P}(N_t > 0). \end{aligned}$$

Finally, since, using Proposition 5.1,

$$\lim_{t \rightarrow \infty} e^{-\alpha t} (2e^{2\alpha t} - \psi'(\alpha)\mathbb{E}_t[N_t^2]) \mathbb{P}(N_t > 0) = \psi'(\alpha)(1 - 6\mu),$$

it follows from (8.2),

$$\begin{aligned} &\lim_{t \rightarrow \infty} \mathbb{E}[(\psi'(\alpha)A(k, t) - e^{\alpha t}c_k\mathcal{E})(\psi'(\alpha)A(l, t) - e^{\alpha t}c_l\mathcal{E})] \\ &= \psi'(\alpha)M_{k,l} + c_kc_l\psi'(\alpha)^2(1 - 6\mu) = \psi'(\alpha)M_{k,l} + c_kc_l\psi'(\alpha)^2 \left(1 - 6\frac{d}{\alpha}\right), \end{aligned}$$

using that $\mu = \frac{1}{b\mathbb{E}V-1}$. In the Yule case, an easy adaptation of the preceding proof leads to

$$\lim_{t \rightarrow \infty} \mathbb{E}[(\psi'(\alpha)A(k, t) - e^{\alpha t}c_k\mathcal{E})(\psi'(\alpha)A(l, t) - e^{\alpha t}c_l\mathcal{E})] = M_{k,l} + c_kc_l.$$

A Formula for the fourth moment of the error

Lemma A.1.

$$\begin{aligned}
\mathbb{E}_t \left[(A(k, t) - c_k N_t)^4 \right] &= 4 \int_{[0, t]} \theta \frac{W(t)}{W(a)} \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} (A(k, a) - c_k N_a)^3 \right] da \\
&+ 48 \int_{[0, t]} \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} N_a A(k, a) \right] \mathbb{E}_a \left[(c_k N_a - A(k, a)) \right] da \\
&+ 24 \int_{[0, t]} \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} N_a^2 \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a) \right] da \\
&+ 24 \int_{[0, t]} \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} A(k, a)^2 \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a) \right] da \\
&+ 8 \int_{[0, t]} \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^3 \right] da \\
&+ 48 \int_{[0, t]} \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} A(k, a) \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a)^2 \right] da \\
&+ 72 \int_{[0, t]} \theta \frac{W(t)^3}{W(a)^3} \left(1 - \frac{W(a)}{W(t)} \right)^2 \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} (A(k, a) - c_k N_a) \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a)^2 \right] da \\
&+ 72 \int_{[0, t]} \theta \frac{W(t)^3}{W(a)^3} \left(1 - \frac{W(a)}{W(t)} \right)^2 \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^2 \right] \mathbb{E}_a \left[A(k, a) - N_a c_k \right] da \\
&+ 96 \int_{[0, t]} \theta \frac{W(t)^4}{W(a)^4} \left(1 - \frac{W(a)}{W(t)} \right)^3 \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^3 \right] da + c_k^4 \mathbb{E}_t N_t^4
\end{aligned}$$

Proof. The proof of this Lemma lies on the calculation of the expectation of each term in the development of

$$(A(k, t) - c_k N_t)^4.$$

To make this, we intensively use the relation (2.11) and the method developed in [5]. We begin by computing

$$\mathbb{E}_t \left[A(k, t)^4 \right].$$

Formula (2.11) gives us,

$$\begin{aligned}
A(k, t)^4 &= 4 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k_i} \sum_{u_{1:3}=1}^{N_{t-a}^{(t)}} \prod_{\substack{j=1 \\ i \neq j}}^3 A^{(u_j)}(k, a) \mathcal{N}(da, di) \\
&= 4 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} A^i(k, a) A^i(k, a) A^i(k, a) \mathcal{N}(da, di) \\
&\quad + 4 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} \sum_{\substack{j_1, j_2, j_3=1 \\ j_1 \neq j_2 \neq j_3 \neq i}}^{N_{t-a}^{(t)}} A^{j_1}(k, a) A^{j_2}(k, a) A^{j_3}(k, a) \mathcal{N}(da, di) \\
&\quad + 12 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} A^i(k, a) A^i(k, a) \sum_{j=1, j \neq i}^{N_{t-a}^{(t)}} A^j(k, a) \mathcal{N}(da, di) \\
&\quad + 4 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} \sum_{j=1, j \neq i}^{N_{t-a}^{(t)}} A^j(k, a)^3 \mathcal{N}(da, di) \\
&\quad + 12 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} A^i(k, a) \sum_{j_1, j_2=1, j_1 \neq j_2 \neq i}^{N_{t-a}^{(t)}} A^{j_1}(k, a) A^{j_2}(k, a) \mathcal{N}(da, di) \\
&\quad + 24 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} A^i(k, a) \sum_{j_1=1, j_1 \neq i}^{N_{t-a}^{(t)}} A^{j_1}(k, a) A^{j_1}(k, a) \mathcal{N}(da, di) \\
&\quad + 12 \int_{[0, t] \times \mathbb{N}} \mathbb{1}_{Z_0^i(a)=k} \sum_{j_1, j_2=1, j_1 \neq j_2 \neq i}^{N_{t-a}^{(t)}} A^{j_1}(k, a)^2 A^{j_2}(k, a) \mathcal{N}(da, di). \tag{A.1}
\end{aligned}$$

The decomposition of the sum in form

$$\sum_{u_{1:3}=1}^{N_{t-a}^{(t)}},$$

has then been made to distinguish independence properties in our calculation. Actually, as soon as, $i \neq j$, $A^i(k, a)$ is independent from $A^j(k, a)$ (see [5] for details). It is essential to note that the expectation of these integrals with respect to the random measure \mathcal{N} are all calculated thanks to

Theorem 3.1 of [5]. So, taking the expectation now leads to,

$$\begin{aligned}
\mathbb{E}_t [A(k, t)^4] &= 4 \int_{[0, t]} \theta \mathbb{E}_a [N_{t-a}^{(t)}] \mathbb{E}_a [\mathbb{1}_{Z_0(a)=k} A(k, a)^3] \theta da \\
&+ 4 \int_{[0, t]} \theta \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[\binom{N_{t-a}^{(t)}}{(4)} \right] \mathbb{E}_a [A(k, a)]^3 da \\
&+ 12 \int_{[0, t]} \theta \mathbb{E}_a [\mathbb{1}_{Z_0(a)=k} A(k, a)^2] \mathbb{E}_a \left[\binom{N_{t-a}^{(t)}}{(2)} \right] \mathbb{E}_a [A(k, a)] da \\
&+ 4 \int_{[0, t]} \theta \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[\binom{N_{t-a}^{(t)}}{(2)} \right] \mathbb{E}_a [A(k, a)^3] da \\
&+ 12 \int_{[0, t]} \theta \mathbb{E}_a [\mathbb{1}_{Z_0(a)=k} A(k, a)] \mathbb{E}_a \left[\binom{N_{t-a}^{(t)}}{(3)} \right] \mathbb{E}_a [A(k, a)]^2 da \\
&+ 24 \int_{[0, t]} \theta \mathbb{E}_a [\mathbb{1}_{Z_0(a)=k} A(k, a)] \mathbb{E}_a \left[\binom{N_{t-a}^{(t)}}{(2)} \right] \mathbb{E}_a [A(k, a)^2] da \\
&+ 12 \int_{[0, t]} \theta \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[\binom{N_{t-a}^{(t)}}{(3)} \right] \mathbb{E}_a [A(k, a)^2] \mathbb{E}_a [A(k, a)] da.
\end{aligned}$$

Using the same method for all the other terms and that, for any positive real number a lower than t ,

$$N_t = \sum_{i=1}^{N_{t-a}^{(t)}} N_a^{(i)},$$

we get Lemma A.1 by reassembling similar terms together. The last term is obtained using the geometric distribution of N_t under \mathbb{P}_t . \square

B Boundedness of the fourth moment

Lemma B.1. *We begin the proof of the boundedness of the fourth moment by some estimates.*

$$\mathbb{E}_t [(A(k, t) - c_k N_t)] = \mathcal{O} \left(e^{-(\theta-\alpha)t} \right), \quad (\text{i})$$

$$\mathbb{E}_t [(A(k, t) - c_k N_t)^3] = \mathcal{O} (W(t)^2), \quad (\text{ii})$$

$$\mathbb{E}_t [(A(k, t) - c_k N_t)^2] = \mathcal{O} (W(t)), \quad (\text{iii})$$

$$\mathbb{E}_t N_t^n = \mathcal{O}(e^{n\alpha t}), \quad n \in \mathbb{N}^*, \quad (\text{iv})$$

$$\mathbb{P}_t (Z_0(t) = k) = \mathcal{O}(e^{(\alpha-\theta)t}). \quad (\text{v})$$

Proof. Relation (i) is easily obtained using the expectation of N_t and $A(k, t)$ using (2.12), (2.14) and the behaviour of W provided by Proposition 5.1. The relation (iii) has been obtained in the proof

of Theorem 6.1 in [5]. The two last relations are easily obtained from (2.5), (2.9) and Lemma 2.2. The relation (ii) is obtained using the following estimation,

$$\left| \mathbb{E}_t \left[(A(k, a) - c_k N_a)^3 \right] \right| \leq \mathbb{E}_t \left[N_a (A(k, a) - c_k N_a)^2 \right].$$

We begin the proof by computing the r.h.s. of the previous inequality using the same techniques as in Appendix A.

$$\begin{aligned} \mathbb{E} [A(k, t)^2 N_t] &= 2 \int_0^t \theta \frac{W(t)}{W(a)} \mathbb{E} [N_a A(k, a) \mathbf{1}_{Z_0(a)=k}] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E} [N_a \mathbf{1}_{Z_0(a)=k}] \mathbb{E} [A(k, a)] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E} [A(k, a) \mathbf{1}_{Z_0(a)=k}] \mathbb{E} [N_a] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{P}_a (Z_0(a) = k) \mathbb{E} [A(k, a) N_a] da \\ &+ 12 \int_0^t \theta \frac{W(t)^3}{W(a)^3} \left(1 - \frac{W(a)}{W(t)} \right)^2 \mathbb{P}_a (Z_0(a) = k) \mathbb{E} [A(k, a)] \mathbb{E} [N_a] da. \end{aligned}$$

$$\begin{aligned} 2\mathbb{E} [A(k, t) N_t^2] &= 2 \int_0^t \theta \frac{W(t)}{W(a)} \mathbb{E} [N_a^2 \mathbf{1}_{Z_0(a)=k}] da \\ &+ 8 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E} [N_a \mathbf{1}_{Z_0(a)=k}] \mathbb{E} [N_a] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{P}_a (Z_0(a) = k) \mathbb{E} [N_a^2] da \\ &+ 12 \int_0^t \theta \frac{W(t)^3}{W(a)^3} \left(1 - \frac{W(a)}{W(t)} \right)^2 \mathbb{P}_a (Z_0(a) = k) \mathbb{E} [N_a]^2 da. \end{aligned}$$

Finally,

$$\begin{aligned} \mathbb{E} [N_t (A(k, t) - c_k N_t)^2] &= 2 \int_0^t \theta \frac{W(t)}{W(a)} \mathbb{E} [N_a (A(k, a) - c_k N_a) \mathbf{1}_{Z_0(a)=k}] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E} [N_a \mathbf{1}_{Z_0(a)=k}] \mathbb{E} [A(k, a) - c_k N_a] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{E} [(A(k, a) - c_k N_a) \mathbf{1}_{Z_0(a)=k}] \mathbb{E} [N_a] da \\ &+ 4 \int_0^t \theta \frac{W(t)^2}{W(a)^2} \left(1 - \frac{W(a)}{W(t)} \right) \mathbb{P}_a (Z_0(a) = k) \mathbb{E} [N_a (A(k, a) - c_k N_a)] da \\ &+ 12 \int_0^t \theta \frac{W(t)^3}{W(a)^3} \left(1 - \frac{W(a)}{W(t)} \right)^2 \mathbb{P}_a (Z_0(a) = k) \mathbb{E} [N_a] \mathbb{E} [A(k, a) - c_k N_a] da \\ &+ c_k^2 \mathbb{E}_t N_t^3. \end{aligned}$$

Now, an analysis similar to the one of Lemma 7.5 leads to the result. \square

Proof of Lemma 7.5. The ideas of the proof, is to analyses one to one every terms of the expression of

$$\mathbb{E}_t \left[(A(k, t) - c_k N_t)^4 \right],$$

given by Lemma A.1 using Lemma B.1 to show that they behave as $\mathcal{O}(W(t)^2)$. Since the ideas are the same for every terms, we just give a few examples.

First of all, we consider

$$\int_{[0,t]} \frac{W(t)}{W(a)} \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} (A(k, a) - c_k N_a)^3 \right] da.$$

Using Lemma B.1 (ii), we have

$$\int_{[0,t]} \frac{W(t)}{W(a)} \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} (A(k, a) - c_k N_a)^3 \right] da = \mathcal{O}(W(t)^2).$$

Now take the term

$$\int_{[0,t]} \frac{W(t)^2}{W(a)^2} \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} N_a^2 \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a) \right] da,$$

we have from Lemma B.1 (i) and (iv),

$$\int_{[0,t]} \frac{W(t)^2}{W(a)^2} \mathbb{E}_a \left[\mathbf{1}_{Z_0(a)=k} N_a^2 \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a) \right] da \leq \int_{[0,t]} \frac{W(t)^2}{W(a)^2} \mathbb{E}_a \left[N_a^2 \right] e^{-(\theta-\alpha)a} da = \mathcal{O}(W(t)^2).$$

Every term in $W(t)$ or $W(t)^2$ are treated this way. Now, we consider the term in $W(t)^4$ which is

$$I := 96 \int_{[0,t]} \frac{W(t)^4}{W(a)^4} \mathbb{P}_a(Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^3 \right] da + 24W(t)^4 c_k^4,$$

since N_t is geometrically distributed under \mathbb{P}_t , and that

$$\mathbb{E}_t N_t^4 = 24W(t)^4 - 36W(t)^3 + \mathcal{O}(W(t)^2). \quad (\text{B.1})$$

On the other hand, using the law of $Z_0(t)$ given by (2.9) and the expectation of $A(k, t)$ given by (2.12) (under \mathbb{P}_t), we have,

$$\begin{aligned} & 96 \int_{[0,t]} \frac{W(t)^4}{W(a)^4} \mathbb{P}_a(Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^3 \right] da \\ &= -96W(t)^4 \int_0^t \frac{\theta e^{-\theta a}}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} \left(\int_0^a \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)} \right)^{k-1} ds \right)^3 da \\ &= -24W(t)^4 \left(\int_0^t \frac{\theta e^{-\theta a}}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} da \right)^4. \end{aligned}$$

Finally,

$$I = 24W(t)^4 \left(\int_t^\infty \frac{\theta e^{-\theta a}}{W_\theta(a)^2} \left(1 - \frac{1}{W_\theta(a)} \right)^{k-1} da \right)^4 = \mathcal{O} \left(W(t)^4 e^{-4\theta t} \right) = o(1).$$

The last example is the most technical and relies with the term in $W(t)^3$, which is, using (B.1) and Lemma A.1,

$$\begin{aligned} J := & 72 \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{E}_a \left[\mathbb{1}_{Z_0(a)=k} (A(k, a) - c_k N_a) \right] \mathbb{E}_a \left[(A(k, a) - c_k N_a)^2 \right] da \\ & + 72 \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^2 \right] \mathbb{E}_a [A(k, a) - N_a c_k] da \\ & - 288 \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^3 \right] da - 36c_k^4 W(t)^3. \end{aligned}$$

On the other hand, using the calculus made in the proof of Theorem 6.3 of [5], we have

$$\begin{aligned} & \mathbb{E}_a \left[(A(k, a) - c_k N_a)^2 \right] \\ & = 4 \int_{[0,a]} \frac{W(a)^2}{W(s)^2} \left(1 - \frac{W(s)}{W(a)} \right) \mathbb{P}_s (Z_0(s) = k) \mathbb{E}_a (A(k, s) - c_k N_s) ds \\ & \quad + 2 \int_{[0,a]} \frac{W(s)}{W(a)} \mathbb{E}_a \left[\mathbb{1}_{Z_0(s)=k} (A(k, s) - c_k N_s) \right] ds + c_k^2 W(a)^2 \left(2 - \frac{1}{W(a)} \right). \end{aligned}$$

Substituting this last expression in J leads to

$$\begin{aligned} J = & -144 \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{E}_a \left[\mathbb{1}_{Z_0(a)=k} (A(k, a) - c_k N_a) \right] \int_{[a,\infty]} \frac{\mathbb{P}(Z_0(a) = k)}{W(s)^2} \mathbb{E}_a \left[(A(k, s) - c_k N_s) \right] ds da \\ & + 144W(t)^3 \int_{[0,t]} \frac{1}{W(a)} \mathbb{E}_a \left[\mathbb{1}_{Z_0(a)=k} (A(k, a) - c_k N_a) \right] \int_{[a,t]} \frac{1}{W(s)^2} \mathbb{P}_s (Z_0(s) = k) \mathbb{E}_a [A(k, s) - N_s c_k] da \\ & - 144c_k^2 \int_{[0,t]} \frac{W(t)^3}{W(a)} \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a [A(k, a) - N_a c_k] da \\ & + 144 \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{P} (Z_0(a) = k) \mathbb{E}_a [A(k, a) - N_a c_k]^3 da \\ & - 288 \int_{[0,t]} \frac{W(t)^3}{W(a)^2} \mathbb{P}_a (Z_0(a) = k) \int_{[0,a]} \frac{1}{W(s)} \mathbb{P}_s (Z_0(s) = k) \mathbb{E}_a (A(k, s) - c_k N_s) ds \mathbb{E}_a [A(k, a) - N_a c_k] da \\ & + 72 \int_{[0,t]} \frac{W(t)^3}{W(a)} \mathbb{P}_a (Z_0(a) = k) c_k^2 \left(2 - \frac{1}{W(a)} \right) \mathbb{E}_a [A(k, a) - N_a c_k] da \\ & - 288 \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{P}_a (Z_0(a) = k) \mathbb{E}_a \left[(A(k, a) - c_k N_a)^3 \right] da - 36c_k^4 W(t)^3. \end{aligned}$$

Using many times that,

$$\begin{aligned}
& \int_{[0,t]} \frac{\theta \mathbb{P}(Z_0(a) = k)}{W(s)^2} \mathbb{E}_a [A(k, s) - c_k N_s] ds \\
&= - \int_{[0,t]} \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)}\right)^{k-1} \int_{[s,\infty]} \frac{\theta e^{-\theta u}}{W_\theta(u)^2} \left(1 - \frac{1}{W_\theta(u)}\right)^{k-1} dud s \\
&= \frac{c_k^2}{2} - \frac{1}{2} \left(\int_{[t,\infty]} \frac{\theta e^{-\theta s}}{W_\theta(s)^2} \left(1 - \frac{1}{W_\theta(s)}\right)^{k-1} ds \right)^2,
\end{aligned}$$

thanks to (2.9), (2.12), and (2.7), we finally get

$$\begin{aligned}
J &= -144 (c_k^2 - c_k(t)^2) \int_{[0,t]} \frac{W(t)^3}{W(a)^3} \mathbb{E}_a [\mathbf{1}_{Z_0(a)=k} (A(k, a) - c_k N_a)] da \\
&+ 36W(t)^3 \left(c_k^2 \left(\int_{[t,\infty]} \frac{W(t)^3}{W(a)^3} \mathbb{E}_a [A(k, a) - N_a c_k]^3 da \right)^2 - \left(\int_{[t,\infty]} \frac{W(t)^3}{W(a)^3} \mathbb{E}_a [A(k, a) - N_a c_k]^3 da \right)^4 \right) \\
&+ 144 (c_k - c_k(t))^2 \int_{[0,t]} \frac{W(t)^3}{W(a)} \mathbb{E}_a [A(k, a) - N_a c_k] da \\
&+ 36W(t)^3 (c_k - c_k(t))^4.
\end{aligned}$$

This shows that J is $\mathcal{O}(W(t)^2)$. □

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