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BOUNDED HEIGHT IN PENCILS OF FINITELY GENERATED SUBGROUPS.

F. AMOROSO, D. MASSER AND U. ZANNIER

ABSTRACT. We prove height bounds concerning intersections of finitely generated subgroups in a torus with algebraic subvarieties, all varying in a pencil. This vastly extends the previously treated constant case and involves entirely different, and more delicate, techniques.

The paper complements and sharpens Mordell-Lang by replacing *finiteness* by *emptiness*, for the intersection of varieties and subgroups, all moving in a pencil, except for bounded height values of the parameters (and excluding identical relations).

More precisely, an instance of the results is as follows. Consider the torus scheme $\mathbb{G}_{m/\mathcal{C}}^r$ over a curve \mathcal{C} defined over $\overline{\mathbb{Q}}$, and let Γ be a subgroup-scheme generated by finitely many sections (satisfying some necessary conditions). Further, let V be any subscheme. Then there is a bound for the height of the points $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that, for some $\gamma \in \Gamma$ which does not generically lie in V , $\gamma(P)$ lies in the fiber V_P .

The context studied in previous papers concerned only the highly special case when V is the origin of $\mathbb{G}_{m/\mathcal{C}}^r$; on the contrary, here V (or V_P) is not assumed to have any kind of additional structure; this feature heavily prevents the previously known proof pattern to apply.

We offer also more precise quantitative bounds, which in particular show that certain linear successive minima are almost equal.

We further offer some direct diophantine applications, to illustrate once again that the results implicitly contain information absent from the previous bounds in this context.

1. INTRODUCTION

Let \mathcal{C} be a projective smooth curve defined over $\overline{\mathbb{Q}}$, with function field denoted $\mathbb{F} := \overline{\mathbb{Q}}(\mathcal{C})$. We choose once and for all a height h on $\mathcal{C}(\overline{\mathbb{Q}})$.¹

Let now $g_1, \dots, g_\kappa \in \mathbb{G}_m(\mathbb{F})$ be non-zero rational functions on \mathcal{C} , supposed to be multiplicatively independent modulo constants. By this we mean as usual that, for integers a_1, \dots, a_κ , we may have $g_1^{a_1} \cdots g_\kappa^{a_\kappa} \in \overline{\mathbb{Q}}^*$ only if all a_i vanish. This is equivalent to suppose that, for any point $P_0 \in \mathcal{C}$ not a zero or pole of any g_i , the functions $g_i/g_i(P_0)$ are multiplicatively independent in the usual sense.

An issue which appears natural is to study the set of points $P \in \mathcal{C}$ such that the values $g_1(P), \dots, g_\kappa(P)$ become multiplicatively dependent, thus destroying the generic independence property of the functions.

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¹For instance, we may choose a system of Weil's functions associated to a divisor of degree 1 and a corresponding height, as explained e.g. in [6].

The generic independence implies that all such points P lie in $\mathcal{C}(\overline{\mathbb{Q}})$, and it is a result of [7] that their height is bounded above (in terms only of g_1, \dots, g_κ). Here is a formulation of Theorem 1' therein:

Theorem 1.1 ([7], Theorem 1'). *Let $\Gamma \subset \mathbb{G}_m(\mathbb{F})$ be a finitely generated subgroup of non-zero rational functions on \mathcal{C} such that the only constants in Γ are roots of unity. Then the set of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that for some $\mathbf{x} \in \Gamma \setminus \{1\}$ we have $\mathbf{x}(P) = 1$ is a set of bounded height.*

A simple example is with \mathcal{C} the affine line, $\mathbb{F} = \overline{\mathbb{Q}}(t)$ and Γ the subgroup generated by t and $1 - t$. It implies that if t is algebraic with

$$t^n(1 - t)^m = 1$$

(for integers n, m not both zero) then its height is bounded (independently of n, m).

It is easily seen that the condition on Γ is necessary: take a non-torsion constant $c \in \Gamma$ and a non-constant $\gamma \in \Gamma$ and consider the equation $c^n \gamma(P) = 1$, whose solutions have unbounded height.

Theorem 1.1 was then used in the same paper [7] as a crucial tool to obtain a finiteness result over the whole $\overline{\mathbb{Q}}$ (valid when we impose an additional multiplicative relation). As in several other circumstances, this provides evidence that bounded-height conclusions, in addition to their possible relevance in themselves, may be quite useful for applications.

Here we prove a more general bounded-height result for specializations in finitely generated subgroups, this time varying in families.

Before stating this, note that Theorem 1.1 may be also phrased as a kind of toric analogue of (a corollary of) Silverman's Specialization Theorem ([22], Theorem C). To illustrate this link, let us consider the 'trivial' (or 'constant') family $\mathbb{G}_{m/\mathcal{C}} := \mathbb{G}_m \times \mathcal{C}$ and the sections $\gamma_i : \mathcal{C} \rightarrow \mathbb{G}_m \times \mathcal{C}$, given by $P \mapsto (g_i(P), P)$, where g_i are generators for Γ , independent modulo constants. The above conclusion then means that *the set of points P where the values of the sections are multiplicatively dependent has bounded height.*

Remark 1.2. Silverman's specialization results in fact concern mainly abelian families, actually non-isotrivial, whereas algebraic tori have only (iso)trivial² families; this introduces some differences in the assumptions (as in results by Manin and Demjanenko, see [21]); for instance the assumption of mere *independence* has to be strengthened, as above, in the sense *modulo constants*, in order to obtain bounded height.

Further differences with respect to the abelian case are due to the fact that heights in abelian varieties behave like quadratic forms, so somewhat more 'regularly' than in the toric case, when this lack may introduce difficulties in some parts of the proofs for the toric case.

Now, multiplicative dependence (at P) means that some nontrivial monomial attains the value 1 at P . Then, rather more generally, given a constant family (scheme) $\pi : \mathbb{G}_{m/\mathcal{C}}^r := \mathbb{G}_m^r \times \mathcal{C} \rightarrow \mathcal{C}$, a subvariety $V \subset \mathbb{G}_m^r \times \mathcal{C}$ and

²By 'isotrivial' we mean that the family becomes trivial, *i. e.* a product, after suitable extension of the base.

sections $\gamma_i : \mathcal{C} \rightarrow \mathbb{G}_m^r/\mathcal{C}$ generating a group Γ , we may ask the following (somewhat vague)

Question: *What can be said about those points P such that some nontrivial element of the group Γ when specialized at P lies on the fiber $V_P = \pi^{-1}(P) \subset \mathbb{G}_m^r$.*

The previous situation is obtained in the very special case when V_P is constantly equal to the origin, *i. e.* $V = \text{origin} \times \mathcal{C}$. By contrast, we stress that here neither V nor the V_P are assumed to have any kind of group structure; especially this feature heavily prevents the previously known proof-pattern to apply.

As a further motivation for such question, note that for a fixed subvariety $V_0 \subset \mathbb{G}_m^r$, the intersection of V_0 with a finitely generated subgroup Γ_0 has a very special structure (its Zariski closure is a finite union of translates of algebraic subgroups contained in V_0), due to theorems tracing back to Lang, Evertse, van der Poorten-Schlickewei, Laurent, which represent (the toric case of) Lang’s celebrated conjectures; in particular, we may say that for ‘general’ V_0 this intersection is finite, and hence, in a sense, unlikely. Thus it looks sensible to ask about the intersection when both V_0 and Γ_0 vary in an algebraic family, and for instance ask: *when is the intersection nonempty* ? In the case of a pencil of varieties, this amounts to the above question.

The present paper offers in a sense a complete solution to this issue, proving that on the appropriate assumptions we have generally bounded height for *any* proper family of subvarieties. In particular, the said intersection is empty except for a ‘sparse’ set of points.

For simplicity, we phrase this conclusion in the language of Theorem 1.1. Namely, we consider a power \mathbb{G}_m^r of the multiplicative algebraic group and we let V be a subvariety of \mathbb{G}_m^r defined over \mathbb{F} ; so we may view V as a family of varieties parameterised by \mathcal{C} . Then we denote by V_P , for almost all $P \in \mathcal{C}$, a specialized variety defined e.g. by specializing at P a given system of defining equations for V . We also need a simple

Definition Given a subgroup Γ of \mathbb{G}_m^r defined over \mathbb{F} we say that Γ is constant-free if its image Γ' by any surjective homomorphism $\mathbb{G}_m^r \rightarrow \mathbb{G}_m^r$ satisfies the assumption $\Gamma' \cap \overline{\mathbb{Q}}^* = \Gamma'_{\text{tors}}$ of Theorem 1.1.

With such notation, we have the following uniform complement to (the toric case of) Lang’s conjecture:

Theorem 1.3. *Let $\Gamma \subset \mathbb{G}_m^r(\mathbb{F})$ be a finitely generated constant-free subgroup and let V be a subvariety of \mathbb{G}_m^r defined over \mathbb{F} . Then the set of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that for some $\mathbf{x} \in \Gamma \setminus V$ the value $\mathbf{x}(P)$ is defined and lies in V_P is a set of bounded height.*

Remark 1.4. Consider a “generic” situation when $\Gamma \cap V$ is empty and V_P does not contain coset of positive dimension. Mordell-Lang tell us that $\Gamma_P \cap V_P$ is finite for all P . Theorem 1.3 gives the following complement: for P of large height $\Gamma_P \cap V_P$ is *empty*.

A simple example is again with \mathcal{C} the affine line as above, now with $\Gamma \subset \mathbb{G}_m^2$ the subgroup generated by $(t, 1)$ and $(1, 1 - t)$. Let V be the

hypersurface of \mathbb{G}_m^2 defined by the equation $x + y = 1$; here we obtain bounded height for the equation

$$t^n + (1 - t)^m = 1$$

(this time with n, m not both 1). But the method applies much more generally to equations like

$$t^n + (1 - t)^m + (1 + t)^l = 1$$

and even

$$t^n(1 - t)^m + \alpha(1 + t)^l(2 - t)^k = \beta$$

for any algebraic α, β (which for $\alpha = 0, \beta = 1$ reduces to the equation above and for $\alpha = 0, \beta = 2$ is the new $t^n(1 - t)^m = 2$ now for any n, m).

Remark 1.5. For a subvariety V defined over the constant field $\overline{\mathbb{Q}}$ (*i. e.* we have a ‘trivial’ family with $V_P = V$ for all P), the conclusion of Theorem 1.3 still holds for a subgroup Γ which is not necessarily constant-free, but such that $\Gamma/\Gamma \cap \mathbb{G}_m^r(\overline{\mathbb{Q}})$ is of rank 1.

Note that some assumption on Γ is needed, since Theorem 1.3 contains Theorem 1.1 as a special case. As a non-trivial example, we may take as \mathcal{C} the affine line as above, and $\Gamma \subset \mathbb{G}_m^2$ the subgroup generated by $\gamma_1 = (t, 1)$ and $\gamma_2 = (1, 2t)$ (note that these vectors are multiplicatively independent modulo constants). Let V be the hypersurface of \mathbb{G}_m^2 defined by the equation $x + y = 0$. Then for $n \in \mathbb{N}$ the element $\mathbf{x}^{(n)} = \mathbf{x}^{(n)}(t) = \gamma_1^{n+1}\gamma_2^n = (t^{n+1}, 2^n t^n) \in \Gamma$ is not generically in V but its specialization at $t = -2^n$ is. Of course the image of Γ under the isogeny sending (x, y) to xy contains 2.

Choosing $\Gamma = \langle (g_1, \dots, g_r) \rangle$, where g_1, \dots, g_r are coordinate functions on \mathcal{C} , and V a subvariety defined over the constant field $\overline{\mathbb{Q}}$, we obtain a bounded height result for certain ‘(semi)unlikely intersections’. To state this, we denote as usual by $[n]: \mathbb{G}_m^r \rightarrow \mathbb{G}_m^r$ the morphism $\mathbf{x} \mapsto [n]\mathbf{x} := \mathbf{x}^n$ of multiplication by $n \in \mathbb{Z}$.

Corollary 1.6. *Let $\mathcal{C} \subseteq \mathbb{G}_m^r$ be a curve and let $V \subseteq \mathbb{G}_m^r$ be a subvariety, both defined over $\overline{\mathbb{Q}}$. Then the height of the points $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that there exists an integer n with $[n]\mathcal{C} \not\subseteq V$ and $[n]P \in V(\overline{\mathbb{Q}})$ is bounded.*

Corollary 1.6 immediately follows from Theorem 1.3 and Remark 1.5. Indeed, let n be an integer such that $[n]\mathcal{C} \not\subseteq V$. Then $\gamma := (g_1^n, \dots, g_r^n) \in \Gamma \setminus V$. By Theorem 1.3, and since now $V_P = V$ for all P , the height of the points $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that $[n]P \in V(\overline{\mathbb{Q}})$ is bounded independently of n .

Note that by the Skolem-Mahler-Lech theorem, for a fixed point $P_0 \in \mathbb{G}_m^r$ the set of integers n with $[n]P_0 \in V$ is a union of a finite number of points and arithmetic progressions, and is ‘usually’ finite.³ As in a comment above, the corollary says that when we move P_0 along a curve \mathcal{C} the corresponding set is ‘usually’ empty, except for points $P_0 \in \mathcal{C}(\overline{\mathbb{Q}})$ of bounded height.

³If the set is not finite, V must contain the Zariski-closure of a set $[nq + s]P_0$, $n \in \mathbb{N}$, for suitable integers $q \neq 0, s$, and in particular must contain a coset of an algebraic subgroup of positive dimension unless P_0 is torsion.

A relevant issue is to detect the set of integers n such that $[n]\mathcal{C} \subseteq V$. Now, this amounts to $[n]\mathbf{x} \in V$ where the coordinates of \mathbf{x} are the restrictions to \mathcal{C} of the coordinate functions on \mathbb{G}_m^r . Classical results describe the Zariski closure of these $[n]\mathbf{x}$ as a finite union of cosets of algebraic subgroups. Thus, if \mathcal{C} is not contained in any translate of a proper subtorus, no proper algebraic subgroup can contain a multiple $[m]\mathbf{x}$ for $m \neq 0$, proving that the said set is finite unless V is the whole space. (See also the paper [23] by Silverman and Voloch for more general finiteness results in this direction.) Actually, using for instance results in [10], it is not difficult to reach directly this finiteness conclusion, moreover determining effectively the set.

A similar remark holds for Theorem 1.3: we may compute ‘effectively’ the intersection $\mathbf{x} \in \Gamma \cap V$, which however may be infinite in general.⁴

1.1. Two simple applications. We show by means of two simple examples that our results, actually already very special cases of them, are capable of applications to diophantine issues, recovering certain finiteness statements. The result achieved in the first example is known and may be obtained by a variety of techniques, but here we reach it directly as a consequence of the above corollary. The second example appears to be new. Probably it can be sharpened, but our aim here is merely to illustrate possible applications, not to develop them in depth. For this same reason we shall be sketchy in the arguments.

Example 1: A family of Thue’s equations. Consider the Thue’s equation

$$x^3 - (t^3 - 1)y^3 = 1,$$

where $t > 1$ is an integer, to be solved in integers x, y .

Our results easily imply that *there are only finitely many integers t such that the equation has an integer solution with $y \neq 0, 1$.*

In fact, let $u = \sqrt[3]{t^3 - 1}$, e.g. the real determination. It is very easy to see that the group of units of the ring $\mathbb{Z}[u]$ is generated by ± 1 and the unit $t - u$.

Let $(x, y) \in \mathbb{Z}^2$ be a solution, so $x - uy \in \mathbb{Z}[u]$ is a unit of norm 1, whence $= (t - u)^n$ for some integer n , and on taking conjugates over \mathbb{Q} we have

$$(t - u)^n + \omega(t - \omega u)^n + \omega^2(t - \omega^2 u)^n = 0,$$

where ω is a primitive cube root of 1.

If $n > 1$, one may easily show that the left side does not vanish identically. (Even without *abc* over function fields, note that on dividing by the first term we would find an equation $1 + \omega\rho^n + \omega^2(-\omega - \omega^2\rho)^n = 0$, where $\rho = (t - \omega u)/(t - u)$ is nonconstant, which is impossible.) Hence, by the corollary (with \mathcal{C} a curve with function field $\overline{\mathbb{Q}}(z, \sqrt[3]{z^3 - 1})$ and $r = 3$), the height of t is bounded and finiteness follows.

This particular result (even that there are no t) has been known for nearly a century, but our method allows substantial generalizations; for example

⁴We do not pause here to give proofs of these statements, which fall in well-known treatments, not otherwise related to the present context.

it may be applied to a conjecture of Thomas about Thue equations, and extended to non-Thue equations such as

$$x^3 - (t^3 - 1)y^3 + 3(t^3 - 1)xy + (t^3 - 1)^2 = 1$$

(which cannot be obtained from a Thue equation by an inhomogeneous linear transformation), and even to more variables such as

$$x^4 + (4t^4 - 1)y^4 + (4t^4 - 1)^2 z^4 + 2(4t^4 - 1)x^2 z^2 - 4(4t^4 - 1)xy^2 z = 1.$$

Example 2: Zeros of polynomial recurrences. Consider a linear recurrence sequence $(u_n(t))_{n \in \mathbb{N}}$ of polynomials in t , defined by prescribing polynomial initial data (not all zero) $u_0(t), \dots, u_{r-1}(t) \in \mathbb{Q}[t]$ and imposing

$$u_{n+r} = c_1(t)u_{n+r-1} + \dots + c_r(t)u_n, \quad n = 0, 1, \dots,$$

where $c_i(t)$ are polynomials with coefficients say in \mathbb{Q} . We assume for simplicity that the characteristic polynomial $Z^r - c_1(t)Z^{r-1} - \dots - c_r(t)$ has no multiple roots in an algebraic closure of $\mathbb{Q}(t)$.

We assert that: *The set of algebraic numbers ξ such that for some n we have $u_n(t) \neq 0$ but $u_n(\xi) = 0$ has bounded height. In particular, there are only finitely many such ξ having bounded degree over \mathbb{Q} .*

Again, this follows rather immediately from the corollary and Northcott's theorem (see [6]), after expressing $u_n(t)$ as a linear combination of n -th powers of the roots of the recurrence (*i. e.* the roots of the characteristic polynomial).

A rephrasing of the last conclusion is that *for every given D there are only finitely many monic polynomials in $\mathbb{Q}[t]$ of degree $\leq D$ which may divide some $u_n(t)$.*

In several cases we may further sharpen this conclusion. Assume for instance that $r \geq 3$ and that the characteristic polynomial has Galois group S_r over $\bar{\mathbb{Q}}(t)$ (which is the 'generic' case).⁵

Then, in particular, the recurrence is non-degenerate, meaning that no ratio of its roots is a root of unity. Under this assumption, and by the Skolem-Mahler-Lech theorem, only finitely many u_n may vanish identically.

Suppose now that an algebraic number ξ_0 is a root of u_n for infinitely many n . Then, again by the Skolem-Mahler-Lech theorem 'several' ratios of the specialised roots become roots of unity, so the n -th powers of the roots of the recurrence collapse and vanishing of $u_n(\xi_0)$ gives rise to linear relations among these roots of 1. If the order of these roots of unity is eventually unbounded, then known theorems on torsion points on curves in \mathbb{G}_m^r (see [24]) imply that the roots of the recurrence may be grouped in subsets of cardinality ≥ 2 such that all the ratios of two roots in the same subset are multiplicatively dependent as algebraic functions; but the assumption on the Galois group then easily yields a contradiction. On the other hand, if the order of the relevant roots of unity is bounded, then we find only finitely many ξ_0 .

Combining this with the previous conclusion, we have:

⁵A doubly transitive Galois group would suffice for the conclusion below. However, as stressed above, it is beyond the scope of these examples to push the analysis further.

On the above assumptions, apart from a finite set of polynomial factors, the degrees of the irreducible factors (over \mathbb{Q}) of the polynomials u_n tends to infinity with n .

Note that some assumption as above is needed for this conclusion; an example in this direction is given by the Chebishev polynomials T_n defined by $T_0(t) = 2, T_1(t) = t$ and $T_{n+2}(t) = tT_{n+1}(t) - T_n(t)$ for $n \geq 0$. It turns out that, for odd m , T_q divides T_{mq} for all odd q , providing an example when the last conclusion is not true. Similarly for polynomials like $u_n(t) := T_n(t)T_{n+h}(t)$, where $r = 4$ but the Galois group is too small.

There are also other applications to common zeros of polynomials in two linear recurrences (e.g. as outlined in Remark 1.11 below). We plan to develop this in a subsequent paper.

1.2. A relation with Unlikely Intersections. The boundedness of the height in the set of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that $[n]P \in V(\overline{\mathbb{Q}})$ is related also to the context of Unlikely Intersections, and more precisely to degenerate cases of the Bounded Height Conjecture⁶ of Bombieri-Masser-Zannier [8] (nowadays a theorem of Habegger [12]), as in Example 1.3 of [24]. To describe this, let $X = \mathcal{C} \times V \subset \mathbb{G}_m^r \times \mathbb{G}_m^r$. If $P \in \mathcal{C}(\overline{\mathbb{Q}})$ is such that $[n]P \in V(\overline{\mathbb{Q}})$, then $(P, [n]P)$ is in the intersection of X with the algebraic subgroup $H_n = \{(\mathbf{x}, \mathbf{x}^n), \mathbf{x} \in \mathbb{G}_m^r\}$ of dimension $\dim H_n = r \leq \text{codim} X$.

Thus it would be tempting to apply the former Bounded Height Conjecture = Habegger's theorem. Nevertheless, such result gives us no information here, since X is *degenerate*: in the notation of [8] we have $X^{\text{oa}} = \emptyset$.

This failure is not surprising, because in the degenerate cases we cannot hope to have in general bounded height in the whole union $\cup_n (X \cap H_n)$. However, our Corollary shows that, at least, bounded height is recovered in the projection of the whole union to the first factor.

In the special case $V = \mathcal{C}$, this kind of problem has been intensively studied by Bays and Habegger in [1], who show (under suitable assumptions) the *finiteness* of the set of $P \in \mathcal{C}$ such that $[n]P \in \mathcal{C}$ for some $n \geq 2$.⁷ As an important tool they first observe that the height in this set is bounded, except in the trivial case when \mathcal{C} is a translate of a subtorus (op.cit., Lemma 6). This last result is a rather direct consequence of a generalized Vojta's inequality due to Rémond [18]. This finiteness result provides further evidence of the usefulness of height bounds as in this paper (see Remark 1.11 at the the end of this introduction to more on finiteness).

⁶Let $X \subseteq \mathbb{G}_m^r$ be a subvariety. Define X^{oa} as the complement in X of the union of unlikely intersections of positive dimension, namely the components of some positive dimension $\delta > 0$ of some intersection $X \cap B$, where B is a translate of an algebraic subgroup and $\dim B \leq \delta + \text{codim} X - 1$. By the former Bounded Height Conjecture, the Weil height is bounded in the intersection of X^{oa} with the union of algebraic subgroups of dimension $\leq \text{codim} X$.

⁷This question was originated by A. Levin; see [24].

1.3. A special case. As another special case, let us consider a simple instance of Corollary 1.6. Let \mathcal{C} be the line in the plane defined by $y - x = 1$ and let $V = \mathcal{C}$. Corollary 1.6 for these data asserts that the height of the roots of $(t + 1)^n - t^n - 1$ is uniformly bounded for $n \geq 2$. All of these polynomials have the simple root $t = 0$ and may have cyclotomic factors $t + 1, t^2 + t + 1$ (depending on the congruence class of $n \bmod 2$ and 3). For $n \geq 2$, the n -th Mirimanov polynomial ([16]) F_n is defined as $(t + 1)^n - t^n - 1$ deprived of these factors. Beukers [2] proved that the polynomials F_n and F_m do not have non-trivial common factors for $n \neq m$. As a crucial intermediate result (see *op.cit.*, Lemma 3.5), he shows that for a root t of F_n the (non-logarithmic, projective) height of $(1 : t : t + 1) \in \mathbb{P}^2(\overline{\mathbb{Q}})$ is at most 216, thus giving a positive answer to our bounded-height question in this special case.

Remark 1.7. HEIGHTS AND MAHLER'S MEASURE. For these polynomials $(t + 1)^n - t^n - 1$ studied by Beukers, and similarly for polynomials of the shape $\sum_{i=1}^r c_i P_i(t)^n$, $n \in \mathbb{N}$, the Mahler's measure (see [6]) is easily estimated as being $\ll C^n$. By standard comparison results, this would imply bounded height for the roots, as soon as we knew irreducibility of the polynomial, or at least a lower bound $\gg n$ for the degrees of irreducible factors (apart from fixed factors). However this (quasi)irreducibility, if at all holding for all large n , seems very hard to prove, and indeed in Beukers' case is a crucial feature behind and beyond his main issue.⁸

Remark 1.8. A PLEASANT EXERCISE. The mentioned proof of Beukers as it stands does not apply to substantially more general cases, e.g. on replacing $1, t, 1 + t$ with higher degree polynomials and/or working with a larger number of polynomials, or even rational or algebraic functions. We shall further comment on this point below, now let us observe that in such extended cases we would very soon meet for instance equations of the shape

$$c_1 P_1(t)^n + \dots + c_r P_r(t)^n = 0,$$

where the c_i are given algebraic numbers, the P_i are given polynomials over $\overline{\mathbb{Q}}$ and n varies through all integers such that the left side is not identically zero. It does not appear easy to analyse the union U of the sets of solutions, for varying n ; it does not seem obvious to us even *to prove that U has infinite complement in $\overline{\mathbb{Q}}$* (or at least to exclude that U equals the whole $\overline{\mathbb{Q}}$). At least when no ratio P_i/P_j , $i \neq j$, is constant, this can be indeed proved⁹ by looking at the absolute values of the terms, a pleasant exercise for the interested reader; but we do not see other easy methods, still less how to prove in some standard direct way more stringent properties of U .

⁸See [2] also for the distribution of the zeros of F_n ; in particular, one finds that there are several zeros with unbounded complex absolute value, which would imply unbounded height if these large zeros were somewhat 'densely distributed' in a possible factor over \mathbb{Q} .

⁹For certain abelian analogues, even to establish this mere fact does not seem even effectively decidable with known methods; see the appendix by Victor Flynn in [15], where an example is given (discussed also in 1.10 below) for which one can exclude certain infinite sets of values of the parameter. However that method, a modern version of Chabauty's, is not guaranteed to apply generally.

Our present results imply in particular that U in fact has bounded height except in well classified cases.

Let us now comment further on Beukers' proof. The starting point is an explicit construction of Padé approximants to powers of the linear polynomial $1 - t$, which goes back to [3] and involves hypergeometric polynomials. This leads to identities of the form $A(t)t^n + B(t)(1 - t)^n = C(t)$ (suitable for $x + y = 1$). In our general situation we cannot hope to use such an explicit construction; just to mention one indication in this respect, results of Bombieri-P.B. Cohen (see [5]) suggest that in general the coefficients of these Padé approximants have a height growing much faster (*i. e.* like $\exp(cn^2)$) than in the case of Beukers (when it grows like $\exp(cn)$), and this would destroy the basic estimates necessary for the method to go through.

We use instead Thue's Method for avoiding such explicit constructions; this involves divided derivatives, Siegel's Lemma and a zero estimate based on Wronskians. However, to deal with certain unexpected vanishings we have to introduce a quite intricate descent (whose structure is different with respect to other investigations using Wronskians). This allows us to prove the following explicit version of a special case of Theorem 1.3.

Theorem 1.9. *Let $r \geq 2$ and $f_1, \dots, f_r \in \mathbb{F}$ be non-zero rational functions such that f_i/f_j is non-constant for some i and j . Then there exist positive real numbers C depending only on f_1, \dots, f_r , having the following properties. Let $\alpha = (\alpha_1 : \dots : \alpha_r) \in \mathbb{P}^{r-1}(\overline{\mathbb{Q}})$. Consider, for a natural number n , a solution $P \in \mathcal{C}(\overline{\mathbb{Q}})$ of the equation*

$$\alpha_1 f_1(P)^n + \dots + \alpha_r f_r(P)^n = 0.$$

Then, if $n \geq C$ and if there are no proper vanishing subsums, we have

$$(1.1) \quad h(P) \leq \frac{rh(\alpha)}{n} + C.$$

Our method provides even more explicit bounds: see Theorem 3.1 in Section 3.

We remark that the dependence in α in equation (1.1) above is essentially sharp, as we can easily see using Siegel's Lemma (*cf* remark 3.2, iii)).

We finally remark that the assumption that n is sufficiently large is necessary: we may have $\alpha_1 f_1^n + \dots + \alpha_r f_r^n = 0$ identically for some small values of n .

Reciprocally, our results implies that $\alpha_1 f_1^n + \dots + \alpha_r f_r^n$ cannot in fact be zero for *infinitely many* n . This last fact may be also derived directly on using the main result of [10] (*i. e.* a general form of the *abc*-inequality over function fields).¹⁰

Of course the use of Thue's Method in classical diophantine approximation is well-known to lead to results which are usually not effective. By contrast all the results of this paper are effective. For example with $t^n + (1 - t)^n = 1$ (and $n \geq 2$) they lead quickly with hardly any effort for precision to $H(t) \leq 10^{120}$ for the (non-logarithmic) height (in fact Beukers's

¹⁰It is no coincidence that the proof of this uses Wronskians, which also appear in our arguments; on the other hand, no consideration of heights appears in [10].

result implies the bound 216).

We shall prove Theorem 1.9 in Section 3, and we shall deduce Theorem 1.3 from it in Section 4.

Remark 1.10. ABELIAN ANALOGUES? In this paper we limit ourselves to the toric case, but analogous questions and statements can be naturally formulated in the abelian context, thus extending Silverman's very setting.

A specific example of what would be the analogue of our main issue for that case appears in the paper [15], where we find the pencil \mathcal{J} of Jacobians J_t of the curves $H_t : y^2 = x^6 + x + t$, of generic genus 2, parametrized by $t \in \mathbb{A}^1$; we also find the section $\sigma : \mathbb{A}^1 \rightarrow \mathcal{J}$, obtained by defining $\sigma(t) \in J_t$ as the class of the divisor $\infty_+ - \infty_-$, difference of the two poles of x on H_t (let us forget here about the values for which H_t has genus < 2). One can now consider the subvariety V of \mathcal{J} obtained as the union of H_t (embedding H_t in J_t e.g. through ∞_+).

The question now is: *What can be said about the points $t_0 \in \mathbb{A}^1$ such that $[n]\sigma(t_0) \in H_{t_0}$, for some $n = n(t_0) > 3$?*¹¹ Now, similarly to Remark 1.8, even to prove that this set of t_0 (necessarily algebraic) has infinite complement in $\overline{\mathbb{Q}}$ is far from being evident; the Appendix by Flynn to [15] in particular achieves this, and much more, providing nontrivial congruence conditions on the suitable t_0 . However Flynn's method (a version of Chabauty's) is not guaranteed to work generally, and moreover the question of whether the set of these numbers has or not bounded height remains open. In fact, we do not know if and how the present methods can be adapted to the abelian context. It seems to us a rather interesting issue to obtain such an abelian analogue.

Remark 1.11. FINITENESS? As already explained at the beginning of this introduction, Theorem 1.1 was a main ingredient to prove the finiteness result of [7]. We expect to prove more general statements as natural complements to our Theorem 1.3. Let us discuss only two special and relevant issues.

Let \mathcal{C} be a curve of \mathbb{G}_m^N , not contained in any translate of a proper subtorus, and $V \subsetneq \mathbb{G}_m^N$ be a proper subvariety. Consider, as in Corollary 1.6, the set of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that there exists an integer n with $[n]\mathcal{C} \not\subseteq V$ and $[n]P \in V$. If $\text{codim}(V) > 1$ we expect not only bounded height but moreover *finiteness*. A special instance is a conjecture stated by Levin, which predicts finiteness if V is a curve (and $N \geq 3$). As already mentioned at the end of § 1.2, Bays and Habegger [1] prove (with an additional technical assumption) Levin's conjecture in the further special case $V = \mathcal{C}$. The ingredients of their proof was a bounded height result of Rémond, Baker lower bound for linear forms in logarithms and Pila-Wilkie Theorem on integral points. It is our intention to apply our main theorem in order to extend this kind of result to a more general setting.

In a direction similar to example 2 of § 1.1, consider now two linear recurrent sequences $(u_m(t))$, $(v_n(t))$ of polynomials, e.g. over \mathbb{Q} . 'Generically'

¹¹We exclude here $n = 3$ because $[3]\sigma(t) \in H_t$ identically.

one should expect u_m, v_n to be usually coprime, except in special circumstances. Now, our results indeed open the way to a complete classification of the cases when u_m and v_n have instead a common root, by bounding the height of such possible roots; this allows other arguments to go through. The above mentioned result of Beukers (§ 1.3) would appear as the very special case when $u_m = v_m =$ the m -th Mirimanov polynomial.

Naturally, a number of details are necessary to obtain precise statements and proofs, and these are the object of work in preparation.

2. NOTATION AND AUXILIARY RESULTS.

2.1. Rational functions. Given $f_1, \dots, f_r \in \mathbb{F}$ not all zero, we put¹²

$$\operatorname{div}(f_1, \dots, f_r) := \sum_P \min_j \operatorname{ord}_P(f_j) P.$$

Note that $\deg(\operatorname{div}(f_1, \dots, f_r)) = \deg(\operatorname{div}(gf_1, \dots, gf_r))$ for any nonzero $g \in \mathbb{F}$.

Let us denote $d := -\deg(\operatorname{div}(f_1, \dots, f_r))$. We remark that $d \geq 0$, since for example if $f_1 \neq 0$ we have $\sum_P \min_j \operatorname{ord}_P(f_j) \leq \sum_P \operatorname{ord}_P(f_1) = 0$. Moreover $d > 0$ if and only if some f_i/f_j is non-constant.

As a special case, let f be a non-zero rational function on \mathcal{C} . We define as usual its degree $d(f)$ as the the degree of the polar divisor $\operatorname{div}(f)_\infty = -\operatorname{div}(1, f)$. This is the geometric height of $(1 : f) \in \mathbb{P}^1(\mathbb{F})$.

An arithmetic height on \mathbb{F} . We define an arithmetic height $h(\cdot)$ of a rational function f on \mathcal{C} as follows. We choose once and for all a non-constant $t \in \mathbb{F} \setminus \overline{\mathbb{Q}}$. Let $F(X, Y) \in \overline{\mathbb{Q}}[X, Y]$ be the irreducible polynomial such that $F(t, f) = 0$ (note that F has degree at most $d(f)$ in X and at most $d(t)$ in Y).

Definition 2.1. For a function $f \in \mathbb{F}$, we define the height $h(f)$ as the projective Weil height of the vector of the coefficients of F .

Clearly $h(1/f) = h(f)$. Also, this coincides with the affine height on $\overline{\mathbb{Q}}[t]$ (if \mathcal{C} is the affine line and $f = P(t)$ is a polynomial then $F(X, Y) = P(X) - Y$.)

We shall need the following elementary estimates for this height.

Lemma 2.2. *There is a constant c depending only on \mathcal{C} and t with the following properties. Let $f \in \mathbb{F}$. Then we have*

- (i) $h(f^n) \leq nh(f) + cnd(f)$ for any positive integer n ,
- (ii) $h(f') \leq c(h(f) + d(f)), d(f') \leq cd(f)$ for $f' = df/dt$,
- (iii) $h(\operatorname{tr} f) \leq h(f) + \log d(t), d(\operatorname{tr} f) \leq d(t)d(f)$ for the trace tr from \mathbb{F} to $\overline{\mathbb{Q}}(t)$,
- (iv) for $g \in \mathbb{F}$ we have

$$\max\{h(f + g), h(fg)\} \leq c(h(f) + h(g) + d(f) + d(g)),$$

- (v) for any non-constant $s \in \mathbb{F} \setminus \overline{\mathbb{Q}}$ there is C depending on s (and \mathcal{C}, t) such that the height of f with respect to s is at most $C(h(f) + d(f))$.

¹²Since we have chosen a smooth projective model of \mathcal{C} , the closed points over $\overline{\mathbb{Q}}$ of the curve correspond to the places of its function field $\mathbb{F} = \overline{\mathbb{Q}}(\mathcal{C})$.

Proof. We use resultants. Let $F(X, Y)$ be the irreducible polynomial for f over $\overline{\mathbb{Q}}(t)$: $F(t, f) = 0$. For (i) we can assume $d(f) \geq 1$, and we take the resultant of $F(X, Y)$ and $Y^n - Z$ with respect to Y to get a non-zero polynomial $F_n(X, Z)$ with $F_n(t, f^n) = 0$. Its degrees in X, Z are at most $nd(f), d(t)$ respectively. And its height is at most $n(h(f) + c)$; here one must be careful to avoid a factorial in the number N of terms in the Sylvester determinant, but it is easy to see that $N \leq (d(t) + 1)^n 2^{d(t)} \leq c^n$. Now F_n might not be irreducible, but by well-known estimates the height of any factor is at most

$$n(h(f) + c) + c(nd(f) + d(t)) \leq nh(f) + cnd(f);$$

and (i) follows.

A similar argument works with (v), now taking the resultant of $F(X, Y)$ and $S(X, Z)$ with respect to X , where $S(t, s) = 0$. Also with (iv): now say $G(t, g) = 0$ and then for $f + g$ we take the resultant of $F(X, Y)$ and $G(X, Z - Y)$ with respect to Y ; then we do g/f with $F(X, Y)$ and $G(X, ZY)$ and deduce fg using $h(1/f) = h(f)$.

And (iii) is rather easy: if

$$F(X, Y) = F_0(X)Y^e + F_1(X)Y^{e-1} + \cdots + F_e(X)$$

then $\text{tr} f = -\frac{d(t)}{e} \frac{F_1(t)}{F_0(t)}$. Finally for (ii) we can also assume $d(f) \geq 1$ and then we note that

$$f' = -\frac{F_0'(t)f^e + \cdots + F_e'(t)}{eF_0(t)f^{e-1} + \cdots + F_{e-1}(t)}$$

so that we can use (iv). □

As at the beginning, we choose once and for all a system of Weil's functions associated to a divisor of degree 1 and a corresponding height h on $\mathcal{C}(\overline{\mathbb{Q}})$. By a well-known result of Néron (see [17]), the height of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ differs from $h(t(P))/d(t)$ by an error term bounded by a constant multiple of $1 + h(P)^{1/2}$.

We need the following functorial bound for the arithmetic height associated to values of (f_1, \dots, f_r) , which is an easy consequence of "Weil's Height Machine":

Lemma 2.3. *For $r \geq 2$ let $f_1, \dots, f_r \in \mathbb{F}$ and $P \in \mathcal{C}(\overline{\mathbb{Q}})$, not a pole or a common zero of f_1, \dots, f_r . Put $d := -\deg \text{div}(f_1, \dots, f_r)$. Then the projective Weil height*

$$h(f_1(P) : \cdots : f_r(P)) = dh(P) + O(1 + h(P)^{1/2})$$

where the implicit constant in the big- O may depend on f_1, \dots, f_r but not on P .

Proof. Let $E' = -\text{div}(f_1, \dots, f_r)$. We may assume f_1, \dots, f_r linearly independent over $\overline{\mathbb{Q}}$; indeed, we may select a basis, say f_1, \dots, f_s ($s \geq 2$), of the $\overline{\mathbb{Q}}$ -vector space generated by f_1, \dots, f_r and observe that $\text{div}(f_1, \dots, f_s) = \text{div}(f_1, \dots, f_r)$ and $h(f_1(P) : \cdots : f_r(P)) = h(f_1(P) : \cdots : f_s(P)) + O(1)$.

Given a divisor D , we denote by h_D the height on $\mathcal{C}(\overline{\mathbb{Q}})$ associated to it, which is defined up to the addition of a bounded term: see [13], Part B for details.

Let $\phi: \mathcal{C} \rightarrow \mathbb{P}^{r-1}$ be the morphism $P \mapsto (f_1(P) : \dots : f_r(P))$ and H be an hyperplane of \mathbb{P}^{r-1} . Then ϕ^*H is linearly equivalent to E' (indeed, for $i = 1, \dots, r-1$, set $g_i = f_i/f_r$; then $\phi^*\{x_r = 0\} = -\text{div}(g_1, \dots, g_{r-1}, 1) \sim E'$).

Thus $h(f_1(P) : \dots : f_r(P)) = h_{E'}(P) + O(1)$. Here and in the rest of this proof, the big- O depend on the divisors.

We now apply Theorem B.5.9 of [13]) (which goes back to Néron), taking as the ample divisor the divisor D of degree 1 such that $h = h_D + O(1)$ and as the divisor equivalent to zero the divisor $E = dD - E'$. We obtain $h_E(P) \leq c(1 + h_D(P)^{1/2})$. Moreover, by Theorem B.3.2 (d) of [13], $h_E = dh - h_{E'} + O(1)$. Thus

$$\begin{aligned} h(f_1(P) : \dots : f_r(P)) &= h_{E'}(P) + O(1) \\ &= dh(P) - dh_E(P) + O(1) \\ &= dh(P) + O(1 + h(P)^{1/2}). \end{aligned}$$

□

UNIFORMITY. The question of how the implicit constants in the O -terms in the lemma depend on the functions is a subtle one, and has been treated in number of papers, which control this dependence in data as the degree and heights of the functions. Here we can prove (at the expense of extra complication) the upper bound

$$dh(P) + O((d + \max h(f_i))(1 + h(P)^{1/2}))$$

with an implicit constant in the big- O which depends only on \mathcal{C} and t . If some sort of refined Height Machine could deliver the analogous lower bound, even for $r = 2$, then it would imply at once some significant cases of our Theorem 1.9. For example with fixed different F_1, \dots, F_r in $\overline{\mathbb{Q}}[t]$ of degree $p \geq 1$ and fixed sufficiently general $\alpha_1, \dots, \alpha_r$ in $\overline{\mathbb{Q}}$ we would have $d = np$ for

$$f(t) = \frac{\alpha_1 F_1(t)^n + \dots + \alpha_{r-1} F_{r-1}(t)^n}{\alpha_r F_r(t)^n}.$$

Thus when $f(t) = -1$ we would deduce

$$0 = h(f(t)) \geq dh(t) - O(n(1 + h(t)^{1/2}))$$

so $h(t) = O(1)$.

However such a lower bound is false in general, as the example $f(t) = (t - 2^d)t^{d-1} - 1$ with $f(2^d) = -1$ shows; the lower bound would be

$$dh(2^d) - O(d(1 + h(2^d)^{1/2})) = d^2 \log 2 - O(d^{3/2}),$$

a contradiction for sufficiently large d .

We shall need also some good bounds for the values of higher derivatives $f^{(l)} = (d/dt)^l f$; or rather those of the divided derivatives $\delta_l f = f^{(l)}/l!$ essential to the success of Thue's Method. Iterating part (ii) of Lemma 2.2 does not suffice. In fact we have to consider certain monomial expressions whose curious weighting will soon be justified.

Lemma 2.4. *For any $f \in \mathbb{F}$ there is c , depending only on f and t , with the following property. Suppose f and t are regular at some $P \in \mathcal{C}(\overline{\mathbb{Q}})$ with $dt(P) \neq 0$. For any non-negative integer L let δ be any vector with components*

$$\delta_0 f(P)^{a_0} \delta_1 f(P)^{a_1} \cdots \delta_l f(P)^{a_l}$$

for non-negative exponents satisfying

$$a_0 + a_1 + \cdots + a_l \leq L, \quad a_1 + \cdots + l a_l \leq L.$$

Then the affine Weil height $h(\delta)$ is at most $cL(h(P) + 1)$.

Proof. This falls into the circle of Eisenstein-related ideas. We can assume that f is not constant. With $F(t, f) = 0$ as above and $\alpha_j = \delta_j f(P)$ the power series $y = \sum_{j=0}^{\infty} \alpha_j x^j$ satisfies $F_0(x, y) = 0$ with $F_0(X, Y) = F(t(P) + X, Y)$. We may therefore apply Theorem 1 (p.162) of Schmidt's paper [19]. He needs a number field k over which F_0 is defined. As he notes, the α_j lie in an extension K of k of relative degree at most the degree $e \geq 1$ of F_0 in Y ; thus $[K : k] \leq d(t)$. We find for each valuation v on k some $A_v \geq 1$, with $A_v = 1$ for all but finitely many v , such that

$$|\alpha_j|_w \leq A_v^{m+j} \quad (j = 0, 1, 2, \dots)$$

for any valuation w on K over v , where $m \geq 1$ is the degree of F_0 in X ; thus $m \leq d(f)$. Thus

$$|\alpha_0^{a_0} \alpha_1^{a_1} \cdots \alpha_l^{a_l}|_w \leq A_v^{m(a_0+a_1+\cdots+a_l)+(a_1+\cdots+l a_l)} \leq A_v^{2mL}.$$

It follows for the non-logarithmic height

$$H(\delta)^{[K:\mathbb{Q}]} \leq \prod_v A_v^{2mL[K:k]}.$$

The v are split into two sets $S_{\infty 1}, S_2$ with

$$\prod_{v \in S_{\infty 1}} A_v \leq ((m+1)(e+1)\sqrt{e})^{(2e+1)[k:\mathbb{Q}]} H(F_0)^{2e[k:\mathbb{Q}]} \leq (2H(F_0))^{c[k:\mathbb{Q}]},$$

$$\prod_{v \in S_2} A_v \leq (16m)^{11e^3[k:\mathbb{Q}]} H(F_0)^{(2e^3+2e)[k:\mathbb{Q}]} \leq (2H(F_0))^{c[k:\mathbb{Q}]},$$

where $H(F_0)$ is still projective (and absolute). Thanks to the crucial (but nowadays natural) linear dependence on $[k:\mathbb{Q}]$ in the exponents we deduce $H(\delta) \leq (2H(F_0))^{4cmL}$. Finally $H(F_0) \leq cH(t(P))^m$, and the result we want follows by relating $h(t(P))$ to $h(P)$ as described above.

We remark that the extra precision of [19] (especially concerning the set S_2) is not really necessary for us; thus by putting harmless additional restrictions on the point P we could have got ourselves into the ‘‘non-singular’’ situation (apparently all that was originally considered by Eisenstein), where the proofs are much easier (as for example in [6] p.360).

□

Given a divisor D we denote by $L(D)$ the finite-dimensional $\overline{\mathbb{Q}}$ -vector space

$$L(D) = \{f \in \mathbb{F}^*, \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

and by $l(D)$ its dimension. We shall need a good basis of $L(NQ)$ for fixed Q and large N . It is convenient to talk also of $L(\infty Q) = \sum_{N=1}^{\infty} L(NQ)$ the vector space of $f \in \mathbb{F}$ which are regular outside Q .

Lemma 2.5. *For any $Q \in \mathcal{C}(\overline{\mathbb{Q}})$ there is a positive integer Δ and real c together with $g, g_0, g_1, \dots, g_{\Delta-1}$ in $L(\infty Q)$, depending only on Q and t , such that the following hold for any $N \geq 1$.*

(i) *We have $d(g) = \Delta$, and the elements*

$$g_j g^k \quad (j = 0, 1, \dots, \Delta - 1, \quad d(g_j) + kd(g) \leq N)$$

form a basis for $L(NQ)$, with

$$N - c \leq l(NQ) \leq N + 1.$$

(ii) *For any f in $L(NQ)$ we have*

$$f = \sum \alpha_{jk} g_j g^k$$

with affine height

$$h(\dots \alpha_{jk} \dots) \leq c(h(f) + N),$$

as well as

$$h(f) \leq c(h(\dots \alpha_{jk} \dots) + N).$$

Proof. It is well known (for example by Riemann-Roch) that as $f \neq 0$ varies over $L(\infty Q)$ the $-\operatorname{ord}_Q(f)$ (which are none other than the degrees $d(f)$) take all sufficiently large values. Let $\Delta \geq 1$ be the smallest positive value, attained by some g in $L(\infty Q)$. If $\Delta = 1$ we are done, as a standard argument of killing poles shows (which will be repeated below).

So we may and shall assume $\Delta \geq 2$. For $j = 1, \dots, \Delta - 1$ pick g_j in $L(\infty Q)$ with $n_j = -\operatorname{ord}_Q(g_j) \equiv j$ modulo Δ and also as small as possible; here $n_j > 0$ is automatic and even $n_j > \Delta$. We define $g_0 = 1$ and $n_0 = 0$. We show by induction on N that these do the trick in (i).

Pick any $f \neq 0$ in $L(NQ)$, so that $n = -\operatorname{ord}_Q(f) \leq N$. If $n \equiv j$ modulo Δ ($j = 0, 1, \dots, \Delta - 1$) then $n \geq n_j$ and we can find α in $\overline{\mathbb{Q}}$ with $f - \alpha g_j g^{(n-n_j)/\Delta}$ in $L((N-1)Q)$. As

$$d(g_j) + \frac{n - n_j}{\Delta} d(g) = n \leq N$$

this shows by induction that the elements in (i) span $L(NQ)$. They are certainly linearly independent, as the

$$-\operatorname{ord}_Q(g_j g^k) = n_j + k\Delta \quad \left(j = 0, 1, \dots, \Delta - 1, \quad k = 0, 1, \dots, \left\lfloor \frac{N - n_j}{\Delta} \right\rfloor \right)$$

are all different (even for all k). In particular

$$l(NQ) = \sum_{j=0}^{\Delta-1} \left(1 + \left\lfloor \frac{N - n_j}{\Delta} \right\rfloor \right),$$

which leads easily to the required estimates using $n_j \geq \Delta$ for $j > 0$. So (i) is proved. Incidentally it is not difficult to estimate the constants so far solely in terms of the genus of \mathcal{C} .

For (ii) we note that this last argument even shows that $g_0, g_1, \dots, g_{\Delta-1}$ are linearly independent over $\overline{\mathbb{Q}}(g)$. As $\Delta = d(g) = [\mathbb{F} : \overline{\mathbb{Q}}(g)]$ it follows that they form a basis of \mathbb{F} over $\overline{\mathbb{Q}}(g)$. Now we can write

$$f = \sum_{j=0}^{\Delta-1} g_j G_j$$

with $G_j = \sum_k \alpha_{jk} g^k$ in $\overline{\mathbb{Q}}[g]$. In the standard way we multiply by $g_0, g_1, \dots, g_{\Delta-1}$ and take the trace from \mathbb{F} to $\overline{\mathbb{Q}}(g)$. The resulting equations can be solved for $G_0, G_1, \dots, G_{\Delta-1}$. It follows easily from Lemma 2.2 (iii) (with g not t), (iv), (v) that

$$\sum_{j=0}^{\Delta-1} h(G_j) \leq c(h(f) + N).$$

But the affine height in (ii) is at most the analogous sum with heights taken with respect to g . So the first of the two required inequalities follows with another appeal to Lemma 2.2 (v). The second is similar but easier. \square

Before we go further we record the following identity for divided derivatives. Namely

$$\delta_l(f^n) = f^{n-l} \sum_{\mathbf{a}} C(\mathbf{a}) \delta_0(f)^{a_0} \delta_1(f)^{a_1} \dots \delta_l(f)^{a_l}$$

where the sum is taken over all $\mathbf{a} = (a_0, a_1, \dots, a_l)$ with non-negative coordinates satisfying

$$|\mathbf{a}| = a_0 + a_1 + \dots + a_l = l, \quad a_1 + \dots + la_l = l$$

(see earlier) and the $C(\mathbf{a})$ are non-negative integers. Some version for undivided derivatives is attributed to the Blessed Francesco Faà di Bruno (who even has $\phi(f)$ instead of f^n), but in this divided form we get an immediate proof by formally writing $\tilde{f} = \sum_{m=0}^{\infty} \delta_m(f) T^m$. Note that, formally, $\tilde{f}(x) = f(x+T)$.

Taking this into account one may then go ahead by picking out the coefficient of T^l in \tilde{f}^n . We take $\delta_0(f) = f$ in a'_0 of the factors \tilde{f} , and then $\delta_1(f)$ in a_1 of the factors, and so on. Then $a'_0 + a_1 + \dots + a_l = n$ and $a_1 + \dots + la_l = l$ making it clear that $a_0 = a'_0 - (n-l) \geq 0$. We need also good estimates for the $C(\mathbf{a})$, but it is similarly clear that their sum is majorized by the coefficient of T^l in $(1+T+T^2+\dots)^n = (1-T)^{-n}$, which is

$$(-1)^l \binom{-n}{l} = \binom{n+l-1}{l} \leq 2^{n+l}$$

and in particular factorial-free.

Now comes our basic ‘‘auxiliary polynomial’’.

Lemma 2.6. *Let f_1, \dots, f_r be in $\overline{\mathbb{Q}}(\mathcal{C})$, let Q in $\mathcal{C}(\overline{\mathbb{Q}})$ be not a pole or zero of f_1, \dots, f_r also with $dt(Q) \neq 0$, and write as before $d = -\deg \operatorname{div}(f_1, \dots, f_r) \geq$*

0. Then there are c_0, c depending only on $\mathcal{C}, f_1, \dots, f_r, t$ and Q with the following property. For any non-negative integers n, M_1, \dots, M_r with $S > M + nd + c_0$ for

$$S = M_1 + \dots + M_r, \quad M = \max\{M_1, \dots, M_r\}$$

define the “Dirichlet exponent”

$$\varrho = \frac{M + dn}{S - M - dn - c_0}.$$

Then there are $A_1 \in L(M_1Q), \dots, A_r \in L(M_rQ)$, not all zero and with heights at most $c(\varrho + 1)(M + n)$, such that $A_1 f_1^n + \dots + A_r f_r^n = 0$.

Proof. By Lemma 2.5 (i) we can take

$$(2.1) \quad A_i = \sum \alpha_{ijk} g_j g^k \quad (i = 1, \dots, r)$$

with j, k satisfying

$$j = 0, 1, \dots, \Delta - 1, \quad d(g_j) + kd(g) \leq M_i$$

and algebraic numbers α_{ijk} to be determined. We first find them such that if $\phi = \sum_{i=1}^r A_i f_i^n \neq 0$ then

$$(2.2) \quad \text{ord}_Q \phi > T$$

where the integer T is nearly as large as linear algebra allows.

If U is the number of unknowns α_{ijk} then their vector α must lie in a certain subspace V of $\overline{\mathbb{Q}}^U$. Here $U = \sum_{i=1}^r l(M_iQ)$ so Lemma 2.5(i) gives

$$(2.3) \quad S - c \leq U \leq S + r.$$

We have $E = T + 1$ equations, so the dimension D of V satisfies

$$(2.4) \quad D \geq U - E \geq S - T - c_0.$$

Thus we assume $T < S - c_0$ for solvability. But if T is too near S then as in Thue’s Method we would lose control of the heights. To regain this we use the version of the Absolute Siegel Lemma proved by David and Philippon [11]; for example taking $\epsilon = 1$ in the estimate at the bottom of page 523 we find non-zero $\alpha \in V$ with

$$(2.5) \quad h(\alpha) \leq \frac{h(V)}{D} + \frac{1}{2} \log D + 1$$

where $h(V)$ is the euclidean height.

To estimate $h(V)$ we note that it is defined by certain equations, and by Hadamard (here the factorials don’t matter) we get

$$(2.6) \quad h(V) \leq cE(\log E + h_{\text{eq}})$$

where h_{eq} is an upper bound for the logarithmic euclidean height of each equation. These are $\delta_l \phi(Q) = 0$ or more explicitly

$$\sum_{i=1}^r \sum_{j,k} \alpha_{ijk} \beta_{ijkl} = 0 \quad (l = 0, 1, \dots, T)$$

with $\beta_{ijkl} = \delta_l(g_j g^k f_i^n)(Q)$. Now

$$\delta_l(g_j g^k f_i^n) = \sum \delta_s(g_j) \delta_p(g^k) \delta_q(f_i^n)$$

taken over all non-negative integers s, p, q with $s + p + q = l$. By Faà di Bruno

$$\delta_p(g^k) = g^{k-p} \sum_{\mathbf{a}} C(\mathbf{a}) \delta_0(g)^{a_0} \delta_1(g)^{a_1} \cdots \delta_p(g)^{a_p},$$

$$\delta_q(f_i^n) = f_i^{n-q} \sum_{\mathbf{b}} C(\mathbf{b}) \delta_0(f_i)^{b_0} \delta_1(f_i)^{b_1} \cdots \delta_q(f_i)^{b_q}.$$

Now we see without difficulty thanks to Lemma 2.4 that $h_{\text{eq}} \leq c(T + M + n)$. Then (2.5) and (2.6) lead to

$$h(\boldsymbol{\alpha}) \leq c \frac{(T + M + n)(T + 1)}{S - T - c} + \log(S + c)$$

because $S - T - c_0 \leq D \leq U \leq S + r$ by (2.3) and (2.4).

We now choose T so large that the condition (2.2) forces after all $\phi = 0$ in the sense that $\text{ord}_Q \phi = \infty$. In fact (2.2) holds also for $\tilde{\phi} = \phi / f_r^n$ because $f_r(Q) \neq 0$, and since A_1, \dots, A_r are in $L(MQ)$ it is easy to see that $d(\tilde{\phi}) \leq M + dn$. So $T = M + dn$ will do, leading to

$$h(\boldsymbol{\alpha}) \leq c\rho(M + dn) + \log(S + c).$$

Finally Lemma 2.5 (ii) gets us to $h(A_i)$ by (2.1); and then we use $S \leq rM$. \square

2.2. Orthogonal spaces and key lemma.

Let \mathbf{w} be a fixed vector of \mathbb{C}^n with all entries non-zero. Given a subset Λ of $\{1, \dots, r\}$ we consider the vector space

$$V_\Lambda = V_{\Lambda, \mathbf{w}} = \{\mathbf{v} \in \mathbf{w}^\perp \mid \forall j \notin \Lambda, v_j = 0\}.$$

Thus $V_\emptyset = \{0\}$ and $\dim V_\Lambda = |\Lambda| - 1$ if $\Lambda \neq \emptyset$. We clearly have $V_{\Lambda_1} \cap V_{\Lambda_2} = V_{\Lambda_1 \cap \Lambda_2}$.

Remark 2.7. Let Λ_1, Λ_2 be non-empty subsets of $\{1, \dots, r\}$. If $\Lambda_1 \cap \Lambda_2 \neq \emptyset$ we have

$$V_{\Lambda_1} + V_{\Lambda_2} = V_{\Lambda_1 \cup \Lambda_2}$$

while $V_{\Lambda_1} + V_{\Lambda_2}$ is a subspace of $V_{\Lambda_1 \cup \Lambda_2}$ of codimension 1 if $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Proof. Let us assume $\Lambda_1 \cap \Lambda_2 \neq \emptyset$. The displayed formula follows from the trivial inclusions $V_{\Lambda_i} \subseteq V_{\Lambda_1 \cup \Lambda_2}$ and from the equality of dimensions:

$$\begin{aligned} \dim(V_{\Lambda_1} + V_{\Lambda_2}) &= \dim(V_{\Lambda_1}) + \dim(V_{\Lambda_2}) - \dim(V_{\Lambda_1} \cap V_{\Lambda_2}) \\ &= \dim(V_{\Lambda_1}) + \dim(V_{\Lambda_2}) - \dim(V_{\Lambda_1 \cap \Lambda_2}) \\ &= (|\Lambda_1| - 1) + (|\Lambda_2| - 1) - (|\Lambda_1 \cap \Lambda_2| - 1) \\ &= |\Lambda_1 \cup \Lambda_2| - 1 = \dim(V_{\Lambda_1 \cup \Lambda_2}). \end{aligned}$$

The last assertion follows similarly. \square

In order to generalise this simple remark, we introduce the following definition.

Definition 2.8. Let $\Gamma = \{\Lambda_1, \dots, \Lambda_s\}$ be a collection of subsets of $\{1, \dots, r\}$. We say that a subset C of $\{1, \dots, r\}$ is a connected component of Γ , if, after renumbering $\Lambda_1, \dots, \Lambda_s$ if necessary, there exists an integer k with $1 \leq k \leq s$ such that

- i) $C = \Lambda_1 \cup \dots \cup \Lambda_k$;
- ii) for $j = 1, \dots, k-1$ we have $\Lambda_{j+1} \cap (\Lambda_1 \cup \dots \cup \Lambda_j) \neq \emptyset$;
- iii) for $j = k+1, \dots, s$ we have $C \cap \Lambda_j = \emptyset$.

We say that Γ is connected, if it has only one connected component.

We may also rephrase this definition, as follows: consider first the graph on $\{1, \dots, s\}$ defined by joining i, j if and only if $\Lambda_i \cap \Lambda_j \neq \emptyset$. Then a connected component in our sense is a union $C = \bigcup_{i \in U} \Lambda_i$, where U is a connected component, in the usual sense, of the graph just defined.

As an example, $\Gamma = \{\{1, 2\}, \{3, 4, 5\}, \{2, 5\}, \{6, 7, 8\}, \{7, 8, 9\}\}$ has two connected components, $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8, 9\}$.

By Remark 2.7 we easily see that:

Remark 2.9. Let $\Gamma = \{\Lambda_1, \dots, \Lambda_s\}$ be a collection of subsets of $\{1, \dots, r\}$ and let C_1, \dots, C_p be the connected components of Γ . Then,

- i) $V_{\Lambda_1} + \dots + V_{\Lambda_s} = V_{C_1} + \dots + V_{C_p}$.
- ii) $\dim(V_{\Lambda_1} + \dots + V_{\Lambda_s}) = |\Lambda_1 \cup \dots \cup \Lambda_s| - p$.

The following definition is crucial for our purposes.

Definition 2.10. Let V be a $\overline{\mathbb{Q}}$ -vector space, v_1, \dots, v_s be $s \geq 2$ vectors of V . Let $a_1 v_1 + \dots + a_s v_s = 0$ be a non-trivial linear relation. We say that this relation is minimal if there are no non-trivial relations $\sum b_i v_i = 0$ over a proper non-empty subset of $\{1, \dots, s\}$.

We remark that the relation $a_1 v_1 + \dots + a_s v_s = 0$ is minimal if and only if $a_1, \dots, a_s \in \overline{\mathbb{Q}}^*$ and $\dim\langle v_1, \dots, v_s \rangle = s - 1$.

We also remark that, given $v_1, \dots, v_s \in V$ linearly dependent and not all zero, there exists a subset $\Lambda \subseteq \{1, \dots, s\}$ such that $\{v_i\}_{i \in \Lambda}$ satisfy a minimal linear relation.

We now agree on some conventions which will be followed in the rest of this section and in the next section.

We fix as above $r \geq 2$ rational functions $f_1, \dots, f_r \in \mathbb{F} \setminus \{0\}$ and a non-constant $t \in \mathbb{F}$. We define S_0 as the finite set consisting of all zeros and poles of f_1, \dots, f_r, dt . We fix a point $Q \in \mathcal{C}(\overline{\mathbb{Q}}) \setminus S_0$ and we define $S = S_0 \cup \{Q\}$. We choose once and for all a positive integer Δ , a real c and $g, g_0, g_1, \dots, g_{\Delta-1} \in L(\infty Q)$ depending only on Q and t and satisfying the statement of Lemma 2.5. The implicit constants in the big- O below will depend only on these data.

The next lemma is the main tool in our construction.

Lemma 2.11. *Let $\Lambda \subseteq \{1, \dots, r\}$ be of cardinality ≥ 2 and $(M'_i)_{i \in \Lambda}$ be positive integers with maximum M' . Define*

$$d_\Lambda = -\deg \operatorname{div}(f_i)_{i \in \Lambda}, \quad \Theta = \max \left(1, \sum_{i \in \Lambda} M'_i - (M' + nd_\Lambda) \right).$$

Let also $P \in \mathcal{C}(\overline{\mathbb{Q}}) \setminus S$, $n \in \mathbb{N}$ and put

$$\mathbf{w} = (f_1(P)^n, \dots, f_r(P)^n).$$

Let finally $\{A_i\}_{i \in \Lambda} \subset \mathbb{F}$ not all zero, such that $A_i \in L(M'_i Q)$. Let us assume that $\{A_i f_i^n\}_{i \in \Lambda}$ satisfy a minimal linear relation. Then there exists a basis of algebraic vectors $\mathbf{v}_1, \dots, \mathbf{v}_{|\Lambda|-1}$ of $V_{\Lambda, \mathbf{w}}$ which satisfies

$$h(\mathbf{v}_j) \leq M' h(P) + O \left(\Theta(h(P) + 1) + (n + M')(1 + h(P)^{1/2}) + \max h(A_i) \right).$$

Proof. In the proof we use a Wronskian argument. Let us first recall some basic facts on it. The derivative $\frac{d}{dt}$ on $\overline{\mathbb{Q}}(t)$ can be uniquely extended to \mathbb{F} . For $F \in \mathbb{F}$ and $R \in \mathcal{C}(\overline{\mathbb{Q}})$ we have

$$(2.7) \quad \begin{cases} \operatorname{ord}_R(dF/dt) = \operatorname{ord}_R(F) - 1 - \operatorname{ord}_R(dt), & \text{if } \operatorname{ord}_R(F) \neq 0; \\ \operatorname{ord}_R(dF/dt) \geq -\operatorname{ord}_R(dt), & \text{if } \operatorname{ord}_R(F) = 0. \end{cases}$$

The (normalized) *Wronskian* of $F_1, \dots, F_k \in \mathbb{F}$ with respect to t is the determinant

$$W(F_1, \dots, F_k) = \det \left(\frac{1}{j!} \frac{d^j F_i}{dt^j} \right)_{\substack{i=1, \dots, k \\ j=0, \dots, k-1}}.$$

It is well known that $W = 0$ if and only if the F_i 's are linearly dependent over $\overline{\mathbb{Q}}$.

Let us now go on with the proof of our lemma. We may assume $\Lambda = \{1, \dots, s\}$. For short we put $F_i = A_i f_i^n$ and $W_i = W(F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_s)$ for $i = 1, \dots, s$.

For later reference, we remark that F_1, \dots, F_s are S -integers (as elements of the function field \mathbb{F}): indeed the $\operatorname{div}(f_i)$ are supported in S and the A_i are also S -integers since $A_i \in L(M'_i Q)$. Moreover the zeros and the poles of dt are in S as well. Thus $d^l F_i / dt^l$ are S -integers (cf (2.7)), and so also W_s .

By assumption, we have a minimal linear relation $a_1 F_1 + \dots + a_s F_s = 0$. Thus F_1, \dots, F_{s-1} are linearly independent and $a_1, \dots, a_s \neq 0$. This proves that $W_s \neq 0$.

Let $1 \leq i \leq s-1$. Since $a_i \neq 0$ we can replace F_i by

$$-\frac{a_1}{a_i} F_1 - \dots - \frac{a_{i-1}}{a_i} F_{i-1} - \frac{a_{i+1}}{a_i} F_{i+1} - \dots - \frac{a_s}{a_i} F_s$$

in W_s . This shows that $W_s = \pm(a_s/a_i)W_i$.

We want to obtain a suitable upper bound for $m_0 := \operatorname{ord}_P(W_s)$. For this, we shall use the fact that W_s has already a big multiplicity at the zeros of F_i , since these functions are essentially n -th powers.

For $i = 1, \dots, s$ and for $l = 0, \dots, s-1$ we have (cf (2.7))

$$\operatorname{ord}_R \left(\frac{d^l F_i}{dt^l} \right) \geq \operatorname{ord}_R(F_i) - l(1 + \operatorname{ord}_R(dt)) = \operatorname{ord}_R(A_i) + n \operatorname{ord}_R(f_i) + O(1)$$

for any $R \in \mathcal{C}(\overline{\mathbb{Q}})$. Thus, for $i = 1, \dots, s$,

$$\begin{aligned} \text{ord}_R(W_s) &= \text{ord}_R(W_i) \geq \sum_{j \neq i} \left(\text{ord}_R(A_j) + n \text{ord}_R(f_j) + O(1) \right) \\ &= \sum_{j=1}^s n \text{ord}_R(f_j) - \lambda_{R,i} + O(1). \end{aligned}$$

where $\lambda_{R,i} := n \text{ord}_R(f_i) - \sum_{j \neq i} \text{ord}_R(A_j)$. We deduce:

$$\begin{aligned} \text{ord}_R(W_s) &\geq \max_{i=1, \dots, s} \left(\sum_{j=1}^s n \text{ord}_R(f_j) - \lambda_{R,i} + O(1) \right) \\ &= \sum_{j=1}^s n \text{ord}_R(f_j) - \lambda_R + O(1) \end{aligned}$$

where we have defined, for $R \in S$,

$$\lambda_R := \min_{i=1, \dots, s} \lambda_{R,i} = \min_{i=1, \dots, s} \left\{ n \text{ord}_R(f_i) - \sum_{j \neq i} \text{ord}_R(A_j) \right\}.$$

We shall use this inequality, for $R \in S$, in the functional (*i. e.* in \mathbb{F}) product formula, applied to W_s , namely the formula $\sum_R \text{ord}_R(W_s) = 0$.

In this formula, for $R = P \notin S$ we find the quantity $\text{ord}_P(W_s)$ that we have to estimate, whereas for R outside $S \cup \{P\}$ we use the trivial bound $\text{ord}_R(W_s) \geq 0$. Also, since the $\text{div}(f_i)$ are supported in $S_0 \subseteq S$, we have¹³ $\sum_{R \in S} \text{ord}_R(f_i) = \deg(\text{div}(f_i)) = 0$. Moreover $P \notin S$. Thus

$$0 = \sum_{\text{all } R} \text{ord}_R(W_s) \geq \text{ord}_P(W_s) + \sum_{R \in S} \text{ord}_R(W_s) \geq \text{ord}_P(W_s) - \sum_{R \in S} \lambda_R + O(1).$$

We now recall that $A_i \in L(M'_i Q)$ and that the f_i are supported in S_0 . Thus

$$\lambda_Q = \min_j \left\{ - \sum_{i \neq j} \text{ord}_Q(A_i) \right\} \leq \min_j \sum_{i \neq j} M'_i = \sum_{i=1}^s M'_i - M'$$

and, for $R \in S_0$,

$$\lambda_R \leq \min_i \{ n \text{ord}_R(f_i) \} = n d_\Lambda.$$

Collecting together these last three inequalities, we get the following sought upper bound for $m_0 = \text{ord}_P(W_s)$:

$$(2.8) \quad m_0 \leq \sum_{R \in S} \lambda_R + O(1) \leq \sum_{i \in \Lambda} M'_i - (M' + n d_\Lambda) + O(1) = \Theta + O(1).$$

We can now construct the desired basis $\mathbf{v}_1, \dots, \mathbf{v}_{|\Lambda|-1}$ of $V_\Lambda = V_{\Lambda, \mathbf{w}}$. For a non-negative integer ρ we put as before

$$\delta_\rho = \frac{1}{\rho!} \frac{d^\rho}{dt^\rho}$$

Given a vector of non-negative integers $\boldsymbol{\rho} = (\rho_1, \dots, \rho_{s-1})$ we let

$$W_{\boldsymbol{\rho}} = \det(\delta_{\rho_j} F_i)_{i,j=1, \dots, s-1}.$$

¹³We use the fact that S contains not only the poles but also the zeros of the f_i 's.

Thus $W_s = W_{(0,1,\dots,s-1)}$. It is also easily seen that

$$(2.9) \quad \delta_{m_0} W_s \in \sum_{|\rho|=e} \mathbb{Z} W_\rho$$

where $|\rho| = \rho_1 + \dots + \rho_{s-1}$ and $e = m_0 + 1 + \dots + (s-2)$. By (2.8) we have

$$(2.10) \quad e \leq \Theta + O(1).$$

Since $P \notin S$ and the zeros of dt are in S , we have $\text{ord}_P(dt) = 0$. By definition of m_0 and by (2.7) we have $\text{ord}_P(\delta_{m_0} W_s) = 0$. Again by (2.7) and since F_i are S -integers, for all ρ we have $\text{ord}_P(\delta_{m_0} W_\rho) \geq 0$. By (2.9), this implies that there exists ρ' with $|\rho'| = e$ such that $\text{ord}_P(W_{\rho'}) = 0$. For $i = 1, \dots, s$ and $j = 1, \dots, s-1$, let

$$(2.11) \quad B_{ij} = a_i f_i^{-n} \delta_{\rho'_j} F_i.$$

Recall that: $P \notin S$, the f_i 's are supported in S , the F_i have all their poles in S , the zeros of dt are in S . By (2.7) we see that $\text{ord}_P(B_{ij}) \geq 0$. Thus $B_{ij}(P) \in \overline{\mathbb{Q}}$.

Since $a_1 F_1 + \dots + a_s F_s = 0$, we have $B_{1j} f_1^n + \dots + B_{sj} f_s^n = 0$ for $j = 1, \dots, s-1$. Thus, for $i = 1, \dots, s$,

$$\mathbf{v}_j = (B_{1j}(P), \dots, B_{sj}(P), 0, \dots, 0) \in V_{\{1, \dots, s\}}.$$

The important fact that we have achieved is that. since P is not a zero of $W_{\rho'}$ the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{s-1}$ are linearly independent and form a basis of $V_{\{1, \dots, s\}}$.

By Lemma 2.2, $h(F_i) = O(n + M'_i + h(A_i))$. In order to deduce an upper bound for the height of B_{ij} we still need a bound for the height of the coefficients a_1, \dots, a_s of the minimal linear relation $a_1 F_1 + \dots + a_s F_s = 0$. Obviously, we may assume $a_s = -1$. We differentiate the relation up to order $s-2$. Since the rational functions F_1, \dots, F_{s-1} are linearly independent over $\overline{\mathbb{Q}}$, their Wronskian is not zero and so we can solve the resulting system for a_1, \dots, a_{s-1} . Using Cramér's Rule and $h(F_i) = O(n + M'_i + h(A_i))$ together with Lemma 2.2 especially (i) and (ii) we find without difficulty that $h(a_i) = O(n + M' + \max h(A_i))$.

Now to simplify the notation we write \mathbf{v}_j as $(B_1(P), \dots, B_s(P), 0, \dots, 0)$ for $B_i = a_i f_i^{-n} \delta_\rho(A_i f_i^n)$ as in (2.11); here $\rho \leq e \leq \Theta + O(1)$ by (2.10). As before

$$f_i^{-n} \delta_\rho(A_i f_i^n) = f_i^{-n} \sum_{l+m=\rho} \delta_l(A_i) \delta_m(f_i^n)$$

which by Faà di Bruno is

$$\sum_{l+m=\rho} \delta_l(A_i) f_i^{-m} \sum_{\mathbf{a}} C(\mathbf{a}) \delta_0(f_i)^{a_0} \delta_1(f_i)^{a_1} \dots \delta_m(f_i)^{a_m}$$

(note the changed power of f_i). As in Lemma 2.5(ii) we write

$$A_i = \sum_{j,k} \alpha_{ijk} g_j g^k \quad (i = 1, \dots, s)$$

and it suffices here to take $k \leq M'_i/\Delta \leq M'/\Delta$. Again Faà di Bruno on $\delta_q(g^k)$ gives

$$f_i^{-n}\delta_\rho(A_i f_i^n) = \sum_{l+m=\rho} f_i^{-m} \sum_{p+q=l} \sum_{j,k} \alpha_{ijk} \delta_p(g_j) g^{k-q} E_{qm},$$

where

$$(2.12) \quad E_{qm} = \sum_{\mathbf{b}} C(\mathbf{b}) \delta_0(g)^{b_0} \delta_1(g)^{b_1} \cdots \delta_q(g)^{b_q} \sum_{\mathbf{a}} C(\mathbf{a}) \delta_0(f_i)^{a_0} \delta_1(f_i)^{a_1} \cdots \delta_m(f_i)^{a_m}.$$

We evaluate all this at P and we want a main term $M'h(P)$ “uniformly in i ”, so that when we bundle the coordinates into the vector we don't get $sM'h(P)$.

In fact the terms g^{k-q} with $k \geq q$ already give

$$h(g^{k-q}(P)) = (k-q)h(g(P)) \leq \frac{M'}{\Delta} h(g(P)) \leq \frac{M'}{\Delta} (\Delta h(P) + O(1 + \sqrt{h(P)}))$$

by the upper bound in Lemma 2.3 with just two functions. The right-hand side is

$$M'h(P) + O(M'(1 + \sqrt{h(P)}))$$

so we already have the main term, clearly uniformly. Thus the rest had better be small. The point here is

$$\begin{aligned} b_0 + b_1 + \cdots + b_q &= b_1 + \cdots + qb_q = q \leq l \leq \rho \leq \Theta + O(1), \\ a_0 + a_1 + \cdots + a_m &= a_1 + \cdots + ma_m = m \leq \rho \leq \Theta + O(1). \end{aligned}$$

So if $k < q$ in g^{k-q} then $|k - q| \leq q$ so we get

$$h(g^{k-q}(P)) = |k - q|h(g(P)) \leq qh(g(P)) = O(\Theta(h(P) + 1)).$$

And by Lemma 2.4 we get for the δ -terms in (2.12), as well as the f_i -term, a height of order at most

$$(q + m)(h(P) + 1) \leq \rho(h(P) + 1) \leq (\Theta + O(1))(h(P) + 1),$$

also uniformly. The C -terms contribute logarithmically to order at most

$$k + q + n + m \leq k + n + \rho \leq M' + n + \Theta + O(1).$$

So this deals uniformly with the E_{qm} . As $p \leq \rho$ the $\delta_p(g_j)$ ($j = 0, 1, \dots, \Delta$) give nothing new, and by Lemma 2.5(ii) the α -terms contribute $O(M' + \max h(A_i))$.

Collecting everything up, we get

$$h(\mathbf{v}_j) \leq M'h(P) + O\left(\Theta(h(P) + 1) + n + \max h(A_i) + M'(1 + h(P)^{1/2})\right)$$

for $j = 1, \dots, s - 1$, slightly better than required. □

3. PROOF OF THEOREM 1.9.

In this section we prove the main technical result of this paper, which immediately implies Theorem 1.9 stated in the introduction.

Theorem 3.1. *Let $r \geq 2$ and $f_1, \dots, f_r \in \mathbb{F}$ be non-zero rational functions such that f_i/f_j is non-constant for some $i \neq j$. Let $d = -\deg \operatorname{div}(f_1, \dots, f_r)$, $K > 0$ be sufficiently large with respect to f_1, \dots, f_r and $\alpha = (\alpha_1 : \dots : \alpha_r) \in \mathbb{P}^{r-1}(\overline{\mathbb{Q}})$. Consider, for a natural number n , a solution $P \in \mathcal{C}(\overline{\mathbb{Q}})$ of the equation*

$$\alpha_1 f_1(P)^n + \dots + \alpha_r f_r(P)^n = 0.$$

Then, if $n \geq K$ and if there are no proper vanishing subsums, we have

$$(3.1) \quad h(P) \leq \left(\frac{r-1}{d} + O(1/K) \right) \frac{h(\alpha)}{n} + O(K^2)$$

where the implicit constant in the big- O depends only on f_1, \dots, f_r .

Theorem 1.9 easily follows from Theorem 3.1. Indeed, choosing K sufficiently large, we have $(r-1)/d + O(1/K) \leq (r-1)/d + 1 \leq r$.

Remark 3.2.

i) In the proof we shall show:

$$(3.2) \quad \frac{h(\alpha)}{n} \geq \frac{dh(P)}{r-1} + O\left(\frac{1}{K}h(P) + h(P)^{1/2} + K\right).$$

which immediately implies (3.1), since either $h(P) \leq K^2$ or $h(P)^{1/2} + K \leq 2h(P)/K$.

Note that here we do not assert uniformity in K . Note also that we will not use the assumption $f_i/f_j \neq \text{constant}$ in the proof of (3.2). This assumption is equivalent to $d \neq 0$ and (3.2) is trivially satisfied if $d = 0$.

ii) Let $\beta = (f_1(P) : \dots : f_r(P)) \in \mathbb{P}^{r-1}(\overline{\mathbb{Q}})$. By Weil's Height Machine Lemma 2.3 (and since $d > 0$), inequality (3.2) is equivalent to

$$(3.3) \quad \frac{h(\alpha)}{n} \geq \frac{h(\beta)}{r-1} + O\left(\frac{1}{K}h(\beta) + h(\beta)^{1/2} + K\right)$$

(which in turn implies $h(\beta) \leq (r-1 + O(1/K))\frac{h(\alpha)}{n} + O(K^2)$).

iii) A standard application of Siegel's lemma to the linear equation

$$(3.4) \quad \alpha_1 \beta_1^n + \dots + \alpha_r \beta_r^n = 0$$

in the unknowns $\alpha_1, \dots, \alpha_r$, shows that there exists a solution with

$$\frac{h(\alpha)}{n} \leq \frac{h(\beta)}{r-1} + O(1).$$

Thus inequality (3.3) is essentially sharp. More precisely, Theorem 3.1 gives a lower bound for the first minimum (with respect to the height) of the linear equation (4.1), and hence shows that the successive minima are close to each other for large n .

iv) If we ask that f_i/f_j is non-constant for all $i \neq j$, the assumption on vanishing subsums can be easily removed (by induction on r).

v) We finally remark that we have a result even if $d = 0$ (*i. e.* f_i/f_j constant for all i, j), but now the lower bound $n \geq K$ must depend also on α . Indeed, if $f_i = c_i f_1$ with c_i constants, our equation becomes $(\alpha_1 c_1^n + \cdots + \alpha_r c_r^n) f_1(P)^n = 0$. By the Skolem-Mahler-Lech, if $f_1(P) \neq 0$ then n is bounded by a constant depending on $\alpha_1, \dots, \alpha_r$ and on c_1, \dots, c_r .

Proof of Theorem 3.1. Let $f_1, \dots, f_r \in \mathbb{F}$ be as in the statement of the theorem. We recall that we have chosen a non-constant rational function $t \in \mathbb{F}$ and that S is the finite set consisting of all zeros and poles of f_1, \dots, f_r, dt and of an extra point Q (which is neither a zero nor a pole of f_1, \dots, f_r, dt).

We fix algebraic numbers $\alpha_1, \dots, \alpha_r$, not all zero. In order to prove (3.1) we may suppose that P does not lie in any prescribed finite set of points. We thus choose $P \in \mathcal{C}(\overline{\mathbb{Q}}) \setminus S$ satisfying our equation

$$\alpha_1 f_1(P)^n + \cdots + \alpha_r f_r(P)^n = 0$$

for some $n \geq K$. We shall also assume $h(P) \geq 1$.

Put now $\mathbf{w} = (f_1^n(P), \dots, f_r^n(P))$. Thus $\alpha \in \mathbf{w}^\perp$, the orthogonal space of \mathbf{w} . Our strategy is the following:

We shall first construct a basis of function-vectors (with controlled heights) for the orthogonal of the vector $(f_1^n, \dots, f_r^n) \in \mathbb{F}^r$. Then we shall specialise at P , in order to obtain a basis of \mathbf{w}^\perp , again with controlled heights. All of this shall involve an induction, necessary to take into account certain unexpected linear relations, *i. e.* relations with certain special properties in addition to those imposed by the construction.

At this stage we shall get a new basis, on replacing one of the vectors of the previous basis with α . By well-known facts, \mathbf{w} and \mathbf{w}^\perp have the same height. Lemma 2.3 gives a lower bound for the height of \mathbf{w} . The height of \mathbf{w}^\perp is bounded from above by the sum of the heights of the vectors of our new basis. Comparing these bounds, we shall get the desired conclusion.

Let us now perform this programme in detail.

Let N_1 be the minimum of the set of integers $m \geq 0$ such that there exist a non-empty $\Lambda \subseteq \{1, \dots, r\}$ and rational functions $A_i \in \mathbb{F}$ ($i \in \Lambda$) not all zero, satisfying

$$(3.5) \quad \begin{cases} A_i \in L(mQ), & \text{for } i \in \Lambda; \\ h(A_i) \leq nK, & \text{for } i \in \Lambda; \\ (A_i f_i^n)_{i \in \Lambda} \text{ are linearly dependent over } \overline{\mathbb{Q}}. \end{cases}$$

We provide an upper bound for N_1 using Lemma 2.6, as we are going to illustrate. Let $0 < \varepsilon < 1/2$ and for this argument define N as the smallest integer such that

$$(3.6) \quad (r - 1 - \varepsilon)N \geq (1 + \varepsilon)nd + c_0.$$

Then $N = O(n)$ and, with $M_1 = \dots = M_r = N$, the Dirichlet exponent ϱ of Lemma 2.6 satisfies

$$\varrho = \frac{N + dn}{(r-1)N - dn - c_0} \leq \frac{1}{\varepsilon}.$$

Thus, by that lemma, there exist $A_1, \dots, A_r \in L(NQ)$ not all zero such that

$$(3.7) \quad A_1 f_1^n + \dots + A_r f_r^n = 0$$

and

$$h(A_i) = O(n/\varepsilon + N/\varepsilon) = O(n/\varepsilon).$$

Choosing $\varepsilon = c/K$, where c is a sufficiently large constant to kill the implicit constant in the last $O()$, we see that there exists a non-trivial solution of (3.7) with $A_i \in L(NQ)$ and $h(A_i) \leq nK$. Not all A_1, \dots, A_r are zero, and we see that the non-zero ones among $A_1 f_1^n, \dots, A_r f_r^n$ sum up to zero, and so they are linearly dependent, as required by (3.5) (on choosing Λ simply as the set of i such that $A_i \neq 0$).

This shows that $N_1 \leq N$. Since N is the smallest integer satisfying (3.6) and since $\varepsilon = c/K$, we have $(r-1)N = nd + O(n/K)$. Thus

$$(3.8) \quad (r-1)N_1 \leq nd + O(n/K).$$

Among all subsets Λ which realize the minimum defining N_1 in (3.5), we choose a subset which is minimal. We denote by Λ_1 such a set, by $l_1 = |\Lambda_1|$ its cardinality (necessarily $l_1 \geq 2$) and by $\{A_i^{(1)}\}_{i \in \Lambda_1}$ the corresponding rational functions. This implies in particular that $\{A_i^{(1)} f_i^n\}_{i \in \Lambda_1}$ satisfy a minimal linear relation.

To go ahead we want to apply Lemma 2.11 to find a suitable basis of V_{Λ_1} . In that lemma, let us put $M'_i = N_1$ for $i \in \Lambda_1$, so $M' = \max M'_i = N_1 = O(n)$ and

$$\Theta = \max \left(1, \sum_{i \in \Lambda_1} M'_i - (M' + nd_1) \right) = \max \left(1, (l_1 - 1)N_1 - nd_1 \right)$$

with $d_1 = d_{\Lambda_1} = -\deg \operatorname{div}(f_i)_{i \in \Lambda_1}$.

We contend that $\Theta = O(n/K)$. To prove this, we use again Lemma 2.6, this time on the f_i ($i \in \Lambda_1$), with $M_i = N_1 - 1$ for $i \in \Lambda_1$. The Dirichlet exponent ϱ is then

$$\varrho = \frac{N_1 - 1 + nd_1}{(l_1 - 1)(N_1 - 1) - nd_1 - c_0}.$$

By Lemma 2.6, there exist rational functions B_i not all zero such that $B_i \in L((N_1 - 1)Q)$ for $i \in \Lambda_1$, $\sum_{i \in \Lambda_1} B_i f_i^n = 0$ and

$$h(B_i) = O((\varrho + 1)n).$$

By the minimality of N_1 , we cannot have $\max_i h(B_i) \leq nK$. Thus $\varrho \geq K/c$, where c is a sufficiently large constant to kill the implicit constant in the last $O()$. This implies

$$(l_1 - 1)N_1 - nd_1 \leq \frac{c}{K}(N_1 + nd_1) + O(1) = O(n/K)$$

as required. This concludes the proof of the contention.

Using Lemma 2.11 and the inequalities

$$h(P) \geq 1, \quad h(A_i) \leq nK, \quad M'_i = N_1 = O(n), \quad \Theta = O(n/K),$$

we find a basis $\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{l_1-1}^{(1)}$ of V_{Λ_1} satisfying

$$\begin{aligned} h(\mathbf{v}_i^{(1)}) &\leq M'h(P) \\ (3.9) \quad &+ O\left(\Theta(h(P) + 1) + (n + M')(1 + h(P)^{1/2}) + \max h(A_i)\right) \\ &= N_1 h(P) + O\left(\frac{n}{K}h(P) + nh(P)^{1/2} + nK\right). \end{aligned}$$

We are going to start a certain induction which shall be made explicit very soon. To outline the general strategy behind such a step, let us assume for the moment $\Lambda_1 = \{1, \dots, r\}$ (which is the generic case). Then we shall obtain directly the desired conclusion of Theorem 1.9. Recall that $\mathbf{w} = (f_1^n(P), \dots, f_r^n(P))$. Since $\boldsymbol{\alpha}$ is a non-zero vector in \mathbf{w}^\perp we may assume (reordering $\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{r-1}^{(1)}$ if necessary) that

$$\boldsymbol{\alpha}, \mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{r-2}^{(1)}$$

is a basis of \mathbf{w}^\perp . Let us denote by h_2 the logarithmic euclidean height (defined on choosing the L_2 -norm at the infinite places). By well-known facts on the height of subspaces (see [9] and [20]) and by the previous upper bounds for the height of these vectors.

$$\begin{aligned} h(\mathbf{w}) &\leq h_2(\mathbf{w}) = h_2(\mathbf{w}^\perp) \leq h_2(\boldsymbol{\alpha}) + \sum_{i=1}^{r-2} h_2(\mathbf{v}_i^{(1)}) + \log(r-1) \\ &\leq (r-2)N_1 h(P) + h(\boldsymbol{\alpha}) + O\left(\frac{n}{K}h(P) + nh(P)^{1/2} + nK\right). \end{aligned}$$

Moreover, by the functorial lower bound for the height Lemma 2.3, we have:

$$h(\mathbf{w}) \geq \left(h(P) + O(h(P)^{1/2})\right) nd.$$

Thus

$$0 \leq \lambda h(P) + \frac{h(\boldsymbol{\alpha})}{n} + O\left(\frac{1}{K}h(P) + h(P)^{1/2} + K\right)$$

with

$$\lambda = (r-2)N_1/n - d.$$

By (3.8) we have

$$\lambda \leq (r-2)\frac{d}{r-1} - d + O(1/K) = -\frac{d}{r-1} + O(1/K).$$

Inequality (3.2) follows.

The main obstacle in this approach is that Λ_1 may be smaller than $\{1, \dots, r\}$. In this case we can somewhat take advantage of the fact that we have an ‘unexpected’ dependence relation. To use this, let us sketch how we intend to argue, by an induction procedure.

We shall freely use notation and remarks from Section 2.2, choosing for \mathbf{w} the vector $(f_1^n(P), \dots, f_r^n(P))$. Since P is not a zero of the f_i 's, the vector \mathbf{w} has non-zero entries.

We construct an integer s with $1 \leq s \leq r$, positive integers N_1, \dots, N_s and non-empty subsets $\Lambda_1, \Lambda_2, \dots, \Lambda_s$ of $\{1, \dots, r\}$ of cardinalities resp. l_1, l_2, \dots, l_s satisfying:

Claim 3.3.

- i) $N_1 \leq N_2 \leq \dots \leq N_s$
- ii) For $j = 2, \dots, s$ the set Λ_j is contained in no connected component of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$
- iii) The collection $\{\Lambda_1, \dots, \Lambda_s\}$ is connected and its union is the full set $\{1, \dots, r\}$.
- iv) Let $t_1 = \dim(V_{\Lambda_1})$ and, for $j = 2, \dots, s$ let

$$t_j = \dim(V_{\Lambda_1} + \dots + V_{\Lambda_j}) - \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-1}}).$$

Then for $j = 1, \dots, s$ we have

$$t_1 N_1 + \dots + t_{j-1} N_{j-1} + \left(r - 1 - \sum_{i=1}^{j-1} t_i \right) N_j \leq nd + O(n/K).$$

- v) For $j = 1, \dots, s$ there exists a basis $\mathbf{v}_1^{(j)}, \dots, \mathbf{v}_{l_j-1}^{(j)}$ of V_{Λ_j} satisfying

$$\max_i h(\mathbf{v}_i^{(j)}) \leq N_j h(P) + O\left(\frac{n}{K} h(P) + nh(P)^{1/2} + nK\right).$$

Before explaining how to perform this construction, we shall show how it allows us to conclude the proof of Theorem 1.9.

We remark that

$$(3.10) \quad V_{\Lambda_1} + \dots + V_{\Lambda_s} = \mathbf{w}^\perp. \quad \text{In particular, } t_1 + \dots + t_s = r - 1,$$

by Claim 3.3 iii) and by Remark 2.9 i). Thus, taking into account i) of Claim 3.3, we may assume, after reordering for each j the vectors $\mathbf{v}_1^{(j)}, \dots, \mathbf{v}_{l_j-1}^{(j)}$ and possibly omitting some of them, that

$$\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{t_1}^{(1)}, \dots, \mathbf{v}_1^{(s-1)}, \dots, \mathbf{v}_{t_{s-1}}^{(s-1)}, \mathbf{v}_1^{(s)}, \dots, \mathbf{v}_{t_s}^{(s)}$$

is a basis of \mathbf{w}^\perp .

We also remark that $t_s \geq 1$. Otherwise $\dim(V_{\Lambda_1} + \dots + V_{\Lambda_{s-1}}) = t_1 + \dots + t_{s-1} = r - 1$ by (3.10). By Remark 2.9 ii), this implies that $\{\Lambda_1, \dots, \Lambda_{s-1}\}$ is connected and its union is the full set $\{1, \dots, r\}$, which contradicts Claim 3.3 ii) with $j = s$.

By assumption there are no proper vanishing subsums in $\alpha_1 f_1(P)^n + \dots + \alpha_r f_r(P)^n = 0$. This implies that $\boldsymbol{\alpha} \notin V_{\Lambda_1} + \dots + V_{\Lambda_{s-1}}$. Thus we may assume that

$$\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{t_1}^{(1)}, \dots, \mathbf{v}_1^{(s-1)}, \dots, \mathbf{v}_{t_{s-1}}^{(s-1)}, \boldsymbol{\alpha}, \mathbf{v}_1^{(s)}, \dots, \mathbf{v}_{t_s}^{(s)}$$

is a basis of \mathbf{w}^\perp . Arguing as we did before, we deduce that

$$(3.11) \quad 0 \leq \lambda h(P) + \frac{h(\boldsymbol{\alpha})}{n} + O\left(\frac{1}{K} h(P) + h(P)^{1/2} + K\right)$$

with

$$\lambda = t_1 \frac{N_1}{n} + \cdots + t_{s-1} \frac{N_{s-1}}{n} + (t_s - 1) \frac{N_s}{n} - d.$$

We now pause for the following computation:

Lemma 3.4. *Let $\tau, \rho, a_1, \dots, a_s$ be positive real numbers such that $a_s \geq 1$ and $a_1 + \dots + a_s = \rho$. Let also $x_1 \leq \dots \leq x_s$ be positive real numbers such that*

$$\sum_{i=1}^s a_i x_i \leq \tau.$$

Then

$$\sum_{j=1}^{s-1} a_j x_j + (a_s - 1)x_s - \tau \leq -\tau/\rho.$$

Proof. Set $\sigma := \sum_{i=1}^s a_i x_i$, so $\sigma \leq \tau$ and also $\sigma \leq \rho x_s$. Hence $\sigma - x_s \leq \sigma(1 - \frac{1}{\rho}) \leq \tau(1 - \frac{1}{\rho})$, since $\rho \geq a_s \geq 1$.

Now, on subtracting τ from both sides we obtain $\sigma - x_s - \tau \leq -\frac{\tau}{\rho}$, as required. \square

To go ahead, we recall that $t_1 + \cdots + t_s = r - 1$ (see (3.10)). Thus, Claim 3.3 iv) for $j = s$ reads

$$t_1 \frac{N_1}{n} + \cdots + t_{s-1} \frac{N_{s-1}}{n} + t_s \frac{N_s}{n} \leq d + O(1/K).$$

We apply the last lemma with

$$a_j = t_j, \quad \rho = r - 1, \quad x_j = N_j/n \quad \text{and} \quad \tau = d + O(1/K).$$

We find

$$\lambda = t_1 \frac{N_1}{n} + \cdots + t_{s-1} \frac{N_{s-1}}{n} + (t_s - 1) \frac{N_s}{n} - d \leq -d/(r - 1) + O(1/K).$$

Thus, by (3.11),

$$0 \leq -\frac{d}{r-1} h(P) + \frac{h(\alpha)}{n} + O\left(\frac{1}{K} h(P) + h(P)^{1/2} + K\right).$$

Inequality (3.2) follows. This concludes the proof of Theorem 1.9, assuming the truth of Claim 3.3.

Now, as promised, we detail the inductive process, verifying all the assertions of Claim 3.3.

We construct by induction an integer s with $1 \leq s \leq r$ and, for each $j = 1, \dots, s$,

- a positive integer N_j ;
- a subset Λ_j of $\{1, \dots, r\}$ of cardinality denoted $l_j := |\Lambda_j|$;
- a subset J_j of $\Lambda_1 \cup \cdots \cup \Lambda_{j-1}$ such that:
 - $|J_j \cap C| = 1$ for each connected component of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$
 - J_j is disjoint from $\Lambda_1 \cup \cdots \cup \Lambda_{j-2} \setminus J_{j-1}$ if $j \geq 2$.
- rational functions $\{A_i^{(j)}\}_{i \in \Lambda_j}$.
- a function $\varphi_j: \{1, \dots, r\} \rightarrow \{1, \dots, j\}$;

The role of the functions φ_j shall appear along the discussion.

For $j = 1$, we let N_1 be the minimum of the set of integers $N \geq 0$ such that there exist $\Lambda \subseteq \{1, \dots, r\}$ and rational functions $A_i \in \mathbb{F}$ ($i \in \Lambda$) not all zero, of height $\leq nK$, with $(A_i f_i^n)_{i \in \Lambda}$ linearly dependent and such that

$$A_i \in L(NQ), \text{ for } i \in \Lambda.$$

Among all subsets Λ which realize the minimum defining N_1 , we choose a subset which is minimal. We denote Λ_1 such set, $l_1 = |\Lambda_1|$ its cardinality and $\{A_i^{(1)}\}_{i \in \Lambda_1}$ the corresponding rational functions. This implies in particular that $\{A_i^{(1)} f_i^n\}_{i \in \Lambda_1}$ satisfy a minimal linear relation. As we have already shown earlier (see (3.8) and (3.9)), we have

$$(r-1)N_1 \leq nd + O(n/K)$$

and there exists a basis $\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{l_1-1}^{(1)}$ of V_{Λ_1} satisfying

$$h(\mathbf{v}_i^{(1)}) \leq N_1 h(P) + O\left(\frac{n}{K} h(P) + nh(P)^{1/2} + nK\right).$$

We also set $J_1 = \emptyset$ and $\varphi_1(i) = 1$ for $i = 1, \dots, r$.

For $j \geq 2$ we go ahead similarly, but modifying somewhat the requirements for the A_j , taking into account the previous steps. More precisely, let $j \geq 2$ and assume to have already constructed N_1, \dots, N_{j-1} , $\Lambda_1, \dots, \Lambda_{j-1}$ and J_{j-1} , φ_{j-1} . If $\{\Lambda_1, \dots, \Lambda_{j-1}\}$ is connected and $\Lambda_1 \cup \dots \cup \Lambda_{j-1} = \{1, \dots, r\}$ we put $s = j - 1$ and we stop here the process.

Otherwise, we choose a subset J_j of $\Lambda_1 \cup \dots \cup \Lambda_{j-1}$ such that $|J_j \cap C| = 1$ for each connected component of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$. We need to show that we can choose J_j disjoint from $\Lambda_1 \cup \dots \cup \Lambda_{j-2} \setminus J_{j-1}$. If the set Λ_{j-1} does not intersect $\Lambda_1 \cup \dots \cup \Lambda_{j-2}$, the connected components of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$ are the connected components of $\{\Lambda_1, \dots, \Lambda_{j-2}\}$ plus the set Λ_{j-1} itself. Thus we may choose $J_j = J_{j-1} \cup \{i_0\}$ where i_0 is any element of Λ_{j-1} . If otherwise Λ_{j-1} intersects $\Lambda_1 \cup \dots \cup \Lambda_{j-2}$, each connected component of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$ contains at least one connected component of $\{\Lambda_1, \dots, \Lambda_{j-2}\}$. Thus we may choose J_j as a suitable subset of J_{j-1} .

Then we let N_j be the minimum of the set of integers $N \geq 0$ such that there exist $\Lambda \subseteq \{1, \dots, r\}$ and rational functions $A_i \in \mathbb{F}$ ($i \in \Lambda$) not all zero, satisfying¹⁴

$$(3.12) \quad \begin{cases} A_i \in L((N_{\varphi_{j-1}(i)} - 1)Q), & \text{if } i \in (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j) \cap \Lambda, \\ A_i \in L(NQ), & \text{if } i \in \Lambda \text{ and } i \notin \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j, \\ h(A_i) \leq nK, & \text{for } i \in \Lambda, \\ (A_i f_i^n)_{i \in \Lambda} \text{ are linearly dependent.} \end{cases}$$

We remark that the set of such N is indeed not empty, as we easily see since $\{1, \dots, r\} \setminus (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j)$ has cardinality ≥ 2 (for otherwise $\{\Lambda_1, \dots, \Lambda_{j-1}\}$ would be connected and $\Lambda_1 \cup \dots \cup \Lambda_{j-1} = \{1, \dots, r\}$).

¹⁴Note that we are prescribing somewhat more stringent conditions than before on the indices inside the subsets previously defined.

We select a minimal set Λ_j among all sets Λ which realize the minimum defining N_j . We denote by $\{A_i^{(j)}\}_{i \in \Lambda_j}$ the corresponding rational functions. Thus $\{A_i^{(j)} f_i^n\}_{i \in \Lambda_j}$ satisfy a minimal linear relation.

We finally set

$$(3.13) \quad \varphi_j(i) = \begin{cases} \varphi_{j-1}(i) & \text{if } i \in \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j, \\ j & \text{otherwise, i. e. if } i \notin \Lambda_1 \cup \dots \cup \Lambda_{j-1} \text{ or } i \in J_j. \end{cases}$$

Thus, by the minimality of N_j , there are no subsets Λ and rational functions $(A_i)_{i \in \Lambda}$ not all zero such that

$$\begin{cases} A_i \in L((N_{\varphi_j(i)} - 1)Q), & \text{for } i \in \Lambda, \\ h(A_i) \leq nK, & \text{for } i \in \Lambda, \\ (A_i f_i^n)_{i \in \Lambda} \text{ are linearly dependent.} \end{cases}$$

We notice that for $j = 2, \dots, s$ we have

$$(3.14) \quad \varphi_{j-1}(i) = j - 1 \text{ for } i \in J_j,$$

since J_j is disjoint from $\Lambda_1 \cup \dots \cup \Lambda_{j-2} \setminus J_{j-1}$.

We also remark that

$$(3.15) \quad N_1 \leq N_2 \leq \dots \leq N_s$$

as asserted in Claim 3.3 i), since at each step we have added conditions.

We finally remark that for $j = 2, \dots, s$, the set Λ_j is contained in no connected component of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$ as we are going to show. This shall prove assertion ii) of Claim 3.3 and ensures that the inductive process ends somewhere. Thus Claim 3.3 iii) also holds by inductive construction.

To verify this assertion, let us assume by contradiction that Λ_j is contained in a connected component C of $\{\Lambda_1, \dots, \Lambda_{j-1}\}$. Let $C \cap J_j = \{i_0\}$. Then for $i \in \Lambda_j$, $i \neq i_0$, we have $A_i^{(j)} \in L((N_{\varphi_{j-1}(i)} - 1)Q)$. Using the minimal linear relation $\sum_{i \in \Lambda_j} a_i A_i^{(j)} f_i^n = 0$ and the fact that Q is not in the support of any of the divisors $\text{div}(f_i)$ we see that $A_{i_0}^{(j)} \in L(N^*Q)$ with $N^* = \max_i (N_{\varphi_{j-1}(i)} - 1)$. By (3.15) we have $N^* \leq N_{j-1}$ and, by (3.14), $j - 1 = \varphi_{j-1}(i_0)$. Thus $A_i^{(j)} \in L((N_{\varphi_{j-1}(i)} - 1)Q)$ for all $i \in \Lambda_j$. This contradicts the minimality in the definition of N_{j-1} (see the remark after (3.13)).

We still have to check assertions iv) and v) of Claim 3.3. To prove assertion iv) we first need the following lemma. Let, as in Claim 3.3 iv), $t_1 = \dim(V_{\Lambda_1})$ and

$$t_j = \dim(V_{\Lambda_1} + \dots + V_{\Lambda_j}) - \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-1}}).$$

for $j = 2, \dots, s$.

Lemma 3.5.

i) For $j = 2, \dots, s$ we have:

$$|\Lambda_1 \cup \dots \cup \Lambda_{j-1}| - |J_j| = \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-1}}) = \sum_{j'=1}^{j-1} t_{j'}$$

ii) Let j, j' be two integers with $j \geq 2$ and $1 \leq j' \leq j-1$. Then the set of $i \in \{1, \dots, r\}$ such that $\varphi_j(i) = j'$ has cardinality $t_{j'}$.

Proof. The first assertion follows from Remark 2.9 ii), since J_j has cardinality equal to the number of connected components of $\Lambda_1 \cup \dots \cup \Lambda_{j-1}$.

We prove the second assertion by induction on j . For $j = 2$ we have by construction $\#\{i \mid \varphi_2(i) = 1\} = |\Lambda_1| - 1 = \dim(V_{\Lambda_1}) = t_1$.

Let $j > 2$ and assume that $\#\{i \mid \varphi_{j-1}(i) = j'\} = t_{j'}$ for $j' = 1, \dots, j-2$. We want to show that $\#\{i \mid \varphi_j(i) = j'\} = t_{j'}$ for $j' = 1, \dots, j-1$.

Assume first $j' \leq j-2$. Let i be such that $\varphi_j(i) = j'$. Then $\varphi_j(i) \neq j$, thus by (3.13) $\varphi_j(i) = \varphi_{j-1}(i)$. This shows that $\{i \mid \varphi_j(i) = j'\} \subseteq \{i \mid \varphi_{j-1}(i) = j'\}$. On the other hand, let i such that $\varphi_{j-1}(i) = j'$. Then $\varphi_{j-1}(i) \neq j-1$ and (3.13) (with j replaced by $j-1$) shows that

$$i \in \Lambda_1 \cup \dots \cup \Lambda_{j-2} \setminus J_{j-1} \subseteq \Lambda_1 \cup \dots \cup \Lambda_{j-1}.$$

Moreover $i \notin J_j$ by (3.14). Thus, by (3.13), $\varphi_j(i) = \varphi_{j-1}(i)$. This proves that $\{i \mid \varphi_{j-1}(i) = j'\} \subseteq \{i \mid \varphi_j(i) = j'\}$. Putting together the two inclusions we see that

$$\{i \mid \varphi_j(i) = j'\} = \{i \mid \varphi_{j-1}(i) = j'\}.$$

Thus, by induction, $\#\{i \mid \varphi_j(i) = j'\} = \#\{i \mid \varphi_{j-1}(i) = j'\} = t_{j'}$.

Assume now $j' = j-1$. By (3.13)

$$\{i \mid \varphi_j(i) = j-1\} = \{i \in \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j \mid \varphi_{j-1}(i) = j-1\}$$

By (3.14) we have $\varphi_{j-1}(i) = j-1$ on J_j . By (3.13) (again with j replaced by $j-1$) we still have $\varphi_{j-1}(i) = j-1$ outside $\Lambda_1 \cup \dots \cup \Lambda_{j-2}$ and thus, *a fortiori* outside $\Lambda_1 \cup \dots \cup \Lambda_{j-1}$. These facts and the first assertion of the present lemma imply

$$\begin{aligned} \#\{i \mid \varphi_j(i) = j-1\} &= \#\{i \mid \varphi_{j-1}(i) = j-1\} - (r - |\Lambda_1 \cup \dots \cup \Lambda_{j-1}|) - |J_j| \\ &= \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-1}}) - (r - \#\{i \mid \varphi_{j-1}(i) = j-1\}). \end{aligned}$$

By induction,

$$\begin{aligned} r - \#\{i \mid \varphi_{j-1}(i) = j-1\} &= \sum_{j'=1}^{j-2} \#\{i \mid \varphi_{j-1}(i) = j'\} = \sum_{j'=1}^{j-2} t_{j'} \\ &= \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-2}}). \end{aligned}$$

Putting together the last two displayed equations, we get

$$\#\{i \mid \varphi_j(i) = j-1\} = \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-1}}) - \dim(V_{\Lambda_1} + \dots + V_{\Lambda_{j-2}}) = t_{j-1}$$

as desired. \square

We are now ready to prove by induction assertion iv) of Claim 3.3. This assertion was already proved in (3.8) for $j = 1$. Let $j \geq 2$. In a similar way as we have done for N_1 , we are going to provide an upper bound for N_j using Lemma 2.6. Let $\varepsilon = c/K$ for a sufficiently large constant c and choose N as the smallest integer such that

$$(3.16) \quad \sum_{j'=1}^{j-1} t_{j'}(N_{j'} - 1) + \left(r - 1 - \sum_{j'=1}^{j-1} t_{j'} - \varepsilon \right) N \geq (1 + \varepsilon)dn + c_0.$$

We notice that $N = O(n)$.

Remark 3.6. *We have*

$$N_{j-1} \leq N.$$

Proof. By Claim 3.3 iv) with j replaced by $j - 1$ (which holds by the present inductive assumption) we have

$$\sum_{j'=1}^{j-2} t_{j'} N_{j'} + \left(r - 1 - \sum_{j'=1}^{j-2} t_{j'} \right) N_{j-1} \leq nd + O(n/K).$$

Thus

$$\begin{aligned} & \left(r - 1 - \sum_{j'=1}^{j-1} t_{j'} - \varepsilon \right) (N_{j-1} - N) \\ &= \left(r - 1 - \sum_{j'=1}^{j-2} t_{j'} \right) N_{j-1} - t_{j-1} N_{j-1} - \varepsilon N_{j-1} - \left(r - 1 - \sum_{j'=1}^{j-1} t_{j'} - \varepsilon \right) N \\ &\leq \left(nd + O(n/K) \right) - \sum_{j'=1}^{j-2} t_{j'} N_{j'} - t_{j-1} N_{j-1} - \left(r - 1 - \sum_{j'=1}^{j-1} t_{j'} - \varepsilon \right) N \\ &\leq \left(nd + O(n/K) \right) - \left((1 + \varepsilon)dn + c_0 \right) \\ &= \left(-cd + O(1) \right) \frac{n}{K} < 0, \end{aligned}$$

if c is a sufficiently large constant to kill the implicit constant in the $O(1)$.

□

We are going to apply Lemma 2.6 with f_1, \dots, f_r and with

$$M_i = \begin{cases} N_{\varphi_{j-1}(i)} - 1, & \text{if } i \in (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j); \\ N, & \text{if } i \notin \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j \end{cases}$$

($i = 1, \dots, r$). Since Q is neither a zero nor a pole of f_1, \dots, f_r and since $N_1 \leq \dots \leq N_{j-1} \leq N$ (by Claim 3.3 i) and by Remark 3.6), we have

$$(3.17) \quad M = \max_i M_i = N.$$

We recall that, by Lemma 3.5 i),

$$|\Lambda_1 \cup \dots \cup \Lambda_{j-1}| - |J_j| = \sum_{j'=1}^{j-1} t_{j'}$$

and, by (3.13) and by Lemma 3.5 ii),

$$\#\{i \in (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j) \mid \varphi_{j-1}(i) = j'\} = \#\{i \mid \varphi_j(i) = j'\} = t_{j'}$$

for $j' = 1, \dots, j-1$. Thus

$$(3.18) \quad S = \sum_{i=1}^r M_i = \sum_{j'=1}^{j-1} t_{j'}(N_{j'} - 1) + \left(r - \sum_{j'=1}^{j-1} t_{j'}\right)N.$$

By (3.16), (3.17) and (3.18), $S \geq (1 + \varepsilon)(N + dn) + c_0$ and the Dirichlet exponent ϱ of Lemma 2.6 satisfies

$$\varrho = \frac{N + dn}{S - N - dn - c_0} \leq \frac{1}{\varepsilon}.$$

By that lemma (if c is a sufficiently large constant) there exist rational functions A_1, \dots, A_r not all zero and of height $\leq nK$ such that

$$\begin{cases} A_i \in L((N_{\varphi_{j-1}(i)} - 1)Q), & \text{if } i \in (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j); \\ A_i \in L(NQ), & \text{if } i \notin \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j \end{cases}$$

satisfying

$$A_1 f_1^n + \dots + A_r f_r^n = 0.$$

Not all A_1, \dots, A_r are zero, and we see that the non-zero ones among $A_1 f_1^n, \dots, A_r f_r^n$ sum up to zero, and so they are linearly dependent. Choosing Λ as the set of i such that $A_i \neq 0$ we see that (3.12) is satisfied. By minimality of N_j we have $N_j \leq N$. Since N is the smallest integer satisfying (3.16) and since $\varepsilon = c/K$, we have

$$\sum_{j'=1}^{j-1} t_{j'} N_{j'} + \left(r - 1 - \sum_{j'=1}^{j-1} t_{j'}\right) N_j \leq nd + O(n/K).$$

as required.

We now prove assertion v) of Claim 3.3. For $j = 1$ this assertion was already proved in (3.9). Let $j \geq 2$. As we did for $j = 1$, we apply Lemma 2.11 to find a suitable basis of V_{Λ_j} . Let, for $i \in \Lambda_j$,

$$M'_i = \begin{cases} N_{\varphi_{j-1}(i)} - 1, & \text{if } i \in (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j) \cap \Lambda_j, \\ N_j, & \text{if } i \in \Lambda_j \text{ and } i \notin \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j \end{cases}$$

and $M' = \max_{i \in \Lambda_j} M'_i = N_j$. Let also, as in Lemma 2.11,

$$\Theta = \max \left(1, \sum_{i \in \Lambda_j} M'_i - (M' + nd_j) \right) = \max \left(1, \sum_{i \in \Lambda_j} M'_i - (N_j + nd_j) \right)$$

with $d_j = d_{\Lambda_j} = -\deg \operatorname{div}(f_i)_{i \in \Lambda_j}$. We contend that $\Theta = O(n/K)$. As for $j = 1$, we prove this assertion using Lemma 2.6, now for the f_i ($i \in \Lambda_j$) with M_i the same as M'_i except for $N_j - 1$ in place of N_j , *i. e.*

$$M_i = \begin{cases} N_{\varphi_{j-1}(i)} - 1, & \text{if } i \in (\Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j) \cap \Lambda_j, \\ N_j - 1, & \text{if } i \in \Lambda_j \text{ and } i \notin \Lambda_1 \cup \dots \cup \Lambda_{j-1} \setminus J_j. \end{cases}$$

Let $S = \sum_{i \in \Lambda_j} M_i = \sum_{i \in \Lambda_j} M'_i + O(1)$ and $M = \max_{i \in \Lambda_j} M_i = N_j - 1 = O(n)$. The Dirichlet exponent ϱ is then

$$\varrho = \frac{N_j - 1 + nd_j}{S - N_j - nd_j - c_0 + 1}.$$

By Lemma 2.6, there exist rational functions B_i not all zero such that $B_i \in L(M_i Q)$ for $i \in \Lambda_j$, $\sum_{i \in \Lambda_j} B_i f_i^n = 0$ and

$$h(B_i) = O((\varrho + 1)n).$$

By the minimality of N_j , we cannot have $\max_i h(B_i) \leq nK$. Thus $\varrho \geq K/c$, where c is a sufficiently large constant to kill the implicit constant in the last $O()$. This implies

$$\sum_{i \in \Lambda_j} M'_i - (N_j + nd_j) \leq \frac{c}{K}(N_j + nd_j) + O(1) = O(n/K)$$

as required.

We apply Lemma 2.11 to the rational functions $\{A_i^{(j)}\}_{i \in \Lambda_j}$, taking into account:

$$h(P) \geq 1, \quad h(A_i) \leq nK, \quad M'_i = N_j = O(n), \quad \Theta = O(n/K).$$

By this lemma, there exists a basis $\mathbf{v}_1^{(j)}, \dots, \mathbf{v}_{l_j-1}^{(j)}$ of V_{Λ_j} satisfying

$$\begin{aligned} h(\mathbf{v}_i^{(j)}) &\leq M' h(P) + O\left(\Theta(h(P) + 1) + (n + M')(1 + h(P)^{1/2}) + \max h(A_i)\right) \\ &= N_j h(P) + O\left(\frac{n}{K} h(P) + nh(P)^{1/2} + nK\right). \end{aligned}$$

This proves assertion v) of Claim 3.3. The proof of Theorem 1.9 is now complete. \square

4. PROOF OF THEOREM 1.3

Let $\Gamma \subset \mathbb{G}_m^r(\mathbb{F})$ be a finitely generated constant-free subgroup, and let V be an algebraic subvariety of \mathbb{G}_m^r , defined over \mathbb{F} . By writing V as an intersection of hypersurfaces, we see that it is enough to prove Theorem 1.3 for a hypersurface V . We may further assume that V is a hyperplane. The case of a general hypersurface can indeed be easily deduced using an isogeny. For technical reasons, it is convenient to homogenize the statement in the linear case:

Proposition 4.1. *Let $\Gamma \subset \mathbb{G}_m^r(\mathbb{F})$ be a finitely generated subgroup such that*

$$(4.1) \quad \forall f \in \Gamma, \quad \forall i, j = 1, \dots, r, \quad f_i/f_j \in \overline{\mathbb{Q}} \implies f_i/f_j \in \overline{\mathbb{Q}}_{\text{tors}}^*.$$

Let $\gamma = (\gamma_1, \dots, \gamma_r) \in \Gamma$ and $\theta_1, \dots, \theta_r \in \mathbb{F}$ such that $\theta_1 \gamma_1 + \dots + \theta_r \gamma_r \neq 0$. Then the height of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that

$$(4.2) \quad \theta_1(P) \gamma_1(P) + \dots + \theta_r(P) \gamma_r(P) = 0$$

is bounded from above in terms only on Γ and $\theta_1, \dots, \theta_r$, independently of $\gamma_1, \dots, \gamma_r$. The same conclusion holds without the assumption (4.1), if $\theta_1, \dots, \theta_r \in \overline{\mathbb{Q}}$ and $\Gamma/\Gamma \cap \mathbb{G}_m^r(\overline{\mathbb{Q}})$ is of rank 1.

Proof. Replacing $\{1, \dots, r\}$ by a subset (and Γ by its projection on the coordinates in the subset) we may assume that there are no proper vanishing subsums in (4.2). Dividing (4.2) by $\gamma_1(P)$ (and replacing Γ by $\{(1, f_2/f_1, \dots, f_r/f_1) \mid \mathbf{f} \in \Gamma\}$) we may also assume $\gamma_1 = 1$ and

$$(4.3) \quad \Gamma \subseteq \{x_1 = 1\}.$$

Assume first that $\Gamma/\Gamma \cap \mathbb{G}_m^r(\overline{\mathbb{Q}})$ is of rank 1 and $\theta_i \in \overline{\mathbb{Q}}$. Thus there exists $\mathbf{f} = (1, f_2, \dots, f_r) \in \Gamma$ such that $\Gamma = \Gamma \cap \mathbb{G}_m^r(\overline{\mathbb{Q}}) \oplus \langle \mathbf{f} \rangle$. Then $\gamma = (1, c_2 f_2^n, \dots, c_r f_r^n)$ for some $c_i \in \overline{\mathbb{Q}}$. We may assume that there exists some $i > 1$ such that f_i is non constant, since otherwise $\theta_1 \gamma_1 + \dots + \theta_r \gamma_r$ is a non zero constant and equation (4.2) does not have solutions. Thus we can apply Theorem 1.9, and we find that the solutions of (4.2) have bounded height.

Assume now that Γ satisfies (4.1). Then $\Gamma = \Gamma_{\text{tors}} \oplus \Gamma'$ where Γ' is freely generated by, say, $\mathbf{g}_1, \dots, \mathbf{g}_\kappa$. We now use Dirichlet's Theorem in a way which is inspired by a method appearing already in Bombieri's paper [4], especially Lemma 4 therein. There exist $\boldsymbol{\omega} \in \Gamma_{\text{tors}}$ and integers $\lambda_1, \dots, \lambda_\kappa$ such that

$$\boldsymbol{\gamma} = \boldsymbol{\omega} \mathbf{g}_1^{\lambda_1} \cdots \mathbf{g}_\kappa^{\lambda_\kappa}.$$

Let $A = \max |\lambda_j|$. Since the equation (4.2) is not trivial (and since Γ_{tors} is finite), we can obviously assume A unbounded. Let $Q \geq 1$ be an integer which will be fixed later, independently of A . By Dirichlet's Theorem on simultaneous approximation, there exists a positive integer $q \leq Q^\kappa$ and integers p_j such that

$$\left| q \frac{\lambda_j}{A} - p_j \right| < \frac{1}{Q}.$$

Let n be the integral part of A/q . We write $\lambda_j = np_j + r_j$ and we set, for $i = 1, \dots, r$,

$$\boldsymbol{\rho} = \prod_{j=1}^{\kappa} \mathbf{g}_j^{r_j} \in \Gamma, \quad \mathbf{f} = \prod_{j=1}^{\kappa} \mathbf{g}_j^{p_j} \in \Gamma, \quad \alpha_i = \omega_i \theta_i(P) \rho_i(P).$$

Since $\gamma_i = \omega_i \rho_i f_i^n$, equation (4.2) can be rewritten as $\alpha_1 f_1(P)^n + \dots + \alpha_r f_r(P)^n = 0$ (without proper vanishing subsums).

We have

$$|p_j| \leq \left| q \frac{\lambda_j}{A} \right| + \frac{1}{Q} \leq q + Q^{-1} \leq 2Q^\kappa.$$

Thus, \mathbf{f} belongs to a finite set, depending only on the generators $\mathbf{g}_1, \dots, \mathbf{g}_\kappa$ and on Q , as the exponents λ_j vary. Moreover

$$|r_j| = \left| \frac{A}{q} \left(q \frac{\lambda_j}{A} - p_j \right) - \left(n - \frac{A}{q} \right) p_j \right| \leq (n+1)Q^{-1} + 2Q^\kappa.$$

Thus $d(\theta_i \rho_i) = O(n/Q + Q^{r\kappa})$, where the implicit constant in the big- O depends only on $\theta_1, \dots, \theta_r$ and on $\mathbf{g}_1, \dots, \mathbf{g}_\kappa$.

We recall that $\alpha_i = \omega_i \theta_i(P) \rho_i(P)$, with ω_i root of unity. Using Lemma 2.3 (for the two functions 1 and $\theta_i \rho_i$), we see easily that

$$h(\boldsymbol{\alpha}) \leq h(\alpha_1) + \cdots + h(\alpha_r) \leq C_1(n/Q + Q^\kappa)h(P)$$

where the constant C_1 depends only on $\theta_1, \dots, \theta_r$ and on $\mathbf{g}_1, \dots, \mathbf{g}_\kappa$ (and neither on Q nor on A). We now choose $Q = [4rC_1] + 1$.

There is a j such that $\lambda_j = \pm A$ and thus some of the p_j are not zero. Since g_1, \dots, g_κ is a basis of Γ' we have $\mathbf{f} \notin \Gamma_{\text{tors}}$. By (4.3) we have $f_1 = 1$. Thus, by (4.1), $f_i = f_i/f_1$ is non-constant for some $i > 1$ and we can apply Theorem 1.9.

Let C be the constant appearing in this theorem, which depends on f_1, \dots, f_r and thus only on $\theta_1, \dots, \theta_r$ and on $\mathbf{g}_1, \dots, \mathbf{g}_\kappa$ (since the rational functions f_i belong to a finite set, depending only on the generators $\mathbf{g}_1, \dots, \mathbf{g}_\kappa$ and on Q and Q has already been fixed as $Q = [4rC_1] + 1$). Since A is unbounded and Q does not depend on A , we may assume $n = [A/q] \geq C$ and $n \geq 4rC_1Q^\kappa$. By Theorem 1.9 and by the inequalities $h(\boldsymbol{\alpha}) \leq C_1(n/Q + Q^\kappa)h(P)$, $Q \geq 4rC_1$ and $n \geq 4rC_1Q^\kappa$,

$$h(P) \leq \frac{rh(\boldsymbol{\alpha})}{n} + C \leq rC_1(1/Q + Q^\kappa/n)h(P) + C \leq h(P)/2 + C.$$

We deduce that $h(P)$ is bounded, as claimed. □

REFERENCES

- [1] M. Bays and P. Habegger, “A note on divisible points of curves”. *Trans. Amer. Math. Soc.* **367** (2015), 1313–1328.
- [2] F. Beukers, “On a sequence of polynomials.” In *Algorithms for algebra. J. Pure Appl. Algebra* **117-118** (1997), 97–103.
- [3] F. Beukers and R. Tijdeman, “On the multiplicities of binary complex recurrences”. *Compositio Math.* **51** (1984), no. 2, 193–213.
- [4] E. Bombieri, “Effective Diophantine approximation on \mathbb{G}_m ”, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **20** (1993), 61–89.
- [5] E. Bombieri and P. B. Cohen, Siegel’s lemma, Padé approximations and Jacobians, *Ann. Sc. Normale Sup. Cl. Sci. (4)* **25** (1997), 155–178.
- [6] E. Bombieri and W. Gubler, “Heights in diophantine geometry”, New Math. Monographs **4**, Cambridge University Press 2006.
- [7] E. Bombieri, D. Masser, and U. Zannier, “Intersecting a curve with algebraic subgroups of multiplicative groups”, *Internat. Math. Res. Notices* **1999**, no. 20, 1119–1140.
- [8] E. Bombieri, D. Masser, and U. Zannier, “Anomalous Subvarieties - Structure Theorems and Applications”, *Internat. Math. Res. Notices* **2007**, no. 19, 1–33.
- [9] E. Bombieri and J. Vaaler, “On Siegel’s lemma”, *Invent. Math.*, **73** (1983), 11–32.
- [10] W. D. Brownawell and D. W. Masser, “Vanishing sums in function fields”. *Math. Proc. Cambridge Philos. Soc.* **100** (1986), no. 3, 427–434.
- [11] S. David and P. Philippon, Minorations des hauteurs normalisées des sous-variétés des tores, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **28** (1999), 489–543.
- [12] P. Habegger, “On the Bounded Height Conjecture”, *Internat. Math. Res. Notices* **2009**, no. 5, 860–886.
- [13] M. Hindry and J. H. Silverman, “Diophantine geometry. An introduction”. GTM 201. Springer-Verlag, Berlin, 2000.

- [14] S. Lang, “Introduction to Algebraic and Abelian Functions”, Springer-Verlag, GTM **89**, 1982.
- [15] D. Masser and U. Zannier, Torsion points on families of simple abelian surfaces and Pell’s equation over polynomial rings, with an appendix by V. Flynn, *J. European Math. Soc.* **17** (2015), 2379–2416.
- [16] D. Mirimanov, “Sur l’équation $(x + 1)^l - x^l - 1 = 0$ ”. *Nouv. Ann. Math.* **3** (1903), 385–397.
- [17] A. Néron, “Quasi-fonctions et hauteurs sur les variétés abéliennes”, *Ann. of Math. (2)* **82** (1965), 249–331.
- [18] G. Rémond, “Sur les sous-variétés des tores”, *Compositio Math.* **134** (2002), no. 3, 337–366.
- [19] W. M. Schmidt, “Eisenstein’s theorem on power series expansions of algebraic functions”, *Acta Arith.* **56** (1990), 161–179.
- [20] W. M. Schmidt, “Diophantine approximations and Diophantine equations”. LNM 1467, Springer-Verlag, Berlin, 1991.
- [21] J-P. Serre, Lectures on the Mordell-Weil Theorem, 2-nd Ed., Vieweg, 1990.
- [22] J. H. Silverman, “Heights and the specialization map for families of abelian varieties”, *J. Reine Angew. Math.* **342** (1983), 197–211.
- [23] J. H. Silverman and F. Voloch, “Multiple Weierstrass Points”, *Compositio Math.*, **79** (1991), 123–134.
- [24] U. Zannier, “Some problems of unlikely intersections in arithmetic and geometry”, *Annals of Mathematics Studies*, vol. **181**, Princeton University Press, 2012, With appendixes by David Masser.
- [25] S. Zhang, “Positive line bundles on arithmetic surfaces”, *Ann. Math. (2)* **136** (1992), no. 3, 569–587.