

# A new family of bivariate discrete distributions on $\mathbb{Z}^2$

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**Abstract** In this paper we introduce a new family of bivariate discrete distributions on  $\mathbb{Z}^2$ , called the Rademacher( $\alpha_1, \alpha_2$ ) –  $\mathbb{N}^2$  class. Its main feature is to generate bivariate random variables with possible negative or positive values for the covariance. This new family can also be considered as an extension (on  $\mathbb{Z}^2$ ) of some standard bivariate discrete (non negative valued) distribution.

**Keywords** Bivariate Poisson distribution; Bivariate Geometric distribution ; Rademacher random variable ; Maximum likelihood estimator.

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## 1 Introduction

Paired count observations appear in many situations and research fields. For example, in insurance, one can be interested by studying the evolution of the number of accidents in a site before and after some new regulations. Bivariate discrete distributions were introduced to fit such data. Over the last three decades, a large contribution on discrete bivariate distributions has been accumulated. For a literature review, we refer to Kocherlakota and Kocherlakota (1992), Johnson et al. (1997), Kocherlakota and Kocherlakota (1997), Papa-georgiou (1997), Balakrishnan (2005), Lai (2006), Bishop et al. (2007), and Sarabia and Gómez-Déniz (2008).

In contrast to the well-known situation when the paired data are counts, i.e. observed on  $\mathbb{N}^2$ , sometimes the data are observed on  $\mathbb{Z}^2$ . For example,

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when analyzing intra-daily stock prices the changes take both positive and negative integer values, known also as ticks (the price can go up or down on certain predefined ranges of value). The price change is therefore characterized by discrete jumps. To our best knowledge there is a shortage of bivariate discrete distribution defined on  $\mathbb{Z}^2$ . Karlis and Ntzoufras (2012) defined a bivariate distribution using copulas, while Bulla et al. (2015) constructed bivariate Skellam distributions by a simple trivariate reduction scheme with additive functions. In a recent article, Chesneau et al. (2015) introduced a family of distributions based on the generalized trivariate (multivariate) reduction technique and the Rademacher distribution. In this paper, we intend to contribute to this literature, by presenting a new family of bivariate discrete distributions, based also on the Rademacher distribution. This new family, denoted by  $\text{Rademacher}(\alpha_1, \alpha_2) - \mathbb{N}^2$ , is defined on  $\mathbb{Z}^2$  and with possible negative or positive values for the covariance. Furthermore, it can be considered as an extension on  $\mathbb{Z}^2$  of some standard bivariate distributions such as the Poisson or the Geometric.

The remainder of the paper proceeds as follows. Firstly, in Section 2, we present the considered  $\text{Rademacher}(\alpha_1, \alpha_2) - \mathbb{N}^2$  family and investigate some of its properties. Section 3 describes statistical approach to estimate the unknown parameters efficiently. Finally, in section 4, we apply our general distributions to the bivariate Poisson and Geometric distributions.

## 2 The $\text{Rademacher}(\alpha_1, \alpha_2) - \mathbb{N}^2$ class

In this section, we introduce a new class of distribution in  $\mathbb{Z}^2$ , denoted by  $\text{Rademacher}(\alpha_1, \alpha_2) - \mathbb{N}^2$  and investigate some of its basic properties.

**Definition 1** Let  $(\alpha_1, \alpha_2) \in (0, 1)^2$ ,  $(X, Y)$  be a bivariate random variables with  $(X, Y)(\Omega) \subseteq \mathbb{N}^2$  and, for any  $j \in \{1, 2\}$ ,  $R_j \sim \mathcal{R}(\alpha_j)$ , i.e.

$$\mathbb{P}(R_j = -1) = 1 - \alpha_j, \quad \mathbb{P}(R_j = 1) = \alpha_j.$$

Let us suppose that  $R_1$  and  $R_2$  are mutually independent and independent of  $X$  and  $Y$ . We say that a bivariate random variable  $(S, T)$  belongs to the  $\text{Rademacher}(\alpha_1, \alpha_2) - \mathbb{N}^2$  class, if and only if

$$(S, T) = (R_1 X, R_2 Y).$$

*Remark 1* The  $\text{Rademacher}(\alpha_1, \alpha_2) - \mathbb{N}^2$  class can be interpreted as the extension in  $\mathbb{Z}^2$  of the class of bivariate discrete distributions defined on  $\mathbb{N}^2$ .

Let us now introduce some basic properties of  $(S, T)$ :

- The support of  $(S, T)$  is given by  $(S, T)(\Omega) = \{(s, t) \in \mathbb{Z}^2; (|s|, |t|) \in (X, Y)(\Omega)\}$ .

– The distribution of  $(S, T)$  is given by

$$\begin{aligned} \mathbb{P}(\{S = s\} \cap \{T = t\}) = \\ \alpha_1^{\mathbf{1}_{\{s>0\}}} (1 - \alpha_1)^{\mathbf{1}_{\{s<0\}}} \alpha_2^{\mathbf{1}_{\{t>0\}}} (1 - \alpha_2)^{\mathbf{1}_{\{t<0\}}} \mathbb{P}(\{X = |s|\} \cap \{Y = |t|\}), \\ (s, t) \in \mathbb{Z}^2. \end{aligned}$$

– The marginal distributions are given by

$$\mathbb{P}(S = s) = \alpha_1^{\mathbf{1}_{\{s>0\}}} (1 - \alpha_1)^{\mathbf{1}_{\{s<0\}}} \mathbb{P}(X = |s|), \quad s \in \mathbb{Z},$$

and

$$\mathbb{P}(T = t) = \alpha_2^{\mathbf{1}_{\{t>0\}}} (1 - \alpha_2)^{\mathbf{1}_{\{t<0\}}} \mathbb{P}(Y = |t|), \quad t \in \mathbb{Z}.$$

– Moreover, we have

$$\mathbb{E}(S^m) = \begin{cases} \mathbb{E}(X^m) & \text{if } m \text{ even,} \\ (2\alpha_1 - 1)\mathbb{E}(X^m) & \text{if } m \text{ odd,} \end{cases}$$

and

$$\mathbb{E}(T^m) = \begin{cases} \mathbb{E}(Y^m) & \text{if } m \text{ even,} \\ (2\alpha_2 - 1)\mathbb{E}(Y^m) & \text{if } m \text{ odd.} \end{cases}$$

– The covariance of  $S$  and  $T$  is given by

$$\mathbb{C}(S, T) = (2\alpha_1 - 1)(2\alpha_2 - 1)\mathbb{C}(X, Y),$$

where  $\mathbb{C}(X, Y)$  denotes the covariance of  $X$  and  $Y$ .

*Remark 2* Note that  $\mathbb{C}(S, T)$  can be positive or negative according to the values of  $(\alpha_1, \alpha_2)$ .

– The characteristic function of  $(S, T)$  is given by

$$\begin{aligned} \phi_*(s, t) &= \mathbb{E}(e^{isS} e^{itT}) \\ &= \alpha_1 \alpha_2 \phi(s, t) + (1 - \alpha_1) \alpha_2 \phi(-s, t) + \alpha_1 (1 - \alpha_2) \phi(s, -t) \\ &\quad + (1 - \alpha_1)(1 - \alpha_2) \phi(-s, -t), \quad (s, t) \in \mathbb{R}^2, \end{aligned}$$

where  $\phi(x, y) = \mathbb{E}(e^{ixX} e^{iyY})$  denotes the characteristic function of  $(X, Y)$ .

### 3 Parameters estimation

From the general definition of the Rademacher( $\alpha_1, \alpha_2$ ) –  $\mathbb{N}^2$  class, we are able to determine the Maximum Likelihood estimator (MLE) of the unknown parameters. The details are given below.

Let  $\theta = (\theta_1, \dots, \theta_m)$  be the vector of unknown parameters in  $(X, Y)$  and  $v = (\alpha_1, \alpha_2, \theta)$ . For any  $(s, t) = ((s_1, t_1), \dots, (s_n, t_n))$  with  $(s_i, t_i) \in \mathbb{Z}^2$ , the likelihood function associated to  $(S, T)$  is given by

$$L_v(s, t) = \prod_{i=1}^n \mathbb{P}(\{S = s_i\} \cap \{T = t_i\}) = \alpha_1^{\sum_{i=1}^n \mathbf{1}_{\{s_i > 0\}}} (1 - \alpha_1)^{\sum_{i=1}^n \mathbf{1}_{\{s_i < 0\}}} \alpha_2^{\sum_{i=1}^n \mathbf{1}_{\{t_i > 0\}}} (1 - \alpha_2)^{\sum_{i=1}^n \mathbf{1}_{\{t_i < 0\}}} \tilde{L}_\theta(|s|, |t|),$$

where  $\tilde{L}_\theta$  denotes the likelihood function associated to  $(X, Y)$ . The log-likelihood function is given by

$$\begin{aligned} \ell_v(s, t) &= \ln L_v(s, t) = \\ &= \ln(\alpha_1) \sum_{i=1}^n \mathbf{1}_{\{s_i > 0\}} + \ln(1 - \alpha_1) \sum_{i=1}^n \mathbf{1}_{\{s_i < 0\}} + \ln(\alpha_2) \sum_{i=1}^n \mathbf{1}_{\{t_i > 0\}} \\ &+ \ln(1 - \alpha_2) \sum_{i=1}^n \mathbf{1}_{\{t_i < 0\}} + \tilde{\ell}_\theta(|s|, |t|), \end{aligned}$$

where  $\tilde{\ell}_\theta$  is the log-likelihood function associated to  $(X, Y)$ .

We have

$$\frac{\partial \ell_v}{\partial \alpha_1}(s, t) = \frac{1}{\alpha_1} \sum_{i=1}^n \mathbf{1}_{\{s_i > 0\}} - \frac{1}{1 - \alpha_1} \sum_{i=1}^n \mathbf{1}_{\{s_i < 0\}}.$$

Hence

$$\frac{\partial \ell_v}{\partial \alpha_1}(s, t) = 0 \Leftrightarrow \alpha_1 = \frac{\sum_{i=1}^n \mathbf{1}_{\{s_i > 0\}}}{n - \sum_{i=1}^n \mathbf{1}_{\{s_i = 0\}}}$$

and this value is a maximum for  $\alpha_1 \rightarrow \hat{\alpha}_1(s, t)$ .

Then the MLE of  $\alpha_1$  and  $\alpha_2$  are given by

$$\hat{\alpha}_1 = \frac{\sum_{i=1}^n \mathbf{1}_{\{S_i > 0\}}}{n - \sum_{i=1}^n \mathbf{1}_{\{S_i = 0\}}} \quad (1)$$

and

$$\hat{\alpha}_2 = \frac{\sum_{i=1}^n \mathbf{1}_{\{T_i > 0\}}}{n - \sum_{i=1}^n \mathbf{1}_{\{T_i = 0\}}}. \quad (2)$$

On the other hand, since  $|S| = X$  and  $|T| = Y$  and

$$\frac{\partial \ell_v}{\partial \theta_i}(s, t) = \frac{\partial \tilde{\ell}_\theta}{\partial \theta_i}(|s|, |t|),$$

the MLE of  $\theta$  have the same definitions of those associated to  $(X, Y)$  but defined with  $(|S_1|, |T_1|), \dots, (|S_n|, |T_n|)$  instead of  $(X_1, Y_1), \dots, (X_n, Y_n)$ .

## 4 Applications

In this section, we apply our general distributions to the bivariate Poisson and Geometric distributions. We then obtain two new bivariate distributions on  $\mathbb{Z}^2$  having possible negative or positive values for the covariance function (contrary to the originals bivariate Poisson and Geometric distributions).

### 4.1 Application to the bivariate Poisson distribution

Let  $(\lambda_0, \lambda_1, \lambda_2) \in (0, \infty)^3$  and  $U_0, U_1$  and  $U_2$  be three independent random variables such that, for any  $j \in \{0, 1, 2\}$ ,  $U_j \sim \mathcal{P}(\lambda_j)$ , i.e.

$$\mathbb{P}(U_j = u) = e^{-\lambda_j} \frac{\lambda_j^u}{u!}, \quad u \in \mathbb{N}.$$

Let us now consider the two random variables:

$$X = U_1 + U_0, \quad Y = U_2 + U_0.$$

We say thus that  $(X, Y)$  follows a bivariate Poisson distribution (for more details, see e.g. Holgate (1964) and Karlis and Ntzoufras (2005)). The probability mass function of  $(X, Y)$  is given by

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = e^{-(\lambda_0 + \lambda_1 + \lambda_2)} \frac{\lambda_1^x \lambda_2^y}{x! y!} \sum_{i=0}^{\min(x, y)} \binom{x}{i} \binom{y}{i} i! \left( \frac{\lambda_0}{\lambda_1 + \lambda_2} \right)^i, \\ (x, y) \in \mathbb{N}^2.$$

Note that

$$X \sim \mathcal{P}(\lambda_1 + \lambda_0), \quad Y \sim \mathcal{P}(\lambda_2 + \lambda_0), \quad \mathbb{C}(X, Y) = \mathbb{V}(U_0) = \lambda_0.$$

Set now

$$(S, T) = (R_1 X, R_2 Y),$$

with  $R_j \sim \mathcal{R}(\alpha_j)$ , for any  $j \in \{1, 2\}$ . Moreover, we suppose that  $R_1$  and  $R_2$  are mutually independent and independent of  $X$  and  $Y$ .

Let us now investigate the estimation of the unknown parameters  $(\alpha_1, \alpha_2, \lambda_0, \lambda_1, \lambda_2)$  from a n-sample  $(S_1, T_1), \dots, (S_n, T_n)$  via two estimation methods: the method of moments and the MLE.

Method of moments. Let  $\theta_1 = \lambda_1 + \lambda_0$ . Observe that

$$\mathbb{E}(S^2) = \theta_1^2 + \theta_1 \quad \Leftrightarrow \quad \theta_1 = \frac{\sqrt{1 + 4\mathbb{E}(S^2)} - 1}{2}$$

which yields the following estimator:

$$\hat{\theta}_1 = \frac{\sqrt{1 + 4\overline{S^2}} - 1}{2}, \quad \overline{S^2} = \frac{1}{n} \sum_{i=1}^n S_i^2.$$

Since

$$\mathbb{E}(S) = (2\alpha_1 - 1)\theta_1 \quad \Leftrightarrow \quad \alpha_1 = \frac{1}{2} \left( \frac{\mathbb{E}(S)}{\theta_1} + 1 \right)$$

an estimator for  $\alpha_1$  is

$$\hat{\alpha}_1 = \frac{1}{2} \left( \frac{\bar{S}}{\hat{\theta}_1} + 1 \right), \quad \bar{S} = \frac{1}{n} \sum_{i=1}^n S_i.$$

Let  $\theta_2 = \lambda_2 + \lambda_0$ . Hence, using similar arguments, we have

$$\hat{\theta}_2 = \frac{\sqrt{1 + 4\bar{T}^2} - 1}{2}, \quad \bar{T} = \frac{1}{n} \sum_{i=1}^n T_i^2$$

and

$$\hat{\alpha}_2 = \frac{1}{2} \left( \frac{\bar{T}}{\hat{\theta}_2} + 1 \right), \quad \bar{T} = \frac{1}{n} \sum_{i=1}^n T_i.$$

Let us now investigate the estimations of  $\lambda_j$  with  $j \in \{0, 1, 2\}$ . Observe that

$$\begin{aligned} \mathbb{P}(\{S = 0\} \cap \{T = 0\}) &= e^{-(\lambda_1 + \lambda_2 + \lambda_0)}, \\ \mathbb{P}(S = 0) &= e^{-(\lambda_1 + \lambda_0)}, \quad \mathbb{P}(T = 0) = e^{-(\lambda_2 + \lambda_0)}. \end{aligned}$$

Therefore natural estimators for  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_0$  are respectively given by

$$\begin{aligned} \hat{\lambda}_1 &= -\log \left( \frac{\sum_{i=1}^n \mathbf{1}_{\{(S_i, T_i) = (0, 0)\}}}{\sum_{i=1}^n \mathbf{1}_{\{T_i = 0\}}} \right), \\ \hat{\lambda}_2 &= -\log \left( \frac{\sum_{i=1}^n \mathbf{1}_{\{(S_i, T_i) = (0, 0)\}}}{\sum_{i=1}^n \mathbf{1}_{\{S_i = 0\}}} \right) \end{aligned}$$

and

$$\hat{\lambda}_0 = -\log \left( \frac{\sum_{i=1}^n \mathbf{1}_{\{(S_i, T_i) = (0, 0)\}}}{ne^{-(\hat{\lambda}_1 + \hat{\lambda}_2)}} \right).$$

**MLE.** The MLE of  $\alpha_1$  (resp.  $\alpha_2$ ) is given by (1) (resp. (2)). To calculate the MLE of  $\lambda = (\lambda_0, \lambda_1, \lambda_2)$  we use the EM-algorithm for the bivariate Poisson models, originally proposed by Karlis and Ntzoufras (2005).

*Remark 3* Note that, Holgate (1964) shows the asymptotic property of MLE for the bivariate Poisson distribution.

#### *A short simulation experiment*

We simulate  $n = 1000$  observations of  $(S, T)$  (where  $(X, Y)$  follows a bivariate Poisson distribution) and we compute both moment method (MM) and maximum likelihood (ML) estimators. After 1000 independent replications, we calculate the average and the standard deviation of the sequence of the obtained estimates. Results are summarized in the following table:

One can see from the above table that the ML estimation gives a smaller standard deviation than the MM estimation for all parameters. Fitting to a Gaussian distribution is illustrated in Figure 1, for the ML estimator.

Parameter	actual value	Mean (MM)	Sd (MM)	Mean (ML)	Sd (ML)
$\alpha_1$	0.4	0.3998928	0.01849841	0.3999116	0.01739304
$\alpha_2$	0.7	0.6994876	0.01677994	0.7001840	0.0145807
$\lambda_0$	1	0.9731342	0.18982206	0.9989756	0.06292195
$\lambda_1$	1	1.0313487	0.19963600	1.0006694	0.06299054
$\lambda_2$	2	2.0355057	0.23531690	1.9981334	0.07110126

**Table 1** Parametric estimation of the unknown parameters of the distribution of  $(S, T)$  via the Method of Moments (MM) and the Maximum Likelihood Estimation (MLE).

#### 4.2 Application to the bivariate Geometric distribution

Let  $(p_0, p_1, p_2) \in (0, 1)^3$  and  $U_0, U_1$  and  $U_2$  be three independent random variables such that, for any  $j \in \{0, 1, 2\}$ ,  $U_j \sim \mathcal{G}_0(p_j)$ , i.e.

$$\mathbb{P}(U_j = u) = p_j q_j^u, \quad u \in \mathbb{N},$$

where  $q_j = 1 - p_j$ . Let now

$$X = U_1 + U_0, \quad Y = U_2 + U_0.$$

Thus, one can say that  $(X, Y)$  follows a bivariate Geometric distribution (modified in 0), for more details, we refer to Dhar and Balaji (2006). The probability mass function of  $(X, Y)$  is given by

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = (p_0 p_1 p_2) q_1^x q_2^y \sum_{k=0}^{\min\{x, y\}} \left( \frac{q_0}{q_1 q_2} \right)^k, \quad (x, y) \in \mathbb{N}^2.$$

*Remark 4* Remark that

– for any  $k \in \mathbb{N}$ ,

$$\mathbb{P}(X = x) = \frac{p_1 p_0}{q_0 - q_1} (q_0^{x+1} - q_1^{x+1}), \quad \mathbb{P}(Y = y) = \frac{p_2 p_0}{q_0 - q_2} (q_0^{y+1} - q_2^{y+1}).$$

– We have

$$\mathbb{C}(X, Y) = \mathbb{V}(U_0) = \frac{q_0}{p_0^2}.$$

Set now

$$(S, T) = (R_1 X, R_2 Y),$$

with  $R_j \sim \mathcal{R}(\alpha_j)$ , for any  $j \in \{1, 2\}$ . Moreover, we suppose that  $R_1$  and  $R_2$  are mutually independent and independent of  $X$  and  $Y$ .

Let us now investigate the estimation of the unknown parameters  $(\alpha_1, \alpha_2, p_0, p_1, p_2)$  from a  $n$ -sample  $(S_1, T_1), \dots, (S_n, T_n)$  via two estimation methods : the method of moments and the MLE.

Method of moments. One can observe that

$$\mathbb{P}(\{S = 0\} \cap \{T = 0\}) = p_0 p_1 p_2, \quad \mathbb{P}(S = 0) = p_0 p_1, \quad \mathbb{P}(T = 0) = p_0 p_2.$$

Natural estimators for  $p_1$ ,  $p_2$  and  $p_0$  are then respectively given by

$$\hat{p}_1 = \frac{\sum_{i=1}^n \mathbf{1}_{\{(S_i, T_i)=(0,0)\}}}{\sum_{i=1}^n \mathbf{1}_{\{T_i=0\}}}, \quad \hat{p}_2 = \frac{\sum_{i=1}^n \mathbf{1}_{\{(S_i, T_i)=(0,0)\}}}{\sum_{i=1}^n \mathbf{1}_{\{S_i=0\}}}$$

and

$$\hat{p}_0 = \frac{\sum_{i=1}^n \mathbf{1}_{\{(S_i, T_i)=(0,0)\}}}{n \hat{p}_1 \hat{p}_2}.$$

Now remark that

$$\mathbb{E}(S) = (2\alpha_1 - 1) \left( \frac{q_1}{p_1} + \frac{q_0}{p_0} \right), \quad \mathbb{E}(T) = (2\alpha_2 - 1) \left( \frac{q_2}{p_2} + \frac{q_0}{p_0} \right).$$

Therefore natural estimators for  $\alpha_1$  and  $\alpha_2$  are respectively given by

$$\hat{\alpha}_1 = \frac{1}{2} \left( \frac{\hat{p}_1 + \hat{p}_0 - (2 - \bar{S}) \hat{p}_1 \hat{p}_0}{\hat{p}_1 + \hat{p}_0 - 2 \hat{p}_1 \hat{p}_0} \right)$$

and

$$\hat{\alpha}_2 = \frac{1}{2} \left( \frac{\hat{p}_2 + \hat{p}_0 - (2 - \bar{T}) \hat{p}_2 \hat{p}_0}{\hat{p}_2 + \hat{p}_0 - 2 \hat{p}_2 \hat{p}_0} \right),$$

where

$$\bar{S} = \frac{1}{n} \sum_{i=1}^n S_i, \quad \bar{T} = \frac{1}{n} \sum_{i=1}^n T_i.$$

MLE. The MLE of  $\alpha_1$  and  $\alpha_2$  are given by (1) and (2). To calculate the MLE of  $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ , it suffices to program a function that calculates the MLE of a bivariate Geometric distribution, which is straightforward with standard statistical packages.

#### *A short simulation experiment*

We simulate  $n = 1000$  observations of  $(S, T)$  (where  $(X, Y)$  follows a bivariate Geometric distribution) and we compute both moment method (MM) and maximum likelihood (ML) estimators. After 1000 independent replications, we calculate the average and the standard deviation of the sequence of the estimates obtained. Results are summarized in the following table:

Remark that from the above table the ML estimation gives a smaller standard deviation than the MM estimation for all parameters. Fitting to a Gaussian distribution is illustrated in Figure 2, for the ML estimator.



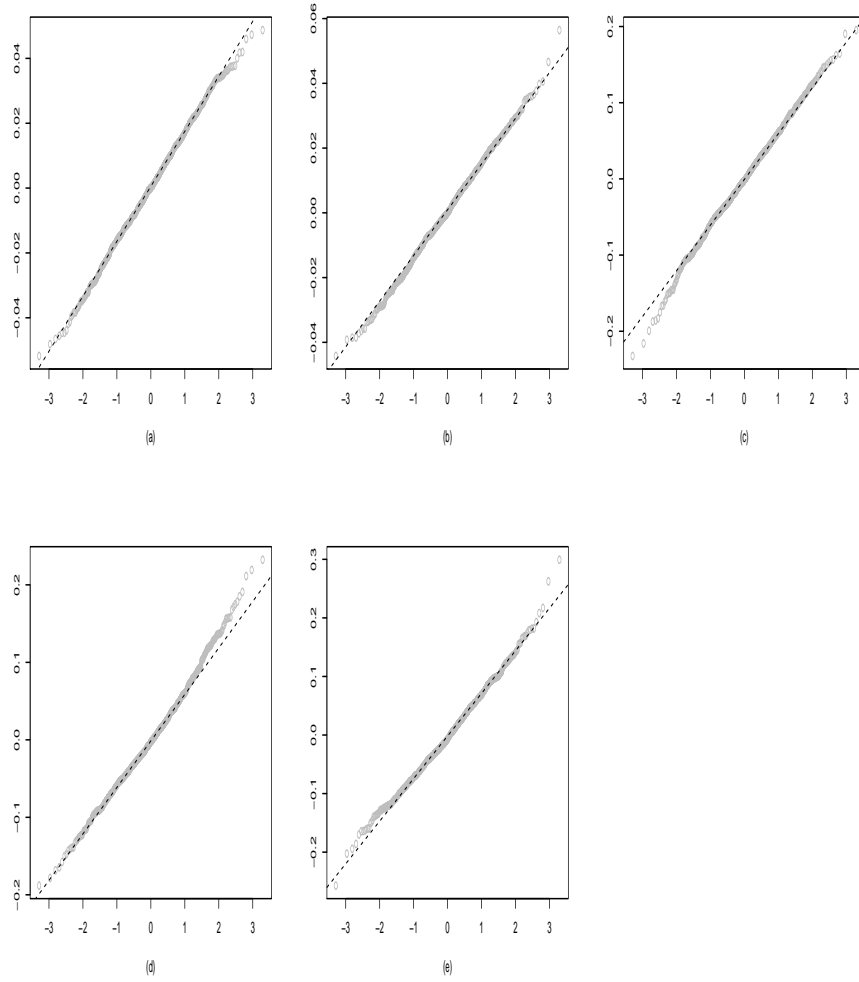
Parameter	actual value	Mean (MM)	Sd (MM)	Mean (ML)	Sd (ML)
$\alpha_1$	0.1	0.09994431	0.027364810	0.10058339	0.010938576
$\alpha_2$	0.6	0.59918124	0.021549369	0.59987719	0.016319864
$p_0$	0.3	0.29942310	0.01728777	0.29933596	0.008043778
$p_1$	0.8	0.80401244	0.037735241	0.80076329	0.015544756
$p_2$	0.4	0.39954472	0.032864295	0.39998300	0.010265953

**Table 2** Parametric estimation of the unknown parameters of the distribution of  $(S, T)$  via the Method of Moments (MM) and the Maximum Likelihood Estimation (MLE).

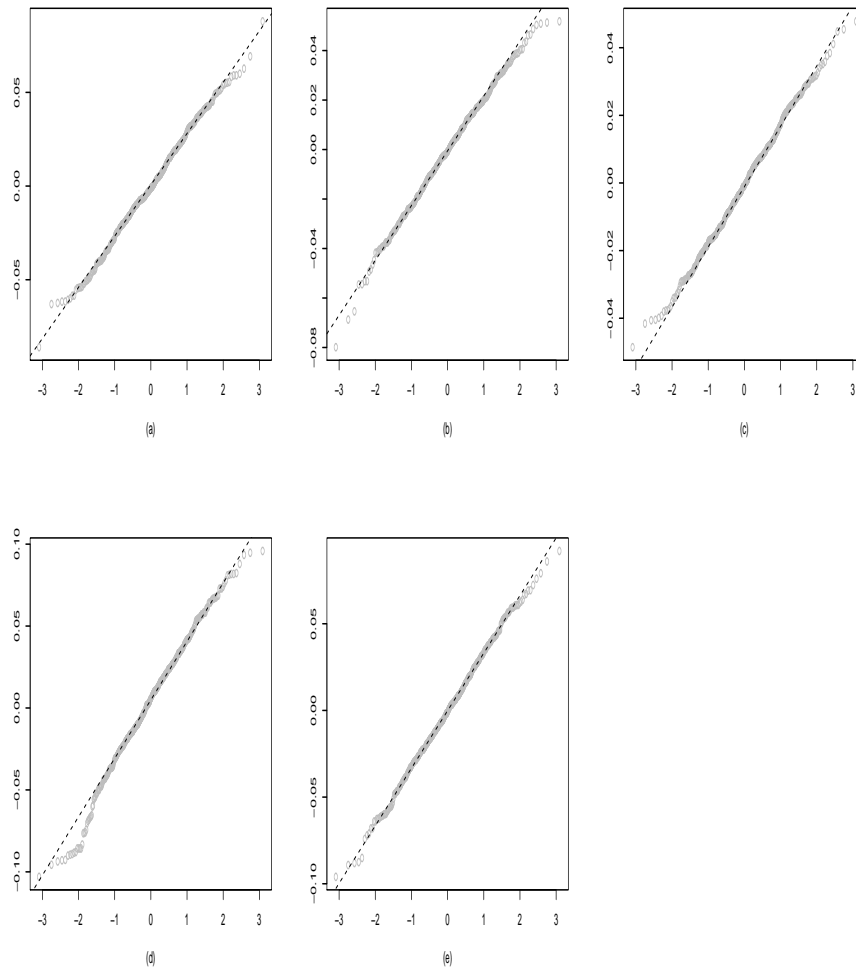
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## Figures



**Fig. 1** Normal Q-Q plots for the errors (ML estimator), when  $(X, Y)$  follows bivariate Poisson distribution and  $n = 1000$ . (a) Normal Q-Q plots of  $\hat{\alpha}_1 - \alpha_1$ . (b) Normal Q-Q plots of  $\hat{\alpha}_2 - \alpha_2$ . (c) Normal Q-Q plots of  $\hat{\lambda}_0 - \lambda_0$ . (d) Normal Q-Q plots of  $\hat{\lambda}_1 - \lambda_1$ . (e) Normal Q-Q plots of  $\hat{\lambda}_2 - \lambda_2$ .



**Fig. 2** Normal Q-Q plots for the errors (ML estimator), when  $(X, Y)$  follows bivariate Geometric distribution and  $n = 1000$ . (a) Normal Q-Q plots of  $\hat{\alpha}_1 - \alpha_1$ . (b) Normal Q-Q plots of  $\hat{\alpha}_2 - \alpha_2$ . (c) Normal Q-Q plots of  $\hat{p}_0 - p_0$ . (d) Normal Q-Q plots of  $\hat{p}_1 - p_1$ . (e) Normal Q-Q plots of  $\hat{p}_2 - p_2$ .