HAL open science

Expressing an observer in preferred coordinates by transforming an injective immersion into a surjective diffeomorphism

Pauline Bernard, Vincent Andrieu, Laurent Praly

To cite this version:

Pauline Bernard, Vincent Andrieu, Laurent Praly. Expressing an observer in preferred coordinates by transforming an injective immersion into a surjective diffeomorphism. SIAM Journal on Control and Optimization, 2018, 56 (3), pp.2327-2352. 10.1137/15M1037755 . hal-01199791v6

HAL Id: hal-01199791

https://hal.science/hal-01199791v6

Submitted on 19 Feb 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EXPRESSING AN OBSERVER IN PREFERRED COORDINATES BY TRANSFORMING AN INJECTIVE IMMERSION INTO A SURJECTIVE DIFFEOMORPHISM

PAULINE BERNARD, VINCENT ANDRIEU AND LAURENT PRALY *

Abstract. When designing observers for nonlinear systems, the dynamics of the given system and of the designed observer are usually not expressed in the same coordinates or even have states evolving in different spaces. In general, the function, denoted τ (or its inverse, denoted τ^*) giving one state in terms of the other is not explicitly known and this creates implementation issues.

We propose to round this problem by expressing the observer dynamics in the the same coordinates as the given system. But this may impose to add extra coordinates, problem that we call *augmentation*. This may also impose to modify the domain or the range of the "augmented" τ or τ^*, problem that we call *extension*.

We show that the augmentation problem can be solved partly by a continuous completion of a free family of vectors and that the extension problem can be solved by a function extension making the image of the extended function the whole space. We also show how augmentation and extension can be done without modifying the observer dynamics and therefore with maintaining convergence.

Several examples illustrate our results.

1. Introduction.

1.1. Context. In many applications, estimating the state of a dynamical system is crucial either to build a controller or simply to obtain real time information on the system. Satisfactory solutions are known for systems the dynamics of which are linear in the preferred coordinates. But when they are nonlinear, we are aware of only two "general purpose" observer design methodologies guaranteeing "non local" convergence under merely some basic observability properties: the high gain observers ([17, 24, 12, 13, 16, 7], ...) and the nonlinear Luenberger observers ([23, 15, 2]). For both, the observer state is living in a space different from the system state one and the system state estimate is obtained typically by solving on-line a nonlinear equation.

As an illustration, consider an harmonic oscillator with unknown frequency with dynamics

$$\dot{x}_1 = x_2 \ , \ \dot{x}_2 = -x_1 x_3 \ , \ \dot{x}_3 = 0 \ , \ y = x_1 \tag{1.1}$$

with state $x = (x_1, x_2, x_3)$ in $\left(\mathbb{R}^2 \setminus \{(0,0)\}\right) \times \mathbb{R}_{>0}$ and measurement y. We are interested in estimating the state x from the only knowledge of y and the fact that x evolves in some known set $\mathcal{A}$. By following in a very orthodox way (see [1] for details) the high gain observer design we get a "raw" observer with dynamics

$$\dot{\hat{\xi}} \ = \ \varphi(\hat{\xi}, \hat{x}, y) \ = \ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathtt{sat}(\hat{x}_1 \hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{\xi}_1] \ , \tag{1.2}$$

with state $\hat{\xi}$ in $\mathbb{R}^4$, where $\mathtt{sat}$ is a saturation function (see (1.12)), and from which the system state estimate $\hat{x}$ is given as $\hat{x} = \tau(\hat{\xi})$ where τ is any continuous function which satisfies

$$\tau(x_1, x_2, -x_1 x_3, -x_2 x_3) = (x_1, x_2, x_3) \qquad \forall x = (x_1, x_2, x_3) \in \mathcal{A} \ . \tag{1.3}$$

*P. Bernard Pauline.Bernard@mines-paristech.fr and L. Praly Laurent.Praly@mines-paristech.fr are with MINES ParisTech, PSL Research University, CAS - France,

V. Andrieu vincent.andrieu@gmail.com is with LAGEP, CNRS, CPE, Université Lyon 1, France

The construction of the mapping τ relies on the inversion to the mapping $\tau^*(x) = (x_1, x_2, -x_1x_3, -x_2x_3)$ which in general has no explicit solution and is not uniquely defined outside of $\tau^*(\mathcal{A})$. The commonly used implicit solution is given as the solution to an optimization problem which may be

$$\hat{x} = \tau(\hat{\xi}) = \operatorname*{Argmin}_{\hat{x}} \left|\hat{\xi} - \tau^*(\hat{x})\right|^2 .$$

Note however that some other forms are possible. For instance in [22], the authors propose to build another implicit solution based on an optimization procedure which yields a global Lipschitz function τ. The drawback of all these optimization based approaches being that they may be costly to solve in practice. Another path is to rely on the Rank theorem, as in [19] and take advantage of the local existence of diffeomorphism ϕ_x and ϕ_ξ such that

$$\phi_\xi \circ \tau^* \circ \phi_x = (x, 0, \ldots, 0) .$$

In this case, one can pick $\hat{x} = \phi_x^{-1}(\pi(\phi_\xi(\hat{\xi}))$ where π is the projection on the set of the first n components. In our example, ϕ_x could be the identity and

$$\phi_\xi(\xi) = \left(\xi_1, \xi_2, -\frac{\xi_1\xi_3 + \xi_2\xi_4}{\xi_1^2 + \xi_2^2}, (\xi_1\xi_4 - \xi_2\xi_3)\right) .$$

But, besides the local nature of this technique, finding expressions for ϕ_x^{-1} and ϕ_ξ may be a very difficult task in practice (see [18] for instance). And unfortunately $\hat{x}$ is needed to evaluate the term $\mathtt{sat}(\hat{x}_1\hat{x}_3^2)$ in (1.2) since the observer dynamics depend on τ.

Instead of a high gain observer design as above, we may use a Luenberger non linear observer design (see [23, 15, 2]). It leads to :

$$\dot{\hat{\xi}} \;=\; \varphi(\hat{\xi}, y) \;=\; A\,\hat{\xi} \;+\; B\,y \tag{1.4}$$

with $\hat{\xi}$ in $\mathbb{R}^4$, A a Hurwitz matrix and (A, B) a controllable pair. The state estimate $\hat{x}$ is again given as $\hat{x} = \tau(\hat{\xi})$ where τ is any continuous function which satisfies $\tau(\tau^*(x)) = x$ for all x in $\mathcal{A}$ where this time,

$$\tau^*(x) \;=\; -(A^2 + x_3 I)^{-1}[ABx_1 + Bx_2] . \tag{1.5}$$

A difference with the high gain observer is that $\hat{x}$ is not involved in (1.4), i.e. the observer dynamics do not depend on τ.

In the following, instead of constructing the (implicit) function τ by a minimization of a criterion introduced as a design tool, we explicitly construct a diffeomorphism τ_e allowing us to express the dynamics of the observer in the x-coordinates[1]. This has been suggested by several researchers [9, 20, 3] in the case where the observer state $\hat{\xi}$ and the state estimate $\hat{x}$ are related by a diffeomorphism. We remove this restriction and complete the preliminary results presented in [1].

In the example above, pulling the observer dynamics in the ξ-coordinates back in the x-coordinates is seemingly impossible since x has dimension 3 whereas $\hat{\xi}$ has dimension 4. To overcome this difficulty, one could think of using again some kind

[1] We will refer to the x-coordinates as the "preferred coordinates" or "given coordinates" because they are chosen by the user to describe the model dynamics.

of projection/restriction. Our proposition is actually of a completely different kind. Instead of considering $\hat{\xi}$ as the estimation of the image by an immersion τ^* of the state x, we see it as the estimation of the image by a diffeomorphism τ_e^* of an augmented state (x, w). Fortunately with such a diffeomorphism τ_e^*, we can use all what has been proposed for expressing the observer dynamics in the preferred coordinates in that case. So with this augmentation of x into (x, w), the design of the commonly used projection/restriction is replaced by the construction of the diffeomorphism τ_e^*. We show in Section 2 that τ_e^* can be obtained by "augmenting" the function $x \mapsto \tau^*(x)$ given in (1.3) or (1.5). For this, it turns out that it is sufficient to complement a full column rank Jacobian into an invertible matrix.

The drawback of this approach however is that, because it is linked to particular coordinate systems, the obtained diffeomorphism may not be defined everywhere. Also, its image could be only a subset of the observer accessibility set (for $\hat{\xi}$), namely the trajectories of $\hat{\xi}$ may leave the image of the diffeomorphism or equivalently the trajectories of $(\hat{x}, \hat{w})$ may leave the domain of definition of the diffeomorphism. We show in Section 3 how this new problem can be overcome via an extension of the image of the diffeomorphism. The key point here is that the given observer dynamics (1.2) remain unchanged. Hence we deal with constraints on the observer state without any kind of projection/restriction as commonly proposed (see [20, 3] for example). A benefit of this is that, to preserve the convergence property, we do not require extra assumptions such as convexity .

To illustrate our results, we continue the example of the harmonic oscillator with unknown frequency and add one based on the bioreactor presented in [12]. We use a high-gain observer as starting point. But, as shown in [5], the same tools can be used with a nonlinear Luenberger observer.

Our contribution relies on, or is inspired by ideas of some known analysis results such as continuously completing an independent set of vectors to a basis [25, 11], diffeotopies [14] or h-cobordism [21]. We rephrase part of them when it is constructive and therefore useful for observer design. Similarly, the constructive part of our proofs are in the main body of our text, those which are not constructive and never used/commented in remarks or examples are in appendix or omitted to save space. This is the long version of a paper which has been submitted for publication in SIAM Journal of Control and optimization. The parts of the paper which are in blue are those which are modified with respect to the journal version.

1.2. Problem statement. We consider the given system with dynamics :

$$\dot{x} = f(x) \quad , \qquad y = h(x) \ , \tag{1.6}$$

with x in $\mathbb{R}^n$ and y in $\mathbb{R}^q$. Its solution at time t, with initial condition x_0 at time 0 is denoted $X(x_0, t)$ and the corresponding output $y_{x_0}(t)$. The observation problem is to construct a dynamical system with input y and output $\hat{x}$, supposed to be an estimate of the system state x as long as the latter is in a specific set of interest denoted $\mathcal{A} \subseteq \mathbb{R}^n$. As starting point, we assume this problem is (formally) already solved but with maybe some implementation issues such as finding an expression of τ. More precisely,

Assumption $\mathbb{A}$ (Converging observer) : There exist an open subset $\mathcal{O}$ of $\mathbb{R}^n$ containing $\mathcal{A}$, a C^1 injective immersion $\tau^* : \mathcal{O} \to \mathbb{R}^m$, and a set[2] *$\varphi\mathcal{T}$ of pairs (φ, τ) of*

[2]The symbol $\varphi\mathcal{T}$ is pronounced *phitau*.

locally Lipschitz functions such that we have

$$\tau(\tau^*(x)) \;=\; x \qquad \forall x \in \mathcal{A} \tag{1.7}$$

and, for any solution $X(x_0,t)$ of (1.6) which is defined and remains in $\mathcal{A}$ for t in $[0,+\infty)$, the solution $(X(x_0,t),\hat{\Xi}(\hat{\xi}_0,t;y_{x_0}))$ of the cascade system :

$$\dot{x} \;=\; f(x) \quad , \qquad y \;=\; h(x) \quad , \qquad \dot{\hat{\xi}} = \varphi(\hat{\xi},\hat{x},y) \quad , \qquad \hat{x} = \tau(\hat{\xi}) \;, \tag{1.8}$$

with initial condition $(x_0,\hat{\xi}_0)$ in $\mathcal{A}\times\mathbb{R}^m$ at time 0, is also defined on $[0,+\infty)$ and satisfies :

$$\lim_{t\to+\infty}\left|\hat{\Xi}(\hat{\xi}_0,t;y_{x_0}) - \tau^*(X(x_0,t))\right| \;=\; 0 \;. \tag{1.9}$$

Remark 1.

1. The convergence property given by (1.9) is in the observer state space only. Property (1.7) is a necessary condition for this convergence to be transferred from the observer state space to the system state space.
2. The need for pairing φ and τ comes from the dependence on $\hat{x} = \tau(\hat{\xi})$ of φ in (1.8). This may imply to change φ whenever we change τ. In the high-gain approach, as in (1.2), when $\mathcal{A}$ is bounded, thanks to the gain ℓ which can be chosen arbitrarily large, φ can be paired with any locally Lipschitz function τ provided its values are saturated whenever they are used as arguments of φ. On another hand, if, as in (1.4), φ does not depend on $\hat{x}$, then it can be paired with any τ.

Example 1. For System (1.1), for any solution with initial condition $x_1 = x_2 = 0$, we have no information on x_3 from the only knowledge of (1.1) and the function $t \mapsto y(t) = X_1(x,t)$. This explains the restriction of our attention to the set

$$\mathcal{A} \;=\; \left\{x\in\mathbb{R}^3 \;:\; x_1^2 + x_2^2 \in \left]\frac{1}{r}, r\right[\;,\; x_3 \in]0,r[\right\} \;, \tag{1.10}$$

where r is some arbitrary strictly positive real number. This set is invariant by (1.1), and the function (1.3) being an injective immersion on $\left(\mathbb{R}^2\setminus\{(0,0)\}\right)\times\mathbb{R}_{>0}$, the system is strongly differentially observable[3] of order 4 on this set. Let $\mathcal{O}$ be any open subset such that $\mathtt{cl}(\mathcal{A}) \subset \mathcal{O} \subseteq \left(\mathbb{R}^2\times\mathbb{R}_{>0}\right)\setminus\left(\{(0,0)\}\times\mathbb{R}_{>0}\right)$, with $\mathtt{cl}$ denoting the set closure. Then, $\mathtt{cl}(\mathcal{A})$ being a compact set, a set $\mathcal{FT}$ satisfying Assumption $\mathbb{A}$ is made of pairs of a locally Lipschitz function τ satisfying (see [16] for example)

$$x = \tau(x_1, x_2, -x_1x_3, -x_2x_3) \qquad \forall x \in \mathcal{A} \tag{1.11}$$

and the function φ defined in (1.2) where

$$\mathtt{sat}(s) \;=\; \min\left\{r^3, \max\left\{s, -r^3\right\}\right\} \tag{1.12}$$

with the gain ℓ in (1.2) adapted to the properties of τ. $\triangle$

Although the problem of observer design seems already solved under Assumption $\mathbb{A}$, it can be difficult to find a left-inverse τ of τ^*. In the following, we consider that the function τ^* and the set $\mathcal{FT}$ are given and we aim at avoiding the left-inversion of

[3]The system is said to be strongly differentially observable of order m if the function $x \mapsto (h(x), L_f h(x), ..., L_f^{m-1}h(x))$ is an injective immersion.

τ^* by expressing the observer for x in the, maybe augmented, x-coordinates. More precisely we aim at solving the following problem.

Our problem (Observer in the x-coordinates) : Assume that Assumption $\mathbb{A}$ is satisfied, we wish to find an open set $\mathcal{O}_a \subseteq \mathbb{R}^m$ and two mappings k and ℓ such that the system defined in $\mathbb{R}^m$

$$\dot{\hat{x}} = k(\hat{x}, \hat{w}, y) \ , \ \dot{\hat{w}} = \ell(\hat{x}, \hat{w}, y) \ , \tag{1.13}$$

defines an observer in $\mathcal{A}$. In other words, for any initial condition x_0 in $\mathcal{A}$ such that the solution $X(x_0, t)$ of (1.6) is defined and remains in $\mathcal{A}$ for t in $[0, +\infty)$, the solution $(X(x_0, t), \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0}))$, with initial condition $(\hat{x}_0, \hat{w}_0)$ in $\mathcal{O}_a$, of the cascade of system (1.6) with the observer (1.13) is also defined on $[0, +\infty)$ and satisfies :

$$\lim_{t \to +\infty} \left| X(x_0, t) - \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}) \right| = 0 \ . \tag{1.14}$$

1.3. A sufficient condition allowing us to express the observer in the given x-coordinates. For the simpler case where the raw observer state $\hat{\xi}$ has the same dimension as the system state x, i.e. $m = n$, τ^*, in Assumption $\mathbb{A}$, is a diffeomorphism on $\mathcal{O}$ and we can express the observer in the given x-coordinates as :

$$\dot{\hat{x}} = \left(\frac{\partial \tau^*}{\partial x}(\hat{x}) \right)^{-1} \varphi(\tau^*(\hat{x}), \hat{x}, y) \tag{1.15}$$

which requires a Jacobian inversion only. However, although, by assumption, the system trajectories remain in $\mathcal{O}$ where the Jacobian is invertible, we have no guarantee the ones of the observer do. Therefore, to obtain convergence and completeness of solutions, we must find means to ensure the estimate $\hat{x}$ does not leave the set $\mathcal{O}$, or equivalently that $\tau^*(\hat{x})$ remains in the image set $\tau^*(\mathcal{O})$. We address this point by modifying τ^* "marginally" in order to get $\tau^*(\mathcal{O}) = \mathbb{R}^m$.

In the more complex situation where $m > n$, τ^* is only an injective immersion. In [1], it is proposed to augment the given x-coordinates in $\mathbb{R}^n$ with extra ones, say w, in $\mathbb{R}^{m-n}$ and correspondingly to augment the given injective immersion τ^* into a diffeomorphism $\tau_e^* : \mathcal{O}_a \to \mathbb{R}^m$, where $\mathcal{O}_a$ is an open subset of $\mathbb{R}^m$, considered as an augmentation of $\mathcal{O}$, i.e. its Cartesian projection on $\mathbb{R}^n$ is contained in $\mathcal{O}$ and contains $\mathtt{cl}(\mathcal{A})$.

To help us find such an appropriate augmentation, we have the following sufficient condition.

PROPOSITION 1.1. *Assume Assumption $\mathbb{A}$ holds and $\mathcal{A}$ is bounded. Assume also the existence of an open subset $\mathcal{O}_a$ of $\mathbb{R}^m$ containing $\mathtt{cl}(\mathcal{A} \times \{0\})$ and of a diffeomorphism $\tau_e^* : \mathcal{O}_a \to \mathbb{R}^m$ satisfying*

$$\tau_e^*(x, 0) \ = \ \tau^*(x) \qquad \forall x \in \mathcal{A} \tag{1.16}$$

and

$$\tau_e^*(\mathcal{O}_a) = \mathbb{R}^m \ . \tag{1.17}$$

and such that, with letting τ_{ex} denote the x-component of the inverse of τ_e^*, there exists a function φ such that the pair (φ, τ_{ex}) is in the set $\varphi\mathcal{T}$ given by Assumption $\mathbb{A}$. Under these conditions, for any initial condition x_0 in $\mathcal{A}$ such that the solution $X(x_0, t)$ of (1.6) is defined and remains in $\mathcal{A}$ for t in $[0, +\infty)$, the solution

$(X(x_0,t), \hat{X}(\hat{x}_0,\hat{w}_0,t;y_{x_0}), \hat{W}(\hat{x}_0,\hat{w}_0,t;y_{x_0}))$, *with initial condition* $(\hat{x}_0,\hat{w}_0)$ *in* $\mathcal{O}_a$, *of the cascade of system (1.6) with the observer :*

$$\dot{\overbrace{\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}}} = \left(\frac{\partial \tau_e^*}{\partial(\hat{x},\hat{w})}(\hat{x},\hat{w})\right)^{-1} \varphi(\tau_e^*(\hat{x},\hat{w}),\hat{x},y) \tag{1.18}$$

is also defined on $[0,+\infty)$ *and satisfies :*

$$\lim_{t\to+\infty} \left|\hat{W}(\hat{x}_0,\hat{w}_0,t;y_{x_0})\right| + \left|X(x_0,t) - \hat{X}(\hat{x}_0,\hat{w}_0,t;y_{x_0})\right| = 0 \ . \tag{1.19}$$

The key point in the observer (1.18) is that, instead of left-inverting the function τ^* via τ as in (1.7), we invert only a matrix.

Proof. See Appendix A. □

With Proposition 1.1, we are left with finding a diffeomorphism τ_e^* satisfying the conditions listed in the statement :

- Equation (1.16) is about the fact that τ_e^* is an augmentation, with adding coordinates, of the given injective immersion τ^*. It motivates the following problem.
  Problem 1 (Immersion augmentation into a diffeomorphism). *Given a set* $\mathcal{A}$, *an open subset* $\mathcal{O}$ *of* $\mathbb{R}^n$ *containing* $\mathtt{cl}(\mathcal{A})$, *and an injective immersion* $\tau^* : \mathcal{O} \to \tau^*(\mathcal{O}) \subset \mathbb{R}^m$, *the pair* $(\tau_a^*, \mathcal{O}_a)$ *is said to solve the problem of immersion augmentation into a diffeomorphism if* $\mathcal{O}_a$ *is an open subset of* $\mathbb{R}^m$ *containing* $\mathtt{cl}(\mathcal{A}\times\{0\})$ *and* $\tau_a^* : \mathcal{O}_a \to \tau_a^*(\mathcal{O}_a) \subset \mathbb{R}^m$ *is a diffeomorphism satisfying*

  $$\tau_a^*(x,0) \ = \ \tau^*(x) \qquad \forall x \in \mathcal{A} \ .$$

  We will present in Section 2 conditions under which Problem 1 can be solved via complementing a full column rank Jacobian of τ^* into an invertible matrix, i.e. via what we call Jacobian complementation.

- The condition expressed in (1.17), is about the fact that τ_e^* is surjective onto $\mathbb{R}^m$. This motivates us to introduce the surjective diffeomorphism extension problem
  Problem 2 (Surjective diffeomorphism extension). *Given an open subset* $\mathcal{O}_a$ *of* $\mathbb{R}^m$, *a compact subset* K *of* $\mathcal{O}_a$, *and a diffeomorphism* τ_a^*: $\mathcal{O}_a \to \mathbb{R}^m$, *the diffeomorphism* $\tau_e^* : \mathcal{O}_a \to \mathbb{R}^m$ *is said to solve the surjective diffeomorphism extension problem if it satisfies*

  $$\tau_e^*(\mathcal{O}_a) = \mathbb{R}^m \quad , \qquad \tau_e^*(z) = \tau_a^*(z) \quad \forall z \in K.$$

  This Problem 2 will be addressed in Section 3.

When Assumption $\mathbb{A}$ holds and $\mathcal{A}$ is bounded, by successively solving Problem 1 and Problem 2 with $\mathtt{cl}(\mathcal{A}\times\{0\}) \subset K \subset \mathcal{O}_a$, we get a diffeomorphism τ_e^* guaranteed to satisfy all the conditions of Proposition 1.1 except maybe the fact that the pair (φ, τ_{ex}) is in $\varphi\mathcal{T}$. How this last condition can be satisfied will be discussed in Section 4 mainly via a list of remarks.

Throughout Sections 2-3, we will show how, step by step, we can express a high gain observer in the x-coordinates for the harmonic oscillator with unknown frequency. We will also show that our approach enables to ensure completeness of solutions of the observer presented in [12] for the bioreactor. The various difficulties we shall encounter on this road will be discussed in Section 5. In particular, we shall see how they can be overcome thanks to a better choice of τ^* and of the pair (φ, τ) given by Assumption $\mathbb{A}$.

2. About Problem 1 : Augmentation of an immersion into a diffeomorphism. In [1], we find the following sufficient condition for the augmentation of an immersion into a diffeomorphism.

LEMMA 2.1 ([1]). *Let $\mathcal{A}$ be a bounded set, $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ containing $\mathtt{cl}(\mathcal{A})$, and $\tau^* : \mathcal{O} \to \tau^*(\mathcal{O}) \subset \mathbb{R}^m$ be an injective immersion. If there exists a bounded open set $\tilde{\mathcal{O}}$ satisfying $\mathtt{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \mathtt{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$ and a C^1 function $\gamma : \mathcal{O} \to \mathbb{R}^{m\times(m-n)}$ the values of which are $m \times (m-n)$ matrices satisfying :*

$$\det\left(\frac{\partial \tau^*}{\partial x}(x) \quad \gamma(x)\right) \neq 0 \qquad \forall x \in \mathtt{cl}(\tilde{\mathcal{O}}) \ , \tag{2.1}$$

then there exists a strictly positive real number ε such that the following pair[4] $(\tau_a^*, \mathcal{O}_a)$ solves Problem 1

$$\tau_a^*(x,w) \ = \ \tau^*(x) \ + \ \gamma(x)\, w \ , \ \mathcal{O}_a = \tilde{\mathcal{O}} \times \mathcal{B}_\varepsilon(0) \ . \tag{2.2}$$

In other words, an injective immersion τ^* can be augmented into a diffeomorphism τ_a^* if we are able to find $m-n$ columns γ which are C^1 in x and which complement the full column rank Jacobian $\frac{\partial \tau^*}{\partial x}(x)$ into an invertible matrix.

Proof. See Appendix B. □

REMARK 2. Complementing a $m \times n$ full-rank matrix into an invertible one is equivalent to finding $m-n$ independent vectors orthogonal to that matrix. Precisely the existence of γ satisfying (2.1) is equivalent to the existence of a C^1 function $\tilde{\gamma} : \mathtt{cl}(\tilde{\mathcal{O}}) \to \mathbb{R}^{m\times(m-n)}$ the values of which are full rank matrices satisfying :

$$\tilde{\gamma}(x)^\top \frac{\partial \tau^*}{\partial x}(x) \ = \ 0 \qquad \forall x \in \mathtt{cl}(\tilde{\mathcal{O}}) \ . \tag{2.3}$$

Indeed, $\tilde{\gamma}$ satisfying (2.3) satisfies also (2.1) since the following matrices are invertible

$$\begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x)^\top \\ \tilde{\gamma}(x)^\top \end{pmatrix} \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \tilde{\gamma}(x) \end{pmatrix} \ = \ \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x)^\top \frac{\partial \tau^*}{\partial x}(x) & 0 \\ 0 & \tilde{\gamma}(x)^\top \tilde{\gamma}(x) \end{pmatrix} \ .$$

Conversely, given γ satisfying (2.1), $\tilde{\gamma}$ defined by the identity below satisfies (2.3) and has full column rank

$$\begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \tilde{\gamma}(x) \end{pmatrix} \ = \ \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \gamma(x) \end{pmatrix} \begin{pmatrix} I & -\left[\frac{\partial \tau^*}{\partial x}(x)^\top \frac{\partial \tau^*}{\partial x}(x)\right]^{-1} \frac{\partial \tau^*}{\partial x}(x)^\top \gamma(x) \\ 0 & I \end{pmatrix} \ .$$

2.1. Submersion case.

PROPOSITION 2.2 (Completion when $\tau^*(\mathtt{cl}(\tilde{\mathcal{O}}))$ is a level set of a submersion). *Let $\mathcal{A}$ be a bounded set, $\tilde{\mathcal{O}}$ be a bounded open set and $\mathcal{O}$ be an open set satisfying*

$$\mathtt{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \mathtt{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O} \ .$$

Let also $\tau^* : \mathcal{O} \to \tau^*(\mathcal{O}) \subset \mathbb{R}^m$ be an injective immersion. Assume there exists a C^2 function $F : \mathbb{R}^m \to \mathbb{R}^{m-n}$ which is a submersion at least on a neighborhood of $\tau^*(\tilde{\mathcal{O}})$ satisfying:

$$F(\tau^*(x)) = 0 \qquad \forall x \in \tilde{\mathcal{O}} \ , \tag{2.4}$$

[4] For a positive real number ε and z_0 in $\mathbb{R}^p$, $\mathcal{B}_\varepsilon(z_0)$ is the open ball centered at z_0 and with radius ε.

then, with the C^1 function $x \mapsto \gamma(x) = \frac{\partial F}{\partial \xi}^T(\tau^*(x))$, the matrix in (2.1) is invertible for all x in $\tilde{\mathcal{O}}$ and the pair $(\tau_a^*, \mathcal{O}_a)$ defined in (2.2) solves Problem 1.

Proof. For all x in $\mathtt{cl}(\tilde{\mathcal{O}})$, $\frac{\partial \tau^*}{\partial x}(x)$ is right invertible and we have $\frac{\partial F}{\partial \xi}(\tau^*(x))\frac{\partial \tau^*}{\partial x}(x) = 0$. Thus, the rows of $\frac{\partial F}{\partial \xi}(\tau^*(x))$ are orthogonal to the column vectors of $\frac{\partial \tau^*}{\partial x}(x)$ and are independent since F is a submersion. The Jacobian of τ^* can therefore be completed with $\frac{\partial F}{\partial \xi}^T(\tau^*(x))$. The proof is completed with Lemma 2.1. ∎

Remark 3. Since $\frac{\partial \tau^*}{\partial x}$ is of constant rank n on $\mathcal{O}$, the existence of such a function F is guaranteed at least locally by the constant rank Theorem.

Example 2 (Continuation of Example 1). Elimination of the $\hat{x}_i$ in the 4 equations given by the injective immersion τ^* defined in (1.3) leads to the function $F(\xi) = \xi_2\xi_3 - \xi_1\xi_4$ satisfying (2.4). It follows that a candidate for complementing:

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \end{pmatrix} \tag{2.5}$$

is

$$\gamma(x) \;=\; \frac{\partial F}{\partial \xi}(\tau^*(x))^T = (x_2x_3, -x_1x_3, x_2, -x_1)^T \;.$$

This vector is nothing but the column of the minors of the matrix (2.5). It gives as determinant $(x_2x_3)^2 + (x_1x_3)^2 + x_2^2 + x_1^2$ which is never zero on $\mathcal{O}$.

Then, it follows from Lemma 2.1, that, for any bounded open set $\tilde{\mathcal{O}}$ such that $\mathcal{A} \subset \mathtt{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$ the following function is a diffeomorphism on $\tilde{\mathcal{O}} \times \mathcal{B}_\epsilon(0)$ for ε sufficiently small

$$\tau_a^*(x, w) = (x_1 + x_2x_3w, x_2 - x_1x_3w, -x_1x_3 + x_2w, -x_2x_3 - x_1w) \;.$$

With picking $\tau_e^* = \tau_a^*$, (1.18) gives us the following observer written in the given x-coordinates augmented with w :

$$\dot{\overbrace{\begin{pmatrix} \hat{x}_1 \\ \hat{x}_3 \\ \hat{x}_2 \\ \hat{w} \end{pmatrix}}} = \begin{pmatrix} 1 & \hat{x}_3\hat{w} & \hat{x}_2\hat{w} & \hat{x}_2\hat{x}_3 \\ -\hat{x}_3\hat{w} & 1 & -\hat{x}_1\hat{w} & -\hat{x}_1\hat{x}_3 \\ -\hat{x}_3 & \hat{w} & -\hat{x}_1 & \hat{x}_2 \\ -\hat{w} & -\hat{x}_3 & -\hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1} \left[\begin{pmatrix} \hat{x}_2 - \hat{x}_1\hat{x}_3\hat{w} \\ -\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{w} \\ -\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{w} \\ \mathtt{sat}(\hat{x}_1\hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{x}_1] \right]$$

Unfortunately the matrix to be inverted is non singular for $(\hat{x}, \hat{w})$ in $\tilde{\mathcal{O}} \times \mathcal{B}_\varepsilon(0)$ only and we have no guarantee that the trajectories of this observer remain in this set. This shows that a further modification transforming τ_a^* into τ_e^* is needed to make sure that ${\tau_e^*}^{-1}(\hat{\xi})$ belongs to this set whatever $\hat{\xi}$ in $\mathbb{R}^4$. This is Problem 2. △

The drawback of this Jacobian complementation method is that it asks for the knowledge of the function F. It would be better to simply have a universal formula relating the entries of the columns to be added to those of $\frac{\partial \tau^*}{\partial x}$.

2.2. The $\tilde{P}[m, n]$ problem. Finding a universal formula for the Jacobian complementation problem amounts to solving the following problem.

Definition 2.3. *($\tilde{P}[m, n]$ problem) For a pair of integers (m, n) such that $0 < n < m$, a C^1 matrix function $\tilde{\gamma} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times(m-n)}$ solves the $\tilde{P}[m, n]$ problem if for any $m \times n$ matrix $\mathfrak{T} = (\mathfrak{T}_{ij})$ of rank n, the matrix $\begin{pmatrix} \mathfrak{T} & \tilde{\gamma}(\mathfrak{T}) \end{pmatrix}$ is invertible, or equivalently, the matrix $\tilde{\gamma}(\mathfrak{T})$ has rank $m - n$ and satisfies $\tilde{\gamma}(\mathfrak{T})^\top \mathfrak{T} \;=\; 0$.*

As a consequence of a theorem due to Eckmann [11, §1.7 p. 126] and Lemma 2.1, we have

THEOREM 2.4. *The $\tilde{P}[m,n]$ problem is solvable by a C^1 function $\tilde{\gamma}$ if and only if the pair (m,n) is one of the following 3 pairs*

$$(>2, m-1) \qquad \text{or} \qquad (4,1) \qquad \text{or} \qquad (8,1)\ . \tag{2.6}$$

Moreover, for each of these pairs and for any bounded set $\mathcal{A}$, bounded open set $\tilde{\mathcal{O}}$ and open set $\mathcal{O}$ satisfying

$$\mathtt{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \mathtt{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}\ ,$$

and any injective immersion $\tau^* : \mathcal{O} \to \tau^*(\mathcal{O}) \subset \mathbb{R}^m$, the pair $(\tau_a^*, \mathcal{O}_a)$ defined in (2.2) with $\gamma(x) = \tilde{\gamma}\left(\frac{\partial \tau_a^*}{\partial x}(x)\right)$ solves Problem 1.

Proof only if. See Appendix C. □

Proof if. For (m,n) equal to $(4,1)$ or $(8,1)$ respectively, possible solutions are

$$\tilde{\gamma}(\mathfrak{T}) = \begin{pmatrix} -\mathfrak{T}_2 & \mathfrak{T}_3 & \mathfrak{T}_4 \\ \mathfrak{T}_1 & -\mathfrak{T}_4 & \mathfrak{T}_3 \\ -\mathfrak{T}_4 & -\mathfrak{T}_1 & -\mathfrak{T}_2 \\ \mathfrak{T}_3 & \mathfrak{T}_2 & -\mathfrak{T}_1 \end{pmatrix},\ \tilde{\gamma}(\mathfrak{T}) = \begin{pmatrix} \mathfrak{T}_2 & \mathfrak{T}_3 & \mathfrak{T}_4 & \mathfrak{T}_5 & \mathfrak{T}_6 & \mathfrak{T}_7 & \mathfrak{T}_8 \\ -\mathfrak{T}_1 & \mathfrak{T}_4 & -\mathfrak{T}_3 & \mathfrak{T}_6 & -\mathfrak{T}_5 & -\mathfrak{T}_8 & \mathfrak{T}_7 \\ -\mathfrak{T}_4 & -\mathfrak{T}_1 & \mathfrak{T}_2 & \mathfrak{T}_7 & \mathfrak{T}_8 & -\mathfrak{T}_5 & -\mathfrak{T}_6 \\ \mathfrak{T}_3 & -\mathfrak{T}_2 & -\mathfrak{T}_1 & \mathfrak{T}_8 & -\mathfrak{T}_7 & \mathfrak{T}_6 & -\mathfrak{T}_5 \\ -\mathfrak{T}_6 & -\mathfrak{T}_7 & -\mathfrak{T}_8 & -\mathfrak{T}_1 & \mathfrak{T}_2 & \mathfrak{T}_3 & \mathfrak{T}_4 \\ \mathfrak{T}_5 & -\mathfrak{T}_8 & \mathfrak{T}_7 & -\mathfrak{T}_2 & -\mathfrak{T}_1 & -\mathfrak{T}_4 & \mathfrak{T}_3 \\ \mathfrak{T}_8 & \mathfrak{T}_5 & -\mathfrak{T}_6 & -\mathfrak{T}_3 & \mathfrak{T}_4 & -\mathfrak{T}_1 & -\mathfrak{T}_2 \\ -\mathfrak{T}_7 & \mathfrak{T}_6 & \mathfrak{T}_5 & -\mathfrak{T}_4 & -\mathfrak{T}_3 & \mathfrak{T}_2 & -\mathfrak{T}_1 \end{pmatrix}$$

where $\mathfrak{T}_j$ is the jth component of the vector $\mathfrak{T}$. For $n = m-1$, we have the identity

$$\det\left(\mathfrak{T} \quad \tilde{\gamma}(\mathfrak{T})\right) = \sum_{j=1}^{m} \tilde{\gamma}_j(\mathfrak{T}_{ij})\, M_{j,m}(\mathfrak{T}_{ij})$$

where $\tilde{\gamma}_j$ is the jth component of the vector-valued function $\tilde{\gamma}$ and the $M_{j,m}$, being the cofactors of $(\mathfrak{T}\ \tilde{\gamma}(\mathfrak{T}))$ computed along the last column, are polynomials in the given components $\mathfrak{T}_{ij}$. At least one of the $M_{j,m}$ is non-zero (because they are minors of dimension n of $\mathfrak{T}$ which is full-rank). So it is sufficient to take $\tilde{\gamma}_j(\mathfrak{T}_{ij}) = M_{j,m}(\mathfrak{T}_{ij})$. □

In the following example we show how by exploiting some structure we can reduce the problem to one of these 3 pairs.

EXAMPLE 3 (Continuation of Example 2). In Example 2, we have complemented the Jacobian (2.5) with the gradient of a submersion and observed that the components of this gradient are actually cofactors. We now know that this is consistent with the case $n = m-1$. But we can also take advantage from the upper triangularity of the Jacobian (2.5) and complement only the vector $(-x_1, -x_2)$ by for instance $(x_2, -x_1)$. The corresponding vector γ is $\gamma(x) = (0, 0, x_2, -x_1)$. Here again, with Lemma 2.1, we know that, for any bounded open set $\tilde{\mathcal{O}}$ such that $\mathtt{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \mathtt{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$ the function

$$\tau_a^*(x,w) \ = \ (x_1\,,\ x_2\,,\ -x_1x_3 + x_2w\,,\ -x_2x_3 - x_1w)$$

is a diffeomorphism on $\tilde{\mathcal{O}} \times \mathcal{B}_\epsilon(0)$. In fact, in this particular case ε can be arbitrary, no need for it to be small. However, the singularity at $\hat{x}_1 = \hat{x}_2 = 0$ remains and equation (1.17) is still not satisfied. △

Given the very small number of cases where a universal formula exists, we now look for a more general solution to the Jacobian complementation problem.

2.3. Wazewski theorem. Historically, the Jacobian complementation problem was first addressed by Wazewski (see [25]). His formulation was :

Given mn continuous functions $\mathfrak{T}_{ij} : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$, look for $m(m-n)$ continuous functions $\gamma_{kl} : \mathcal{O} \to \mathbb{R}$ such that the following matrix is invertible for all x in $\mathcal{O}$:

$$P(x) \;=\; \begin{pmatrix} \mathfrak{T}(x) & \gamma(x) \end{pmatrix} \;. \tag{2.7}$$

The difference with the previous section, is that here, we look for continuous functions γ of x in $\mathbb{R}^n$ instead of continuous functions γ of $\mathfrak{T}$ in $\mathbb{R}^{m\times n}$.

Wazewski established that this other version of the problem admits a far more general solution :

THEOREM 2.5 ([25, Theorems 1 and 3] and [11, page 127]). *If $\mathcal{O}$, equipped with the subspace topology of $\mathbb{R}^n$, is a contractible space, then there exists a C^∞ function γ making the matrix $P(x)$ in (2.7) invertible for all x in $\mathcal{O}$.*

The reader is referred to [11, page 127] or [10, pages 406-407] and to [25, Theorems 1 and 3] for the complete proof of existence of a continuous function γ. We give the main constructive points of this proof below. Also, in appendix D, we show, by using a partition of unity, how this continuous function γ making P invertible can be modified into a smoother one giving the same invertibility property. But before this, let us give the following corollary obtained as a consequence of Lemma 2.1.

COROLLARY 2.6. *Let $\mathcal{A}$ be a bounded set, $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ containing $\mathtt{cl}(\mathcal{A})$ and which, equipped with the subspace topology of $\mathbb{R}^n$, is a contractible space. Let also $\tau^* : \mathcal{O} \to \tau^*(\mathcal{O}) \subset \mathbb{R}^m$ be an injective immersion. There exists a C^1 function γ such that, for any bounded open set $\tilde{\mathcal{O}}$ satisfying*

$$\mathtt{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \mathtt{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$$

we can find a strictly positive real number ε such that the pair $(\tau^*_a, \mathcal{O}_a)$ defined in (2.2) solves Problem 1.

About the construction of γ: The proof of Theorem 2.5 given by Wazevski is based on Remark 2, noting that, if we have the decomposition

$$\mathfrak{T}(x) = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}$$

with $A(x)$ invertible on some given subset $\Re$ of $\mathcal{O}$, then

$$\gamma(x) = \begin{pmatrix} C(x) \\ D(x) \end{pmatrix}$$

satisfies (2.3) on $\Re$ if and only if $D(x)$ is invertible on $\Re$ and we have

$$C(x) = -(A^T(x))^{-1}B(x)^T D(x) \qquad \forall x \in \Re \;. \tag{2.8}$$

Thus, C is imposed by the choice of D and choosing D invertible is enough to build γ on $\Re$.

Also, if we already have a candidate

$$P(x) = \begin{pmatrix} A(x) & C_0(x) \\ B(x) & D_0(x) \end{pmatrix}$$

on a boundary $\partial\Re$ of $\Re$, then, necessarily, if $A(x)$ is invertible for all x in $\partial\Re$, then $D_0(x)$ is invertible and $C_0(x) = -(A^T(x))^{-1}B(x)^T D_0(x)$ all x in $\partial\Re$. Thus, to extend the construction of a continuous function γ inside $\Re$ from its knowledge on the boundary $\partial\Re$, it suffices to pick D as any invertible matrix satisfying $D = D_0$ on $\partial\Re$.

Because we can propagate continuously γ from one boundary to the other, Wazewski deduces from these two observations that, it is sufficient to partition the set $\mathcal{O}$ into adjacent sets $\Re_i$ where a given $n \times n$ minor A_i is invertible. This is possible since $\mathfrak{T}$ is full-rank on $\mathcal{O}$. When $\mathcal{O}$ is a parallelepiped, he shows that there exists an ordering of the $\Re_i$ such that the continuity of each D_i can be successively ensured. We illustrate this construction in Example 4 below.

EXAMPLE 4. Consider the function

$$\mathfrak{T}(x) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \\ \frac{\partial \wp}{\partial x_1} x_3 & \frac{\partial \wp}{\partial x_2} x_3 & \wp \end{pmatrix} \quad , \qquad \wp(x_1, x_2) \;=\; \max\left\{0, \frac{1}{r^2} - (x_1^2 + x_2^2)\right\}^4 .$$

$\mathfrak{T}(x)$ has full rank 3 for any x in $\mathbb{R}^3$, since $\wp(x_1, x_2) \neq 0$ when $x_1 = x_2 = 0$. To follow Wazewski's construction, let δ be a strictly positive real number and consider the following 5 regions of $\mathbb{R}^3$ (see Figure 2.1)

$$\Re_1 \;=]-\infty, -\delta] \times \mathbb{R}^2 \quad , \qquad \Re_2 \;=\; [-\delta, \delta] \times [\delta, +\infty] \times \mathbb{R},$$
$$\Re_3 \;=\; [-\delta, \delta]^2 \times \mathbb{R} \quad , \qquad \Re_4 \;=\; [-\delta, \delta] \times [-\infty, -\delta] \times \mathbb{R} \quad , \qquad \Re_5 \;=\; [\delta, +\infty[\times \mathbb{R}^2.$$

We select δ sufficiently small in such a way that $\wp$ is not 0 in $\Re_3$.

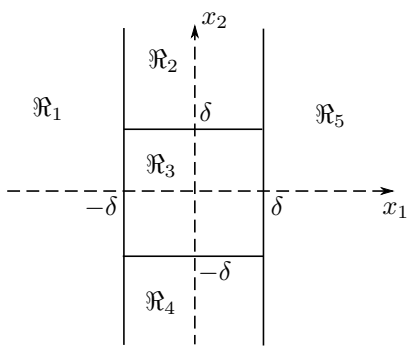

FIG. 2.1. *Projections of the regions $\Re_i$ on $\mathbb{R}^2$.*

We start Wazewski's algorithm in $\Re_3$. Here, the invertible minor A is given by rows 1, 2 and 5 of $\mathfrak{T}$ (full-rank lines of $\mathfrak{T}$) and B by rows 3 and 4. With picking D as the identity, C is $(A^T)^{-1}B$ according to (2.8). D gives rows 3 and 4 of γ and C gives rows 1, 2 and 5 of γ.
Then we move to the region $\Re_2$. There the matrix A is given by rows 1, 2 and 4 of $\mathfrak{T}$, B by rows 3 and 5. Also D, along the boundary between $\Re_3$ and $\Re_2$, is given by rows 3 and 5 of γ obtained in the previous step. We extrapolate this inside $\Re_2$ by keeping D constant in planes x_1 =constant. An expression for C and therefore for γ follows.
We do exactly the same thing for $\Re_4$.
Then we move to the region $\Re_1$. There the matrix A is given by rows 1, 2 and 3 of $\mathfrak{T}$, B by rows 4 and 5. Also D, along the boundary between $\Re_1$ and $\Re_2$, between $\Re_1$ and $\Re_3$ and between $\Re_1$ and $\Re_4$, is given by rows 4 and 5 of γ obtained in the previous steps. We extrapolate this inside $\Re_1$ by keeping D constant in planes x_2 =constant. An expression for C and therefore for γ follows.
We do exactly the same thing for $\Re_5$.

Note that this construction produces a continuous γ, but we could have extrapolated D in a smoother way to obtain γ as smooth as necessary. $\triangle$

Although Wazewski's method provides a more general answer to the problem of Jacobian complementation than the few solvable $\tilde{P}[m,n]$ problems, the explicit expressions of γ given in Section 2.2 are preferred in practice (when the couple (m,n) is appropriate) to Wazewski's costly computations.

3. About Problem 2 : Image extension of a diffeomorphism. We study now how a diffeomorphism can be augmented to make its image be the whole set $\mathbb{R}^m$, i.e. to make it surjective.

3.1. A sufficient condition. There is a rich literature reporting very advanced results on the diffeomorphism extension problem. In the following some of the techniques are inspired from [14, Chapter 8] and [21, pages 2, 7 to 14 and 16 to 18](among others). Here we are interested in the particular aspect of this topic which is the diffeomorphism image extension as described by Problem 2. A very first necessary condition about this problem is in the following remark.

Remark 4. Since τ_e^*, obtained solving Problem 2, makes the set $\mathcal{O}_a$ diffeomorphic to $\mathbb{R}^m$, $\mathcal{O}_a$ must be contractible.

One of the key technical property which will allow us to solve Problem 2 can be phrased as follows.

Definition 3.1 (Condition $\mathbb{B}$). *An open subset E of $\mathbb{R}^m$ is said to verify condition $\mathbb{B}$ if there exist a C^1 function $\kappa : \mathbb{R}^m \to \mathbb{R}$, a bounded*[5] *$C^1$ vector field χ, and a closed set K_0 contained in E such that:*

1. *$E = \{z \in \mathbb{R}^n, \kappa(z) < 0\}$*
2. *K_0 is globally attractive for χ*
3. *we have the following transversality property:*

$$\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \qquad \forall z \in \mathbb{R}^m : \ \kappa(z) = 0.$$

The two main ingredients of this condition are the function κ and the vector field χ which, both, have to satisfy the transversality property $\mathbb{B}$.3. In the case where only the function κ is given satisfying $\mathbb{B}$.1 and with no critical point on the boundary of E, its gradient could play the role of χ. But then for K_0 to be globally attractive we need at least to remove all the possible critical points that κ could have outside K_0. This task is performed for example on Morse functions in the proof of the h-Cobordism Theorem [21]. We are in a much simpler situation when χ is given and makes E forward invariant.

Lemma 3.2. *Let E be a bounded open subset of $\mathbb{R}^m$, χ be a bounded C^1 vector field , and K_0 be a compact set contained in E such that:*

1. *K_0 is globally asymptotically stable for χ*
2. *E is forward invariant for χ.*

For any strictly positive real number $\overline{d}$, there exists a bounded set $\mathcal{E}$ such that

$$\mathtt{cl}(E) \subset \mathcal{E} \subset \{z \in \mathbb{R}^m, \ \inf_{z_E \in E} |z - z_E| \leq \overline{d}\}$$

and $\mathcal{E}$ verifies condition $\mathbb{B}$.

This Lemma says roughly that if E does not satisfy conditions $\mathbb{B}$.1 or $\mathbb{B}$.3 but is forward invariant for χ, then Condition $\mathbb{B}$ is satisfied by an arbitrarily close superset of E. Its proof is given in Appendix G.

[5] If not replace χ by $\frac{\chi}{\sqrt{1+|\chi|^2}}$.

Our main result on the diffeomorphism image extension problem is:

THEOREM 3.3 (Image extension). *Let $\mathcal{O}_a$ be an open subset of $\mathbb{R}^m$ and τ_a^*: $\mathcal{O}_a \to \mathbb{R}^m$ be a diffeomorphism. If*

a) either $\tau_a^*(\mathcal{O}_a)$ verifies condition $\mathbb{B}$,

b) or $\mathcal{O}_a$ is C^2-diffeomorphic to $\mathbb{R}^m$ and τ_a^* is C^2,

then for any compact set K in $\mathcal{O}_a$, there exists a diffeomorphism $\tau_e^* : \mathcal{O}_a \to \mathbb{R}^m$ solving Problem 2.

The proof of case a) of this theorem is given in Section 3.2. It provides an explicit construction of τ_e^*. The proof of case b) can be found in Appendix F. For the time being, we observe that a direct consequence is :

COROLLARY 3.4. *Let $\mathcal{A}$ be a bounded subset of $\mathbb{R}^n$, $\mathcal{O}_a$ be an open subset of $\mathbb{R}^m$ containing $\mathtt{cl}(\mathcal{A} \times \{0\})$ and $\tau_a^* : \mathcal{O}_a \to \tau_a^*(\mathcal{O}_a)$ be a diffeomorphism such that*

a) either $\tau_a^*(\mathcal{O}_a)$ verifies condition $\mathbb{B}$,

b) or $\mathcal{O}_a$ is C^2-diffeomorphic to $\mathbb{R}^m$ and τ_a^* is C^2.

Then, there exists a diffeomorphism $\tau_e^* : \mathcal{O}_a \to \mathbb{R}^m$, such that

$$\tau_e^*(\mathcal{O}_a) = \mathbb{R}^m \quad , \quad \tau_e^*(x, 0) \;=\; \tau_a^*(x, 0) \qquad \forall x \in \mathcal{A} \; .$$

Thus, if besides the pair $(\tau_a^*, \mathcal{O}_a)$ solves Problem 1, then $(\tau_e^*, \mathcal{O}_a)$ solves Problems 1 and 2.

3.2. Proof of part a) of Theorem 3.3. We have the following technical lemma a constructive proof of which is given in Appendix E.

LEMMA 3.5. *Let E be an open strict subset of $\mathbb{R}^m$ verifying Condition $\mathbb{B}$. For any closed subset K of E, lying at a strictly positive distance of the boundary of E, there exists a diffeomorphism ϕ: $\mathbb{R}^m \to E$, such that ϕ is the identity function on K.*

In the case a) of Theorem 3.3, we suppose that $\tau_a^*(\mathcal{O}_a)$ satisfies $\mathbb{B}$. Now, τ_a^* being a diffeomorphism on an open set $\mathcal{O}_a$, the image of any compact subset K of $\mathcal{O}_a$ is a compact subset of $\tau_a^*(\mathcal{O}_a)$. According to Lemma 3.5, there exists a diffeomorphism ϕ from $\mathbb{R}^m$ to $\tau_a^*(\mathcal{O}_a)$ which is the identity on $\tau_a^*(K)$. Thus, the function $\tau_e^* \;=\; \phi^{-1} \circ \tau_a^*$ solves Problem 2 and the theorem is proved.

EXAMPLE 5 (Continuation of Example 2). In Example 2, we have introduced the function

$$F(\xi) = \xi_2\xi_3 - \xi_1\xi_4 \;\triangleq\; \frac{1}{2}\xi^T M \xi$$

as a submersion on $\mathbb{R}^4 \backslash \{0\}$ satisfying

$$F(\tau^*(x)) = 0, \tag{3.1}$$

where τ^* is the injective immersion given in (1.3). With it we have augmented τ^* as

$$\tau_a^*(x, w) = \tau^*(x) + \frac{\partial F}{\partial \xi}^T (\tau^*(x))\, w = \tau^*(x) + M\tau^*(x)\, w$$

which is a diffeomorphism on $\mathcal{O}_a = \tilde{\mathcal{O}} \times]-\varepsilon, \varepsilon[$ for some strictly positive real number ε.

To modify τ_a^* in τ_e^* satisfying $\tau_e^*(\mathcal{O}_a) = \mathbb{R}^4$, we let K be the compact set

$$K = \mathtt{cl}(\tau_a^*(\mathcal{A} \times \{0\})) \;\subset \tau_a^*(\mathcal{O}_a) \subset \mathbb{R}^4 \; .$$

With Lemma 3.5, we know that, if $\tau_a^*(\mathcal{O}_a)$ verifies condition $\mathbb{B}$, there exists a diffeomorphism ϕ defined on $\mathbb{R}^4$ such that ϕ is the identity function on the compact set K

and $\phi(\mathbb{R}^4) = \tau_e^*(\mathcal{O}_a)$. In that case, as seen above, the diffeomorphism $\tau_e^* = \phi^{-1} \circ \tau_a^*$ defined on $\mathcal{O}_a$ is such that $\tau_e^* = \tau_a^*$ on $\mathcal{A} \times \{0\}$ and $\tau_e^*(\mathcal{O}_a) = \mathbb{R}^4$, i.e. would be a solution to Problems 1 and 2. Unfortunately this is impossible. Indeed, due to the observability singularity at $x_1 = x_2 = 0$, $\tilde{\mathcal{O}}$ (and thus $\mathcal{O}_a$) is not contractible. Therefore, there is no diffeomorphism τ_e^* such that $\tau_e^*(\mathcal{O}_a) = \mathbb{R}^4$. We will see in Section 5 how this problem can be overcome. For the time being, we show that it is still possible to find τ_e^* such that $\tau_e^*(\mathcal{O}_a)$ covers "almost all" $\mathbb{R}^4$. The idea is to find an approximation E of $\tau_a^*(\mathcal{O}_a)$ verifying condition $\mathbb{B}$ and apply the same method on E. Indeed, if E is close enough to $\tau_a^*(\mathcal{O}_a)$, one can expect to have $\tau_e^*(\mathcal{O}_a)$ "almost equal to" $\mathbb{R}^4$.

With (3.1) and since $M^2 = I$, we have, $F(\tau_a^*(x, w)) = |\tau^*(x)|^2\, w$. Since $\mathcal{O}_a$ is bounded, there exists $\delta > 0$ such that the set $E = \left\{\xi \in \mathbb{R}^4 : F(\xi)^2 < \delta\right\}$ contains $\tau_a^*(\mathcal{O}_a)$ and thus the compact set K. Let us show that E verifies condition $\mathbb{B}$. We pick

$$\kappa(\xi) \;=\; F(\xi)^2 - \delta \;=\; \left(\frac{1}{2}\xi^T M \xi\right)^2 - \delta \;.$$

and consider the vector field χ

$$\chi(\xi) = -2\frac{\partial \kappa}{\partial \xi}(\xi) = -[\xi^T M \xi]\, M\xi \qquad \text{or more simply} \qquad \chi(\xi) = -\xi \;.$$

The latter implies the transversality property $\mathbb{B}.3$ is verified. Besides, the closed set $K_0 = \{0\}$ is contained in E and is globally attractive for the vector field χ.

Then Lemma 3.5 gives the existence of a diffeomorphism $\phi : \mathbb{R}^4 \to E$ which is the identity on K and verifies $\phi(\mathbb{R}^4) = E$. We obtain an expression of ϕ by following the constructive proof of this Lemma (see Appendix E). Let E_ε be the set

$$E_\varepsilon = \left\{\xi \in \mathbb{R}^4 : \left(\frac{1}{2}\xi^T M \xi\right)^2 < e^{-4\varepsilon}\,\delta\right\} \;.$$

It contains K. Let also $\nu : [-\varepsilon, +\infty[\to \mathbb{R}$ and $\mathfrak{t} : \mathbb{R}^4 \setminus E_\varepsilon \to \mathbb{R}$ be the functions defined as

$$\nu(t) \;=\; \frac{(t+\varepsilon)^2}{2\varepsilon + t} \quad , \qquad \mathfrak{t}(\xi) = \frac{1}{4}\ln\frac{\left(\frac{1}{2}\xi^T M \xi\right)^2}{\delta} \;. \tag{3.2}$$

$\mathfrak{t}(\xi)$ is the time that a solution of $\dot{\xi} = \chi(\xi) = -\xi$ with initial condition ξ needs to reach the boundary of E i.e. $e^{-\mathfrak{t}(\xi)}\xi$ belongs to the boundary of E. From the proof Lemma 3.5, we know the function $\phi : \mathbb{R}^4 \to E$ defined as :

$$\phi(\xi) = \begin{cases} \xi & , \quad \text{if } \left(\frac{1}{2}\xi^T M \xi\right)^2 \le e^{-4\varepsilon}\delta, \\ e^{-\nu(\mathfrak{t}(\xi))}\xi & , \quad \text{otherwise,} \end{cases} \tag{3.3}$$

is a diffeomorphism $\phi : \mathbb{R}^4 \to E$ which is the identity on K and verifies $\phi(\mathbb{R}^4) = E$.

As explained above, we use ϕ to replace τ_a^* by the diffeomorphism $\tau_e^* = \phi^{-1} \circ \tau_a^*$ also defined on $\mathcal{O}_a$. But, because $\tau_a^*(\mathcal{O}_a)$ is a strict subset of E, $\tau_e^*(\mathcal{O}_a)$ is a strict subset of $\mathbb{R}^4$, i.e. equation (1.17) is not satisfied. Nevertheless, for any trajectory of the observer $t \mapsto \hat{\xi}(t)$ in $\mathbb{R}^4$, our estimate defined by $(\hat{x}, \hat{w}) = \tau_e^{*-1}(\hat{\xi})$ will be such that $\tau_a^*(\hat{x}, \hat{w})$ remains in E, along this trajectory i.e. $|\tau^*(\hat{x})|^2\, \hat{w} < \delta$. This ensures that, far from the observability singularity where $|\tau^*(\hat{x})| = 0$, $\hat{w}$ remains sufficiently small to keep the invertibility of the Jacobian of τ_e^*. But we still have a problem

close to the observability singularity, i.e. when $(\hat{x}_1, \hat{x}_2)$ is close to the origin. We shall see in Section 5 how to avoid this difficulty via a better choice of the initial injective immersion τ^*. $\triangle$

3.3. Application : bioreactor. As a more practical illustration we consider the model of bioreactor presented in [12] :

$$\dot{x}_1 = \frac{a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_1 \ , \ \dot{x}_2 = -\frac{a_3 a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_2 + u a_4 \ , \ y = x_1$$

where the a_i's are strictly positive real numbers and the control u verifies : $0 < u_{min} < u(t) < u_{max} < a_1$. This system evolves in the set $\mathcal{O} = \left\{ x \in \mathbb{R}^2 \ : \ x_1 > \varepsilon_1 \ , \ x_2 > -a_2 x_1 \right\}$ which is forward invariant. A high gain observer design leads us to consider the function $\tau^* : \mathcal{O} \to \mathbb{R}^2$ defined as :

$$\tau^*(x_1, x_2) \ = \ (x_1, \dot{x}_1|_{u=0}) \ = \ \left(x_1, \frac{a_1 x_1 x_2}{a_2 x_1 + x_2} \right) \ .$$

It is a diffeomorphism onto

$$\tau^*(\mathcal{O}) \ = \ \left\{ \xi \in \mathbb{R}^2 \ : \ \xi_1 > 0 \ , \ a_1 \xi_1 > \xi_2 \right\} \ .$$

The image by τ^* of the bioreactor dynamics is of the form

$$\dot{\xi}_1 \ = \ \xi_2 + g_1(\xi_1) u \quad , \qquad \dot{\xi}_2 \ = \ \varphi_2(\xi_1, \xi_2) + g_2(\xi_1, \xi_2) u$$

for which the following high gain observer can be built:

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 + g_1(\hat{\xi}_1) u - k_1 \ell (\hat{\xi}_1 - y) \quad , \quad \dot{\hat{\xi}}_2 = \varphi_2(\hat{\xi}_1, \hat{\xi}_2) + g_2(\hat{\xi}_1, \hat{\xi}_2) u - k_2 \ell (\hat{\xi}_1 - y) \ , \quad (3.4)$$

where k_1 and k_2 are strictly positive real numbers and ℓ sufficiently large. As in [12], τ^* being a diffeomorphism the dynamics of this observer in the x-coordinates are

$$\dot{\hat{x}} \ = \ \begin{pmatrix} \frac{a_1 \hat{x}_1 \hat{x}_2}{a_2 \hat{x}_1 + \hat{x}_2} - u \hat{x}_1 \\ -\frac{a_3 a_1 \hat{x}_1 \hat{x}_2}{a_2 \hat{x}_1 + \hat{x}_2} - u \hat{x}_2 + u a_4 \end{pmatrix} + \ell \begin{pmatrix} 1 & 0 \\ -1 & \frac{(a_2 \hat{x}_1 + \hat{x}_2)^2}{a_1 a_2 \hat{x}_1^2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} (\hat{\xi}_1 - y) \ . \quad (3.5)$$

Unfortunately the right hand side is singular at $\hat{x}_1 = 0$ and $\hat{x}_2 = -a_1 \hat{x}_1$. $\mathcal{O}$ being forward invariant, the system trajectories stay away from the singularity. But nothing guarantees the same property holds for the observer trajectories given by (3.5). In other words, since τ^* is already a diffeomorphism, Problem 1 is solved with $m = n$, $\tau_a^* = \tau^*$ and $\mathcal{O}_a = \mathcal{O}$. But (1.17) is not satisfied, i.e. Problem 2 must be solved.

To construct the extension τ_e^* of τ_a^*, we view the image $\tau_a^*(\mathcal{O}_a)$ as the intersection $\tau_a^*(\mathcal{O}_a) = E_1 \cap E_2$ with :

$$E_1 \ = \ \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2, \ \xi_1 > \varepsilon_1 \right\} \quad , \qquad E_2 \ = \ \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2, \ a_1 \xi_1 > \xi_2 \right\} \ .$$

This exhibits the fact that $\tau_a^*(\mathcal{O}_a)$ does not satisfy the condition $\mathbb{B}$ since its boundary is not C^1. We could smoothen this boundary to remove its "corner". But we prefer to exploit its particular "shape" and proceed as follows :

1. We build a diffeomorphism $\phi_1 : \mathbb{R}^2 \to E_1$ which acts on ξ_1 without changing ξ_2.
2. We build a diffeomorphism $\phi_2 : \mathbb{R}^2 \to E_2$ which acts on ξ_2 without changing ξ_1.
3. Denoting $\phi = \phi_2 \circ \phi_1 : \mathbb{R}^2 \to E_1 \cap E_2$, we take $\tau_e^* = \phi^{-1} \circ \tau_a^* : \mathcal{O}_a \to \mathbb{R}^2$.

To build ϕ_1 and ϕ_2, we follow the procedure given in the proof of Lemma 3.5 since E_1 and E_2 satisfy condition $\mathbb{B}$ with :

$$\kappa_1(\xi) = \varepsilon_1 - \xi_1 \ , \ \kappa_2(\xi) = \xi_2 - a_1\xi_1 \ , \ \chi_1(\xi) = \begin{pmatrix} -(\xi_1 - 1) \\ 0 \end{pmatrix} \ , \ \chi_2(\xi) = \begin{pmatrix} 0 \\ -(\xi_2 + 1) \end{pmatrix}.$$

By following the same steps as in Example 5, with ε an arbitrary small strictly positive real number and ν defined in (3.2), we obtain :

$$\left| \begin{array}{l} \mathfrak{t}_1(\xi) = \ln \frac{1-\xi_1}{1-\varepsilon} \\ E_{\varepsilon,1} = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2, \ \xi_1 > 1 - \frac{1-\varepsilon}{e^{\varepsilon}} \right\} \\ \phi_1(\xi) = \begin{cases} \xi & , \quad \text{if } \xi \in E_{\varepsilon,1} \\ \frac{\xi_1 - 1}{e^{\nu(\mathfrak{t}_1(\xi))}} + 1 \ , & \text{otherwise} \end{cases} \end{array} \right| \quad \left| \begin{array}{l} \mathfrak{t}_2(\xi) = \ln \frac{\xi_2+1}{a_1\xi_1+1} \ , \\ E_{\varepsilon,2} = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2, \ \xi_2 \leq \frac{a_1\xi_1+1}{e^{\varepsilon}} - 1 \right\} \\ \phi_2(\xi) = \begin{cases} \xi & , \quad \text{if } \xi \in E_{\varepsilon,2} \\ \frac{\xi_2+1}{e^{\nu(\mathfrak{t}_2(\xi))}} - 1 \ , & \text{otherwise} \end{cases} \end{array} \right. \tag{3.6}$$

We remind the reader that, in the $\hat{\xi}$-coordinates, the observer dynamics are not modified. The difference between using τ^* or τ_e^* is seen in the $\hat{x}$-coordinates only. And, by construction it has no effect on the system trajectories since we have

$$\tau^*(x) = \tau_e^*(x) \quad \forall x \in \mathcal{O} \text{ “} - \varepsilon \text{”} \ .$$

As a consequence the difference between τ^* and τ_e^* is significant only during the transient, making sure, for the latter, that $\hat{x}$ never reaches a singularity of the Jacobian of τ_e^*.

We present in Figure 3.1 the results in the $\hat{\xi}$ coordinates (to allow us to see the effects of both τ^* and τ_e^*) of a simulation with (similar to [12]) :

$$\begin{gathered} a_1 = a_2 = a_3 = 1 \ , \ a_4 = 0.1 \\ u(t) = 0.08 \text{ for } t \leq 10 \ , \quad = 0.02 \text{ for } 10 \leq t \leq 20 \ , \quad = 0.08 \text{ for } t \geq 20 \\ x(0) = (0.04, 0.07), \quad \hat{x}(0) = (0.03, 0.09), \quad \ell = 5. \end{gathered}$$

The solid black curves are the singularity locus. The red (= solid dark) curve represents the bioreactor solution. The magenta (= light grey dashdot) curve represents the solution of the observer built with τ_e^*. It evolves freely in $\mathbb{R}^2$ according to the dynamics (3.4), not worried by any constraints. The blue (= dark dashed) curve represents its image by ϕ which brings it back inside the constrained domain where τ^{*-1} can then be used. This means these two curves represent the same object but viewed in different coordinates.

The solution of the observer built with τ^* would coincide with the magenta (= light grey dashdot) curve up to the point it reaches one solid black curve of a singularity locus. At that point it leaves $\tau^*(\mathcal{O})$ and consequently stop existing in the x-coordinates.

As proposed in [20, 3], instead of keeping the raw dynamics (3.4) untouched as above, another solution would be to modify them to force $\hat{\xi}$ to remain in the set $\tau^*(\mathcal{O})$. For instance, taking advantage of the convexity of this set, the modification proposed in [3] consists in adding to (3.4) the term

$$\mathcal{M}(\hat{\xi}) = -g\, S_\infty \frac{\partial \mathfrak{h}}{\partial \hat{\xi}}(\hat{\xi})^T \, \mathfrak{h}(\hat{\xi}) \quad , \qquad \mathfrak{h}(\hat{\xi}) \ = \ \begin{pmatrix} \max\{\kappa_1(\hat{\xi}) + \varepsilon, 0\}^2 \\ \max\{\kappa_2(\hat{\xi}) + \varepsilon, 0\}^2 \end{pmatrix} \tag{3.7}$$

with S_∞ a symetric positive definite matrix depending on (k_1, k_2, ℓ), ε an arbitrary small real number and g a sufficiently large real number. The solution corresponding to this modified observer dynamics is shown in Figure 3.1 with the dotted black curve.

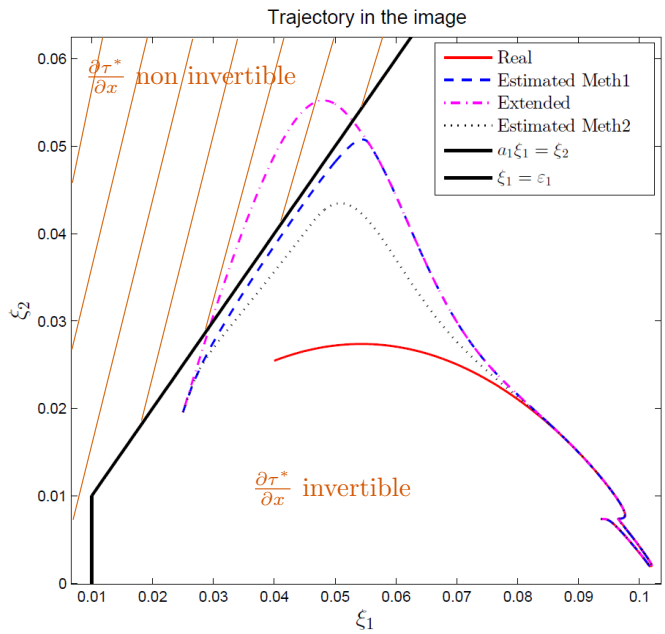

FIG. 3.1. *Bioreactor and observers solutions in the $\hat{\xi}$-coordinates*

As expected it stays away from the the singularities locus in a very efficient way. But, for this method to apply, we have the restriction that $\tau^*(\mathcal{O})$ should be convex, instead of satisfying the less restrictive condition $\mathbb{B}$. Moreover, to guarantee that $\hat{\xi}$ is in $\tau^*(\mathcal{O})$, g has to be large enough and even larger when the measurement noise is larger. On the contrary, when the observer is built with τ_e^*, there is no need to tune properly any parameter to obtain convergence, at least theoretically. Nevertheless there maybe some numerical problems when $\hat{\xi}$ becomes too large or equivalently $\phi(\hat{\xi})$ is too close to the boundary of $\tau^*(\mathcal{O})$. To overcome this difficulty we can select the "thickness" of the layer (parameter ε in (3.6)) sufficiently large. Actually instead of "opposing" the two methods, we suggest to combine them. The modification (3.7) makes sure $\hat{\xi}$ does not go too far outside the domain, and τ_e^* makes sure that $\hat{x}$ does not cross the singularity locus.

4. About the requirement that (τ_{ex}, φ) is in $\varphi\mathcal{T}$ in Proposition 1.1 . Throughout Sections 2 and 3, we have given conditions under which it is possible to solve Problem 1 and Problem 2 when Assumption $\mathbb{A}$ holds and $\mathcal{A}$ is bounded.

However, to apply Proposition 1.1 we need τ_{ex}, the x-component of the inverse τ_e of τ_e^*, solution of Problem 2, to be associated with a function φ such that the pair (φ, τ_{ex}) is in the set $\varphi\mathcal{T}$ given by assumption $\mathbb{A}$.

Fortunately pairing a function φ with a function τ_{ex} obtained from a left inverse of τ_e^* is not as difficult as it seems, at least for general purpose observer designs such as high gain observers or nonlinear Luenberger observers.

Indeed, we have already observed in point 2 of Remark 1 that if, as for Luenberger observers, there is a pair, in the set $\varphi\mathcal{T}$, the component φ of which does not depend on τ, then we can associate this φ to any τ_{ex}.

Also, for high gain observers, we need only that τ_{ex}, used as argument of φ, be globally Lipschitz. This is obtained by modifying, if needed, this function outside a

compact set, as the saturation function does in (1.2).

5. Modifying τ^* and $\varphi\mathcal{T}$ given by Assumption $\mathbb{A}$.
The sufficient conditions, given in Sections 2 and 3, to solve Problem 1 and Problem 2 in order to fulfill the requirements of Proposition 1.1, impose conditions on the dimensions or on the domain of injectivity $\mathcal{O}$ which are not always satisfied : contractibility for Jacobian complementation and diffeomorphism extension, limited number of pairs (m, n) for the $\tilde{P}[m, n]$ problem, etc. Expressed in terms of our initial problem, these conditions are limitations on the data f, h and τ^* that we considered. In the following, we show by means of examples that, in some cases, these data can be modified in such a way that our various tools apply and give a satisfactory solution. Such modifications are possible since we restrict our attention to system solutions which remain in $\mathcal{A}$. Therefore we can modify arbitrarily the data f, h and τ^* outside this set. For example we can add arbitrary "fictitious" components to the measured output y as long as their value is known on $\mathcal{A}$.

5.1. For contractibility. It may happen that the set $\mathcal{O}$ attached to τ^* is not contractible, for example due to an observability singularity. We have seen that Jacobian complementation and image extension may be prevented by this (see Theorem 2.5 and Remark 4). A possible approach to overcome this difficulty when we know the system trajectories stay away from the singularities is to add a fictitious output traducing this information :

Example 6 (Continuation of Example 3). The observer we have obtained at the end of Example 3 for the harmonic oscillator with unknown frequency is not satisfactory in particular because of the singularity at $\hat{x}_1 = \hat{x}_2 = 0$. To overcome this difficulty we add, to the given measurement $y = x_1$, the following

$$y_2 = h_2(x) \;=\; \wp(x_1, x_2)\, x_3$$

with

$$\wp(x_1, x_2) \;=\; \max\left\{0, \frac{1}{r^2} - (x_1^2 + x_2^2)\right\}^4 \;.$$

By construction this function is zero on $\mathcal{A}$ and y_2 can thus be considered as an extra measurement. The interest of y_2 is to give access to x_3 even at the singularity $x_1 = x_2 = 0$. Indeed, consider the new function τ^* defined as

$$\tau^*(x) \;=\; (x_1\,,\; x_2\,,\; -x_1x_3\,,\; -x_2x_3\,,\; \wp(x_1, x_2)\, x_3) \;. \tag{5.1}$$

τ^* is C^1 on $\mathbb{R}^3$ and its Jacobian is :

$$\frac{\partial \tau^*}{\partial x}(x) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \\ \frac{\partial \wp}{\partial x_1} x_3 & \frac{\partial \wp}{\partial x_2} x_3 & \wp \end{pmatrix} \;, \tag{5.2}$$

which has full rank 3 on $\mathbb{R}^3$, since $\wp(x_1, x_2) \neq 0$ when $x_1 = x_2 = 0$. It follows that the singularity has disappeared and this new τ^* is an injective immersion on the entire $\mathbb{R}^3$ which is contractible.

We have shown in Example 4 how Wazewski's algorithm allows us to get in this case a C^2 function $\gamma : \mathbb{R}^3 \to \mathbb{R}^4$ satisfying :

$$\det\left(\frac{\partial \tau^*}{\partial x}(x)\; \gamma(x)\right) \;\neq\; 0 \qquad \forall x \in \mathbb{R}^3 \;.$$

This gives us $\tau_a^*(x,w) \;=\; \tau^*(x)+\gamma(x)w$ which is a C^2-diffeomorphism on $\mathbb{R}^3\times\mathcal{B}_\varepsilon(0)$, with ε sufficiently small.

Furthermore, $\mathcal{O}_a \;=\; \mathbb{R}^3\times\mathcal{B}_\varepsilon(0)$ being now diffeomorphic to $\mathbb{R}^5$, Corollary 3.4 applies and provides an extension τ_e^* of τ_a^* satisfying Problems 1 and 2. $\triangle$

5.2. For a solvable $\tilde{P}[m,n]$ problem. If we are in a case that cannot be reduced to a solvable $\tilde{P}[m,n]$ problem, we may try to modify m by adding arbitrary rows to $\frac{\partial\tau^*}{\partial x}$. We illustrate this technique with the following example.

Example 7 (Continuation of Example 6). In Example 6, by adding the fictitious measured output $y_2 = h_2(x)$, we have obtained another function τ^* for the harmonic oscillator with unknown frequency which is an injective immersion on $\mathbb{R}^3$. In this case, we have $n=3$ and $m=5$ which gives a pair not in (2.6). But, as already exploited in Example 3, the first 2 rows of the Jacobian $\frac{\partial\tau^*}{\partial x}$ in (5.2) are independent for all x in $\mathbb{R}^3$. It follows that our Jacobian complementation problem reduces to complement the vector $(-x_1,-x_2,\wp(x_1,x_2))$. This is a problem with pair $(3,1)$ which is not in (2.6) either. Instead, the pair $(4,1)$ is, meaning that the following vector can be complemented via a universal formula $(-x_1,-x_2,\wp(x_1,x_2),0)$. We have added a zero component, without changing the full rank property. Actually this vector is extracted from the Jacobian of

$$\tau^*(x) \;=\; (x_1\,,\;x_2\,,\;-x_1x_3\,,\;-x_2x_3\,,\;\wp(x_1,x_2)\,x_3\,,\;0)\;. \tag{5.3}$$

In the high gain observer paradigm, this zero we add can come from another (fictitious) measured output $y_3 = 0$. A complement of $(-x_1,-x_2,\wp(x_1,x_2),0)$ is

$$\begin{pmatrix} x_2 & -\wp & 0\\ -x_1 & 0 & -\wp\\ 0 & -x_1 & -x_2\\ \wp & x_2 & -x_1 \end{pmatrix}$$

It gives the function

$$\tau_a^*(x,w) = \Big(\; x_1\,,\;x_2\,,\;[-x_1x_3+x_2w_1-\wp(x_1,x_2)w_2]\,,\;[-x_2x_3-x_1w_1-\wp(x_1,x_2)w_3]\,,$$
$$[\wp(x_1,x_2)x_3-x_1w_2-x_2w_3]\,,\;[\wp(x_1,x_2)w_1+x_2w_2-x_1w_3)]\Big)\,.$$

the Jacobian determinant of which is $(x_1^2+x_2^2+\wp(x_1,x_2)^2)^2$ which is nowhere 0 on $\mathbb{R}^6$. Hence τ_a^* is locally invertible. Actually it is diffeomorphism from $\mathbb{R}^6$ onto $\mathbb{R}^6$ since we can express $\hat{\xi} = \tau_a^*(x,w)$ as

$$\begin{pmatrix} x_1\\ x_2 \end{pmatrix} = \begin{pmatrix} \hat{\xi}_1\\ \hat{\xi}_2 \end{pmatrix}\;,\quad \begin{pmatrix} -\hat{\xi}_1 & \hat{\xi}_2 & -\wp(\hat{\xi}_1,\hat{\xi}_2) & 0\\ -\hat{\xi}_2 & -\hat{\xi}_1 & 0 & -\wp(\hat{\xi}_1,\hat{\xi}_2)\\ \wp(\hat{\xi}_1,\hat{\xi}_2) & 0 & -\hat{\xi}_1 & -\hat{\xi}_2\\ 0 & \wp(\hat{\xi}_1,\hat{\xi}_2) & \hat{\xi}_2 & -\hat{\xi}_1 \end{pmatrix}\begin{pmatrix} x_3\\ w_1\\ w_2\\ w_3 \end{pmatrix} = \begin{pmatrix} \hat{\xi}_3\\ \hat{\xi}_4\\ \hat{\xi}_5\\ \hat{\xi}_6 \end{pmatrix}\;,$$

where the matrix on the left is invertible by construction. Since $\tau_a^*(\mathbb{R}^6) = \mathbb{R}^6$, there is no need of an image extension and we simply take $\tau_e^* = \tau_a^*$. To have all the assumptions of Proposition 1.1 satisfied, it remains to find a function φ such that (τ_{ex},φ) is in the set $\varphi\mathcal{T}$, the function τ_{ex} being the x-component of the inverse of τ_e^*. Exploiting the fact that, for x in $\mathcal{A}$, we have

$$\dot{y}_2 = \dot{\overline{\wp(x_1,x_2)x_3}} \;=\; 0 \quad,\qquad \dot{y}_3 \;=\; 0\;,$$

the high gain observer paradigm gives the function

$$\varphi(\hat{\xi},\hat{x},y) \;=\; \begin{pmatrix} \hat{\xi}_2+\ell k_1(y-\hat{x}_1) \\ \hat{\xi}_3+\ell^2 k_2(y-\hat{x}_1) \\ \hat{\xi}_4+\ell^3 k_3(y-\hat{x}_1) \\ \mathtt{sat}(\hat{x}_1\hat{x}_3^2)+\ell^4 k_4(y-\hat{x}_1) \\ -a\,\hat{\xi}_5 \\ -b\,\hat{\xi}_6 \end{pmatrix}$$

where the function $\mathtt{sat}$ is defined in (1.12) and a and b are arbitrary strictly positive real numbers. With picking ℓ large enough, it can be paired with any function $\tau:\mathbb{R}^6\to\mathbb{R}^6$ which is locally Lipschitz, and thus in particular with τ_{ex}. Therefore, Proposition 1.1 applies and gives the following observer for the harmonic oscillator with unknown frequency

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{w}}_1 \\ \dot{\hat{w}}_2 \\ \dot{\hat{w}}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\hat{x}_3-\frac{\partial\wp}{\partial\hat{x}_1}\hat{w}_2 & \hat{w}_1-\frac{\partial\wp}{\partial x_2}\hat{w}_2 & -\hat{x}_1 & \hat{x}_2 & -\wp & 0 \\ -\hat{w}_1-\frac{\partial\wp}{\partial\hat{x}_1}\hat{w}_3 & -\hat{x}_3-\frac{\partial\wp}{\partial x_2}\hat{w}_3 & -\hat{x}_2 & -\hat{x}_1 & 0 & -\wp \\ \frac{\partial\wp}{\partial x_1}\hat{x}_3-\hat{w}_2 & \frac{\partial\wp}{\partial x_2}\hat{x}_3-\hat{w}_3 & \wp & 0 & -\hat{x}_1 & -\hat{x}_2 \\ \frac{\partial\wp}{\partial x_1}\hat{w}_1-\hat{w}_3 & \frac{\partial\wp}{\partial x_2}\hat{w}_1+\hat{w}_2 & 0 & \wp & \hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1} \times \tag{5.4}$$

$$\times \begin{pmatrix} \hat{x}_2+\ell k_1(y-\hat{x}_1) \\ [-\hat{x}_1\hat{x}_3+\hat{x}_2\hat{w}_1-\wp(\hat{x}_1,\hat{x}_2)\hat{w}_2]+\ell^2 k_2(y-\hat{x}_1) \\ [-\hat{x}_2\hat{x}_3-\hat{x}_1\hat{w}_1-\wp(\hat{x}_1,\hat{x}_2)\hat{w}_3]+\ell^3 k_3(y-\hat{x}_1) \\ \mathtt{sat}(\hat{x}_1\hat{x}_3^2)+\ell^4 k_4(y-\hat{x}_1) \\ -a\,[\wp(\hat{x}_1,\hat{x}_2)\hat{x}_3-\hat{x}_1\hat{w}_2-\hat{x}_2\hat{w}_3] \\ -b\,[\wp(\hat{x}_1,\hat{x}_2)\hat{w}_1+\hat{x}_2\hat{w}_2-\hat{x}_1\hat{w}_3)] \end{pmatrix}.$$

It is globally defined and globally convergent for any solution of the oscillator initialized in the set $\mathcal{A}$ given in (1.10).

Observer (5.4) is an illustration of what can be obtained by using in a very nominal way our tools. We do not claim any property for it. For example, by using another design, an observer of dimension 2, globally convergent on $\mathcal{A}$, can be obtained. $\triangle$

In this example we have made the Jacobian complementation possible by increasing m with augmenting the number of coordinates of τ^*. Actually if we augment τ^* with n zeros the possibility of a Jacobian complementation is guaranteed. Indeed pick any C^1 function B the values of which are $m\times m$ matrices with positive definite symmetric part, we can complement $\begin{pmatrix}\frac{\partial\tau^*}{\partial x}\\ 0\end{pmatrix}$ which is full column rank with $\gamma \;=\; \begin{pmatrix}-B\\ \frac{\partial\tau^*}{\partial x}^\top\end{pmatrix}$. This follows from the identity (Schur complement) involving invertible matrices

$$\begin{pmatrix} \frac{\partial\tau^*}{\partial x} & -B \\ 0 & \frac{\partial\tau^*}{\partial x}^\top \end{pmatrix}\begin{pmatrix} 0 & I \\ I & B^{-1}\frac{\partial\tau^*}{\partial x} \end{pmatrix} \;=\; \begin{pmatrix} -B & 0 \\ \frac{\partial\tau^*}{\partial x}^\top & \frac{\partial\tau^*}{\partial x}^\top B^{-1}\frac{\partial\tau^*}{\partial x} \end{pmatrix}.$$

So we have here a universal method to solve our Problem 1. Its drawback is that the dimension of the state increases by m, instead of $m-n$.

6. Conclusion. We have presented a method to express the dynamics of an observer in preferred coordinates enlarging its domain of validity and possibly avoiding

the difficult left-inversion of an injective immersion. It assumes the knowledge of an injective immersion and a converging observer for the immersed system.

The idea is not to modify this observer dynamics but to map it back to the preferred coordinates in a different way. Our construction involves two tools : the augmentation of an injective immersion into a diffeomorphism through a Jacobian complementation and the extension of the image of the obtained diffeomorphism to enlarge the domain where the observer solutions can go without encountering singularities.

For the Jacobian complementation we rely on results by Wazewski [25] and Eckmann [11]. They allows us to build a diffeomorphism by augmenting the preferred coordinates with new ones and to write the given observer dynamics in these augmented coordinates.

For the diffeomorphism extension, we have proposed our own method inspired from diffeotopies [14, Chapter 8] and h-cobordism [21, pages 2, 7 to 14 and 16 to 18].

We have assumed the system is time-invariant and autonomous. Adding time-variations is not a problem but dealing with exogenous inputs is more complex. This is in part due to the fact that, as far as we know, the theory of observers, in presence of such inputs, relying on immersion into a space of larger dimension, as high gain observers or nonlinear Luenberger observers, is not satisfactory enough yet. Progress on this topic has to be made before trying to extend our results.

One very important question which remains to be addressed is about optimizing the observer performance. In our framework it consists in an appropriate selection of the given "raw" observer, i.e. the functions φ and τ^* in (1.2), and the diffeomorphism τ_e for optimizing a cost expressing the quality of the estimated quantities with respect to what they are made for. For such a task, remaining in an ideal context with no modelling error and no measurement disturbance, allows only to address the transient behavior of the state estimate. To be interesting for practice, at least as important if not more important is the long range dependence of the state estimates on unmodelled effects.

Appendix A. Proof of Proposition 1.1. Let $(x_0, (\hat{x}_0, \hat{w}_0))$ be arbitrary in $\mathcal{A} \times \mathcal{O}_a$ but such that $X(x_0, t)$ solution of (1.6) is defined and remains in $\mathcal{A}$ for t in $[0, +\infty)$. Let $[0, T[$ be the right maximal interval of definition of the solution $(X(x_0, t), \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0}))$ when considered with values in $\mathcal{A} \times \mathcal{O}_a$. Assume for the time being T is finite. Then, when t goes to T, either $(\hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0}))$ goes to infinity or to the boundary of $\mathcal{O}_a$. By construction $t \mapsto \hat{\Xi}(t) := \tau_e^* \left(\hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0})\right)$ is a solution of (1.8) on $[0, T[$ with $\tau = \tau_{ex}$. From assumption $\mathbb{A}$ and since (φ, τ_{ex}) is in $\varphi\mathcal{T}$, it can be extended as a solution defined on $[0, +\infty[$ when considered with values in $\mathbb{R}^m = \tau_e^*(\mathcal{O}_a)$. This implies that $\hat{\Xi}(T)$ is well defined in $\mathbb{R}^m$. Since, with (1.17), the inverse τ_e of τ_e^* is a diffeomorphism defined on $\mathbb{R}^m$, we obtain $\lim_{t \to T} \left(\hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0})\right) = \tau_e(\hat{\Xi}(T))$, which is an interior point of $\tau_e(\mathbb{R}^m) = \mathcal{O}_a$. This point being neither a boundary point nor at infinity, we have a contradiction. It follows that T is infinite.

Finally, with assumption $\mathbb{A}$, we have :

$$\lim_{t \to +\infty} \left| \tau_e^* \left(\hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}((\hat{x}_0, \hat{w}_0, t; y_{x_0})\right) - \tau^*(X(x_0, t)) \right| = 0 \ .$$

Since $X(x_0, t)$ remains in $\mathcal{A}$, $\tau^*(X(x_0, t))$ equals $\tau_e^*(X(x_0, t), 0)$ and remains in the compact set $\tau^*(\mathtt{cl}(\mathcal{A}))$. So there exists a compact subset $\mathbf{C}$ of $\mathbb{R}^m$ and a time $t_{\mathbf{C}}$

such that $\tau_e^*\left(\hat{X}(\hat{x}_0,\hat{w}_0,t;y_{x_0}),\hat{W}(\hat{x}_0,\hat{w}_0,t;y_{x_0})\right)$ is in $\mathbf{C}$ for all $t > t_{\mathbf{C}}$. Since τ_e^* is a diffeomorphism, its inverse τ_e is Lipschitz on the compact set $\mathbf{C}$. This implies (1.19).

Appendix B. Proof of Lemma 2.1. The fact that τ_a^* is an immersion for ε small enough is established in [1]. We now prove it is injective. Let ε_0 be a strictly positive real number such that the Jacobian of $\tau_a^*(x,w)$ in (2.2) is invertible for any (x,w) in $\mathtt{cl}(\tilde{\mathcal{O}}\times\mathcal{B}_{\varepsilon_0}(0))$. Since $\mathtt{cl}(\tilde{\mathcal{O}}\times\mathcal{B}_{\varepsilon_0}(0))$ is compact, not to contradict the Implicit function Theorem, there exists a strictly positive real number δ such that any two pairs (x_a,w_a) and (x_b,w_b) in $\mathtt{cl}(\tilde{\mathcal{O}}\times\mathcal{B}_{\varepsilon_0}(0))$ which satisfy

$$\tau_a^*(x_a,w_a) \;=\; \tau_a^*(x_b,w_b) \quad , \quad (x_a,w_a) \;\neq\; (x_b,w_b) \tag{B.1}$$

satisfies also $|x_a-x_b| \;+\; |w_a-w_b| \;\geq\; \delta$. On another hand, since τ^* is continuous and injective on $\mathtt{cl}(\tilde{\mathcal{O}})\subset\mathcal{O}$, it has an inverse which is uniformly continuous on the compact set $\tau^*(\mathtt{cl}(\tilde{\mathcal{O}}))$ (see [4, §16.9]). It follows that there exists a strictly positive real number η such that

$$|x_a-x_b| < \frac{\delta}{2} \qquad \forall\,(\tau^*(x_a),\tau^*(x_b))\in\tau^*(\mathtt{cl}(\tilde{\mathcal{O}}))^2 : \; |\tau^*(x_a)-\tau^*(x_b)| < \eta \;.$$

But if (B.1) holds with w_a and w_b in $\mathcal{B}_\varepsilon(0)$ with $\varepsilon\leq\varepsilon_0$, we have

$$\delta - 2\varepsilon \;\leq\; |x_a-x_b| \;, \qquad |\tau^*(x_a)-\tau^*(x_b)| \;=\; |\gamma(x_a)w_a-\gamma(x_b)w_b| \;\leq\; 2\varepsilon \sup_{x\in\mathtt{cl}(\tilde{\mathcal{O}})}|\gamma(x)| \;.$$

We have a contradiction for all $\varepsilon\leq\min\left\{\frac{3\delta}{4},\frac{\eta}{2\varepsilon\sup_{x\in\mathtt{cl}(\tilde{\mathcal{O}})}|\gamma(x)|}\right\}$. So (B.1) cannot hold for such ε's, i.e. τ_a^* is injective on $\tilde{\mathcal{O}}\times\mathcal{B}_\varepsilon(0)$.

Appendix C. Proof of "only if" in Theorem 2.4. The following theorem is due to Eckmann.

THEOREM C.1 ([11]). *For $m>n$, there exists a continuous function $\mathfrak{x}\in\mathbb{R}^{m\times n}\mapsto\tilde{\gamma}_1(\mathfrak{x})\in\mathbb{R}^m$ with non zero values and satisfying*

$$\tilde{\gamma}_1(\mathfrak{x})^T\mathfrak{x}=0 \qquad \forall\mathfrak{x}\in\mathbb{R}^{m\times n} : \; \mathit{Rank}(\mathfrak{x})=n$$

if and only if (m,n) is in one of the following 4 pairs

$$(\geq 2, m-1) \qquad or \qquad (even,1) \qquad or \qquad (7,2) \qquad or \qquad (8,3) \tag{C.1}$$

With Remark 2, any pair (m,n) for which $\tilde{P}[m,n]$ is solvable must be one in the list (C.1). The pair $(\geq 2, m-1)$ is in the list (2.6). For the pair $(\text{even},1)$, we need to find $m-1$ vectors to complement the given one into an invertible matrix. After normalizing the vector $\mathfrak{x}$ so that it belongs to the unit sphere $\mathbb{S}^{m-1}$ and projecting each vector $\gamma_i(\mathfrak{x})$ of $\gamma(\mathfrak{x})$ onto the orthogonal complement of $\mathfrak{x}$, this complementation problem is equivalent to asking whether $\mathbb{S}^{m-1}$ is parallelizable (since the $\gamma_i(\mathfrak{x})$ will be a basis for the tangent space at $\mathfrak{x}$ for each $\mathfrak{x}\in\mathbb{S}^{m-1}$). It turns out that this problems admits solutions only for $m=4$ or $m=8$ (see [8]). So in the pairs $(\text{even},1)$ only $(4,1)$ and $(8,1)$ are in the list (2.6).

Finally, since $\tilde{P}[6,1]$ has no solution, the pairs $(7,2)$ and $(8,3)$ cannot be in the list (2.6). Indeed let $\mathfrak{x}$ be a full column rank $(m-1)\times(n-1)$ matrix. $\begin{pmatrix}\mathfrak{x} & 0\\ 0 & 1\end{pmatrix}$ is a full column rank $m\times n$ matrix. If if $\tilde{P}[m,n]$ has a solution, there exist a continuous $(m-1)\times(m-n)$ matrix function $\tilde{\gamma}$ and a continuous row vector functions a^T such

that such that $\begin{pmatrix} \tilde{\gamma}(\mathfrak{x}) & \mathfrak{x} & 0 \\ a(\mathfrak{x})^\top & 0 & 1 \end{pmatrix}$ is invertible. This implies that $\begin{pmatrix} \tilde{\gamma}(\mathfrak{x}) & \mathfrak{x} \end{pmatrix}$ is also invertible. So if $\tilde{P}[m,n]$ has a solution, $\tilde{P}[m-1,n-1]$ must have one.

Appendix D. End of proof of Theorem 2.5. We want to show that a continuous function γ making P in (2.7) invertible can be modified into a smoother one giving the same invertibility property. Let γ_i denote the ith column of γ. We start with modifying γ_1 into $\tilde{\gamma}_1$. Since $\mathfrak{x}$, γ and the determinant are continuous, for any x in $\mathcal{O}$, there exists a strictly positive real number r_x, such that, may be after changing γ_1 into $-\gamma_1$,

$$\det\left(\mathfrak{x}(y)\ \gamma_1(x)\ \gamma_{2:m-n}(y)\right) > 0\ , \qquad \forall y \in \mathcal{B}_{r_x}(x)\ , \tag{D.1}$$

where $\gamma_{i:j}$ denotes the matrix composed of the ith to j^{th} columns of γ. The family of sets $(\mathcal{B}_{r_x}(x))_{x\in\mathcal{O}}$ is an open cover of $\mathcal{O}$. Therefore, by [14, Theorem 2.1], there exists a subordinate C^∞ partition of unity, i.e. there exist a family of C^∞ functions $\psi_x : \mathcal{O} \to \mathbb{R}_{\geq 0}$ such that

$$\mathtt{Supp}\,(\psi_x) \subset \mathcal{B}_{r_x}(x) \quad \forall x \in \mathcal{O}\ , \tag{D.2}$$

$$\{\mathtt{Supp}\,(\psi_x)\}_{x\in\mathcal{O}} \text{ is locally finite}\ , \tag{D.3}$$

$$\sum_{x\in\mathcal{O}} \psi_x(y) = 1 \quad \forall y \in \mathcal{O}\ . \tag{D.4}$$

With this, we define the function $\tilde{\gamma}_1$ on $\mathcal{O}$ by

$$\tilde{\gamma}_1(y) = \sum_{x\in\mathcal{O}} \psi_x(y)\gamma_1(x)\ .$$

This function is well-defined and C^∞ on $\mathcal{O}$ because the sum is finite at each point according to (D.3). Using multi-linearity of the determinant, we have, for all y in $\mathcal{O}$,

$$\det\left(\mathfrak{x}(y)\ \tilde{\gamma}_1(y)\ \gamma_{2:m-n}(y)\right) = \sum_{x\in\mathcal{O}} \psi_x(y) \det\left(\mathfrak{x}(y)\ \gamma_1(x)\ \gamma_{2:m-n}(y)\right)\ .$$

Thanks to (D.3), at each point y in $\mathcal{O}$, there is a finite number of $\psi_x(y)$ which are not zero. Also, the right hand side is the sum of non negative terms because of (D.1) and the non negativeness of the ψ_x, and one of these terms is strictly positive because of (D.1) and (D.4). Therefore, we can replace the continuous function γ_1 by the C^∞ function $\tilde{\gamma}_1$ as a first column of γ. Then we follow exactly the same procedure for γ_2 with this modified γ. By proceeding this way, one column after the other, we get our result.

Appendix E. Construction of a diffeomorphism from an open set to $\mathbb{R}^m$. We use the following notations:
The complementary, closure and boundary of a set S are denoted S^c, $\mathtt{cl}(S)$ and ∂S, respectively. The Hausdorff distance d_H between two sets A and B is defined by :

$$d_H(A,B) \ = \ \max\left\{ \sup_{z_A\in A} \inf_{z_B\in B} |z_A - z_B|\,,\, \sup_{z\in A} \inf_{z_B\in B} |z_A - z_B| \right\}\ .$$

$Z(z,t)$ denotes the (unique) solution, at time t, to $\dot{z} = \chi(z)$ going trough z at time 0 and $\Sigma_\varepsilon = \bigcup_{t\in[0,\varepsilon]} Z(\partial E, t)$.

Lemma E.1. *Let E be an open strict subset of $\mathbb{R}^m$ verifying $\mathbb{B}$, with a C^s vector field χ and a C^s mapping κ. There exists a strictly positive (maybe infinite) real*

number ε_∞ such that, for any ε in $[0,\varepsilon_\infty[$, there exists a C^s-diffeomorphism ϕ: $\mathbb{R}^m \to E$, such that

$$\phi(z) = z \quad \forall z \in E_\varepsilon = E \cap (\Sigma_\varepsilon)^{\mathsf{c}} \quad , \qquad d_H(\partial E_\varepsilon, \partial E) \ \leq \ \varepsilon \sup_z |\chi(z)| \ .$$

Proof. According to Condition $\mathbb{B}$, χ is bounded and K_0 is a compact subset of the open set E. It follows that there exists a strictly positive (maybe infinite) real number ε_∞ such that

$$Z(z,t) \notin K_0 \qquad \forall (z,t) \in \partial E \times [0, 2\varepsilon_\infty[\ .$$

In the following ε is a real number in $[0,\varepsilon_\infty[$.

We introduce the notations

$$\Sigma_{2\varepsilon} = \bigcup_{t\in[0,2\varepsilon]} Z(\partial E, t) \quad , \qquad E_{2\varepsilon} = E \cap (\Sigma_{2\varepsilon})^{\mathsf{c}}$$

and establish some properties.

– *E is forward invariant for χ.* This is a direct consequence of points $\mathbb{B}$.1 and $\mathbb{B}$.3.

– *$\Sigma_{2\varepsilon}$ is closed.* Take a sequence (z_k) of points in $\Sigma_{2\varepsilon}$ converging to z^*. By definition of $\Sigma_{2\varepsilon}$, there exists a sequence (t_k), such that :

$$t_k \in [0, 2\varepsilon] \qquad \text{and} \qquad Z(z_k, -t_k) \in \partial E \qquad \forall k \in \mathbb{N} \ .$$

Since $[0,2\varepsilon]$ is compact, one can extract a subsequence $(t_{\sigma(k)})$ converging to t^* in $[0,2\varepsilon]$, and by continuity of the function $(z,t) \mapsto Z(z,-t)$, $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$ tends to $Z(z^*, -t^*)$ which is in ∂E, since ∂E is closed. Finally, because t^* is in $[0,2\varepsilon]$, z^* is in $\Sigma_{2\varepsilon}$ by definition.

– *$\Sigma_{2\varepsilon}$ is contained in $\mathtt{cl}(E)$.* Since, E is forward invariant for χ, and so is $\mathtt{cl}(E)$ (see [**?**, Theorem 16.3]). This implies

$$\partial E \ \subset \ \Sigma_{2\varepsilon} = \bigcup_{t\in[0,2\varepsilon]} Z(\partial E, t) \ \subset \ \mathtt{cl}(E) \ = \ E \cup \partial E \ .$$

At this point, it is useful to note that, because $\Sigma_{2\varepsilon}$ is a closed subset of $\mathtt{cl}(E)$ and E is open, we have $\Sigma_{2\varepsilon} \cap E \ = \ \Sigma_{2\varepsilon} \backslash \partial E$. This implies :

$$E \backslash E_{2\varepsilon} = (E_{2\varepsilon})^{\mathsf{c}} \cap E = (E^{\mathsf{c}} \cup \Sigma_{2\varepsilon}) \cap E = \Sigma_{2\varepsilon} \cap E = \Sigma_{2\varepsilon} \backslash \partial E, \tag{E.1}$$

and $E = E_{2\varepsilon} \underset{\neq}{\cup} (\Sigma_{2\varepsilon} \backslash \partial E)$.

With all these properties at hand, we define now two functions $\mathfrak{t}$ and θ. The assumptions of global attractiveness of the closed set K_0 contained in E open, of transversality of χ to ∂E, and the property of forward-invariance of E, imply that, for all z in E^{c}, there exists a unique non negative real number $\mathfrak{t}(z)$ satisfying:

$$\kappa\left(Z(z, \mathfrak{t}(z))\right) = 0 \quad \Longleftrightarrow \quad Z(z, \mathfrak{t}(z)) \in \partial E.$$

The same arguments in reverse time allow us to see that, for all z in $\Sigma_{2\varepsilon}$, $\mathfrak{t}(z)$ exists, is unique and in $[-2\varepsilon, 0]$. This way, the function $z \to \mathfrak{t}(z)$ is defined on $(E_{2\varepsilon})^{\mathsf{c}}$. Next, for all z in $(E_{2\varepsilon})^{\mathsf{c}}$, we define :

$$\theta(z) = Z(z, \mathfrak{t}(z)).$$

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions $z \mapsto \mathfrak{t}(z)$ and $z \mapsto \theta(z)$ are C^s on $(E_{2\varepsilon})^{\mathsf{c}}$.

Remark 5. κ having constant rank 1 in a neighborhood of ∂E, this set is a closed, regular submanifold of $\mathbb{R}^m$. The arguments above show that $z \mapsto (\theta(z), \mathfrak{t}(z))$ is a diffeomorphism between E^{c} and $\partial E \times [0, +\infty[$. Since ∂E is a deformation retract of E^{c} and the open unit ball is diffeomorphic to $\mathbb{R}^m$ [**?**], if E were bounded, E^{c} could be seen as a h-cobordism between ∂E and the unit sphere $\mathbb{S}^{m-1}$ and $\mathfrak{t}$ as a Morse function with no critical point in E^{c}. See [21] for instance.

Now we evaluate $\mathfrak{t}(z)$ for z in $\partial\Sigma_{2\varepsilon}$. Let z be arbitrary in $\partial\Sigma_{2\varepsilon}$ and therefore in $\Sigma_{2\varepsilon}$ which is closed. Assume its corresponding $\mathfrak{t}(z)$ is in $]-2\varepsilon, 0[$. The Implicit Function Theorem shows that $z \mapsto \mathfrak{t}(z)$ and $z \mapsto \theta(z)$ are defined and continuous on a neighborhood of z. Therefore, there exists a strictly positive real number r satisfying

$$\forall y \in \mathcal{B}_r(z)\ ,\ \exists t_y \in]-2\varepsilon, 0[:\ Z(y, t_y) \in \partial E\ .$$

This implies that the neighborhood $\mathcal{B}_r(z)$ of z is contained in $\Sigma_{2\varepsilon}$, in contradiction with the fact that z is on the boundary of $\Sigma_{2\varepsilon}$. This shows that, for all z in $\partial\Sigma_{2\varepsilon}$, $\mathfrak{t}(z)$ is either 0 or -2ε. We write this as

$$(\partial\Sigma_{2\varepsilon})_i = \{z \in \Sigma_{2\varepsilon}\ :\ \mathfrak{t}(z) = -2\varepsilon\} \quad , \qquad \partial\Sigma_{2\varepsilon} = \partial E \cup (\partial\Sigma_{2\varepsilon})_i\ .$$

Now we want to prove $\partial E_{2\varepsilon} \subset (\partial\Sigma_{2\varepsilon})_i$. To obtain this result, we start by showing :

$$\partial E_{2\varepsilon} \cap \partial E = \emptyset \quad \text{and} \quad \partial E_{2\varepsilon} \subset \partial\Sigma_{2\varepsilon}\ . \tag{E.2}$$

Suppose the existence of z in $\partial E_{2\varepsilon} \cap \partial E$. z being in ∂E, its corresponding $\mathfrak{t}(z)$ is 0. By the Implicit Function Theorem, there exists a strictly positive real number r such that,

$$\forall y \in \mathcal{B}_r(z)\ ,\ \exists t_y \in]-\varepsilon, \varepsilon[\ : Z(y, t_y) \in \partial E\ .$$

But, by definition, any y, for which there exists t_y in $]-\varepsilon, 0]$, is in $\Sigma_{2\varepsilon}$. If instead t_y is strictly positive, then necessarily y is in E^{c}, because E is forward-invariant for χ and a solution starting in E cannot reach ∂E in positive finite time. We have obtained : $\mathcal{B}_r(z) \subset \Sigma_{2\varepsilon} \cup E^{\mathsf{c}} = (E_{2\varepsilon})^{\mathsf{c}}$. $\mathcal{B}_r(z)$ being a neighborhood of z, this contradicts the fact that z is in the boundary of $E_{2\varepsilon}$.

At this point, we have proved that $\partial E_{2\varepsilon} \cap \partial E = \emptyset$, and, because $E_{2\varepsilon}$ is contained in E, this implies $\partial E_{2\varepsilon} \subset E$. With this, (E.2) will be established by proving that we have $\partial E_{2\varepsilon} \subset \partial\Sigma_{2\varepsilon}$. Let z be arbitrary in $\partial E_{2\varepsilon}$ and therefore in E which is open. There exists a strictly positive real number r such that we have :

$$\emptyset \neq \mathcal{B}_r(z) \cap E_{2\varepsilon} = \mathcal{B}_r(z) \cap \left(E \cap (\Sigma_{2\varepsilon})^{\mathsf{c}}\right)\ ,\quad \emptyset \neq \mathcal{B}_r(z) \cap (E_{2\varepsilon})^{\mathsf{c}} = \mathcal{B}_r(z) \cap (E^{\mathsf{c}} \cup \Sigma_{2\varepsilon})\ ,\quad \mathcal{B}_r(z) \subset E\ .$$

This implies $\mathcal{B}_r(z) \cap (\Sigma_{2\varepsilon})^{\mathsf{c}} \neq \emptyset$ and $\mathcal{B}_r(z) \cap \Sigma_{2\varepsilon} \neq \emptyset$ and therefore that z is in $\partial\Sigma_{2\varepsilon}$.

We have established $\partial E_{2\varepsilon} \cap \partial E = \emptyset$, $\partial E_{2\varepsilon} \subset \partial\Sigma_{2\varepsilon}$ and $\partial\Sigma_{2\varepsilon} = \partial E \cup (\partial\Sigma_{2\varepsilon})_i$. This does imply :

$$\partial E_{2\varepsilon} \subset (\partial\Sigma_{2\varepsilon})_i = \{z \in E\ :\ \mathfrak{t}(z) = -2\varepsilon\}\ . \tag{E.3}$$

This allows us to extend by continuity the definition of $\mathfrak{t}$ to $\mathbb{R}^m$ by letting

$$\mathfrak{t}(z) = -2\varepsilon \qquad \forall z \in E_{2\varepsilon}\ .$$

All the properties we have established for $\Sigma_{2\varepsilon}$ and $E_{2\varepsilon}$ hold also for Σ_ε and E_ε. In particular, we have

$$\mathfrak{t}(z) \in [-2\varepsilon, -\varepsilon] \qquad \forall z \in E_\varepsilon \setminus E_{2\varepsilon} \ . \tag{E.4}$$

Thanks to all these preparatory steps, we are finally ready to define a function $\phi : \mathbb{R}^m \to E$. Let $\nu : \mathbb{R} \to \mathbb{R}$ be a function such that the function $t \mapsto \nu(t) - t$ is a C^s (decreasing) diffeomorphism from $\mathbb{R}$ onto $]0, +\infty[$ mapping $[-\varepsilon, +\infty[$ onto $]0, \varepsilon]$ and being "minus" identity on $] - \infty, -\varepsilon]$, i.e.

$$\nu(t) - t \ = \ -t \quad \forall t \leq -\varepsilon \ .$$

We have

$$\nu(t) > t \qquad \forall t \in \mathbb{R} \quad , \qquad \nu(\mathfrak{t}(z)) \ = \ 0 \qquad \forall z \in E_\varepsilon \setminus E_{2\varepsilon} \ . \tag{E.5}$$

We let :

$$\phi(z) = \begin{cases} Z\left(z, \nu(\mathfrak{t}(z))\right), & \text{if } z \in (E_{2\varepsilon})^c \ , \\ z, & \text{if } z \in E_{2\varepsilon} \ . \end{cases}$$

The image of ϕ is contained in E since we have (E.5), $E_{2\varepsilon} \ \subset \ E$ and :

$$Z(z, \mathfrak{t}(z)) \ \in \ \partial E \quad , \qquad Z(z,t) \ \in \ E \qquad \forall (z,t) \in \partial E \times \mathbb{R}_{>0} \ .$$

Like the functions Z, ν, and $\mathfrak{t}$, the function ϕ is C^s on the interior of $(E_{2\varepsilon})^{\mathsf{c}}$. Also, since (E.5) implies

$$\phi(z) \ = \ z \qquad \forall z \in E_\varepsilon \setminus E_{2\varepsilon} \ , \tag{E.6}$$

ϕ is trivially C^s on E_ε and therefore on $(E_{2\varepsilon})^{\mathsf{c}} \cup E_\varepsilon = \mathbb{R}^m$.

We now show that ϕ is invertible. Because of (E.6), this is trivial on E_ε. Let y be arbitrary in $E \cap (E_{2\varepsilon})^{\mathsf{c}} = E \cap \Sigma_{2\varepsilon}$. To y corresponds $\mathfrak{t}(y)$ in the interval $[-2\varepsilon, 0[$. Thus, $-\mathfrak{t}(y)$ is in $]0, 2\varepsilon]$, image of $[-2\varepsilon, +\infty[$ by the C^s diffeomorphism $t \mapsto \nu(t) - t$. Hence there exists $\mathfrak{s}(y)$ in $[-2\varepsilon, +\infty[$ satisfying

$$\nu(\mathfrak{s}(y)) \ - \ \mathfrak{s}(y) \ = \ -\mathfrak{t}(y) \ . \tag{E.7}$$

Moreover, (E.4) implies that for y in $E_\varepsilon \setminus E_{2\varepsilon}$ subset of $E \cap (E_{2\varepsilon})^{\mathsf{c}}$, we have

$$\mathfrak{s}(y) \ = \ \mathfrak{t}(y)$$

So with letting

$$\mathfrak{s}(y) \ = \ \mathfrak{t}(y) \ = \ -2\varepsilon \qquad \forall y \in E_{2\varepsilon}$$

we have defined a function $\mathfrak{s} : E \to [-2\varepsilon, +\infty[$, which thanks to the implicit function theorem, is C^s and satisfies (E.7).

This allows us to define properly $\phi^{-1} : \mathbb{R}^m \to E$ as :

$$\phi^{-1}(y) = Z\left(y, -\nu(\mathfrak{s}(y))\right) \ .$$

By composition, this function is C^s and it is an inverse of ϕ in particular because, with (E.7), we have

$$\mathfrak{t}(Z(y, -\nu(\mathfrak{s}(y)))) = \mathfrak{t}(Z(y, \mathfrak{t}(y) - \mathfrak{s}(y))) = \mathfrak{s}(y) \qquad \forall y \in E \ .$$

This gives

$$\phi(Z(y,-\nu(\mathfrak{s}(y))) = Z(Z(y,-\nu(\mathfrak{s}(y))),\nu(\mathfrak{t}(Z(y,-\nu(\mathfrak{s}(y)))))) \\ = Z(Z(y,-\nu(\mathfrak{s}(y))),\nu(\mathfrak{s}(y))) = y$$

All this implies ϕ is a C^s-diffeomorphism from $\mathbb{R}^m$ to E.

Finally, we note that, for any point z_ε in ∂E_ε, there exists a point z in ∂E satisfying :

$$|z_\varepsilon - z| \;=\; \left|\int_0^\varepsilon \chi(Z(z,s))ds\right| \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \;.$$

And conversely, for any z in ∂E, there exist z_ε in ∂E_ε satisfying :

$$|z_\varepsilon - z| \;=\; \left|\int_0^\varepsilon \chi(Z(z,s))ds\right| \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \;.$$

It follows that, with ε as small as needed,

$$d_H(\partial E_\varepsilon, \partial E) \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \qquad (E.8) \;\square$$

Lemma 3.5 is a direct consequence of Lemma E.1 if we pick ε_∞, maybe infinite, satisfying

$$Z(z,t) \notin K \qquad \forall (z,t) \in \partial E \times [0, 2\varepsilon_\infty[\;.$$

ε_∞ can be chosen strictly positive since $d(K, \partial E)$ is non zero and χ is bounded.

Appendix F. Proof of case b) of Theorem 3.3. To complete the proof of Theorem 3.3, we use another technical result.

LEMMA F.1 (Diffeomorphism extension from a ball). *Consider a C^2 diffeomorphism $\lambda : \mathcal{B}_R(0) \to \lambda(\mathcal{B}_R(0)) \subset \mathbb{R}^m$, with R a strictly positive real number. For any real number ε in $]0,1[$, there exists a diffeomorphism $\lambda_e : \mathbb{R}^m \to \mathbb{R}^m$ satisfying*

$$\lambda_e(z) = \lambda(z) \qquad \forall z \in \mathtt{cl}(\mathcal{B}_{\frac{R}{1+\varepsilon}}(0)) \;.$$

Proof. It sufficient to prove that [14, Theorem 8.1.4] applies. We let

$$U = \mathcal{B}_{\frac{R}{1+\frac{\varepsilon}{2}}}(0) \quad , \qquad A = \mathtt{cl}(\mathcal{B}_{\frac{R}{1+\varepsilon}}(0)) \quad , \qquad I = \left]-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\right[\;,$$

and, without loss of generality we may assume that $\lambda(0) = 0$.

Then, consider the function $F : U \times I \to \mathbb{R}^m$ defined as

$$F(z,t) = \left(\frac{\partial \lambda}{\partial z}(0)\right)^{-1} \frac{\lambda(zt)}{t} \;,\; \forall t \in I \setminus \{0\} \;, \quad F(z,0) = z \;.$$

We start by showing that F is an isotopy of U.

- For any t in I, the function $z \mapsto F_t = F(z,t)$ is an embedding from U onto $F_t(U) \subset \mathbb{R}^m$. Indeed, for any pair (z_a, z_b) in U^2 satisfying $F(z_a,t) = F(z_b,t)$, we obtain $\lambda(z_a t) = \lambda(z_b t)$ where $(z_a t, z_b t)$ is in U^2. The function λ being injective on this set, we have $z_a = z_b$ which establishes F_t is injective. Moreover, we have:

$$\frac{\partial F_t}{\partial z}(z) = \left(\frac{\partial \lambda}{\partial z}(0)\right)^{-1} \frac{\partial \lambda}{\partial z}(zt) \quad \forall t \in I \setminus \{0\} \quad , \qquad \frac{\partial F_0}{\partial z}(z) = Id.$$

  Hence, F_t is full rank on U and therefore an embedding.

- For all z in U, the function $t \mapsto F(z,t)$ is C^1. This follows directly from the fact that, the function λ being C^2, and $\lambda(0)=0$, we have

$$\frac{\lambda(zt)}{t} = \frac{\partial \lambda}{\partial z}(0)z + z'\left(\frac{\partial^2 \lambda}{\partial z \partial z}(0)\right) z\frac{t}{2} + \circ(t) \ .$$

In particular, we obtain $\frac{\partial F}{\partial t}(z,t) = \left(\frac{\partial \lambda}{\partial z}(0)\right)^{-1} \rho(z,t)$ where

$$\rho(z,t) = \frac{1}{t^2}\left[\frac{\partial \lambda}{\partial z}(zt)zt - \lambda(zt)\right] \quad \forall t \in I \setminus \{0\} \ , \quad \rho(z,0) = \frac{1}{2}z'\left(\frac{\partial^2 \lambda}{\partial z \partial z}(0)\right) z \ .$$

Moreover, for all t in I, the function $z \mapsto \frac{\partial F}{\partial t}(z,t)$ is locally Lipschitz and therefore gives rise to an ordinary differential equation with unique solutions.

Also the set $\bigcup_{(z,t)\in U\times I}\{(F(z,t),t)\}$ is open. This follows from Brouwer's Invariance theorem since the function $(z,t) \mapsto (F(z,t),t)$ is a diffeomorphism on the open set $U \times I$. With [14, Theorem 8.1.4], we know there exists a diffeotopy F_e from $\mathbb{R}^m \times I$ onto $\mathbb{R}^m$ which satisfies $F_e = F$ on $A \times [0,1]$. Thus, the diffeomorphism $\lambda_e = F_e(.,1)$ defined on $\mathbb{R}^m$ onto $\mathbb{R}^m$ verifies $\lambda_e(z) = F_e(z,1) = F(z,1) = \lambda(z)$ for all $z \in A$. □

We now place ourselves in the case b) of Theorem 3.3, namely we suppose that τ_a^* is C^2 and $\mathcal{O}_a$ is C^2-diffeomorphic to $\mathbb{R}^m$. Let $\phi_1 : \mathcal{O}_a \to \mathbb{R}^m$ denote the corresponding diffeomorphism. Let R_1 be a strictly positive real number such that the open ball $\mathcal{B}_{R_1}(0)$ contains $\phi_1(K)$. Let R_2 be a real number strictly larger than R_1. With Lemma 3.5 again, and since $\mathcal{B}_{R_2}(0)$ verifies condition $\mathbb{B}$, there exists of C^2-diffeomorphism $\phi_2 : \mathbb{R}^m \to \mathcal{B}_{R_2}(0)$ satisfying $\phi_2(z) = z$ for all z in $\mathcal{B}_{R_1}(0)$. At this point, we have obtained a C^2-diffeomorphism $\phi = \phi_2 \circ \phi_1 : \mathcal{O}_a \to \mathcal{B}_{R_2}(0)$. Consider $\lambda = \tau_a^* \circ \phi^{-1} : \mathcal{B}_{R_2}(0) \to \tau_a^*(\mathcal{O}_a) \quad (= \lambda(\mathcal{B}_{R_2}(0)))$. According to Lemma F.1, we can extend λ to $\lambda_e : \mathbb{R}^m \to \mathbb{R}^m$ such that $\lambda_e = \tau_a^* \circ \phi^{-1}$ on $\mathcal{B}_{R_1}(0)$. Finally, consider $\tau_e^* = \lambda_e \circ \phi_1 : \mathcal{O}_a \to \mathbb{R}^m$. Since, by construction of ϕ_2, $\phi = \phi_1$ on $\phi_1^{-1}(\mathcal{B}_{R_1}(0))$ which contains K, we have $\tau_e^* = \tau_a^*$ on K.

Appendix G. Proof of Lemma 3.2. The compact K_0 being globally asymptotically attractive and interior to E which is forward invariant, E is globally attractive. It is also stable due to the continuity of solutions with respect to initial conditions uniformly on compact time subsets of the domain of definition. So it is globally asymptotically stable. It follows from [26, Theorem 3.2] that there exist C^∞ functions $V_K : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $V_E : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ which are proper on $\mathbb{R}^m$ and a class $\mathcal{K}_\infty$ function α satisfying

$$\begin{gathered}
\alpha(d(z,K_0)) \leq V_K(z) \ , \quad \alpha(d(z,E)) \leq V_E(z) \qquad \forall\, z \in \mathbb{R}^m \ , \\
V_K(z) = 0 \quad \forall z \in K_0 \quad , \quad V_E(z) = 0 \quad \forall\, z \in E \ , \\
\frac{\partial V_K}{\partial z}(z)\,\chi(z) \leq -V_K(z) \ , \quad \frac{\partial V_E}{\partial z}(z)\,\chi(z) \leq -V_E(z) \qquad \forall\, z \in \mathbb{R}^m \ .
\end{gathered}$$

With $\overline{d}$ an arbitrary strictly positive real number, the notations

$$v_E = \sup_{z\in\mathbb{R}^m:\, d(z,E)\leq \overline{d}} V_K(z) \quad , \qquad \mu = \frac{\alpha(\overline{d})}{2v_E} \ ,$$

and since α is of class $\mathcal{K}_\infty$, we obtain the implications

$$V_E(z)+\mu V_K(z)=\alpha(\overline{d}) \quad \Rightarrow \quad \alpha(d(z,E)) \leq V_E(z) \leq \alpha(\overline{d}) \quad \Rightarrow \quad d(z,E) \leq \overline{d} \quad \Rightarrow \quad V_K(z) \leq v_E \ .$$

With our definition of μ, this yields also

$$\alpha(\overline{d}) - \mu\, V_K(z) = V_E(z) \quad \Rightarrow \quad 0 \;<\; \frac{\alpha(\overline{d})}{2} \leq V_E(z) \quad \Rightarrow \quad 0 < d(z,E) \leq \overline{d}\,.$$

On the other hand, with the compact notation $\mathcal{V}(z) = V_E(z) + \mu V_K(z)$, we have $\frac{\partial \mathcal{V}}{\partial z}(z)\,\chi(z) \leq -\mathcal{V}(z)$, for all $z \in \mathbb{R}^m$. All this implies that the sublevel set $\mathcal{E} = \{z \in \mathbb{R}^m : \mathcal{V}(z) < \alpha(\overline{d})\}$ is contained in $\{z \in \mathbb{R}^m \colon d(z,E) \in [0,\overline{d}]\}$ and that $\mathtt{cl}(E)$ is contained in $\mathcal{E}$. Besides, $\mathcal{E}$ verifies condition $\mathbb{B}$ with the vector field χ and the function $\kappa = \mathcal{V} - \alpha(\overline{d})$.

REFERENCES

[1] V. Andrieu, J.-B. Eytard, and L. Praly. Dynamic extension without inversion for observers. *IEEE Conference on Decision and Control*, pages 878–883, 2014.

[2] V. Andrieu and L. Praly. On the existence of a Kazantzis–Kravaris / Luenberger observer. *SIAM Journal on Control and Optimization*, 45(2):432–456, 2006.

[3] D. Astolfi and L. Praly. Output feedback stabilization for siso nonlinear systems with an observer in the original coordinate. *IEEE Conference on Decision and Control*, pages 5927 – 5932, 2013.

[4] R. Bartle. *The elements of real analysis.* John Wiley & Sons, 1964.

[5] P. Bernard, L. Praly, and V. Andrieu. Tools for observers based on coordinate augmentation. *IEEE Conference on Decision and Control*, 2015.

[6] P. Bernard, L. Praly, and V. Andrieu. Expressing an observer in given coordinates by augmenting and extending an injective immersion to a surjective diffeomorphism. *https://arxiv.org/abs/1509.04865*, 2017.

[7] G. Besançon and A. Ticlea. An immersion-based observer design for rank-observable nonlinear systems. *IEEE Transactions on Automatic Control*, 52(1):83–88, 2007.

[8] R. Bott and J. Milnor. On the parallelizability of the spheres. *Bullettin of American Mathematical Society*, 64(3):87–89, 1958.

[9] F. Deza, E. Busvelle, J.P. Gauthier, and D. Rakotopara. High gain estimation for nonlinear systems. *Systems & Control Letters*, 18:295–299, 1992.

[10] J. Dugundgi. *Topology.* Allyn and Bacon, 1966.

[11] B. Eckmann. *Mathematical survey lectures* 1943 − 2004. Springer, 2006.

[12] J.-P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems application to bioreactors. *IEEE Transactions on Automatic Control*, 37(6):875–880, 1992.

[13] J-P. Gauthier and I. Kupka. *Deterministic observation theory and applications.* Cambridge University Press, 2001.

[14] M. Hirsch. *Differential topology.* Springer, 1976.

[15] N. Kazantzis and C. Kravaris. Nonlinear observer design using Lyapunov's auxiliary theorem. *Systems and Control Letters*, 34:241–247, 1998.

[16] H. K. Khalil and L. Praly. High-gain observers in nonlinear feedback control. *Int. J. Robust. Nonlinear Control*, 24, April 2013.

[17] H.K. Khalil and A. Saberi. Adaptive stabilization of a class of nonlinear systems using high-gain feedback. *IEEE Transactions on Automatic Control*, 32(11):1031–1035, 1987.

[18] F. Lafont, E. Busvelle, and J.-P. Gauthier. An adaptive high-gain observer for wastewater treatment systems. *Journal of Process Control*, 21(6):893–900, 2011.

[19] J. Levine and R. Marino. Nonlinear system immersion, observers and finite-dimensional filters. *Systems & Control Letters*, 7:133–142, 1986.

[20] M. Maggiore and K.M. Passino. A separation principle for a class of non uniformly completely observable systems. *IEEE Transactions on Automatic Control*, 48, July 2003.

[21] J. Milnor. *Lectures on the h-cobordism Theorem. Notes by L. Siebenmann and J. Sondow.* Princeton University Press, 1965.

[22] A. Rapaport and A. Maloum. Design of exponential observers for nonlinear systems by embedding. *International Journal of Robust and Nonlinear Control*, 14:273–288, 2004.

[23] A. Shoshitaishvili. On control branching systems with degenerate linearization. *IFAC Symposium on Nonlinear Control Systems*, pages 495–500, 1992.

[24] A. Tornambe. Use of asymptotic observers having high gains in the state and parameter estimation. *IEEE Conference on Decision and Control*, 2:1791–1794, 1989.

[25] T. Wazewski. *Sur les matrices dont les éléments sont des fonctions continues*, volume 2. Composito Mathematica, 1935. p. 63-68.

[26] F. W. Wilson. Smoothing derivatives of functions and applications. *Transactions American Mathematical Society*, 139:413–428, 1969.