HAL
open science

# COPULA'S APPROXIMATIONS: APPLICATION TO QUANTILE ESTIMATION

Andrés Cuberos, Esterina Masiello, Véronique Maume-Deschamps

**To cite this version:**

Andrés Cuberos, Esterina Masiello, Véronique Maume-Deschamps. COPULA'S APPROXIMATIONS: APPLICATION TO QUANTILE ESTIMATION. 2015. hal-01201838v1

**HAL Id: hal-01201838**

**https://hal.science/hal-01201838v1**

Preprint submitted on 18 Sep 2015 (v1), last revised 19 Oct 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# COPULA'S APPROXIMATIONS: APPLICATION TO QUANTILE ESTIMATION

A. CUBEROS, E. MASIELLO, AND V. MAUME-DESCHAMPS

Abstract. Estimating high level quantiles of aggregated variables (mainly sums or weighted sums) is crucial in risk management for many application fields such as finance, insurance, environment... This question has been widely treated but new efficient methods are always welcome; especially if they apply in (relatively) high dimension. We propose an estimation procedure based on the *checkerboard copula*. It allows to get good estimations from a (quite) small sample of the multivariate law and a full knowledge of the marginal laws. This situation is realistic for many applications. Estimations may be improved by including in the checkerboard copula some additional information (on the law of a sub-vector or on extreme probabilities). Our approach is illustrated by numerical examples.

## 1. Introduction

Consider a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ and a measurable function $\Psi : \mathbb{R}^d \to \mathbb{R}^+$, called the aggregation function. In the context of quantitative risk management $\mathbf{X}$ is known as a risk vector and generally represents the profit-losses of a portfolio at a given future date. $\Psi(\mathbf{X})$, the aggregated risk, represents its total future position. The main examples of aggregation functions are: the sum, max, weighted sums or a slightly more complex function that may include stop-loss reinsurance type function on each of the marginals. In this paper, we will be essentially concerned with $\Psi = \sum$, which is the most commonly studied aggregation function. We are interested here in the estimation of $p$-quantiles, $0 < p < 1$, of $\Psi(\mathbf{X})$: $Q_p(\Psi(\mathbf{X})) = \inf\{x \in \mathbb{R}, F_\Psi(x) \geq p\}$. To this purpose, we will assume that the distributions $F_1, \ldots, F_d$ of the marginal variables $X_1, \ldots, X_d$ are known and that some information on the dependence between them is given. Usually this information is available via some observations of the joint distribution and also via expert opinion.

In practice, neither the marginals nor the dependence of the risk vector $\mathbf{X}$ will be known. However, in many cases, the information available on the marginal distributions is much more important than the one on the dependence structure. For example, when some observations of the vector $\mathbf{X}$ are available, inferences one can do on the marginal distributions give better results than inferences one can do on the multivariate distribution. Also, samples available for marginal laws may be much larger than those available for the joint distribution. So, even if the assumption of the knowledge of

*Key words and phrases.* risk aggregation, empirical copulas, checkerboard copulas, quantile estimation.

marginal distributions may seem not realistic, there is, however, in practice much more knowledge on the marginal distributions than on the dependence structure of the random vector. These situations arise e.g. for environmental data, in insurance contexts...

When the marginals are known but the dependence is unknown, the re-arrangement algorithm (introduced in special cases in [20] and [19]) allows to obtain bounds on the distribution of $\Psi(\mathbf{X})$ ([18]) working well for $d \geq 30$. By improving the re-arrangement algorithm, bounds on the VaR are obtained in ([10]) in high dimensional ($d \geq 1000$) inhomogeneous portfolio. Cases in which some kind of dependence information is available lead to narrower bounds ([2, 3]) for the risk measure at hand. Bounds are also derived in ([4]) for dependence structures described by different copula models. A general mathematical framework which interpolates between marginal knowledge and full knowledge of the distribution function of $\mathbf{X}$ is considered in ([9]).

In this paper, we propose to use the checkerboard copula (introduced in [15]) to merge the information given by a small sample of the distribution of $\mathbf{X}$ with the known marginal distributions. Moreover, we introduce the checkerboard copula with information on the tail and with information on a sub-vector, to take into account some additional informations which may improve the quantile estimation (see Section 3.1). Some simulations are provided in Section 4.
We begin (see Section 2) with a brief discussion on the admissible multivariate distribution with fixed marginals and aggregated laws: in other words, given marginal laws and a distribution for $\Psi(\mathbf{X})$, what are the possible multivariate distributions for $\mathbf{X}$? Conclusions are provided in Section 5.

## 2. The invariant aggregation copula class

Let $\mathbf{F}$ be the distribution function of $\mathbf{X} = (X_1, \ldots, X_d)$. By Sklar's Theorem, there exists a copula distribution $C$ on $[0,1]^d$ such that

$$\mathbf{F}(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

When the marginal random variables of $\mathbf{X}$ are absolutely continuous this copula $C$ is unique. We will assume that the marginals of $\mathbf{X}$ are absolutely continuous. The aggregation function $\Psi : \mathbb{R}^d \to \mathbb{R}^+$ is considered to be measurable and non-decreasing on each variable. Let us denote by $F_\Psi(x) = \mathbb{P}(\Psi(\mathbf{X}) \leq x)$. Of course, the copula of the vector $\mathbf{X}$ determines the distribution of $\Psi(\mathbf{X})$. Nevertheless, the copula specification may be redundant, as for any copula $C$ there may exist an infinite set of copulas $\mathcal{C}_{\Psi,C}$ such that $\Psi(\mathbf{X}^C) \overset{\mathcal{L}}{=} \Psi(\mathbf{X}^{C'})$ for any $C' \in \mathcal{C}_{\Psi,C}$, where $\mathbf{X}^C$ denotes a random vector with same marginals as $\mathbf{X}$ with copula $C$.

The Fréchet class of the marginal distributions $F_1, \ldots, F_d$, denoted by $\mathfrak{F}_d(F_1, F_2, \ldots, F_d)$, consists of all $d$-multivariate distributions with $F_1, \ldots, F_d$ as marginals. This class is completely determined by the class of all $d$-copulas, i.e.:

$$\mathfrak{F}_d(F_1, \ldots, F_d) = \{F : F(x_1, \ldots, x_d) = C\left(F_1(x_1), \ldots, F_d(x_d)\right)\}.$$

Moreover, when the marginals are absolutely continuous there is a bijective correspondence between both classes.

In the context of risk aggregation the following more useful class has been introduced in [1].

**Definition 1.** An aggregate risk $S$ is called an admissible risk of marginal distributions $F_1, \ldots, F_d$ if it can be written as $S = \Psi(X_1, \ldots, X_d)$ where $X_i \sim F_i$ for $i = 1, \ldots, d$. The admissible risk class is defined by the set of admissible risks of given marginal distributions but unknown dependence structure:

$$\mathfrak{S}_d(F_1, \ldots, F_d, \Psi) = \{\Psi(X_1, \ldots, X_d) : X_i \sim F_i, i = 1, \ldots, d\}.$$

Some interesting properties of this class have been presented in [1] when $\Psi$ is the sum. Here we present a related class from the copula point of view.

**Definition 2.** Let $\mathbf{X}$ be a random vector and $\Psi$ an aggregation function. The class of copulas

$$\mathcal{C}(\mathbf{X}, \Psi) = \{C \in \mathcal{C} : \Psi(\mathbf{X}^C) \overset{\mathcal{L}}{=} \Psi(\mathbf{X})\}$$

is the invariant aggregation copula class of $\mathbf{X}$ and $\Psi$.

The invariant aggregation copula class is related to the set of admissible risks, in a similar way as the copulas are related to the Fréchet class:

$$F \in \mathfrak{F}_d(F_1, \ldots, F_d) \Leftrightarrow \exists C : F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$
$$S \in \mathfrak{S}_d(F_1, \ldots, F_d, \Psi) \Leftrightarrow \exists \mathcal{C}(\mathbf{X}, \Psi) : \forall C \in \mathcal{C}(\mathbf{X}, \Psi) \quad S \overset{\mathcal{L}}{=} \Psi(\mathbf{X}^C),$$

In what follows, we present an example and some results that show explicitly that this class is not trivial.

**Example of non trivial classes $\mathcal{C}(\mathbf{X}, \Psi)$.** For $\Psi = \sum$ and $\Psi = \max$, it is easy to prove that the classes $\mathcal{C}(\mathbf{X}, \Psi)$ are not trivial.

**Example 1.** We construct explicitly two different random vectors $(X, Y)$ and $(X', Y')$ such that $X \overset{\mathcal{L}}{=} X'$, $Y \overset{\mathcal{L}}{=} Y'$ and $X + Y \overset{\mathcal{L}}{=} X' + Y'$. Let $(X, Y)$ be any random vector in $\mathbb{R}^2$ with density $f$. Suppose that for some $\epsilon > 0$ and some $a < b$ and $c < d$ with $b - a = d - c$ we have that $f(x, y) > \epsilon$ for any $(x, y) \in [a, b] \times [c, d]$. The equality

$$\int_0^{2\pi} \int_0^{2\pi} \sin(x - y)\, dxdy = 0$$

implies that

$$g(x, y) = f(x, y) + \epsilon \sin\left(2\pi \frac{x - a}{b - a} - 2\pi \frac{y - c}{d - c}\right) I_{[a,b]\times[c,d]}(x, y)$$

is a density function. Moreover, as for any $t$ the following equations hold,

$$\int_0^t \int_0^{2\pi} \sin(x - y)\, dxdy = 0 \quad \text{and} \quad \int_0^{2\pi} \int_0^t \sin(x - y)\, dxdy = 0.$$

Thus, the marginal densities of $f$ and $g$ are identical. Furthermore, it can also be checked easily that

$$\int_0^{2\pi}\int_0^{2\pi} \sin(x-y) I_{\{0\le x+y\le t\}}(x,y)\,dxdy = 0$$

for any $t>0$, thus if $(X',Y')$ is a random vector with density $g$, it satisfies that $X'+Y' \overset{\mathcal{L}}{=} X+Y$.

The example above may be generalized in any dimension.

**Proposition 2.1.** *If $\boldsymbol{X}$ admits a density $f$ such that $f(x_1,x_2,\dots,x_d)>\epsilon$ for some $\epsilon>0$ and any $x_1,\dots,x_d$ then $\mathcal{C}(\boldsymbol{X},+)$ has infinite elements.*

*Proof.* Let $f$ be the density of $\mathbf{X}$, and choose $a<b$, $c<d$ such that $b-a=d-c$, then

$$\begin{aligned} g(x_1,x_2,\dots,x_d) &= f(x_1,x_2,\dots,x_d) + \\ &\quad \epsilon_0 \sin\left(2\pi\frac{x_1-a}{b-a} - 2\pi\frac{x_2-c}{d-c}\right) I_{[a,b]\times[c,d]}(x_1,x_2) \end{aligned}$$

is a density function for any $0<\epsilon_0<\epsilon$. If $\mathbf{X}'$ is a random vector with density $g$ then $\mathbf{X}'\in\mathcal{C}(\mathbf{X},+)$. As the same is true for any $0<\epsilon_0<\epsilon$, we have shown that there are infinite elements in $\mathcal{C}(\mathbf{X},+)$. □

By definition, any element of the class $\mathcal{C}(\mathbf{X},\Psi)$ characterizes $\Psi(\mathbf{X})$. The following result shows that in some cases we can always find a symmetrical copula in $\mathcal{C}(\mathbf{X},\Psi)$.

**Proposition 2.2.** *If $\boldsymbol{X}$ admits a density with identical marginals and $\Psi$ is a symmetrical aggregation function then there exists a symmetrical copula $C$ such that $\Psi(\boldsymbol{X}) \overset{\mathcal{L}}{=} \Psi(\boldsymbol{X}^C)$.*

*Proof.* Let $f(x_1,\dots,x_d)$ be the density of $\mathbf{X}$. Define $g(x_1,\dots,x_d)$ as

$$g(x_1,\dots,x_d) = \frac{1}{d!}\sum_{\sigma\in S_d} f(x_{\sigma(1)},\dots,x_{\sigma(d)}),$$

where $S_d$ is the set of all the permutations of $\{1\dots,d\}$. Let $\mathbf{X}'$ be a random vector with density $g$. Then it is easy to check that the marginals of $\mathbf{X}'$ are distributed as the marginals of $\mathbf{X}$. It follows equally, from the symmetry of $\Psi$, that $\Psi(\mathbf{X}) \overset{\mathcal{L}}{=} \Psi(\mathbf{X}')$. As the density of $\mathbf{X}'$ is completely symmetrical so is its copula. □

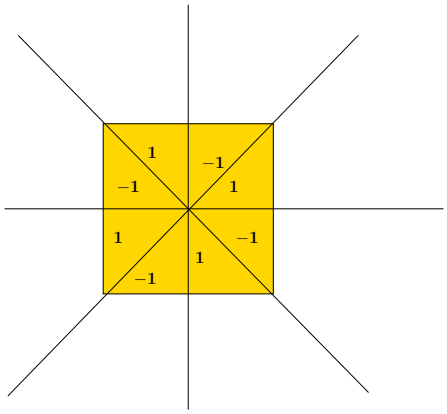
*Remark.* In the case of $d$ dimensional Archimedean copulas, it is known that the copula $C$ is uniquely determined by its diagonal $\delta$, $\delta(t)=C(t,\dots,t)$ if $\delta'(1-)=d$ (see [13, 21] in dimension 2 and [11] in higher dimension). This means that if $\Psi=\max$ or $\Psi=\min$, given a fixed common law for $X_1,\dots,X_d$ and a fixed law for $\Psi(\mathbf{X})$, then there is only one Archimedean copula which leaves $\Psi(\mathbf{X})$ and the marginal laws invariant. Nevertheless, using constructions in [16], infinitely many copulas with a fixed diagonal may be constructed.

Below we provide a construction of infinitely many laws of random vectors with a fixed law for their max and fixed marginal laws (remark that if the marginal laws are not the same, then the law of the max is not determined by the diagonal of the copula).

**Proposition 2.3.** *Assume that $\boldsymbol{X}$ is absolutely continuous with density $f$ such that $\inf_K f > 0$ with $K = \prod_{i=1}^{d}[a_i, b_i]$. If $K$ is symmetric with respect to the diagonal, then there exists a density function $\varphi$ such that $f \equiv \varphi$ outside $K$, $f \neq \varphi$ on $K$ and the random vector $\widetilde{\boldsymbol{X}}$ whose density function is $\varphi$ is such that*

- *for $i = 1, \ldots, d$, $\widetilde{X}_i \overset{\mathcal{L}}{=} X_i$,*
- $\max(\boldsymbol{X}) \overset{\mathcal{L}}{=} \max(\widetilde{\boldsymbol{X}})$.

*Proof.* We sketch the proof in dimension 2. Let $f > \varepsilon$ on $K$ and $\varphi = f + \varepsilon\gamma$ where $\gamma$ has its support in $K$ as shown below:




It is easy to verify that the random vector $\widetilde{\mathbf{X}}$ whose density is $\varphi$ has the same marginal laws as $\mathbf{X}$ and $\max(\widetilde{\mathbf{X}}) \overset{\mathcal{L}}{=} \max(\mathbf{X})$. □

Even if the example above seems trivial it shows that the full knowledge of the copula distribution is unnecessary when studying an aggregation: there is some redundant information.

## 3. Non-parametric estimation of the aggregation distribution when the marginals are known

We have seen in the above section that the exact copula estimation can be considered as a redundant exercise when estimating the distribution of an aggregation of $\mathbf{X}$. The information given by the copula of $\mathbf{X}$, for the study of $\Psi(\mathbf{X})$ is the same as any copula from the class $\mathcal{C}(\mathbf{X}, \Psi)$. This may be seen as a justification of the fact that when the marginals are known, there is some flexibility in the copula estimation in order to estimate the aggregated distribution.

In this section we propose a non-parametric estimator of the distribution of $\Psi(\mathbf{X})$ when marginals $F_1, \ldots, F_d$ are known and an independent and identically distributed (i.i.d.) sample $\mathbf{X}^1, \ldots, \mathbf{X}^n$ is given. The sample size $n$ is quite small. The estimation of $F_\Psi$ will allow us to obtain an estimation

of the $p$-quantile $Q_p(\Psi(\mathbf{X}))$. This estimation will be compared to the one obtained from the empirical cumulative distribution function

$$\widehat{F_\Psi}(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{\Psi(\mathbf{X}^i)\leq t\}}.$$

Before illustrating the estimation procedure (Section 3.3), we need to define three kinds of checkerboard copulas (Section 3.1) as well as an empirical version of each of them (Section 3.2).

3.1. **Checkerboard Copulas.** As in the above sections, let $F$ denote the cumulative distribution function (c.d.f.) of $\mathbf{X}$, $C$ its copula function and $F_i$ the c.d.f of $X_i$, $i = 1, \ldots, d$.
Let $\mu_C$ be the probability measure associated to $C$, i.e such that:

$$\mu_C\left(\prod_{i=1}^{d}[0, u_i]\right) = C(u_1, \ldots, u_d)$$

for any $u = (u_1, \ldots, u_d) \in [0, 1]^d$.
By a $\mu$-decomposition of a set $A \subset \mathbb{R}^d$ we mean a finite family of measurable sets $\{A_i \subset A\}$ such that

(1) $\mu(A_i \cap A_j) = 0$ whenever $i \neq j$
(2) $\sum_i \mu(A_i) = \mu(A)$.

**Definition 3.** A measure $\mu^*$ is a checkerboard approximation for a copula $C$ if there exists a $\lambda$-decomposition $\mathcal{A} = \{(a_i, b_i)\}$ of $I^d$, the $d$-dimensional unit cube, made out of $d$-intervals such that for all $i$,

(1) $\mu^*$ is uniform on $(a_i, b_i)$;
(2) $\mu^*(A) = \mu_C(A)$ for any $A \in \mathcal{A}$.

For $m \in \mathbb{N}$, let us consider the regular $\lambda$-decomposition of the unite cube $[0, 1]^d$ denoted as $\mathcal{I}_m$ and consisting of $m^d$ $d$-cubes with side length $1/m$ :

$$I_{i,m} = \prod_{j=1}^{d}\left[\frac{i_j - 1}{m}, \frac{i_j}{m}\right],\ i = (i_1, \ldots, i_d),\ i_j \in \{1, \ldots, m\}.$$

$\mu^*_m$ is the checkerboard approximation associated to the regular decomposition $\mathcal{I}_m$.

We shall denote by $C^*_m$ the checkerboard copula associated to the measure $\mu^*_m$. The definition of the checkerboard copula may then be rewritten as:

$$C^*_m(x) = \sum_i m^d \mu(I_{i,m})\lambda([0, x] \cap I_{i,m})$$

where $[0, x] = \prod_{i=1}^{d}[0, x_i]$, for $x = (x_1, \ldots, x_d) \in [0, 1]^d$ and $\lambda$ is the $d$-dimensional Lebesgue measure. In [15], it is proved that $C^*_m$ is a copula that approximates $C$. The following proposition gives a more precise bound on the approximation of $C$ by $C^*_m$.

**Proposition 3.1.** *Let $C_m^*$ be the checkerboard copula defined above. We have:*

$$\sup_{x\in[0,1]^d} |C_m^*(x) - C(x)| \leq \frac{d}{2m}.$$

*Proof.* This is clear that for any $x \in [0,1]^d$ with $x = \frac{i}{m}$, $i \in \{1,\dots,m\}^d$, $C_m^*(x) = C(x)$. For $i \in \{1,\dots,m\}^d$ and $k \in \{1,\dots,m\}$, we denote by $B_i^{k+}$ and $B_i^{k-}$ the (half)-strips:

$$B_i^{k+} = \left\{x \in [0,1]^d,\ \frac{i_k}{m} - \frac{1}{2m} < x_k \leq \frac{i_k}{m}\right\} \text{ and}$$

$$B_i^{k-} = \left\{x \in [0,1]^d,\ \frac{i_k - 1}{m} < x_k \leq \frac{i_k}{m} - \frac{1}{2m}\right\}.$$

If $x \in I_{i,m}$ then,

$$\begin{aligned}
|C_m^*(x) - C(x)| &\leq \sum_{k=1}^d |\mu_m^*(B_i^{k-}) - \lambda(B_i^{k-})|\mathbf{1}_{B_i^{k-}}(x) + \\
&\quad \sum_{k=1}^d |\mu_m^*(B_i^{k+}) - \lambda(B_i^{k+})|\mathbf{1}_{B_i^{k+}}(x) \\
&\leq \sum_{k=1}^d \min(\mu_m^*(B_i^{k-}), \lambda(B_i^{k-}))\mathbf{1}_{B_i^{k-}}(x) + \\
&\quad \sum_{k=1}^d \min(\mu_m^*(B_i^{k+}), \lambda(B_i^{k+}))\mathbf{1}_{B_i^{k+}}(x) \\
&= \frac{d}{2m}
\end{aligned}$$

since $\mu_m^*$ and $\lambda$ are both associated to a copula,

$$\mu_m^*(B_i^{k-}) = \lambda(B_i^{k-}) = \mu_m^*(B_i^{k+}) = \lambda(B_i^{k+}) = \frac{1}{2m}.$$

The announced result follows. □

We also define two kinds of *checkerboard copula with additional information.* First of all, we consider the case where the distribution of a sub-vector $\mathbf{X}^J = (X_i)_{i\in J}$, $J \subset \{1,\dots,d\}$, is known, $|J| = k < d$. Denote $C^J$ the copula of $\mathbf{X}^J$. Let $\mu^J$ be the probability measure on $[0,1]^k$ associated to $C^J$. For $i = (i_1,\dots,i_d)$, let $x = (x_1,\dots,x_d) \in [0,1]^d$, $x^J = (x_j)_{j\in J}$, $x^{-J} = (x_j)_{j\notin J}$ and

$$I_{i,m}^J = \left\{x \in [0,1]^d \ /\ x_j \in \left[\frac{i_j - 1}{m}, \frac{i_j}{m}\right],\ j \in J\right\},$$

$$I_{i,m}^{-J} = \left\{x \in [0,1]^d \ /\ x_j \in \left[\frac{i_j - 1}{m}, \frac{i_j}{m}\right],\ j \notin J\right\}.$$

The checkerboard copula with information on $\mathbf{X}^J$ is defined below.

**Definition 4.** Consider the probability measure on $[0,1]^d$ defined by

$$\mu_m^J([0,x]) = \sum_{i\subset\{1,\dots,d\}} \frac{m^{d-k}}{\mu^J(I_{i,m}^J)} \mu(I_{i,m})\lambda([0,x^{-J}] \cap I_{i,m}^{-J})\mu^J([0,x^J] \cap I_{i,m}^J).$$

Let $C_m^J$, the checkerboard copula with additional information on $\mathbf{X}^J$, be defined by $C_m^J(x) = \mu_m^J([0,x])$.

**Proposition 3.2.** *$C_m^J$ is a copula, it approximates $C$:*

$$\sup_{x\in[0,1]^d} |C_m^J(x) - C(x)| \leq \frac{d}{2m}.$$

*If $\boldsymbol{X}^J$ and $\boldsymbol{X}^{-J}$ are independent then,*

$$\sup_{x\in[0,1]^d} |C_m^J(x) - C(x)| \leq \frac{d-k}{2m}.$$

*Proof.* The definition of $C_m^J$ insures that it is a cumulative distribution function on $[0,1]^d$. The fact that $C_m^J$ is a copula then follows from an easy computation to get that $C_m^J(x) = x_k$ whenever $x = (x_j)_{j=1,\dots,d}$, with $x_j = 1$ for $j \neq k$.
The rest of the proof is done as in that of Proposition 3.1. □

We may also add information on the tail and so define the following particular checkerboard copula.

**Definition 5.** For $p \in ]0,1[$, let $E_p = \left(\prod_{i=1}^d [0,p]^d\right)^c$ and $\mathcal{E}_p$ the $\lambda$-decomposition of $E_p$ consisting of the hyper rectangles $[a_1,b_1]\times\cdots\times[a_d,b_d]$ where $[a_i,b_i] = [0,p]$ or $[a_i,b_i] = [p,1]$ for all $i = 1,\dots d$ with at least one of $[a_i,b_i] = [p,1]$. We assume that $\mu_C(A)$ is known for each $A \in \mathcal{E}_p$. Consider the $\lambda$-decomposition of the $d$-cube $[0,p]^d$ given by $\mathcal{J}_m$ consisting of the elements $J_{i,m} = p \cdot I_{m,i}^d$ for $d$-tuple $i = (i_1,\dots,i_d)$ in $\{0, 1/m, \dots, (m-1)/m\}^d$. Define $C_m^{\mathcal{E}_p}$ as the checkerboard copula associated to the $\lambda$-decomposition of the unit $d$-cube $\mathcal{J}_m \cup \mathcal{E}_p$, that is

$$C_m^{\mathcal{E}}(x) = \mu_C(E^c)\mu_m^*([0,x]/t)\mathbf{1}_{E^c}(x) + \sum_{E\in\mathcal{E}_p} \frac{\mu_C(E)}{\lambda(E)}\lambda([0,x]\cap E).$$

This is the checkerboard copula with extra information on the tail.

In what follows, we will define an empirical version of the checkerboard copulas defined above, by using the empirical copula.

3.2. **Empirical checkerboard copulas.** The empirical copula, introduced by Deheuvels ([7]), may be used to estimate non parametrically the copula.

**Definition 6.** Let $\mathbf{X}^1,\dots,\mathbf{X}^n$ be $n$ independent copies of $\mathbf{X}$. Each of them writes $\mathbf{X}^j = (X_1^j,\dots,X_d^j)$. Let $R_i^1,\dots,R_i^n$, $i = 1,\dots,d$ be their marginals ranks, i.e.,

$$R_i^j = \sum_{k=1}^n 1\{X_i^{(j)} \geq X_i^{(k)}\},\ i = 1,\dots,d,\ j = 1,\dots,n$$

where $X_i^{(1)} < \cdots < X_i^{(n)}$ are the order statistics associated to the $i$th coordinate sample $X_i^1,\dots X_i^n$. The empirical copula $\widehat{C}_n$ of $\mathbf{X}^1,\dots\mathbf{X}^n$ is defined

as

$$\widehat{C}_n(u_1,\dots,u_d) = \frac{1}{n}\sum_{k=1}^{n} 1\left\{\frac{1}{n}R_1^k \le u_1,\dots,\frac{1}{n}R_d^k \le u_d\right\}.$$

It is well known (see [12] e.g.) that the empirical copula may be used to estimate $C$. Nevertheless, it is not a copula as its marginal laws are discrete. We shall use the empirical copula $\widehat{C}_n$ and the empirical probability measure $\widehat{\mu}$ associated to $\widehat{C}_n$, to define an empirical version of the checkerboard copulas defined above.

**Definition 7.** Let $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(n)}$ be $n$ independent copies of $\mathbf{X}$.

- The empirical checkerboard copula (ECBC) $\widehat{C}_m^*$ is defined by

$$\widehat{C}_m^*(x) = \sum_i m^d \widehat{\mu}(I_{i,m})\lambda([0,x]\cap I_{i,m}).$$

- The ECBC with information on a sub-vector $\mathbf{X}^J$ is defined by

$$\widehat{C}_m^J(x) = \sum_{i\subset\{1,\dots,d\}} \frac{m^{d-k}}{\mu^J(I_{i,m}^J)}\widehat{\mu}(I_{i,m})\lambda([0,x^{-J}]\cap I_{i,m}^{-J})\mu^J([0,x^J]\cap I_{i,m}^J).$$

- The ECBC with information on the tail is defined by:

$$\widehat{C}_m^{\mathcal{E}}(x) = \mu_C(E^c)\widehat{C}_m^*(x/t)\mathbf{1}_{E^c}(x) + \sum_{E\in E_p}\frac{\mu_C(E)}{\lambda(E)}\lambda([0,x]\cap E).$$

In the next proposition we show that $\widehat{C}_m^*$, defined above, is a copula whenever the integer $m$ from the length size of the partition $\mathcal{I}_m$ divides $n$, the sample size.

**Proposition 3.3.** *Assume that $m \in \mathbb{N}$ divides $n$. Then, the empirical chekerboard copula $\widehat{C}_m^*$ defined on the regular partition $\mathcal{I}_m$ and based on an i.i.d sample of size $n$ is a copula.*

*Proof.* Suppose that $m \le n$ and that $m$ divides $n$. By definition the empirical checkerboard copula is a distribution function, we should simply check that the marginals are uniform. Without losing generality we show only that the projection on the first coordinate of the measure induced by $\widehat{C}_m^*$ is uniform, or equivalently that $\widehat{C}_m^*(x) = x_1$ for any $x \in [0,1]^d$ with $x_j = 1$ for $j \ne 1$.
For $\ell \in \{1,\dots,m\}$, consider the strip $B_\ell^1$:

$$B_\ell^1 = \left\{x\in[0,1]^d,\ \frac{\ell-1}{m} < x_1 \le \frac{\ell}{m}\right\} = \left]\frac{\ell-1}{m},\frac{\ell}{m}\right] \times [0,1]^{d-1}.$$

The empirical copula is concentrated on $n$ points of $[0,1]^d$ whose coordinates are of the form $\frac{j}{n}$, $j = 1,\dots,n$. Moreover, there is exactly one mass on each strip $B_j^1$, $j = 1,\dots,n$. So that if $k = n/m$, then the number of masses of $\widehat{C}_n$ on each strip $B_\ell^1$, $\ell = 1,\dots,m$ is exactly $k$, which means that $\widehat{\mu}(B_\ell^1) = \frac{k}{n} = \frac{1}{m}$. Let $x = (x_1,1,\dots,1)$, $x_1 \in [0,1]$, $x \in B_\ell^1$ with

$\frac{\ell-1}{m} < x_1 \leq \frac{\ell}{m}$.

$$\begin{aligned}
\widehat{C}^*_m(x) &= \sum_{i\in\{1,\dots,m\}^d} m^d \widehat{\mu}(I_{i,m})\lambda([0,x]\cap I_{i,m}) \\
&= \sum_{j<\ell} \widehat{\mu}(B^1_j) + \sum_{I_{i,m}\subset B^1_\ell} m^d \widehat{\mu}(I_{i,m}) \frac{(x_1 - \frac{\ell-1}{m})}{m^{d-1}} \\
&= \sum_{j<\ell} \widehat{\mu}(B^1_j) + m\left(x_1 - \frac{\ell-1}{m}\right)\widehat{\mu}(B^1_\ell) \\
&= \frac{\ell-1}{m} + \left(x_1 - \frac{\ell-1}{m}\right) = x_1,
\end{aligned}$$

which concludes the proof. □

*Remark.* The same kind of calculations shows that the empirical checkerboard copulas with additional information $\widehat{C}^J_m$ and $\widehat{C}^{\mathcal{E}}_m$ are copulas provided that $m$ divides $n$.

Figure 1 shows simulations of different empirical checkerboard copulas with and without information on the tail and two different decompositions.

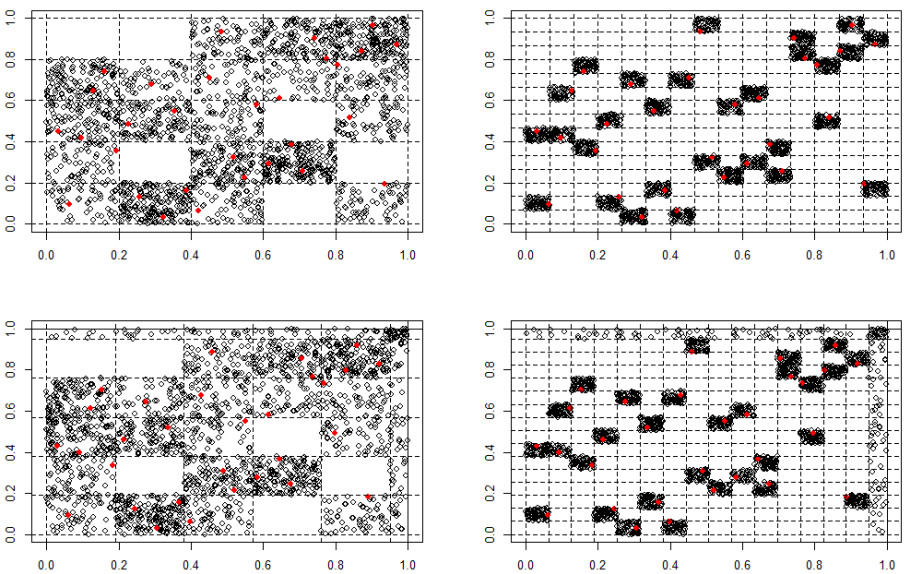



FIGURE 1. Simulation of four checkerboard empirical copulas based on the same sample of size $n = 30$. At the top there is no information on the tail whereas on the bottom information on $\mathcal{E}_p$, with $p = 0.95$ is assumed. On the left the regular decomposition is $\mathcal{I}_5$, on the right $\mathcal{I}_{15}$. The red points are the sample rank points.

*Remark.* Simulating a sample of size $N$ from $\widehat{C}^*_m$ given a sample of size $n$ of $\mathbf{X}$ is very easy because it suffices to

(1) get the sample rank,
(2) determine the number $n_{i,m}$ of points of the rank sample in each $I_{i,m}$,
(3) choose one element of $\mathcal{I}_m$ such that the probability to chose $I_{i,m}$ is $\dfrac{n_{i,m}}{n}$,
(4) take a point in the chosen $I_{i,m}$ at step (3) uniformly,

repeat steps (3) and (4) $N$ times.

3.3. **Estimation procedure.** In what follows $\widehat{C}_m^{cb}$ denotes any of the three ECBCs defined above. We propose the following estimation procedure. Assume the marginal laws are known and a (quite small) sample of size $n$ of $\mathbf{X}$ is available.

(1) Estimate $\mu$ by $\widehat{\mu}$ using the empirical copula.
(2) Obtain the ECBC $\widehat{C}_m^{cb}$ (depending if some additional information is known).
(3) Simulate a sample of size $N$ from the copula $\widehat{C}_m^{cb}$ for $N$ big:
$$(u_1^{(1)}, \dots, u_d^{(1)}), \dots, (u_1^{(N)}, \dots, u_d^{(N)})$$
(4) Get a sample of $\Psi(\mathbf{X})$ using the marginals to transform the above sample:
$$\Psi\left(F_1^{\leftarrow}(u_1^{(1)}), \dots, F_d^{\leftarrow}(u_d^{(1)})\right), \dots, \Psi\left(F_1^{\leftarrow}(u_1^{(N)}), \dots, F_d^{\leftarrow}(u_d^{(N)})\right)$$
(5) Estimate the distribution function $F_\Psi$ of $\Psi(\mathbf{X})$ empirically using the above sample. We will denote $\widehat{F}(\Psi)$ the empirical distribution function from the sample above.

Below, we state a convergence result which may have some interest from a theoretical point of view but which is not our purpose here because we aim at working with (relatively) small samples of size $n$.

**Proposition 3.4.** *For some $A > 0$, let $A\sqrt{n} \leq m \leq n$. Assume that $\Psi(\boldsymbol{X})$ is absolutely continuous and $C$ has continuous partial derivatives. Then,*
$$\|F_\Psi - \widehat{F}(\Psi)\|_\infty = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

*Proof.* Use the convergence result by Fermanian et al ([12]). □

## 4. Numerical Application

In this section we use the estimator of the distribution function $F_\Psi$ to estimate the quantiles $Q_p(S)$ for $S = X_1 + \cdots + X_d$ at different confidence levels $0 < p < 1$. We will consider the Pareto-Clayton model, defined in Section 4.1, because, in that case, the exact value of $Q_p(S)$ can be calculated, so that we may compare our simulation results to the exact one.

4.1. **The Pareto - Clayton model.** We consider $\mathbf{X} = (X_1, \dots, X_d)$ such that:
$$\mathbb{P}(X_1 > x_1, \dots, X_d > x_d \,|\, \Lambda = \lambda) = \prod_{i=1}^{d} e^{-\lambda x_i},$$

that is, conditionally to the value of $\Lambda$ the marginals of $\mathbf{X}$ are independent and exponentially distributed.
If $\Lambda$ is Gamma distributed, then the $X_i$'s are Pareto distributed with dependence given by a survival Clayton copula.
If $\Lambda$ is Levy distributed, then the $X_i$'s are Weibull distributed with a Gumbel survival copula.
These models have been studied by Oakes (1989) and Yeh (2007) [17, 22]. In the context of multivariate risk theory, they have been used e.g. in [14] and [6].
In what follows, we consider that $\Lambda \rightsquigarrow \Gamma(\alpha, \beta)$, so that the $X_i$'s are Pareto $(\alpha, \beta)$ distributed and the dependence structure is described by a survival Clayton copula with parameter $1/\alpha$. In [8], it is shown that, in this case, $S$ follows the so-called Beta prime distribution:

$$F_S(x) = F_\beta\left(\frac{x}{1+x}\right)$$

where $F_\beta$ is the distribution function of the Beta$(d\beta, \alpha)$ distribution. The inverse of $F_S$ (or quantile function of $S$) can also be expressed in terms of the inverse of the Beta distribution

$$F_S^{\leftarrow}(p) = \frac{F_\beta^{\leftarrow}(p)}{1 - F_\beta^{\leftarrow}(p)}.$$

From these results (see also [5]), we may compute $Q_p(S)$.

4.2. **Simulation study in dimension 2 (with and without information on the tail).** We consider a Pareto-Clayton model in dimension 2, with $\beta = 1$ and $\alpha = 2$. The multivariate sample is of size $n = 30$ and for each presented method we performed $N = 1000$ estimations of the $p$-quantile at different confidence levels. Table 1 presents the mean and the root mean squared error of the $N = 1000$ estimations. The estimations were calculated using the ECBC with and without information on the tail on different $\lambda$-regular decompositions $\mathcal{I}_m$ for $m = 6, 15, 30$. The information on the tail is introduced on $\mathcal{E}_p$, for $p = 0.95, 0.99$, by giving to each $E$ in $\mathcal{E}_p$ the measure $\mu_C(E)$ where $C$ is the survival Clayton copula with parameter $1/2$. For comparison a direct estimation from the empirical distribution of $S$ is given.

All the methods we proposed perform much better than the empirical estimation based on the multivariate sample alone. The estimations based on the ECBC with $\lambda$-decomposition $\mathcal{I}_m$, perform better when $m = 6$ and $m = 15$ than when $m = 30$. ECBC with $m = 6$ performs slightly better than ECBC with $m = 15$ for the estimation of the quantiles with confidence levels lower than 99.5% and slightly worst on the higher levels. When the information on the tail is introduced on $\mathcal{E}_p$ with $p = 0.95$ the estimation on the quantile with confidence level $90\%, 95\%$ and $99\%$ is considerably improved. When it is introduced on $\mathcal{E}_p$ with $p = 0.95$ the estimations improve on the higher confidence levels 99.5% and 99.9%.

| | Quantile 80% | Quantile 90% | Quantile 95% | Quantile 99% | Quantile 99.5% | Quantile 99.9% |
|---|---|---|---|---|---|---|
| Exact value | 2.5 | 4.1 | 6.4 | 16.0 | 23.2 | 53.4 |
| Empirical | 2.5 | 4.0 | 6.1 | 12.2 | 13.2 | 14.0 |
| | (26%) | (31%) | (39%) | (72%) | (70%) | (78%) |
| ECBC (m=6) | | | | | | |
| No tail information | 2.6 | 4.4 | 6.6 | 14.8 | 20.8 | 45.7 |
| | (9%) | (8%) | (6%) | (8%) | (11%) | (15%) |
| Information on $\mathcal{E}_p$ | 2.6 | 4.4 | 6.4 | 14.2 | 22.7 | 49.5 |
| p=0.99 | (9%) | (8%) | (5%) | (11%) | (3%) | (8%) |
| Information on $\mathcal{E}_p$ | 2.7 | 4.1 | 6.1 | 15.6 | 21.8 | 46.8 |
| p=0.95 | (10%) | (5%) | (4%) | (3%) | (6%) | (13%) |
| ECBC (m=15) | | | | | | |
| No tail information | 2.5 | 4.2 | 6.8 | 15.5 | 21.5 | 46.4 |
| | (12%) | (13%) | (11%) | (9%) | (10%) | (14%) |
| Information on $\mathcal{E}_p$ | 2.5 | 4.3 | 6.8 | 14.3 | 22.7 | 49.5 |
| p=0.99 | (12%) | (12%) | (12%) | (11%) | (3%) | (8%) |
| Information on $\mathcal{E}_p$ | 2.6 | 4.3 | 6.2 | 15.6 | 21.8 | 46.8 |
| p=0.95 | (11%) | (10%) | (4%) | (3%) | (6%) | (13%) |
| ECBC (m=30) | | | | | | |
| No tail information | 2.5 | 4.2 | 6.6 | 15.8 | 22.0 | 47.0 |
| | (13%) | (15%) | (17%) | (13%) | (12%) | (14%) |
| Information on $\mathcal{E}_p$ | 2.5 | 4.2 | 6.7 | 14.3 | 22.7 | 49.5 |
| p=0.99 | (13%) | (16%) | (16%) | (11%) | (3%) | (8%) |
| Information on $\mathcal{E}_p$ | 2.6 | 4.4 | 6.2 | 15.6 | 21.8 | 46.8 |
| p=0.95 | (13%) | (11%) | (4%) | (3%) | (6%) | (13%) |

TABLE 1. The mean and the RMSE in % of the exact value for 1000 estimations of the Quantile for a Pareto-Clayton sum in dimension 2.

4.3. **Simulation study introducing information on a sub-vector.** In order to assess the gain that the knowledge of the information on a sub-vector may give to the estimation, we performed here the following simulation study. Let $\mathbf{X} = (X_1, X_2, X_3)$ be the model where $X_1 = X_2 = Y/2$, and $X_3 \sim Y$ where $Y$ is Pareto distributed with $\alpha = 2$. We assume that $(Y, X_3)$ is a Pareto-Clayton model as in section 4.2. That is, $X_1$ and $X_2$ are comonotonic (or fully dependent) and the dependence between $X_1$ and $X_3$ is given by a survival Clayton of parameter 1/2. Clearly the distribution of the sum $S = X_1 + X_2 + X_3$ is equal to the distribution of the sum of the Pareto-Clayton model in dimension 2, with parameters $\alpha = 2$ and $\beta = 1$ and thus the exact value of the quantiles can be easily computed. We compare the results on the quantiles estimation using the ECBC method without and with information on the sub-vector $(X_1, X_2)$ and $\lambda-$decompositions $\mathcal{I}_m$ for $m = 6, 15, 30$. As before the multivariate sample is of size $n = 30$ and for each method we performed $N = 1000$ estimations of the quantile at different confidence levels. The results are presented in Table 2.

| | Quantile 80% | Quantile 90% | Quantile 95% | Quantile 99% | Quantile 99.5% | Quantile 99.9% |
|---|---|---|---|---|---|---|
| Exact | 2.5 | 4.1 | 6.4 | 16.0 | 23.2 | 53.4 |
| ECBC (m=6) | | | | | | |
| No information | 2.7 | 4.6 | 6.6 | 14.0 | 19.1 | 40.7 |
| | (13%) | (13%) | (7%) | (13%) | (18%) | (24%) |
| Information on | 2.6 | 4.4 | 6.6 | 14.8 | 20.8 | 45.7 |
| $(X_1, X_2)$ | (9%) | (8%) | (6%) | (8%) | (11%) | (15%) |
| ECBC (m=10) | | | | | | |
| No information | 2.5 | 4.6 | 7.0 | 14.5 | 19.8 | 41.3 |
| | (12%) | (13%) | (12%) | (11%) | (15%) | (23%) |
| Information on | 2.5 | 4.3 | 6.7 | 15.2 | 21.2 | 46.1 |
| $(X_1, X_2)$ | (11%) | (9%) | (9%) | (8%) | (10%) | (15%) |
| ECBC (m=30) | | | | | | |
| No information | 2.5 | 4.2 | 6.8 | 15.9 | 21.4 | 43.3 |
| | (14%) | (16%) | (19%) | (14%) | (14%) | (21%) |
| Information on | 2.5 | 4.2 | 6.6 | 15.8 | 21.9 | 47.1 |
| $(X_1, X_2)$ | (13%) | (16%) | (17%) | (13%) | (13%) | (14%) |

TABLE 2. The mean and the RMSE in % of the exact value for 1000 estimations of the Quantiles in dimension 3, with or without using the knowledge of the comonotonic dependence between $X_1$ and $X_2$.

It can be noticed that the RSME of the quantile estimation is lower when the information on $(X_1, X_2)$ is introduced in the ECBC of dimension 3 and the gap is more important on higher confidence levels.

*Remark.* Simulating with respect to the empirical checkerboard copula with information on a sub-vector may be a difficult task because one has to simulate with respect to a given copula conditionally to belonging to a given set $I_{i,m}$. In the case of a comonotonic sub-vector, this becomes trivial because we only need to simulate one coordinate uniformly.

Simulation results with the same kind of model in dimension 6 are presented in Table 3. We assumed $\mathbf{X} = (X_1, \ldots, X_6)$ with $X_1 = X_2 = Y/2$ and $X_3, X_4, X_5$ and $X_6$ distributed as $Y$, a Pareto r.v. with parameter $\alpha = 2$. The copula of $\mathbf{X}$ is assumed to be a survival Clayton of parameter 1/2. As above, the size of the multivariate sample is $n = 30$ and for each method we performed $N = 1000$ estimations of the quantile at different confidence levels.

Again, by introducing the information on the sub-vector $(X_1, X_2)$ we get a smaller RMSE than in the case where no information is added. On the other hand, we also remark that by increasing the dimension (from $d = 3$ to $d = 6$) we get higher RMSE, for the same sample size, which is an expected behavior.

4.4. **A simulation in higher dimension.** We conclude this simulation section with a simulation in dimension 10. We consider a Pareto-Clayton

| | Quantile 80% | Quantile 90% | Quantile 95% | Quantile 99% | Quantile 99.5% | Quantile 99.9% |
|---|---|---|---|---|---|---|
| Exact | 6.1 | 9.8 | 14.9 | 36.4 | 52.4 | 120.1 |
| ECBC (m=6) | | | | | | |
| No information | 7.2 | 11.0 | 15.0 | 28.2 | 37.3 | 74.3 |
| | (19%) | (14%) | (7%) | (23%) | (29%) | (38%) |
| Information on | 7.1 | 10.9 | 15.0 | 28.9 | 38.4 | 77.5 |
| $(X_1, X_2)$ | (17%) | (13%) | (7%) | (21%) | (27%) | (36%) |
| ECBC (m=10) | | | | | | |
| No information | 6.8 | 11.3 | 16.1 | 30.0 | 39.2 | 76.3 |
| | (16%) | (18%) | (13%) | (19%) | (26%) | (37%) |
| Information on | 6.7 | 11.1 | 16.0 | 30.6 | 40.3 | 79.7 |
| $(X_1, X_2)$ | (15%) | (16%) | (12%) | (17%) | (24%) | (34%) |
| ECBC (m=30) | | | | | | |
| No information | 6.3 | 10.4 | 16.3 | 33.7 | 43.4 | 81.3 |
| | (15%) | (19%) | (20%) | (18%) | (22%) | (33%) |
| Information on | 6.3 | 10.4 | 16.2 | 34.0 | 44.1 | 84.2 |
| $(X_1, X_2)$ | (15%) | (18%) | (19%) | (18%) | (20%) | (31%) |

TABLE 3. The mean and the RMSE in % of the exact value for 1000 estimations of the Quantiles in dimension 6.

model with $\beta = 1$, $\alpha = \frac{1}{2}$. The multivariate sample size is $n = 75$, then $n = 150$. We perform $N = 1000$ runs for the checkerboard method without any kind of information. In this case, we have simply taken $m = n$. Results are presented in Table 4.

| | Quantile 80% | Quantile 90% | Quantile 95% | Quantile 99% | Quantile 99.5% | Quantile 99.9% |
|---|---|---|---|---|---|---|
| Exact value | 12.2 | 19.2 | 29 | 70.1 | 100.8 | 230.5 |
| Empirical, $n = 75$ | 12.6 | 20 | 29.9 | 62.2 | 75.8 | 86.7 |
| | (12%) | (15%) | (19%) | (41%) | (67%) | (181%) |
| Checkerboard, $n = 75$ | 12.5 | 20.1 | 31.2 | 74.8 | 92.4 | 152.6 |
| | (10%) | (13%) | (16%) | (21%) | (22%) | (53%) |
| Empirical, $n = 150$ | 12.4 | 19.6 | 30.3 | 67.3 | 89.9 | 121 |
| | (9%) | (11%) | (14%) | (27%) | (40%) | (108%) |
| Checkerboard, $n = 150$ | 12.4 | 19.6 | 29.8 | 75.4 | 107.6 | 173.9 |
| | (7%) | (9%) | (12%) | (18%) | (22%) | (38%) |

TABLE 4. Mean and RMSE in % of the exact value for the Pareto-Clayton sum in dimension 10.

It is notable that the checkerboard method performs well even in dimension 10. Let us emphasize that $n = 150$ in dimension 10 is not much.

## 5. Conclusion

In this paper, we have constructed empirical checkerboard copulas with and without additional information on the joint law. We have used them to get efficient estimations of the quantiles of the sum when using a (relatively) small sample of the joint law and the knowledge of the marginal laws. We have also proved (in the case of max and $\sum$) that there exist infinitely many copulas for given marginal laws and given aggregation law. This theoretical result indicates that when the marginal laws are known, if we are only interested in the aggregated law, the full knowledge of the joint law is not necessary.

We are aware that many theoretical and practical questions have to be studied further, among which:

- the optimal choice of $m$ with respect to the sample size $n$,
- the quantification of the impact of plugging additional information in the empirical checkerboard copula,
- the number $N$ of simulations necessary to reach a certain precision level,
- developing efficient algorithms to simulate with respect to the empirical checkerboard copula with information on a sub-vector, for other copulas than the comonotonic one,
- developing efficient algorithms to simulate with respect to the empirical checkerboard copula with information on the tail in dimension larger than 2.

**Acknowledgements**

This work has bee partially supported by the MultiRisk LEFE-MANU project and the research chair *Actuariat responsable* sponsored by Generali.

## References

[1] Carole Bernard, Xiao Jiang, and Ruodu Wang. Risk aggregation with dependence uncertainty. *Insurance: Mathematics and Economics*, 54(1):93–108, 2014.

[2] Carole Bernard, Ludger Rüschendorf, and Steven Vanduffel. Value-at-Risk bounds with variance constraints. 2013.

[3] Carole Bernard and Steven Vanduffel. A new approach to assessing model risk in high dimensions. 2014.

[4] Hélène Cossette, Marie-Pier Côté, Mélina Mailhot, and Etienne Marceau. A note on the computation of sharp numerical bounds for the distribution of the sum, product or ratio of dependent risks. *Journal of Multivariate Analysis*, 130:1–20, 2014.

[5] Andrés Cuberos, Esterina Masiello, and Véronique Maume-Deschamps. High level quantile approximations of sums of risks. 2014.

[6] Michel Dacorogna, Leila El Bahtouri, and Marie Kratz. Explicit diversification benefit for dependent risks. preprint, 2014.

[7] Paul Deheuvels. La fonction de dépendance empirique et ses propriétés. *Acad. Roy. Belg. Bull. Cl. Sci.*, 65(5):274 – 292, 1979.

[8] Satya D Dubey. Compound gamma, beta and F distributions. *Metrika*, 16(1):27–31, 1970.

[9] Paul Embrechts and Giovanni Puccetti. Risk aggregation. *Copula Theory and Its Applications*, 2010.

[10] Paul Embrechts, Giovanni Puccetti, and Ludger Rüschendorf. Model uncertainty and var aggregation. *Journal of Banking & Finance*, 37(8):2750–2764, 2013.

[11] Arturo Erdely, José M González-Barrios, and María M Hernández-Cedillo. Frank's condition for multivariate archimedean copulas. *Fuzzy Sets and Systems*, 240:131–136, 2014.

[12] Jean-David Fermanian, Dragan Radulovic, and Marten Wegkamp. Weak convergence of empirical copula processes. *Bernoulli*, 10(5):847–960, 2004.

[13] M.J. Frank. Diagonals of copulas and schröeder's equation. *Procedings Aequ. Math.*, (150), 1996.

[14] Véronique Maume-Deschamps, Didier Rullière, and Khalil Saïd. Impact of dependence on some multivariate risk indicators. 2015.

[15] Piotr Mikusinski and Michael D Taylor. Some approximations of n-copulas. *Metrika*, 72(3):385–414, 2010.

[16] Roger B Nelsen, José Juan Quesada-Molina, José Antonio Rodríguez-Lallena, and Manuel Úbeda-Flores. On the construction of copulas and quasi-copulas with given diagonal sections. *Insurance: Mathematics and Economics*, 42(2):473–483, 2008.

[17] David Oakes. Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, 84(406):487–493, 1989.

[18] Giovanni Puccetti and Ludger Rüschendorf. Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Mathematics*, 236(7):1833–1840, 2012.

[19] Ludger Rüschendorf. Random variables with maximum sums. *Advances in Applied Probability*, pages 623–632, 1982.

[20] Ludger Rüschendorf. Solution of a statistical optimization problem by rearrangement methods. *Metrika*, 30(1):55–61, 1983.

[21] Engin A Sungur and Yimin Yang. Diagonal copulas of archimedean class. *Communications in Statistics-Theory and Methods*, 25(7):1659–1676, 1996.

[22] Hsiaw-Chan Yeh. The frailty and the Archimedean structure of the general multivariate Pareto distributions. *Bulletin Institute of Mathematics Acedemia Sinica*, 2(3):713–729, 2007.

Université de Lyon, Université Lyon 1, Laboratoire SAF EA 2429, SCOR SE
*E-mail address*: acuberos@scor.com

Université de Lyon, Université Lyon 1, Institut Camille Jordan ICJ UMR 5208 CNRS
*E-mail address*: esterina.masiello@univ-lyon1.fr

Université de Lyon, Université Lyon 1, Institut Camille Jordan ICJ UMR 5208 CNRS
*E-mail address*: veronique.maume@univ-lyon1.fr