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► **To cite this version:**

Pauline Bernard, Vincent Andrieu, Laurent Praly. Expressing an observer in preferred coordinates by transforming an injective immersion into a surjective diffeomorphism. 2017. hal-01199791v5

**HAL Id: hal-01199791**

**<https://hal.science/hal-01199791v5>**

Preprint submitted on 16 Feb 2018 (v5), last revised 19 Feb 2018 (v6)

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# EXPRESSING AN OBSERVER IN PREFERRED COORDINATES BY TRANSFORMING AN INJECTIVE IMMERSION INTO A SURJECTIVE DIFFEOMORPHISM

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**Abstract.** When designing observers for nonlinear systems, the dynamics of the given system and of the designed observer are usually not expressed in the same coordinates or even have states evolving in different spaces. In general, the function, denoted  $\tau$  (or its inverse, denoted  $\tau^*$ ) giving one state in terms of the other is not explicitly known and this creates implementation issues.

We propose to round this problem by expressing the observer dynamics in the the same coordinates as the given system. But this may impose to add extra coordinates, problem that we call *augmentation*. This may also impose to modify the domain or the range of the “augmented”  $\tau$  or  $\tau^*$ , problem that we call *extension*.

We show that the augmentation problem can be solved partly by a continuous completion of a free family of vectors and that the extension problem can be solved by a function extension making the image of the extended function the whole space. We also show how augmentation and extension can be done without modifying the observer dynamics and therefore with maintaining convergence.

Several examples illustrate our results.

## 1. Introduction.

**1.1. Context.** In many applications, estimating the state of a dynamical system is crucial either to build a controller or simply to obtain real time information on the system. Satisfactory solutions are known for systems the dynamics of which are linear in the preferred coordinates. But when they are nonlinear, we are aware of only two “general purpose” observer design methodologies guaranteeing “non local” convergence under merely some basic observability properties: the high gain observers ([17, 24, 12, 13, 16, 7], ...) and the nonlinear Luenberger observers ([23, 15, 2]). For both, the observer state is living in a space different from the system state one and the system state estimate is obtained typically by solving on-line a nonlinear equation.

As an illustration, consider an harmonic oscillator with unknown frequency with dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1x_3, \quad \dot{x}_3 = 0, \quad y = x_1 \quad (1.1)$$

with state  $x = (x_1, x_2, x_3)$  in  $(\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}_{>0}$  and measurement  $y$ . We are interested in estimating the state  $x$  from the only knowledge of  $y$  and the fact that  $x$  evolves in some known set  $\mathcal{A}$ . By following in a very orthodox way (see [1] for details) the high gain observer design we get a “raw” observer with dynamics

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \text{sat}(\hat{x}_1 \hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{\xi}_1], \quad (1.2)$$

with state  $\hat{\xi}$  in  $\mathbb{R}^4$ , where  $\text{sat}$  is a saturation function (see (1.12)), and from which the system state estimate  $\hat{x}$  is given as  $\hat{x} = \tau(\hat{\xi})$  where  $\tau$  is any continuous function which satisfies

$$\tau(x_1, x_2, -x_1x_3, -x_2x_3) = (x_1, x_2, x_3) \quad \forall x = (x_1, x_2, x_3) \in \mathcal{A}. \quad (1.3)$$

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The construction of the mapping  $\tau$  relies on the inversion to the mapping  $\tau^*(x) = (x_1, x_2, -x_1x_3, -x_2x_3)$  which in general has no explicit solution and is not uniquely defined outside of  $\tau^*(\mathcal{A})$ . The commonly used implicit solution is given as the solution to an optimization problem which may be

$$\hat{x} = \tau(\hat{\xi}) = \underset{\hat{x}}{\text{Argmin}} \left| \hat{\xi} - \tau^*(\hat{x}) \right|^2 .$$

Note however that some other forms are possible. For instance in [22], the authors propose to build another implicit solution based on an optimization procedure which yields a global Lipschitz function  $\tau$ . The drawback of all these optimization based approaches being that they may be costly to solve in practice. Another path is to rely on the Rank theorem, as in [19] and take advantage of the local existence of diffeomorphism  $\phi_x$  and  $\phi_\xi$  such that

$$\phi_\xi \circ \tau^* \circ \phi_x = (x, 0, \dots, 0) .$$

In this case, one can pick  $\hat{x} = \phi_x^{-1}(\pi(\phi_\xi(\hat{\xi})))$  where  $\pi$  is the projection on the set of the first  $n$  components. In our example,  $\phi_x$  could be the identity and

$$\phi_\xi(\xi) = \left( \xi_1, \xi_2, -\frac{\xi_1\xi_3 + \xi_2\xi_4}{\xi_1^2 + \xi_2^2}, (\xi_1\xi_4 - \xi_2\xi_3) \right) .$$

But, besides the local nature of this technique, finding expressions for  $\phi_x^{-1}$  and  $\phi_\xi$  may be a very difficult task in practice (see [18] for instance). And unfortunately  $\hat{x}$  is needed to evaluate the term  $\text{sat}(\hat{x}_1\hat{x}_3^2)$  in (1.2) since the observer dynamics depend on  $\tau$ .

Instead of a high gain observer design as above, we may use a Luenberger non linear observer design (see [23, 15, 2]). It leads to :

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y) = A\hat{\xi} + By \tag{1.4}$$

with  $\hat{\xi}$  in  $\mathbb{R}^4$ ,  $A$  a Hurwitz matrix and  $(A, B)$  a controllable pair. The state estimate  $\hat{x}$  is again given as  $\hat{x} = \tau(\hat{\xi})$  where  $\tau$  is any continuous function which satisfies  $\tau(\tau^*(x)) = x$  for all  $x$  in  $\mathcal{A}$  where this time,

$$\tau^*(x) = -(A^2 + x_3I)^{-1}[ABx_1 + Bx_2] . \tag{1.5}$$

A difference with the high gain observer is that  $\hat{x}$  is not involved in (1.4), i.e. the observer dynamics do not depend on  $\tau$ .

In the following, instead of constructing the (implicit) function  $\tau$  by a minimization of a criterion introduced as a design tool, we explicitly construct a diffeomorphism  $\tau_e$  allowing us to express the dynamics of the observer in the  $x$ -coordinates<sup>1</sup>. This has been suggested by several researchers [9, 20, 3] in the case where the observer state  $\hat{\xi}$  and the state estimate  $\hat{x}$  are related by a diffeomorphism. We remove this restriction and complete the preliminary results presented in [1].

In the example above, pulling the observer dynamics in the  $\xi$ -coordinates back in the  $x$ -coordinates is seemingly impossible since  $x$  has dimension 3 whereas  $\hat{\xi}$  has dimension 4. To overcome this difficulty, one could think of using again some kind

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<sup>1</sup>We will refer to the  $x$ -coordinates as the “preferred coordinates” or “given coordinates” because they are chosen by the user to describe the model dynamics.

of projection/restriction. Our proposition is actually of a completely different kind. Instead of considering  $\hat{\xi}$  as the estimation of the image by an immersion  $\tau^*$  of the state  $x$ , we see it as the estimation of the image by a diffeomorphism  $\tau_e^*$  of an augmented state  $(x, w)$ . Fortunately with such a diffeomorphism  $\tau_e^*$ , we can use all what has been proposed for expressing the observer dynamics in the preferred coordinates in that case. So with this augmentation of  $x$  into  $(x, w)$ , the design of the commonly used projection/restriction is replaced by the construction of the diffeomorphism  $\tau_e^*$ . We show in Section 2 that  $\tau_e^*$  can be obtained by “augmenting” the function  $x \mapsto \tau^*(x)$  given in (1.3) or (1.5). For this, it turns out that it is sufficient to complement a full column rank Jacobian into an invertible matrix.

The drawback of this approach however is that, because it is linked to particular coordinate systems, the obtained diffeomorphism may not be defined everywhere. Also, its image could be only a subset of the observer accessibility set (for  $\hat{\xi}$ ), namely the trajectories of  $\hat{\xi}$  may leave the image of the diffeomorphism or equivalently the trajectories of  $(\hat{x}, \hat{w})$  may leave the domain of definition of the diffeomorphism. We show in Section 3 how this new problem can be overcome via an extension of the image of the diffeomorphism. The key point here is that the given observer dynamics (1.2) remain unchanged. Hence we deal with constraints on the observer state without any kind of projection/restriction as commonly proposed (see [20, 3] for example). A benefit of this is that, to preserve the convergence property, we do not require extra assumptions such as convexity .

To illustrate our results, we continue the example of the harmonic oscillator with unknown frequency and add one based on the bioreactor presented in [12]. We use a high-gain observer as starting point. But, as shown in [5], the same tools can be used with a nonlinear Luenberger observer.

Our contribution relies on, or is inspired by ideas of some known analysis results such as continuously completing an independent set of vectors to a basis [25, 11], diffeotopies [14] or  $h$ -cobordism [21]. We rephrase part of them when it is constructive and therefore useful for observer design. Similarly, the constructive part of our proofs are in the main body of our text, those which are not constructive and never used/commented in remarks or examples are in appendix or omitted to save space.

[A more complete version with all the proofs is in \[6\].](#)

**1.2. Problem statement.** We consider the given system with dynamics :

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad , \quad (1.6)$$

with  $x$  in  $\mathbb{R}^n$  and  $y$  in  $\mathbb{R}^q$ . Its solution at time  $t$ , with initial condition  $x_0$  at time 0 is denoted  $X(x_0, t)$  and the corresponding output  $y_{x_0}(t)$ . The observation problem is to construct a dynamical system with input  $y$  and output  $\hat{x}$ , supposed to be an estimate of the system state  $x$  as long as the latter is in a specific set of interest denoted  $\mathcal{A} \subseteq \mathbb{R}^n$ . [As starting point](#), we assume this problem is (formally) already solved but with maybe some implementation issues such as finding an expression of  $\tau$ . More precisely,

*Assumption A (Converging observer) : There exist an open subset  $\mathcal{O}$  of  $\mathbb{R}^n$  containing  $\mathcal{A}$ , a  $C^1$  injective immersion  $\tau^* : \mathcal{O} \rightarrow \mathbb{R}^m$ , and a set<sup>2</sup>  $\mathcal{F}$  of pairs  $(\varphi, \tau)$  of locally Lipschitz functions such that we have*

$$\tau(\tau^*(x)) = x \quad \forall x \in \mathcal{A} \quad (1.7)$$

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<sup>2</sup>The symbol  $\mathcal{F}$  is pronounced *phitau*.

and, for any solution  $X(x_0, t)$  of (1.6) which is defined and remains in  $\mathcal{A}$  for  $t$  in  $[0, +\infty)$ , the solution  $(X(x_0, t), \hat{\Xi}(\hat{\xi}_0, t; y_{x_0}))$  of the cascade system :

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad , \quad \dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) \quad , \quad \hat{x} = \tau(\hat{\xi}) \quad , \quad (1.8)$$

with initial condition  $(x_0, \hat{\xi}_0)$  in  $\mathcal{A} \times \mathbb{R}^m$  at time 0, is also defined on  $[0, +\infty)$  and satisfies :

$$\lim_{t \rightarrow +\infty} \left| \hat{\Xi}(\hat{\xi}_0, t; y_{x_0}) - \tau^*(X(x_0, t)) \right| = 0 \quad . \quad (1.9)$$

REMARK 1.

1. The convergence property given by (1.9) is in the observer state space only. Property (1.7) is a necessary condition for this convergence to be transferred from the observer state space to the system state space.
2. The need for pairing  $\varphi$  and  $\tau$  comes from the dependence on  $\hat{x} = \tau(\hat{\xi})$  of  $\varphi$  in (1.8). This may imply to change  $\varphi$  whenever we change  $\tau$ . In the high-gain approach, as in (1.2), when  $\mathcal{A}$  is bounded, thanks to the gain  $\ell$  which can be chosen arbitrarily large,  $\varphi$  can be paired with any locally Lipschitz function  $\tau$  provided its values are saturated whenever they are used as arguments of  $\varphi$ . On another hand, if, as in (1.4),  $\varphi$  does not depend on  $\hat{x}$ , then it can be paired with any  $\tau$ .

EXAMPLE 1. For System (1.1), for any solution with initial condition  $x_1 = x_2 = 0$ , we have no information on  $x_3$  from the only knowledge of (1.1) and the function  $t \mapsto y(t) = X_1(x, t)$ . This explains the restriction of our attention to the set

$$\mathcal{A} = \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in \left] \frac{1}{r}, r \right[ , x_3 \in ]0, r[ \right\} \quad , \quad (1.10)$$

where  $r$  is some arbitrary strictly positive real number. This set is invariant by (1.1), and the function (1.3) being an injective immersion on  $(\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}_{>0}$ , the system is strongly differentially observable<sup>3</sup> of order 4 on this set. Let  $\mathcal{O}$  be any open subset such that  $\text{cl}(\mathcal{A}) \subset \mathcal{O} \subseteq (\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0, 0)\} \times \mathbb{R}_{>0})$ , with  $\text{cl}$  denoting the set closure. Then,  $\text{cl}(\mathcal{A})$  being a compact set, a set  $\mathcal{F}$  satisfying Assumption  $\mathbb{A}$  is made of pairs of a locally Lipschitz function  $\tau$  satisfying (see [16] for example)

$$x = \tau(x_1, x_2, -x_1x_3, -x_2x_3) \quad \forall x \in \mathcal{A} \quad (1.11)$$

and the function  $\varphi$  defined in (1.2) where

$$\text{sat}(s) = \min \{ r^3, \max \{ s, -r^3 \} \} \quad (1.12)$$

with the gain  $\ell$  in (1.2) adapted to the properties of  $\tau$ .  $\triangle$

Although the problem of observer design seems already solved under Assumption  $\mathbb{A}$ , it can be difficult to find a left-inverse  $\tau$  of  $\tau^*$ . In the following, we consider that the function  $\tau^*$  and the set  $\mathcal{F}$  are given and we aim at avoiding the left-inversion of  $\tau^*$  by expressing the observer for  $x$  in the, maybe augmented,  $x$ -coordinates. More precisely we aim at solving the following problem.

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<sup>3</sup>The system is said to be strongly differentially observable of order  $m$  if the function  $x \mapsto (h(x), L_f h(x), \dots, L_f^{m-1} h(x))$  is an injective immersion.

Our problem (Observer in the  $x$ -coordinates) : Assume that Assumption  $\mathbb{A}$  is satisfied, we wish to find an open set  $\mathcal{O}_a \subseteq \mathbb{R}^m$  and two mappings  $k$  and  $\ell$  such that the system defined in  $\mathbb{R}^m$

$$\dot{\hat{x}} = k(\hat{x}, \hat{w}, y) , \quad \dot{\hat{w}} = \ell(\hat{x}, \hat{w}, y) , \quad (1.13)$$

defines an observer in  $\mathcal{A}$ . In other words, for any initial condition  $x_0$  in  $\mathcal{A}$  such that the solution  $X(x_0, t)$  of (1.6) is defined and remains in  $\mathcal{A}$  for  $t$  in  $[0, +\infty)$ , the solution  $(X(x_0, t), \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0}))$ , with initial condition  $(\hat{x}_0, \hat{w}_0)$  in  $\mathcal{O}_a$ , of the cascade of system (1.6) with the observer (1.13) is also defined on  $[0, +\infty)$  and satisfies :

$$\lim_{t \rightarrow +\infty} \left| X(x_0, t) - \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}) \right| = 0 . \quad (1.14)$$

**1.3. A sufficient condition allowing us to express the observer in the given  $x$ -coordinates.** For the simpler case where the raw observer state  $\hat{\xi}$  has the same dimension as the system state  $x$ , i.e.  $m = n$ ,  $\tau^*$ , in Assumption  $\mathbb{A}$ , is a diffeomorphism on  $\mathcal{O}$  and we can express the observer in the given  $x$ -coordinates as :

$$\dot{\hat{x}} = \left( \frac{\partial \tau^*}{\partial x}(\hat{x}) \right)^{-1} \varphi(\tau^*(\hat{x}), \hat{x}, y) \quad (1.15)$$

which requires a Jacobian inversion only. However, although, by assumption, the system trajectories remain in  $\mathcal{O}$  where the Jacobian is invertible, we have no guarantee the ones of the observer do. Therefore, to obtain convergence and completeness of solutions, we must find means to ensure the estimate  $\hat{x}$  does not leave the set  $\mathcal{O}$ , or equivalently that  $\tau^*(\hat{x})$  remains in the image set  $\tau^*(\mathcal{O})$ . We address this point by modifying  $\tau^*$  “marginally” in order to get  $\tau^*(\mathcal{O}) = \mathbb{R}^m$ .

In the more complex situation where  $m > n$ ,  $\tau^*$  is only an injective immersion. In [1], it is proposed to augment the given  $x$ -coordinates in  $\mathbb{R}^n$  with extra ones, say  $w$ , in  $\mathbb{R}^{m-n}$  and correspondingly to augment the given injective immersion  $\tau^*$  into a diffeomorphism  $\tau_e^* : \mathcal{O}_a \rightarrow \mathbb{R}^m$ , where  $\mathcal{O}_a$  is an open subset of  $\mathbb{R}^m$ , considered as an augmentation of  $\mathcal{O}$ , i.e. its Cartesian projection on  $\mathbb{R}^n$  is contained in  $\mathcal{O}$  and contains  $\text{cl}(\mathcal{A})$ .

To help us find such an appropriate augmentation, we have the following sufficient condition.

**PROPOSITION 1.1.** *Assume Assumption  $\mathbb{A}$  holds and  $\mathcal{A}$  is bounded. Assume also the existence of an open subset  $\mathcal{O}_a$  of  $\mathbb{R}^m$  containing  $\text{cl}(\mathcal{A} \times \{0\})$  and of a diffeomorphism  $\tau_e^* : \mathcal{O}_a \rightarrow \mathbb{R}^m$  satisfying*

$$\tau_e^*(x, 0) = \tau^*(x) \quad \forall x \in \mathcal{A} \quad (1.16)$$

and

$$\tau_e^*(\mathcal{O}_a) = \mathbb{R}^m . \quad (1.17)$$

and such that, with letting  $\tau_{ex}$  denote the  $x$ -component of the inverse of  $\tau_e^*$ , there exists a function  $\varphi$  such that the pair  $(\varphi, \tau_{ex})$  is in the set  $\mathcal{F}$  given by Assumption  $\mathbb{A}$ . Under these conditions, for any initial condition  $x_0$  in  $\mathcal{A}$  such that the solution  $X(x_0, t)$  of (1.6) is defined and remains in  $\mathcal{A}$  for  $t$  in  $[0, +\infty)$ , the solution  $(X(x_0, t), \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}), \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0}))$ , with initial condition  $(\hat{x}_0, \hat{w}_0)$  in  $\mathcal{O}_a$ , of the cascade of system (1.6) with the observer :

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \left( \frac{\partial \tau_e^*}{\partial (\hat{x}, \hat{w})}(\hat{x}, \hat{w}) \right)^{-1} \varphi(\tau_e^*(\hat{x}, \hat{w}), \hat{x}, y) \quad (1.18)$$

is also defined on  $[0, +\infty)$  and satisfies :

$$\lim_{t \rightarrow +\infty} \left| \hat{W}(\hat{x}_0, \hat{w}_0, t; y_{x_0}) \right| + \left| X(x_0, t) - \hat{X}(\hat{x}_0, \hat{w}_0, t; y_{x_0}) \right| = 0 . \quad (1.19)$$

The key point in the observer (1.18) is that, instead of left-inverting the function  $\tau^*$  via  $\tau$  as in (1.7), we invert only a matrix.

*Proof.* See Appendix A.  $\square$

With Proposition 1.1, we are left with finding a diffeomorphism  $\tau_e^*$  satisfying the conditions listed in the statement :

- Equation (1.16) is about the fact that  $\tau_e^*$  is an augmentation, with adding coordinates, of the given injective immersion  $\tau^*$ . It motivates the following problem.

**PROBLEM 1** (Immersion augmentation into a diffeomorphism). *Given a set  $\mathcal{A}$ , an open subset  $\mathcal{O}$  of  $\mathbb{R}^n$  containing  $\text{cl}(\mathcal{A})$ , and an injective immersion  $\tau^* : \mathcal{O} \rightarrow \tau^*(\mathcal{O}) \subset \mathbb{R}^m$ , the pair  $(\tau_a^*, \mathcal{O}_a)$  is said to solve the problem of immersion augmentation into a diffeomorphism if  $\mathcal{O}_a$  is an open subset of  $\mathbb{R}^m$  containing  $\text{cl}(\mathcal{A} \times \{0\})$  and  $\tau_a^* : \mathcal{O}_a \rightarrow \tau_a^*(\mathcal{O}_a) \subset \mathbb{R}^m$  is a diffeomorphism satisfying*

$$\tau_a^*(x, 0) = \tau^*(x) \quad \forall x \in \mathcal{A} .$$

We will present in Section 2 conditions under which Problem 1 can be solved via complementing a full column rank Jacobian of  $\tau^*$  into an invertible matrix, i.e. via what we call Jacobian complementation.

- The condition expressed in (1.17), is about the fact that  $\tau_e^*$  is surjective onto  $\mathbb{R}^m$ . This motivates us to introduce the surjective diffeomorphism extension problem
- PROBLEM 2** (Surjective diffeomorphism extension). *Given an open subset  $\mathcal{O}_a$  of  $\mathbb{R}^m$ , a compact subset  $K$  of  $\mathcal{O}_a$ , and a diffeomorphism  $\tau_a^* : \mathcal{O}_a \rightarrow \mathbb{R}^m$ , the diffeomorphism  $\tau_e^* : \mathcal{O}_a \rightarrow \mathbb{R}^m$  is said to solve the surjective diffeomorphism extension problem if it satisfies*

$$\tau_e^*(\mathcal{O}_a) = \mathbb{R}^m \quad , \quad \tau_e^*(z) = \tau_a^*(z) \quad \forall z \in K .$$

This Problem 2 will be addressed in Section 3.

When Assumption A holds and  $\mathcal{A}$  is bounded, by successively solving Problem 1 and Problem 2 with  $\text{cl}(\mathcal{A} \times \{0\}) \subset K \subset \mathcal{O}_a$ , we get a diffeomorphism  $\tau_e^*$  guaranteed to satisfy all the conditions of Proposition 1.1 except maybe the fact that the pair  $(\varphi, \tau_{ex})$  is in  $\varphi\mathcal{F}$ . How this last condition can be satisfied will be discussed in Section 4 mainly via a list of remarks.

Throughout Sections 2-3, we will show how, step by step, we can express a high gain observer in the  $x$ -coordinates for the harmonic oscillator with unknown frequency. We will also show that our approach enables to ensure completeness of solutions of the observer presented in [12] for the bioreactor. The various difficulties we shall encounter on this road will be discussed in Section 5. In particular, we shall see how they can be overcome thanks to a better choice of  $\tau^*$  and of the pair  $(\varphi, \tau)$  given by Assumption A.

**2. About Problem 1 : Augmentation of an immersion into a diffeomorphism.** In [1], we find the following sufficient condition for the augmentation of an immersion into a diffeomorphism.

**LEMMA 2.1** ([1]). *Let  $\mathcal{A}$  be a bounded set,  $\mathcal{O}$  be an open subset of  $\mathbb{R}^n$  containing  $\text{cl}(\mathcal{A})$ , and  $\tau^* : \mathcal{O} \rightarrow \tau^*(\mathcal{O}) \subset \mathbb{R}^m$  be an injective immersion. If there exists a bounded open set  $\tilde{\mathcal{O}}$  satisfying  $\text{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \text{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$  and a  $C^1$  function  $\gamma : \mathcal{O} \rightarrow \mathbb{R}^{m \times (m-n)}$*

the values of which are  $m \times (m - n)$  matrices satisfying :

$$\det \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \gamma(x) \end{pmatrix} \neq 0 \quad \forall x \in \text{cl}(\tilde{\mathcal{O}}) , \quad (2.1)$$

then there exists a strictly positive real number  $\varepsilon$  such that the following pair<sup>4</sup>  $(\tau_a^*, \mathcal{O}_a)$  solves Problem 1

$$\tau_a^*(x, w) = \tau^*(x) + \gamma(x)w , \quad \mathcal{O}_a = \tilde{\mathcal{O}} \times \mathcal{B}_\varepsilon(0) . \quad (2.2)$$

In other words, an injective immersion  $\tau^*$  can be augmented into a diffeomorphism  $\tau_a^*$  if we are able to find  $m - n$  columns  $\gamma$  which are  $C^1$  in  $x$  and which complement the full column rank Jacobian  $\frac{\partial \tau^*}{\partial x}(x)$  into an invertible matrix.

*Proof.* See Appendix B.  $\square$

REMARK 2. Complementing a  $m \times n$  full-rank matrix into an invertible one is equivalent to finding  $m - n$  independent vectors orthogonal to that matrix. Precisely the existence of  $\gamma$  satisfying (2.1) is equivalent to the existence of a  $C^1$  function  $\tilde{\gamma} : \text{cl}(\tilde{\mathcal{O}}) \rightarrow \mathbb{R}^{m \times (m-n)}$  the values of which are full rank matrices satisfying :

$$\tilde{\gamma}(x)^\top \frac{\partial \tau^*}{\partial x}(x) = 0 \quad \forall x \in \text{cl}(\tilde{\mathcal{O}}) . \quad (2.3)$$

Indeed,  $\tilde{\gamma}$  satisfying (2.3) satisfies also (2.1) since the following matrices are invertible

$$\begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x)^\top \\ \tilde{\gamma}(x)^\top \end{pmatrix} \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \tilde{\gamma}(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x)^\top \frac{\partial \tau^*}{\partial x}(x) & 0 \\ 0 & \tilde{\gamma}(x)^\top \tilde{\gamma}(x) \end{pmatrix} .$$

Conversely, given  $\gamma$  satisfying (2.1),  $\tilde{\gamma}$  defined by the identity below satisfies (2.3) and has full column rank

$$\begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \tilde{\gamma}(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \gamma(x) \end{pmatrix} \begin{pmatrix} I & - \left[ \frac{\partial \tau^*}{\partial x}(x)^\top \frac{\partial \tau^*}{\partial x}(x) \right]^{-1} \frac{\partial \tau^*}{\partial x}(x)^\top \gamma(x) \\ 0 & I \end{pmatrix} .$$

### 2.1. Submersion case.

PROPOSITION 2.2 (Completion when  $\tau^*(\text{cl}(\tilde{\mathcal{O}}))$  is a level set of a submersion).  
Let  $\mathcal{A}$  be a bounded set,  $\tilde{\mathcal{O}}$  be a bounded open set and  $\mathcal{O}$  be an open set satisfying

$$\text{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \text{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O} .$$

Let also  $\tau^* : \mathcal{O} \rightarrow \tau^*(\mathcal{O}) \subset \mathbb{R}^m$  be an injective immersion. Assume there exists a  $C^2$  function  $F : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$  which is a submersion at least on a neighborhood of  $\tau^*(\tilde{\mathcal{O}})$  satisfying:

$$F(\tau^*(x)) = 0 \quad \forall x \in \tilde{\mathcal{O}} , \quad (2.4)$$

then, with the  $C^1$  function  $x \mapsto \gamma(x) = \frac{\partial F}{\partial \xi}^T(\tau^*(x))$ , the matrix in (2.1) is invertible for all  $x$  in  $\tilde{\mathcal{O}}$  and the pair  $(\tau_a^*, \mathcal{O}_a)$  defined in (2.2) solves Problem 1.

*Proof.* For all  $x$  in  $\text{cl}(\tilde{\mathcal{O}})$ ,  $\frac{\partial \tau^*}{\partial x}(x)$  is right invertible and we have  $\frac{\partial F}{\partial \xi}(\tau^*(x)) \frac{\partial \tau^*}{\partial x}(x) = 0$ . Thus, the rows of  $\frac{\partial F}{\partial \xi}(\tau^*(x))$  are orthogonal to the column vectors of  $\frac{\partial \tau^*}{\partial x}(x)$  and are

<sup>4</sup>For a positive real number  $\varepsilon$  and  $z_0$  in  $\mathbb{R}^p$ ,  $\mathcal{B}_\varepsilon(z_0)$  is the open ball centered at  $z_0$  and with radius  $\varepsilon$ .

independent since  $F$  is a submersion. The Jacobian of  $\tau^*$  can therefore be completed with  $\frac{\partial F}{\partial \xi}^T(\tau^*(x))$ . The proof is completed with Lemma 2.1.  $\square$

REMARK 3. Since  $\frac{\partial \tau^*}{\partial x}$  is of constant rank  $n$  on  $\mathcal{O}$ , the existence of such a function  $F$  is guaranteed at least locally by the constant rank Theorem.

EXAMPLE 2 (Continuation of Example 1). Elimination of the  $\hat{x}_i$  in the 4 equations given by the injective immersion  $\tau^*$  defined in (1.3) leads to the function  $F(\xi) = \xi_2\xi_3 - \xi_1\xi_4$  satisfying (2.4). It follows that a candidate for complementing:

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \end{pmatrix} \quad (2.5)$$

is 
$$\gamma(x) = \frac{\partial F}{\partial \xi}(\tau^*(x))^T = (x_2x_3, -x_1x_3, x_2, -x_1)^T .$$

This vector is nothing but the column of the minors of the matrix (2.5). It gives as determinant  $(x_2x_3)^2 + (x_1x_3)^2 + x_2^2 + x_1^2$  which is never zero on  $\mathcal{O}$ .

Then, it follows from Lemma 2.1, that, for any bounded open set  $\tilde{\mathcal{O}}$  such that  $\mathcal{A} \subset \text{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$  the following function is a diffeomorphism on  $\tilde{\mathcal{O}} \times \mathcal{B}_\epsilon(0)$  for  $\epsilon$  sufficiently small

$$\tau_a^*(x, w) = (x_1 + x_2x_3w, x_2 - x_1x_3w, -x_1x_3 + x_2w, -x_2x_3 - x_1w) .$$

With picking  $\tau_e^* = \tau_a^*$ , (1.18) gives us the following observer written in the given  $x$ -coordinates augmented with  $w$  :

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_3 \\ \hat{x}_2 \\ \hat{w} \end{pmatrix} = \begin{pmatrix} 1 & \hat{x}_3\hat{w} & \hat{x}_2\hat{w} & \hat{x}_2\hat{x}_3 \\ -\hat{x}_3\hat{w} & 1 & -\hat{x}_1\hat{w} & -\hat{x}_1\hat{x}_3 \\ -\hat{x}_3 & \hat{w} & -\hat{x}_1 & \hat{x}_2 \\ -\hat{w} & -\hat{x}_3 & -\hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1} \left[ \begin{pmatrix} \hat{x}_2 - \hat{x}_1\hat{x}_3\hat{w} \\ -\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{w} \\ -\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{w} \\ \text{sat}(\hat{x}_1\hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{x}_1] \right]$$

Unfortunately the matrix to be inverted is non singular for  $(\hat{x}, \hat{w})$  in  $\tilde{\mathcal{O}} \times \mathcal{B}_\epsilon(0)$  only and we have no guarantee that the trajectories of this observer remain in this set. This shows that a further modification transforming  $\tau_a^*$  into  $\tau_e^*$  is needed to make sure that  $\tau_e^{*-1}(\hat{\xi})$  belongs to this set whatever  $\hat{\xi}$  in  $\mathbb{R}^4$ . This is Problem 2.  $\triangle$

The drawback of this Jacobian complementation method is that it asks for the knowledge of the function  $F$ . It would be better to simply have a universal formula relating the entries of the columns to be added to those of  $\frac{\partial \tau^*}{\partial x}$ .

**2.2. The  $\tilde{P}[m, n]$  problem.** Finding a universal formula for the Jacobian complementation problem amounts to solving the following problem.

DEFINITION 2.3. ( $\tilde{P}[m, n]$  problem) For a pair of integers  $(m, n)$  such that  $0 < n < m$ , a  $C^1$  matrix function  $\tilde{\gamma} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times (m-n)}$  solves the  $\tilde{P}[m, n]$  problem if for any  $m \times n$  matrix  $\varpi = (\varpi_{ij})$  of rank  $n$ , the matrix  $(\varpi \quad \tilde{\gamma}(\varpi))$  is invertible, or equivalently, the matrix  $\tilde{\gamma}(\varpi)$  has rank  $m - n$  and satisfies  $\tilde{\gamma}(\varpi)^\top \varpi = 0$ .

As a consequence of a theorem due to Eckmann [11, §1.7 p. 126] and Lemma 2.1, we have

THEOREM 2.4. The  $\tilde{P}[m, n]$  problem is solvable by a  $C^1$  function  $\tilde{\gamma}$  if and only if the pair  $(m, n)$  is one of the following 3 pairs

$$(> 2, m - 1) \quad \text{or} \quad (4, 1) \quad \text{or} \quad (8, 1) . \quad (2.6)$$

Moreover, for each of these pairs and for any bounded set  $\mathcal{A}$ , bounded open set  $\tilde{\mathcal{O}}$  and open set  $\mathcal{O}$  satisfying

$$\text{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \text{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O} ,$$

and any injective immersion  $\tau^* : \mathcal{O} \rightarrow \tau^*(\mathcal{O}) \subset \mathbb{R}^m$ , the pair  $(\tau_a^*, \mathcal{O}_a)$  defined in (2.2) with  $\gamma(x) = \tilde{\gamma}\left(\frac{\partial \tau_a^*}{\partial x}(x)\right)$  solves Problem 1.

*Proof only if.* This is a direct consequence of Remark 2, of the facts that if  $\tilde{P}[m, n]$  has a solution,  $\tilde{P}[m-1, n-1]$  must have one, and that the only parallelizable spheres are  $\mathbb{S}^1$ ,  $\mathbb{S}^3$  and  $\mathbb{S}^7$  (see [8]) and of

**THEOREM 2.5** ([11, §1.7 p. 126]). *For  $m > n$ , there exists a continuous function  $\varkappa \in \mathbb{R}^{m \times n} \mapsto \tilde{\gamma}_1(\varkappa) \in \mathbb{R}^m$  with non zero values and satisfying*

$$\tilde{\gamma}_1(\varkappa)^T \varkappa = 0 \quad \forall \varkappa \in \mathbb{R}^{m \times n} : \text{Rank}(\varkappa) = n$$

*if and only if  $(m, n)$  is in one of the following 4 pairs*

$$(\geq 2, m-1) \quad \text{or} \quad (\text{even}, 1) \quad \text{or} \quad (7, 2) \quad \text{or} \quad (8, 3) \quad \square$$

A detailed version of the proof can be found in [6].

*Proof if.* For  $(m, n)$  equal to  $(4, 1)$  or  $(8, 1)$  respectively, possible solutions are

$$\tilde{\gamma}(\varkappa) = \begin{pmatrix} -\varkappa_2 & \varkappa_3 & \varkappa_4 \\ \varkappa_1 & -\varkappa_4 & \varkappa_3 \\ -\varkappa_4 & -\varkappa_1 & -\varkappa_2 \\ \varkappa_3 & \varkappa_2 & -\varkappa_1 \end{pmatrix}, \quad \tilde{\gamma}(\varkappa) = \begin{pmatrix} \varkappa_2 & \varkappa_3 & \varkappa_4 & \varkappa_5 & \varkappa_6 & \varkappa_7 & \varkappa_8 \\ -\varkappa_1 & \varkappa_4 & -\varkappa_3 & \varkappa_6 & -\varkappa_5 & -\varkappa_8 & \varkappa_7 \\ -\varkappa_4 & -\varkappa_1 & \varkappa_2 & \varkappa_7 & \varkappa_8 & -\varkappa_5 & -\varkappa_6 \\ \varkappa_3 & -\varkappa_2 & -\varkappa_1 & \varkappa_8 & -\varkappa_7 & \varkappa_6 & -\varkappa_5 \\ -\varkappa_6 & -\varkappa_7 & -\varkappa_8 & -\varkappa_1 & \varkappa_2 & \varkappa_3 & \varkappa_4 \\ \varkappa_5 & -\varkappa_8 & \varkappa_7 & -\varkappa_2 & -\varkappa_1 & -\varkappa_4 & \varkappa_3 \\ \varkappa_8 & \varkappa_5 & -\varkappa_6 & -\varkappa_3 & \varkappa_4 & -\varkappa_1 & -\varkappa_2 \\ -\varkappa_7 & \varkappa_6 & \varkappa_5 & -\varkappa_4 & -\varkappa_3 & \varkappa_2 & -\varkappa_1 \end{pmatrix}$$

where  $\varkappa_j$  is the  $j$ th component of the vector  $\varkappa$ . For  $n = m - 1$ , we have the identity

$$\det(\varkappa \quad \tilde{\gamma}(\varkappa)) = \sum_{j=1}^m \tilde{\gamma}_j(\varkappa_{ij}) M_{j,m}(\varkappa_{ij})$$

where  $\tilde{\gamma}_j$  is the  $j$ th component of the vector-valued function  $\tilde{\gamma}$  and the  $M_{j,m}$ , being the cofactors of  $(\varkappa \quad \tilde{\gamma}(\varkappa))$  computed along the last column, are polynomials in the given components  $\varkappa_{ij}$ . At least one of the  $M_{j,m}$  is non-zero (because they are minors of dimension  $n$  of  $\varkappa$  which is full-rank). So it is sufficient to take  $\tilde{\gamma}_j(\varkappa_{ij}) = M_{j,m}(\varkappa_{ij})$ .  $\square$

In the following example we show how by exploiting some structure we can reduce the problem to one of these 3 pairs.

**EXAMPLE 3** (Continuation of Example 2). In Example 2, we have complemented the Jacobian (2.5) with the gradient of a submersion and observed that the components of this gradient are actually cofactors. We now know that this is consistent with the case  $n = m - 1$ . But we can also take advantage from the upper triangularity of the Jacobian (2.5) and complement only the vector  $(-x_1, -x_2)$  by for instance  $(x_2, -x_1)$ . The corresponding vector  $\gamma$  is  $\gamma(x) = (0, 0, x_2, -x_1)$ . Here again, with Lemma 2.1, we know that, for any bounded open set  $\tilde{\mathcal{O}}$  such that  $\text{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \text{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$  the function

$$\tau_a^*(x, w) = (x_1, x_2, -x_1 x_3 + x_2 w, -x_2 x_3 - x_1 w)$$

is a diffeomorphism on  $\tilde{\mathcal{O}} \times \mathcal{B}_\varepsilon(0)$ . In fact, in this particular case  $\varepsilon$  can be arbitrary, no need for it to be small. However, the singularity at  $\hat{x}_1 = \hat{x}_2 = 0$  remains and equation (1.17) is still not satisfied.  $\triangle$

Given the very small number of cases where a universal formula exists, we now look for a more general solution to the Jacobian complementation problem.

**2.3. Wazewski theorem.** Historically, the Jacobian complementation problem was first addressed by Wazewski (see [25]). His formulation was :

*Given  $mn$  continuous functions  $\varpi_{ij} : \mathcal{O} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , look for  $m(m-n)$  continuous functions  $\gamma_{kl} : \mathcal{O} \rightarrow \mathbb{R}$  such that the following matrix is invertible for all  $x$  in  $\mathcal{O}$  :*

$$P(x) = \begin{pmatrix} \varpi(x) & \gamma(x) \end{pmatrix} . \quad (2.7)$$

The difference with the previous section, is that here, we look for continuous functions  $\gamma$  of  $x$  in  $\mathbb{R}^n$  instead of continuous functions  $\gamma$  of  $\varpi$  in  $\mathbb{R}^{m \times n}$ .

Wazewski established that this other version of the problem admits a far more general solution :

**THEOREM 2.6** ([25, Theorems 1 and 3] and [11, page 127]). *If  $\mathcal{O}$ , equipped with the subspace topology of  $\mathbb{R}^n$ , is a contractible space, then there exists a  $C^\infty$  function  $\gamma$  making the matrix  $P(x)$  in (2.7) invertible for all  $x$  in  $\mathcal{O}$ .*

The reader is referred to [11, page 127] or [10, pages 406-407] and to [25, Theorems 1 and 3] for the complete proof of existence of a continuous function  $\gamma$ . **It can be made smoother by using a partition of unity (see [6]).** We give the main constructive points of this proof below. But before this, let us give the following corollary obtained as a consequence of Lemma 2.1.

**COROLLARY 2.7.** *Let  $\mathcal{A}$  be a bounded set,  $\mathcal{O}$  be an open subset of  $\mathbb{R}^n$  containing  $\text{cl}(\mathcal{A})$  and which, equipped with the subspace topology of  $\mathbb{R}^n$ , is a contractible space. Let also  $\tau^* : \mathcal{O} \rightarrow \tau^*(\mathcal{O}) \subset \mathbb{R}^m$  be an injective immersion. There exists a  $C^1$  function  $\gamma$  such that, for any bounded open set  $\tilde{\mathcal{O}}$  satisfying*

$$\text{cl}(\mathcal{A}) \subset \tilde{\mathcal{O}} \subset \text{cl}(\tilde{\mathcal{O}}) \subset \mathcal{O}$$

*we can find a strictly positive real number  $\varepsilon$  such that the pair  $(\tau_a^*, \mathcal{O}_a)$  defined in (2.2) solves Problem 1.*

*About the construction of  $\gamma$ :* The proof of Theorem 2.6 given by Wazewski is based on Remark 2, noting that, if we have the decomposition

$$\varpi(x) = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}$$

with  $A(x)$  invertible on some given subset  $\mathfrak{R}$  of  $\mathcal{O}$ , then

$$\gamma(x) = \begin{pmatrix} C(x) \\ D(x) \end{pmatrix}$$

satisfies (2.3) on  $\mathfrak{R}$  if and only if  $D(x)$  is invertible on  $\mathfrak{R}$  and we have

$$C(x) = -(A^T(x))^{-1} B(x)^T D(x) \quad \forall x \in \mathfrak{R} . \quad (2.8)$$

Thus,  $C$  is imposed by the choice of  $D$  and choosing  $D$  invertible is enough to build  $\gamma$  on  $\mathfrak{R}$ .

Also, if we already have a candidate

$$P(x) = \begin{pmatrix} A(x) & C_0(x) \\ B(x) & D_0(x) \end{pmatrix}$$

on a boundary  $\partial\mathfrak{R}$  of  $\mathfrak{R}$ , then, necessarily, if  $A(x)$  is invertible for all  $x$  in  $\partial\mathfrak{R}$ , then  $D_0(x)$  is invertible and  $C_0(x) = -(A^T(x))^{-1} B(x)^T D_0(x)$  all  $x$  in  $\partial\mathfrak{R}$ . Thus, to