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NONLINEAR OBSERVER IN THE ORIGINAL COORDINATES WITH DIFFEOMORPHISM EXTENSION AND JACOBIAN COMPLETION

PAULINE BERNARD, VINCENT ANDRIEU AND LAURENT PRALY *

Abstract. Typical difficulties we face in implementing observers in applications are:

- because of physical constraints or observability singularities, the estimated state should remain in some restricted region of the system state manifold ;
- observability may also impose the observer dynamics to live in an observer state manifold with higher dimension than the system state. In such a case, the observer implementation needs a left inverse of an injective immersion. In the approach we propose to round these difficulties, we do not modify the given converging dynamics of the observer, assumed to be globally defined. Instead we select carefully the coordinates to express them. This is done via appropriate diffeomorphisms:
 - one diffeomorphism sending the restricted region to the whole space. This is done by an image extension.
 - one function extending the injective immersion into a diffeomorphism and making the observer state coordinates equivalent to the system state coordinates complemented with extra ones introduced to fill the dimension gap. This is done via a Jacobian completion.

Such a design makes possible the expression of the observer dynamics in the, maybe complemented, original coordinates.

Several examples illustrate our results.

1. Introduction.

1.1. Context. In many applications, estimating the state of a dynamical system is crucial either to build a controller or simply to obtain real time information on the system. A lot of efforts have thus been made in the scientific community to find universal methods for the construction of observers. Although very satisfactory solutions are known for linear systems ([16]), nonlinear observer designs still suffer from a significant lack of generality.

For nonlinear systems, we are aware of two "general purpose" observer design methodologies with guaranteed "non local" convergence: the high gain observers ([15, 20, 8, 9] etc) and the nonlinear Luenberger observers ([19, 13, 2]). Those observers do not demand any particular structure and only require some basic observability properties. However, in both cases, the observer state is in general living in a space different from the system state one and the state estimate is obtained typically by solving on-line a nonlinear equation, which may be very complicated.

As an illustration, consider an harmonic oscillator with unknown frequency with dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 x_3, \quad \dot{x}_3 = 0, \quad y = x_1 \tag{1.1}$$

with state $x = (x_1, x_2, x_3)$ in $\mathbb{R}^2 \times \mathbb{R}_{>0}$ and measurement y . We are interested in estimating as \hat{x} the state x from the only knowledge of y and maybe the fact that x evolves in some known set. We can solve this observer problem by following in a very orthodox way the high gain observer design (see [14] for example for the general theory and [1] for details). This leads to an answer with a dynamical system, with dynamics

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \text{sat}(\hat{x}_1 \hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{\xi}_1], \tag{1.2}$$

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with observer state $\hat{\xi}$ in \mathbb{R}^4 and where **sat** is a saturation function. With this, the system state estimate \hat{x} is obtained as

$$\hat{x} = \tau(\hat{\xi})$$

where the function τ is obtained from solving in \hat{x}

$$\hat{\xi} = \left(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4 \right) = \tau^*(\hat{x}) = (\hat{x}_1, \hat{x}_2, -\hat{x}_1\hat{x}_3, -\hat{x}_2\hat{x}_3) . \quad (1.3)$$

This is a system of 4 equations in 3 unknown which in general has no exact solution. To get an approximate solution, we may have to solve an optimization problem such as

$$\hat{x} = \tau(\hat{\xi}) = \underset{\hat{x}}{\text{Argmin}} \left| \hat{\xi} - \tau^*(\hat{x}) \right|^2 .$$

We are aware that, for the particular expression of τ^* in (1.3), an expression for $\tau(\hat{\xi})$, solution of the above problem, can be obtained¹. But this is not the case in general and we go on in this paper ignoring the data of τ to get \hat{x} from $\hat{\xi}$. Actually we cannot fully ignore τ since the values of φ in (1.2) depend on \hat{x} . This subtlety will be addressed in Assumption A.C below with dealing with pairs (φ, τ) .

To eliminate the minimization step above, we may want to write the observer dynamics directly in the x coordinates. This is not straightforward since x has dimension 3 whereas $\hat{\xi}$ has dimension 4. We round this difficulty by adding one component, say w to x . With this, we are left with writing the dynamics of (\hat{x}, \hat{w}) . This can be done by moving, in the (\hat{x}, \hat{w}) coordinates, the $\hat{\xi}$ dynamics known to solve the observer problem. For this, we need a diffeomorphism from (\hat{x}, \hat{w}) to $\hat{\xi}$ which should “extend” the function $x \mapsto \tau^*(x)$ given in (1.3) We show in Section 2 that this can be done by a Jacobian completion. Unfortunately, in doing so, the obtained diffeomorphism is rarely defined everywhere. Thus, we are facing the new problem of having no guarantee that the trajectory in (\hat{x}, \hat{w}) of the observer remains in the domain of definition of the diffeomorphism. We show in Section 3 how this difficulty can be rounded via a diffeomorphism extension.

In the following, as in the example above, we assume we are given a preliminary, say raw, observer with state $\hat{\xi}$ of dimension possibly different from the one of x and we look for a more satisfactory one written in the, may be complemented, x components.

Writing the observer in the original coordinates has been suggested by several researchers [5, 17, 3] in the case where the dimension of the raw observer state and the system state are the same. Our contribution is to relax this latter condition via Jacobian completion for which preliminary results have already been presented in [1].

As mentioned above in the example, the route via Jacobian completion may suffer problems with the domain of validity of the diffeomorphism. Actually, we may have such a validity problem already with the raw observer (see Section 5). This makes necessary the study of diffeomorphism extension which in the context of observer designs has not been studied yet, as far as we know. Its objective is to force the observer state to remain in a specific set without modifying the given raw observer dynamics. Another path, which has been proposed in [17, 3], is to modify the dynamics. But then, extra assumptions such as convexity are needed to preserve the convergence property.

¹Another possible expression is $\tau(\hat{\xi}) = \left(\hat{\xi}_1, \hat{\xi}_2, -\frac{\hat{\xi}_1\hat{\xi}_3 + \hat{\xi}_4\hat{\xi}_2}{\max\{\hat{\xi}_1^2 + \hat{\xi}_2^2, \frac{1}{\tau^2}\}} \right)$.

We motivate and illustrate our results with continuing the example of the harmonic oscillator with unknown frequency and with the bioreactor presented in [8]. These are done with using a high-gain observer as raw observer only, due to space limitations. But, as shown in [4], exactly the same tools apply to nonlinear Luenberger observers à la [19, 13, 2].

1.2. Problem statement. We consider the given system with dynamics :

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad , \quad (1.4)$$

with x in \mathbb{R}^n and y in \mathbb{R}^q . The observation problem is to construct an algorithm which, from the knowledge of y , estimates the system state x as long as it is in a specific set of interest denoted $\mathcal{A} \subseteq \mathbb{R}^n$. In this paper, our starting point is to assume that this problem is (formally) already solved but with maybe some implementation issues. More precisely, we assume :

A.II=Injective immersion : *There exists an open subset \mathcal{O} of \mathbb{R}^n which contains \mathcal{A} and a C^1 injective immersion $\tau^* : \mathcal{O} \rightarrow \mathbb{R}^m$.*

Actually this injective immersion should be such that the image $L_f \tau^*$ by τ^* of the vector field f , is in a form such that an observer can be designed. We write the corresponding observer as :

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) \quad , \quad \hat{x} = \tau(\hat{\xi}) \quad , \quad (1.5)$$

where the functions $\varphi : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ and $\tau : \mathbb{R}^m \rightarrow \mathbb{R}^n$ are locally Lipschitz. For the image by τ^* of a solution of (1.4) to be a solution of (1.5), at least as long as the former is in \mathcal{A} , we must have :

$$L_f \tau^*(x) = \varphi(\tau^*(x), x, h(x)) \quad , \quad \tau(\tau^*(x)) = x \quad \forall x \in \mathcal{A} \quad (1.6)$$

In the following we do not impose explicitly these identities are satisfied. They are imposed implicitly via the convergence requirement made precise below.

As already noticed with (1.2), we need the function τ to reconstruct \hat{x} , argument of φ in (1.5). This dependence of φ on τ may imply to change φ whenever we change τ . This leads us to consider a set² $\varphi\mathcal{T}$ of pairs (φ, τ) . With this, we can now phrase our assumption concerning the convergence problem mentioned above.

A.C=Convergence : *We know a set $\varphi\mathcal{T}$ of locally Lipschitz functions (φ, τ) such that, for any solution $X(x, t)$ of (1.4) which is defined and remains in \mathcal{A} for t in $[0, +\infty)$, the solution $(X(x, t), \hat{\Xi}((\hat{\xi}, x), t))$ of the cascade system :*

$$\dot{x} = f(x) \quad , \quad \dot{\hat{\xi}} = \varphi(\hat{\xi}, \tau(\hat{\xi}), h(x)) \quad , \quad (1.7)$$

issued from $(x, \hat{\xi})$ in $\mathcal{A} \times \mathbb{R}^m$ at time 0, is also defined on $[0, +\infty)$ and satisfies :

$$\lim_{t \rightarrow +\infty} \left| \tau^*(X(x, t)) - \hat{\Xi}((\hat{\xi}, x), t) \right| = 0 \quad .$$

Very trivially, φ can be paired with any function τ in the particular case in which φ does not depend on τ , i.e. it is in the form $\varphi(\hat{\xi}, \tau(x), y) = \varphi(\hat{\xi}, y)$ as in the case of the nonlinear Luenberger

²The symbol $\varphi\mathcal{T}$ is pronounced *phitau*.

observer for instance (see [4]). In the high-gain approach, when \mathcal{A} is bounded, thanks to the gain ℓ which can be chosen arbitrarily large, φ can be paired with any locally Lipschitz function τ provided its values are saturated whenever they are used as arguments of φ .

In the following, we consider that the function τ^* and the set \mathcal{F} are given and we aim at designing an observer for x in the, maybe complemented, x -coordinates. However, in trying to meet this objective, we shall encounter some difficulties which will lead us at the end to reconsider the data of τ^* and \mathcal{F} .

EXAMPLE 1. For the harmonic oscillator with unknown frequency (1.1), since, for any solution with initial condition $x_1 = x_2 = 0$, we cannot get any information on x_3 from the only knowledge of its dynamics and the function $t \mapsto y(t) = X_1(x, t)$, we define the set \mathcal{A} in which we want the estimation as

$$\mathcal{A} = \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in \left] \frac{1}{r}, r \right[, x_3 \in]0, r[\right\}, \quad (1.8)$$

where r is some arbitrary strictly positive real number. This set is invariant and, the system is strongly differentially observable of order 4 on this set. Indeed, it can be checked that the function (1.3) is an injective immersion on $(\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0, 0)\} \times \mathbb{R}_{>0})$. This implies that Assumption A.II is satisfied for any open subset \mathcal{O} such that $\text{cl}(\mathcal{A}) \subset \mathcal{O} \subseteq (\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0, 0)\} \times \mathbb{R}_{>0})$, with cl denoting the set closure. Then, relying on what is known on high gain observers (see [14] for example), we can claim that, in the present case where \mathcal{A} is compact, a set \mathcal{F} satisfying Assumption A.C is made of pairs of a locally Lipschitz function τ satisfying

$$x = \tau(x_1, x_2, -x_1 x_3, -x_2, x_3) \quad \forall x \in \mathcal{A}$$

and the function φ defined in (1.2) where

$$\text{sat}(s) = \min \{ r^3, \max \{ s, -r^3 \} \} \quad (1.9)$$

and the gain is adjusted to the properties of τ .

We do not need to know an expression for τ , in what follows knowing its existence is sufficient. \triangle

1.3. A sufficient condition allowing us to express the observer in the original coordinates. To motivate the technicalities we shall use to write the observer in the original coordinates, we start with the simpler case where the raw observer state $\hat{\xi}$ has the same dimension as the system state x , i.e. $m = n$. The example of the bioreactor from [8] detailed in Section 5 falls into this category. In this case, τ^* is a diffeomorphism on \mathcal{O} , i.e it is simply a change of coordinates. We can implement the observer in the original coordinates :

$$\dot{\hat{x}} = \left(\frac{\partial \tau^*}{\partial x}(\hat{x}) \right)^{-1} \varphi(\tau^*(\hat{x}), \hat{x}, y). \quad (1.10)$$

which only relies on a Jacobian inversion. However, although we know by assumption that the system trajectories remain in \mathcal{O} where the Jacobian is invertible, we have no guarantee the ones of the observer do. Therefore, we must find means to ensure the estimate \hat{x} does not leave this set in order to obtain convergence and completeness of solutions. To achieve this, we will see that a possible approach is the extension of the image set $\tau^*(\mathcal{O})$ to \mathbb{R}^m .

With this in mind, we now look at the more complicated situation where $m > n$, i.e τ^* is only an injective immersion. The example of the harmonic oscillator with unknown frequency corresponds

to this case. In [1], it is proposed to augment the given injective immersion τ^* into a diffeomorphism $\tau_e^* : \mathcal{O}_e = \mathcal{O} \times \mathcal{S}_w \rightarrow \mathbb{R}^m$, thus adding $m - n$ dimensions to the state through a new variable w . To help us in finding such an appropriate augmentation, we have the following sufficient condition.

PROPOSITION 1.1. *Assume A.II and A.C hold and \mathcal{A} is bounded. Assume also*

P1: Completion to a diffeomorphism. There exists an open subset \mathcal{O}_e of \mathbb{R}^m and a C^1 diffeomorphism $\tau_e^ : \mathcal{O}_e \rightarrow \mathbb{R}^m$ such that \mathcal{O}_e contains $\mathcal{A} \times \{0\}$ and we have*

$$\tau_e^*(x, 0) = \tau^*(x) \quad \forall x \in \mathcal{A} . \quad (1.11)$$

*P2: Surjectivity.*³ *The function τ_e^* is surjective. In other words, $\tau_e^*(\mathcal{O}_e) = \mathbb{R}^m$.*

P3: Couple (\hat{x}, φ) admissible. With denoting τ_{ex} the x -component of τ_e , the inverse of τ_e^ , there exists a function φ such that the pair (τ_{ex}, φ) is in the set \mathcal{F} given by assumption A.C.*

Under these conditions, for any solution $X(x, t)$ of (1.4) which is defined and remains in \mathcal{A} for t in $[0, +\infty)$ the solution $(X(x, t), \hat{X}(x, \hat{x}, \hat{w}, t), \hat{W}(x, \hat{x}, \hat{w}, t))$, with initial condition (\hat{x}, \hat{w}) in \mathcal{O}_e , of the cascade of system (1.4) with the observer :

$$\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} = \left(\frac{\partial \tau_e^*}{\partial (\hat{x}, \hat{w})}(\hat{x}, \hat{w}) \right)^{-1} \varphi(\tau_e^*(\hat{x}, \hat{w}), \hat{x}, y) \quad (1.12)$$

is also defined on $[0, +\infty)$ and satisfies :

$$\lim_{t \rightarrow +\infty} \left| \hat{W}(x, \hat{x}, \hat{w}, t) \right| + \left| X(x, t) - \hat{X}(x, \hat{x}, \hat{w}, t) \right| = 0 . \quad (1.13)$$

The key point in the observer (1.12) is that, instead of left-inverting the function τ^* via τ as in (1.5), we invert only a matrix.

Proof. For any compact set \mathfrak{C} containing $\mathcal{A} \times \{0\}$ and contained in \mathcal{O}_e , there exists a class \mathcal{K} function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that we have

$$|w| + |x_a - x_b| \leq \alpha(|\tau^*(x_a) - \tau_e^*(x_b, w)|) \quad \forall x_a \in \mathcal{A}, \forall (x_b, w) \in \mathfrak{C} . \quad (1.14)$$

Indeed, let:

$$\alpha_0(s) = \max_{\substack{((x_b, w_b), (x_c, w_c)) \in \mathfrak{C}^2 \\ |\tau_e^*(x_b, w_b) - \tau_e^*(x_c, w_c)| \leq s}} |x_b - x_c| + |w_b - w_c|$$

This defines properly a non decreasing function satisfying :

$$|x_b - x_c| + |w_b - w_c| \leq \alpha_0(|\tau_e^*(x_b, w_b) - \tau_e^*(x_c, w_c)|) \quad \forall ((x_b, w_b), (x_c, w_c)) \in \mathfrak{C}^2 .$$

If $\alpha_0(0)$ is not zero, then there exists $((x_b, w_b), (x_c, w_c))$ in \mathfrak{C}^2 satisfying :

$$|\tau_e^*(x_b, w_b) - \tau_e^*(x_c, w_c)| = 0 \quad , \quad |x_b - x_c| + |w_b - w_c| > 0 .$$

³ In the particular case where a subset \mathcal{S} of \mathbb{R}^m is known to be invariant by the dynamics $\dot{\hat{\xi}} = \varphi(\hat{\xi}, \tau(\hat{\xi}), y)$, it is enough to have $\mathcal{S} \subset \tau_e^*(\mathcal{O}_e)$ in P2. But dealing with this case is difficult when \mathcal{S} depends on τ .

But this contradicts the injectivity of τ_e^* (i.e. P1). Then let s_k be a sequence converging to 0. For each k and l in $\mathbb{N}_{>0}$, we can find $((x_{b,kl}, w_{b,kl}), (x_{c,kl}, w_{c,kl}))$ in \mathfrak{C}^2 which satisfies

$$\begin{aligned}\alpha_0(s_k) &\geq |x_{b,kl} - x_{c,kl}| + |w_{b,kl} - w_{c,kl}| \geq \alpha_0(s_k) - \frac{1}{l}, \\ |\tau_e^*(x_{b,kl}, w_{b,kl}) - \tau_e^*(x_{c,kl}, w_{c,kl})| &\leq s_k.\end{aligned}$$

The ‘‘diagonal’’ sequence $((x_{b,kk}, w_{b,kk}), (x_{c,kk}, w_{c,kk}))$ being in a compact set admits an accumulation point $(x_{b,*}, w_{b,*}), (x_{c,*}, w_{c,*})$ which, because of the continuity and the injectivity of τ_e^* must satisfy $(x_{b,*}, w_{b,*}) = (x_{c,*}, w_{c,*})$. This implies that $\alpha_0(s_k)$ tends to 0 and thus α_0 is continuous at 0. Now, consider the function defined by the following Riemann integral

$$\alpha(s) = \frac{1}{s} \int_s^{2s} \alpha_0(\nu) d\nu + s.$$

It is continuous, strictly increasing and zero at zero and we have :

$$|x_b - x_c| + |w_b - w_c| \leq \alpha(|\tau_e^*(x_b, w_b) - \tau_e^*(x_c, w_c)|) \quad \forall ((x_b, w_b), (x_c, w_c)) \in \mathfrak{C}^2.$$

So in particular for $x_c = x_a$ and $w_c = 0$, with x_a in \mathcal{A} and (1.11), we obtain (1.14).

Let $(x, (\hat{x}, \hat{w}))$ be arbitrary in $\mathcal{A} \times \mathcal{O}_e$ but such that $X(x, t)$ solution of (1.4) is defined and remains in \mathcal{A} for t in $[0, +\infty)$. Let $[0, T[$ be the right maximal interval of definition of the solution $(X(x, t), \hat{X}(x, \hat{x}, \hat{w}, t), \hat{W}(x, \hat{x}, \hat{w}, t))$ when considered with values in $\mathcal{A} \times \mathcal{O}_e$. With aiming at showing that T is infinite by contradiction, assume it is finite. Then, when t goes to T , either $(\hat{X}(x, \hat{x}, \hat{w}, t), \hat{W}(x, \hat{x}, \hat{w}, t))$ goes to infinity or to the boundary of \mathcal{O}_e . By construction $t \mapsto \hat{\Xi}(t) := \tau_e^*(\hat{X}(\hat{x}, \hat{w}, t), \hat{W}(\hat{x}, \hat{w}, t))$ is a solution of (1.7) on $[0, T[$. From P3 and assumption A.C it can be extended as a solution defined on $[0, +\infty[$ when considered with values in $\mathbb{R}^m = \tau_e^*(\mathcal{O}_e)$. This implies that $\hat{\Xi}(T)$ is well defined in \mathbb{R}^m . Since, with P2, the inverse τ_e of τ_e^* is a diffeomorphism defined on \mathbb{R}^m , we obtain $\lim_{t \rightarrow T} (\hat{X}(\hat{x}, \hat{w}, t), \hat{W}(\hat{x}, \hat{w}, t)) = \tau_e(\hat{\Xi}(T))$, which is an interior point of $\tau_e(\mathbb{R}^m) = \mathcal{O}_e$. This point being neither a boundary point nor at infinity, we have a contradiction. It follows that T is infinite.

Finally, with assumption A.C, we have :

$$\lim_{t \rightarrow +\infty} \left| \tau_e^*(X(x, t)) - \tau_e^*(\hat{X}(\hat{x}, \hat{w}, t), \hat{W}(\hat{x}, \hat{w}, t)) \right| = 0.$$

Since $\tau_e^*(X(x, t))$ remains in the compact set $\tau_e^*(\mathcal{A})$, there exists a compact subset \mathbf{C} of \mathbb{R}^m and a time $t_{\mathbf{C}}$ such that $\tau_e^*(\hat{X}(\hat{x}, \hat{w}, t), \hat{W}(\hat{x}, \hat{w}, t))$ is in \mathbf{C} for all $t > t_{\mathbf{C}}$. Applying (1.14) to $\mathfrak{C} = \tau_e(\mathbf{C})$ which is a compact subset of \mathcal{O}_e , we obtain (1.13). \square

REMARK 1. *It follows from this proof that the assumptions of boundedness of \mathcal{A} and (1.11) can be replaced by:* For any compact subset \mathfrak{C} of \mathcal{O}_e , there exists a class \mathcal{K} function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that, (1.14) holds. *Also conversely, if (1.14) and P2 hold, then (1.11) holds.*

Addressing problems P1 and P2 which have their own interest outside the observer context is the main topic of this paper. We will present in Section 2 conditions under which the extension of an immersion into a diffeomorphism in the sense of P1 can be done via a Jacobian completion. We will address problem P2 in Section 3 with giving conditions under which we can extend the image of the diffeomorphism τ_e^* to \mathbb{R}^m .

Throughout Sections 2-3, we will show how, step by step, we can express a high gain observer in the original coordinates for the harmonic oscillator with unknown frequency. The various difficulties we shall encounter on this road will be discussed in Section 4. In particular, we shall see how they can be rounded thanks to a better choice of τ^* in Assumption A.II and the pair (τ, φ) given by Assumption A.C. Finally, we will show in Section 5 that our approach enables to ensure completeness of solutions of the observer presented in [8] for the bioreactor.

2. About P1 : Jacobian completion.

2.1. A way to satisfy P1. In [1], we have the following sufficient condition for having P1 satisfied.

PROPOSITION 2.1 ([1]). *Assume \mathcal{A} is bounded and A.II holds. Let \mathcal{S}_x be a bounded open set such that $\mathcal{A} \subset \text{cl}(\mathcal{S}_x) \subset \mathcal{O}$. If there exists a C^1 function $\gamma : \text{cl}(\mathcal{S}_x) \rightarrow \mathbb{R}^{m*(m-n)}$ whose values are $m*(m-n)$ matrices satisfying :*

$$\det \begin{pmatrix} \frac{\partial \tau^*}{\partial x}(x) & \gamma(x) \end{pmatrix} \neq 0 \quad \forall x \in \text{cl}(\mathcal{S}_x), \quad (2.1)$$

then there exists a strictly positive real number ε_\diamond such that, for any ε in $(0, \varepsilon_\diamond]$, a function τ_ε^* satisfying condition P1 of Proposition 1.1 with⁴ $\mathcal{O}_\varepsilon = \mathcal{S}_x \times \mathcal{B}_\varepsilon(0)$ is :

$$\tau_\varepsilon^*(x, w) = \tau^*(x) + \gamma(x) w. \quad (2.2)$$

Proof. The fact that τ_ε^* is an immersion for ε small enough is established in [1]. We now prove it is injective. Let ε_0 be a strictly positive real number such that the Jacobian of $\tau_\varepsilon^*(x, w)$ in (2.2) is invertible for any (x, w) in $\text{cl}(\mathcal{S}_x \times \mathcal{B}_{\varepsilon_0}(0))$. Since $\text{cl}(\mathcal{S}_x \times \mathcal{B}_{\varepsilon_0}(0))$ is compact, not to contradict the Implicit function Theorem, there exists a strictly positive real number δ such that any two pairs (x_a, w_a) and (x_b, w_b) in $\text{cl}(\mathcal{S}_x \times \mathcal{B}_{\varepsilon_0}(0))$ which satisfy

$$\tau_\varepsilon^*(x_a, w_a) = \tau_\varepsilon^*(x_b, w_b) \quad , \quad (x_a, w_a) \neq (x_b, w_b) \quad (2.3)$$

satisfies also

$$|x_a - x_b| + |w_a - w_b| \geq \delta.$$

On another hand, since τ^* is continuous and injective on $\text{cl}(\mathcal{S}_x) \subset \mathcal{O}$, by following the same arguments as those for establishing (1.14), we can prove the existence of a class \mathcal{K} function β such that we have

$$|x_a - x_b| \leq \beta(|\tau^*(x_a) - \tau^*(x_b)|) \quad \forall (x_a, x_b) \in \text{cl}(\mathcal{S}_x)^2.$$

It follows that, if (2.3) holds with w_a and w_b in $\mathcal{B}_\varepsilon(0)$ with $\varepsilon \leq \varepsilon_0$, we have

$$\delta - 2\varepsilon \leq |x_a - x_b| \leq \beta(|\tau^*(x_a) - \tau^*(x_b)|) = \beta(|\gamma(x_a)w_a - \gamma(x_b)w_b|) \leq \beta \left(2\varepsilon \sup_{x \in \text{cl}(\mathcal{S}_x)} |\gamma(x)| \right)$$

⁴For a positive real number ε and z_0 in \mathbb{R}^p , $\mathcal{B}_\varepsilon(z_0)$ is the open ball centered at z_0 and with radius ε .

But there exists a strictly positive real number $\varepsilon_\diamond \leq \varepsilon_0$ such that, for any ε in $(0, \varepsilon_\diamond]$, we have

$$\delta - 2\varepsilon > \beta \left(2\varepsilon \sup_{x \in \text{cl}(\mathcal{S}_x)} |\gamma(x)| \right).$$

So, for all such ε , (2.3) is impossible. This proves that τ_e^* is injective on $\mathcal{S}_x \times \mathcal{B}_\varepsilon(0)$ for any ε in $(0, \varepsilon_\diamond]$. \square

With this proposition, condition P1 is related to the existence of a C^1 function γ such that the matrix (2.1) is invertible.

2.2. Submersion case. We can solve the Jacobian completion problem when $\tau^*(\text{cl}(\mathcal{S}_x))$ is a level set of a submersion.

PROPOSITION 2.2 (Completion from a submersion). *Assume A.II holds. Let \mathcal{S}_x be a bounded open set such that $\text{cl}(\mathcal{S}_x) \subset \mathcal{O}$. Assume there exists a C^2 function $F : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$ which is a submersion at least on a neighborhood of $\tau^*(\mathcal{S}_x)$ satisfying:*

$$F(\tau^*(x)) = 0 \quad \forall x \in \mathcal{S}_x, \quad (2.4)$$

then with the C^1 function $x \mapsto \gamma(x) = \frac{\partial F^T}{\partial \xi}(\tau^*(x))$, the matrix in (2.1) is invertible for all x in \mathcal{S}_x .

Proof. For all x in $\text{cl}(\mathcal{S}_x)$, $\frac{\partial \tau^*}{\partial x}(x)$ is right invertible and we have $\frac{\partial F}{\partial \xi}(\tau^*(x)) \frac{\partial \tau^*}{\partial x}(x) = 0$. Thus, the rows of $\frac{\partial F}{\partial \xi}(\tau^*(x))$ are orthogonal to the column vectors of $\frac{\partial \tau^*}{\partial x}(x)$. Moreover, they are independent due to the fact that it is a submersion. The Jacobian of τ^* can therefore be completed with $\frac{\partial F^T}{\partial \xi}(\tau^*(x))$. \square

REMARK 2. *Since $\frac{\partial \tau^*}{\partial x}$ is of constant rank n on \mathcal{O} , the existence of such a function F is guaranteed at least locally by the constant rank Theorem.*

EXAMPLE 2 (Continuation of Example 1). Elimination of the \hat{x}_i in the 4 equations given by the injective immersion τ^* defined in (1.3) leads to the function $F(\xi) = \xi_2 \xi_3 - \xi_1 \xi_4$ satisfying (2.4). It follows that a candidate for completing:

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \end{pmatrix} \quad (2.5)$$

is

$$\frac{\partial F}{\partial \xi}(\tau^*(x))^T = (x_2 x_3, -x_1 x_3, x_2, -x_1)^T.$$

Notice that this vector is nothing but the column of the minors of the Jacobian (2.5). It gives as determinant

$$(x_2 x_3)^2 + (x_1 x_3)^2 + x_2^2 + x_1^2$$

which is never zero on \mathcal{O} .

With Proposition 2.1, we know that, for any bounded open set \mathcal{S}_x such that $\mathcal{A} \subset \text{cl}(\mathcal{S}_x) \subset \mathcal{O}$ the function

$$\tau_e^*(x, w) = (x_1 + x_2 x_3 w, x_2 - x_1 x_3 w, -x_1 x_3 + x_2 w, -x_2 x_3 - x_1 w)$$

is a diffeomorphism on $\mathcal{S}_x \times \mathcal{B}_\varepsilon(0)$ for ε in $]0, \varepsilon_\circ]$.

With this τ_e^* , (1.12) gives us the following observer written in the original x -coordinates complemented with w :

$$\overbrace{\begin{pmatrix} \hat{x}_1 \\ \hat{x}_3 \\ \hat{x}_2 \\ \hat{w} \end{pmatrix}} = \begin{pmatrix} 1 & \hat{x}_3 \hat{w} & \hat{x}_2 \hat{w} & \hat{x}_2 \hat{x}_3 \\ -\hat{x}_3 \hat{w} & 1 & -\hat{x}_1 \hat{w} & -\hat{x}_1 \hat{x}_3 \\ -\hat{x}_3 & \hat{w} & -\hat{x}_1 & \hat{x}_2 \\ -\hat{w} & -\hat{x}_3 & -\hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1} \times \left[\begin{pmatrix} \hat{x}_2 - \hat{x}_1 \hat{x}_3 \hat{w} \\ -\hat{x}_1 \hat{x}_3 + \hat{x}_2 \hat{w} \\ -\hat{x}_2 \hat{x}_3 - \hat{x}_1 \hat{w} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \text{sat}(\hat{x}_1 \hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{x}_1] \right]$$

Unfortunately the matrix to be inverted is non singular for (\hat{x}, \hat{w}) in $\mathcal{S}_x \times \mathcal{B}_\varepsilon(0)$ only and we have no guarantee that the solutions of this observer remain in this set. This shows that another modification of τ_e^* is needed to make sure that $\tau_e^{*-1}(\hat{\xi})$ belongs to this set whatever $\hat{\xi}$ in \mathbb{R}^4 is, namely to satisfy P2. \triangle

The drawback of the Jacobian completion method is that it asks for the knowledge of the function F . It would be nicer to have simply a universal formula relating the entries of the columns to be added to those of $\frac{\partial \tau_e^*}{\partial x}$.

2.3. The $P[m, n]$ problem. It turns out that the existence of such a universal formula is related to the $P[m, n]$ problem studied by Eckmann (among others). This problem is:

For a continuous $m \times n$ matrix $\varphi = (\varphi_{ij})$ of rank n (with $n < m$), can we find $m(m-n)$ continuous functions $\gamma_{kl}(\varphi_{11}, \dots, \varphi_{mn})$ of the coefficients of φ , such that, with $\gamma(\varphi) = (\gamma_{kl}(\varphi_{11}, \dots, \varphi_{mn}))$, the matrix :

$$(\varphi \quad \gamma(\varphi)) , \tag{2.6}$$

is invertible ?

THEOREM 2.3. *[[7] Eckmann theorem] The $P[m, n]$ problem is solvable if and only if the pair (m, n) is in the following table.*

$m =$	even	≥ 2	7	8
$n =$	1	$m - 1$	2	3

(2.7)

The two most "common" cases are :

- $m = n + 1$: when there is only one column to add, an easy method is two add the vector of the corresponding minors (see Example 2)
- $n = 1, m$ even : when there is an odd number of columns to add to a single one, a solution is to complete with permutations of the elements from the first existing column (see Example 8).

According to Theorem 2.3, the pairs for which the $P[m, n]$ problem admits solutions are rather rare. Fortunately, as illustrated in the following example, we can sometimes "modify" m and n in a way to get one of these good pairs even when it is not directly the case.

EXAMPLE 3 (Continuation of Example 2). In Example 2, we have completed the Jacobian

(2.5) with the gradient of a submersion and observed that the components of this gradient are actually minors. We now know that this is consistent with the case $m = n + 1$. But we can also take advantage from the upper triangularity of the Jacobian (2.5) and complete only the vector $(-x_1, -x_2)$ by for instance $(x_2, -x_1)$. Actually in doing so, we move to the case with $n = 1$ and m even. The corresponding vector γ is $\gamma(x) = (0, 0, x_2, -x_1)$. Here again, with Proposition 2.1, we know that, for any bounded open set \mathcal{S}_x such that $\mathcal{A} \subset \text{cl}(\mathcal{S}_x) \subset \mathcal{O}$ the function the function

$$\tau_\varepsilon^*(x, w) = (x_1, x_2, -x_1x_3 + x_2w, -x_2x_3 - x_1w)$$

is a diffeomorphism on $\mathcal{S}_x \times \mathcal{B}_\varepsilon(0)$, where in this particular case ε can be arbitrary, no need for it to be small. However, the singularity at $\hat{x}_1 = \hat{x}_2 = 0$ remains and P2 is still not satisfied. \triangle

2.4. Wazewski theorem. Historically, the problem of Jacobian completion was first addressed by Wazewski (see [21]). His formulation was :

Given mn continuous functions $\varphi_{ij} : \mathcal{O} \subset \mathbb{R}^n \rightarrow \mathbb{R}$, look for $m(m - n)$ continuous functions $\gamma_{kl} : \mathcal{O} \rightarrow \mathbb{R}$ such that the following matrix is invertible for all x in \mathcal{O} :

$$P(x) = \begin{pmatrix} \varphi(x) & \gamma(x) \end{pmatrix} . \quad (2.8)$$

The difference with Eckmann's problem, is that here, we look for continuous functions of x instead of continuous functions of the components of $\varphi(x)$. This other version of the problem admits a far more general solution than Eckmann's one :

THEOREM 2.4 ([21, Theorems 1 and 3] and [7, page 127]). *If \mathcal{O} , equipped with the subspace topology of \mathbb{R}^n , is a contractible space, then there exists a C^∞ function γ making invertible the matrix $P(x)$ in (2.8) for all x in \mathcal{O} .*

REMARK 3. *A key assumption in Theorem 2.4 is that the set \mathcal{O} , on which the immersion τ^* is defined, is contractible. This is a strong requirement which for instance does not apply to any open set \mathcal{O} contained in $\mathbb{R}^3 \setminus (\{0, 0\} \times \mathbb{R})$ containing the closure of \mathcal{A} defined in (1.8) of Example 2. We will see in Section 4 how we can change the immersion τ^* given by Assumptions A.II to satisfy this constraint.*

Proof. The reader is referred to [7, page 127] or [6, pages 406-407] and to [21, Theorems 1 and 3] for the complete proof of existence of a continuous function γ . Here are some sketches.

The proof given by Eckmann in [7] can be summarized as follows. Let $V_{m,n}$ be the space of orthogonal matrices of m rows and n columns, and let p be the projection which removes the $m - n$ last columns of an orthogonal matrix of dimension m . It is known that $(SO(m), p, V_{m,n})$ is a Hurewicz fibration and thus we can "lift" the application $\varphi : X \rightarrow V_{m,n}$ to an application $P : X \rightarrow SO(m)$ since X is contractible. Unfortunately, we have not been able to extract a "formula" from this proof.

The proof given by Wazewski in [21] for spaces homeomorphic to a parallelepiped is more "workable". It is based on the fact that, if $\varphi(x)$ is a full rank $m \times n$ matrix, then the completed matrix $(\varphi(x) \ \gamma(x))$ is invertible whenever $\gamma(x)$ is a full-rank $m \times (m - n)$ matrix satisfying

$$\varphi(x)^T \gamma(x) = 0 . \quad (2.9)$$

To exploit this remark, we note that, if we have the decomposition

$$\varphi(x) = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}$$

with $A(x)$ invertible on some given subset \mathfrak{R} of \mathcal{O} , then

$$\gamma(x) = \begin{pmatrix} C(x) \\ D(x) \end{pmatrix}$$

satisfies the above properties on \mathfrak{R} if and only if $D(x)$ is invertible on \mathfrak{R} and we have

$$C(x) = -(A^T(x))^{-1}B(x)^T D(x) \quad \forall x \in \mathfrak{R} .$$

Thus, C is imposed by the choice of D and choosing D invertible is enough to build γ on \mathfrak{R} .

Also, if we already have a candidate

$$P(x) = \begin{pmatrix} A(x) & C_0(x) \\ B(x) & D_0(x) \end{pmatrix}$$

on a boundary $\partial\mathfrak{R}$ of \mathfrak{R} , then, necessarily, if $A(x)$ is invertible for all x in $\partial\mathfrak{R}$, then $D_0(x)$ is invertible and $C_0(x) = -(A^T(x))^{-1}B(x)^T D_0(x)$ all x in $\partial\mathfrak{R}$. Thus, to extend the construction of a continuous function γ inside \mathfrak{R} from its knowledge on the boundary $\partial\mathfrak{R}$, it suffices to ensure $D = D_0$ on $\partial\mathfrak{R}$. Because we can propagate continuously γ from one boundary to the other, Wazewski deduces from these two observations that, it is sufficient to partition the set \mathcal{O} into adjacent sets \mathfrak{R}_i where a given $n \times n$ minor A_i is invertible. This is possible since φ is full-rank on \mathcal{O} . When \mathcal{O} is a parallelepiped, he shows that there exists an ordering of the \mathfrak{R}_i such that the continuity of each D_i can be successively ensured. We illustrate this proof in Example 4 below.

It remains to complete the proof of Eckmann or Wazewski by showing how the continuous function γ making P invertible can be modified into a smoother one giving the same invertibility property. Let γ_i denote the i th column of γ .

We start with modifying γ_1 into $\tilde{\gamma}_1$. Since φ , γ and the determinant are continuous, for any x in \mathcal{O} , there exists a strictly positive real number r_x , such that

$$\det(\varphi(y) \ \gamma_1(x) \ \gamma_{2:m-n}(y)) > 0, \quad \forall y \in \mathcal{B}_{r_x}(x), \quad (2.10)$$

where $\gamma_{i:j}$ denotes the matrix composed of the i^{th} to j^{th} columns of γ . The family of sets $(\mathcal{B}_{r_x}(x))_{x \in \mathcal{O}}$ is an open cover of \mathcal{O} . Therefore, by [12, Theorem 2.1], there exists a subordinate C^∞ partition of unity, i.e. there exist a family of C^∞ functions $\psi_x : \mathcal{O} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\text{Supp}(\psi_x) \subset \mathcal{B}_{r_x}(x) \quad \forall x \in \mathcal{O}, \quad (2.11)$$

$$\{\text{Supp}(\psi_x)\}_{x \in \mathcal{O}} \text{ is locally finite}, \quad (2.12)$$

$$\sum_{x \in \mathcal{O}} \psi_x(y) = 1 \quad \forall y \in \mathcal{O}. \quad (2.13)$$

With this, we define the function $\tilde{\gamma}_1$ on \mathcal{O} by

$$\tilde{\gamma}_1(y) = \sum_{x \in \mathcal{O}} \psi_x(y) \gamma_1(x).$$

This function is well-defined and C^∞ on \mathcal{O} because the sum is finite at each point according to (2.12). Using multi-linearity of the determinant, we have, for all y in \mathcal{O} ,

$$\det(\varphi(y) \ \tilde{\gamma}_1(y) \ \gamma_{2:m-n}(y)) = \sum_{x \in \mathcal{S}_x} \psi_x(y) \det(\varphi(y) \ \gamma_1(x) \ \gamma_{2:m-n}(y)).$$

Thanks to (2.12), at each point y in \mathcal{S}_x , there is a finite number of $\psi_x(y)$ which are not zero. Moreover, according to (2.13), there is at least one $\psi_x(y)$ strictly positive. On the other hand, with (2.11) and (2.10), we know the determinant corresponding to any non zero $\psi_x(y)$ is strictly positive. Therefore, we can replace the continuous function γ_1 by the C^∞ function $\tilde{\gamma}_1$ as a first column of γ . Then we follow exactly the same procedure for γ_2 with this modified γ . By proceeding this way, one column after the other, we get our result. \square

EXAMPLE 4. Consider the function

$$\varphi(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \\ \frac{\partial \varphi}{\partial x_1}[x_3 - \varsigma] & \frac{\partial \varphi}{\partial x_2}[x_3 - \varsigma] & \varphi \end{pmatrix},$$

where ς is a real number and $\varphi(x_1, x_2) = \max\{0, 1 - (x_1^2 + x_2^2)\}^2$. $\varphi(x)$ has full rank 3 for any x in \mathbb{R}^3 , since $\varphi(x_1, x_2) \neq 0$ when $x_1 = x_2 = 0$. Hence a continuous function exists to augment φ into an invertible matrix. To follow Wazewski construction, let δ be a strictly positive real number and consider the following 5 regions of \mathbb{R}^3

$$\begin{aligned} \mathfrak{R}_1 &=]-\infty, -\delta] \times \mathbb{R}^2, & \mathfrak{R}_2 &= [-\delta, \delta] \times [\delta, +\infty] \times \mathbb{R}, \\ \mathfrak{R}_3 &= [-\delta, \delta]^2 \times \mathbb{R}, & \mathfrak{R}_4 &= [-\delta, \delta] \times [-\infty, -\delta] \times \mathbb{R}, \\ \mathfrak{R}_5 &= [\delta, +\infty[\times \mathbb{R}^2. \end{aligned}$$

We select δ sufficiently small in such a way that φ is not 0 in \mathfrak{R}_3 .

We start Wazewski's algorithm in \mathfrak{R}_3 . There the matrix A is given by rows 1, 2 and 5 of φ and B by rows 3 and 4. With picking D as the identity, C is $(A^T)^{-1}B$. D gives rows 3 and 4 of γ and C gives rows 1, 2 and 5.

Then we move to the region \mathfrak{R}_2 . There the matrix A is given by rows 1, 2 and 4 of φ , B by rows 3 and 5. Also D , along the boundary between \mathfrak{R}_3 and \mathfrak{R}_2 , is given by rows 3 and 5 of γ obtained in the previous step. We extrapolate this inside \mathfrak{R}_2 by kipping D constant in planes $x_1 = \text{constant}$. An expression for C and therefore for γ follows.

We do exactly the same thing for \mathfrak{R}_4 .

Then we move to the region \mathfrak{R}_1 . There the matrix A is given by rows 1, 2 and 3 of φ , B by rows 4 and 5. Also D , along the boundary between \mathfrak{R}_1 and \mathfrak{R}_2 , between \mathfrak{R}_1 and \mathfrak{R}_3 and between \mathfrak{R}_1 and \mathfrak{R}_4 , is given by rows 4 and 5 of γ obtained in the previous steps. We extrapolate this inside \mathfrak{R}_1 by kipping D constant in planes $x_2 = \text{constant}$. An expression for C and therefore for γ follows.

We do exactly the same thing for \mathfrak{R}_5 .

Note that this construction produces a continuous γ , but we could have extrapolated D in a smoother way to obtain γ as smooth as necessary. \triangle

3. About P2 : Image extension of a diffeomorphism.

3.1. Problem definition and result. To satisfy condition P2 of Proposition 1.1 we may need to extend a given diffeomorphism τ_e^* , satisfying condition P1 with some set \mathcal{O}_e , in such a way that the image of the extended diffeomorphism covers the entire \mathbb{R}^m . In order to keep P1, this new diffeomorphism must still be an extension of τ^* in the sense of (1.11).

There is a rich literature reporting very advanced results on diffeomorphism extension problem. In the following some of the techniques are inspired from [12, Chapter 8] and [18, pages 2, 7 to 14

and 16 to 18](among others). Note however that we are interested in a weak version of this problem since we allow a small deformation of the set on which the given diffeomorphism is to be matched. In the following, we give a solution to this problem when the following property is satisfied.

DEFINITION 3.1 (Conditions \mathbb{B}). *An open subset E of \mathbb{R}^m is said to verify condition \mathbb{B} if there exist a C^1 function $\kappa : \mathbb{R}^m \rightarrow \mathbb{R}$, a bounded⁵ C^1 vector field χ , and a closed set K_0 contained in E such that:*

1. $E = \{z \in \mathbb{R}^m, \kappa(z) < 0\}$
2. K_0 is globally attractive for χ
3. we have the following transversality assumption:

$$\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \quad \forall z \in \mathbb{R}^m : \kappa(z) = 0.$$

THEOREM 3.2 (Image extension). *Let $\psi : \mathcal{D} \subset \mathbb{R}^m \rightarrow \psi(\mathcal{D}) \subset \mathbb{R}^m$ be a diffeomorphism. If $\psi(\mathcal{D})$ verifies condition \mathbb{B} or \mathcal{D} is C^2 -diffeomorphic to \mathbb{R}^m and ψ is C^2 , then for any compact set K in \mathcal{D} there exists a diffeomorphism $\psi_e : \mathcal{D} \rightarrow \mathbb{R}^m$ satisfying :*

$$\psi_e(\mathcal{D}) = \mathbb{R}^m \quad , \quad \psi_e(z) = \psi(z) \quad \forall z \in K.$$

From this theorem, we deduce the following corollary :

COROLLARY 3.3. *Let \mathcal{A} be a bounded subset of \mathbb{R}^n , \mathcal{O}_e an open subset of \mathbb{R}^m containing $\mathcal{A} \times \{0\}$ and $\tau_e^* : \mathcal{O}_e \rightarrow \tau_e^*(\mathcal{O}_e)$ be a diffeomorphism satisfying (1.11) and such that either $\tau_e^*(\mathcal{O}_e)$ verifies condition \mathbb{B} or \mathcal{O}_e is C^2 -diffeomorphic to \mathbb{R}^m and τ_e^* is C^2 . Then, there exists $\bar{\tau}_e^* : \mathcal{O}_e \rightarrow \mathbb{R}^m$, such that $\bar{\tau}_e^*$ satisfies Properties P1 and P2.*

REMARK 4. *For \mathcal{O}_e to be diffeomorphic to \mathbb{R}^m , \mathcal{O}_e must be contractible.*

3.2. Proof of Theorem 3.2. The proof of Theorem 3.2 uses the following two technical lemmas :

LEMMA 3.4 (simplified). *Let E be an open strict subset of \mathbb{R}^m verifying \mathbb{B} . Then, for any compact set K in E , there exists a diffeomorphism $\phi : \mathbb{R}^m \rightarrow E$, such that ϕ is the identity function on K .*

See Appendix A for a proof with a more precise statement. In particular, the set K where ϕ is to be the identity function may not be compact. Besides, ϕ can be made as smooth as the vector field χ in condition \mathbb{B} . This proof is partly constructive as illustrated in the following example. See also Section 5.

EXAMPLE 5. Let M be an $m \times m$ matrix. Consider the function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ $F(\xi) = \frac{1}{2}\xi^T M \xi$ and let the sets E and K be

$$E = \{\xi \in \mathbb{R}^m, F(\xi)^2 < \delta\} \quad , \quad K = \{\xi \in \mathbb{R}^m, F(\xi)^2 = 0\}$$

where δ is some strictly positive real number. E verifies condition \mathbb{B} with

$$\kappa(\xi) = \left(\frac{1}{2}\xi^T M \xi\right)^2 - \delta \quad , \quad \chi(\xi) = -\xi \quad .$$

Indeed, the origin is in E and is asymptotically stable for χ . Moreover, on the boundary of E , where

$$\left(\frac{1}{2}\xi^T M \xi\right)^2 = \delta,$$

⁵If not replace χ by $\frac{\chi}{\sqrt{1+|\chi|^2}}$.

the transversality assumption is verified since we have

$$\frac{\partial \kappa}{\partial \xi}(\xi) \chi(\xi) = -\delta < 0.$$

To get a diffeomorphism, first, we define a "layer" close to the boundary of E as

$$\left\{ \xi \in \mathbb{R}^4, e^{-4\varepsilon} \delta < \left(\frac{1}{2} \xi^T M \xi \right)^2 < \delta \right\}.$$

It is the set of all the points reached at some time t in $(0, \varepsilon)$ by a solution of

$$\dot{\xi} = \chi(\xi) \tag{3.1}$$

issued from a point in the boundary of E . It is contained in E . We want the diffeomorphism to be the identity function on K . For this we consider the complementary set within E of the "layer"

$$E_\varepsilon = \left\{ \xi \in \mathbb{R}^m, \left(\frac{1}{2} \xi^T M \xi \right)^2 \leq e^{-4\varepsilon} \delta \right\}$$

It contains K . To construct the diffeomorphism, we associate to any point ξ in $\mathbb{R}^m \setminus E_\varepsilon$

$$t_\xi = \frac{1}{4} \ln \frac{\left(\frac{1}{2} \xi^T M \xi \right)^2}{\delta}.$$

t_x is the time a solution of (3.1) issued from ξ needs to reach the boundary of E . With this we define the diffeomorphism $\phi : \mathbb{R}^m \rightarrow E$ as :

$$\phi(\xi) = \begin{cases} \xi & , \text{ if } \left(\frac{1}{2} \xi^T M \xi \right)^2 \leq e^{-4\varepsilon} \delta, \\ e^{-t_\xi - \nu(t_\xi)} \xi & , \text{ otherwise,} \end{cases} \tag{3.2}$$

where $\nu : \mathbb{R} \rightarrow \mathbb{R}$ is any C^1 , strictly decreasing function satisfying :

$$\nu(t) = -t \quad \forall t \leq -\varepsilon \quad , \quad \lim_{t \rightarrow \infty} \nu(t) = 0.$$

For example it can be

$$\nu(t) = \frac{\varepsilon^2}{2\varepsilon + t} \quad \forall t \geq -\varepsilon. \tag{3.3}$$

△

LEMMA 3.5. Consider a C^2 diffeomorphism $\psi : \mathcal{B}_R(0) \rightarrow \psi(\mathcal{B}_R(0)) \subset \mathbb{R}^m$. For any ε strictly positive, there exists a diffeomorphism $\psi_\varepsilon : \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfying

$$\psi_\varepsilon(z) = \psi(z) \quad \forall z \in \mathcal{c}\mathcal{I}(\mathcal{B}_{R-\varepsilon}(0)).$$

As shown in Appendix B, this result is a direct consequence of [12, Theorem 8.1.4]. Unfortunately its proof is less constructive than for the previous case.

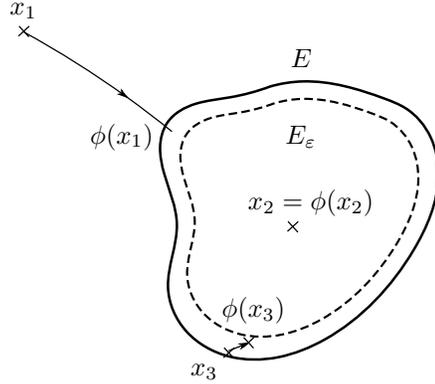


FIG. 3.1. Construction of the diffeomorphism in Example 5.

Proof. of Theorem 3.2.

First case: $\psi(\mathcal{D})$ satisfies \mathbb{B} : ψ being a diffeomorphism on an open set \mathcal{D} , the image of any compact subset K of \mathcal{D} is a compact subset of $\psi(\mathcal{D})$. According to Lemma 3.4, there exists a diffeomorphism ϕ from \mathbb{R}^m to $\psi(\mathcal{D})$ which is the identity on $\psi(K)$. Thus, the extension $\psi_e = \phi^{-1} \circ \psi$ satisfies the required properties.

Second case: \mathcal{D} is C^2 -diffeomorphic to \mathbb{R}^m and ψ is C^2 : Let $\phi_1 : \mathcal{D} \rightarrow \mathbb{R}^m$ denote the corresponding diffeomorphism. Let R_1 be a strictly positive real number such that the open ball $\mathcal{B}_{R_1}(0)$ contains $\phi_1(K)$. Let R_2 be a real number strictly larger than R_1 . With Lemma 3.4 again, and since $\mathcal{B}_{R_2}(0)$ verifies condition \mathbb{B} , there exists of C^2 -diffeomorphism $\phi_2 : \mathcal{B}_{R_2}(0) \rightarrow \mathbb{R}^m$ satisfying

$$\phi_2(z) = z \quad \forall z \in \mathcal{B}_{R_1}(0).$$

At this point, we have obtained a C^2 -diffeomorphism $\phi = \phi_2^{-1} \circ \phi_1 : \mathcal{D} \rightarrow \mathcal{B}_{R_2}(0)$. Consider $\lambda = \psi \circ \phi^{-1} : \mathcal{B}_{R_2}(0) \rightarrow \psi(\mathcal{D})$. According to Lemma 3.5, we can extend λ to $\lambda_e : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $\lambda_e = \psi \circ \phi^{-1}$ on $\mathcal{B}_{R_1}(0)$. Finally, consider $\psi_e = \lambda_e \circ \phi_1 : \mathcal{D} \rightarrow \mathbb{R}^m$. Since, by construction of ϕ_2 , $\phi = \phi_1$ on $\phi_1^{-1}(\mathcal{B}_{R_1}(0))$ which contains K , we have $\psi_e = \psi$ on K . \square

3.3. Relaxing condition \mathbb{B} . The representation of E in terms of the C^1 function κ in condition \mathbb{B} for Lemma 3.4 can be replaced by the following weaker condition \mathbb{B}^{rel} .

DEFINITION 3.6 (Relaxed condition \mathbb{B}^{rel}). *An open bounded subset E of \mathbb{R}^m is said to verify the relaxed condition \mathbb{B}^{rel} if there exist a bounded C^1 vector field χ , and a compact set K_0 contained in E such that:*

1. K_0 is globally asymptotically stable for χ
2. E is forward invariant by χ .

The following lemma shows that the relaxed condition \mathbb{B}^{rel} implies the condition \mathbb{B} up to a small deformation of E :

LEMMA 3.7. *Let E be an open bounded subset of \mathbb{R}^m verifying the relaxed condition \mathbb{B}^{rel} . For any strictly positive real number \bar{d} , there exists a bounded set \mathcal{E} such that*

$$cl(E) \subset \mathcal{E} \subset \{z \in \mathbb{R}^m, d(z, E) \in [0, \bar{d}]\}$$

and \mathcal{E} verifies condition \mathbb{B} .

Proof. The compact K_0 being globally asymptotically attractive and interior to E which is forward invariant, E is globally attractive. It is also stable due to the continuity of solutions with respect to initial conditions uniformly on compact time subsets of the domain of definition. So it is globally asymptotically stable. It follows from [22, Theorem 3.2] that there exist C^∞ functions $V_K : \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ and $V_E : \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ which are proper on \mathbb{R}^m and a class \mathcal{K}_∞ function α satisfying

$$\begin{aligned} \alpha(d(z, K_0)) &\leq V_K(z), V_K(z) = 0 \quad \forall z \in K_0, \\ \alpha(d(z, E)) &\leq V_E(z), V_E(z) = 0 \quad \forall z \in E, \\ \frac{\partial V_K}{\partial z}(z) \chi(z) &\leq -V_K(z) \quad \forall z \in \mathbb{R}^m \\ \frac{\partial V_E}{\partial z}(z) \chi(z) &\leq -V_E(z) \quad \forall z \in \mathbb{R}^m. \end{aligned}$$

With the notations $v_E = \sup_{z \in \mathbb{R}^m : d(z, E) \leq \bar{d}} V_K(z)$ and $\mu = \frac{\alpha(\bar{d})}{2v_E}$, and since α is of class \mathcal{K}_∞ , we obtain the implications

$$\begin{aligned} V_E(z) + \mu V_K(z) = \alpha(\bar{d}) &\Rightarrow \alpha(d(z, E)) \leq V_E(z) \leq \alpha(\bar{d}), \\ &\Rightarrow d(z, E) \leq \bar{d}, \\ &\Rightarrow V_K(z) \leq v_E. \end{aligned}$$

With our definition of μ , this yields also

$$\begin{aligned} \alpha(\bar{d}) - \mu V_K(z) = V_E(z) &\Rightarrow 0 < \frac{\alpha(\bar{d})}{2} \leq V_E(z), \\ &\Rightarrow 0 < d(z, E) \leq \bar{d}. \end{aligned}$$

On the other hand, with the compact notation $\mathcal{V}(z) = V_E(z) + \mu V_K(z)$, we have $\frac{\partial \mathcal{V}}{\partial z}(z) \chi(z) \leq -\mathcal{V}(z)$, for all $z \in \mathbb{R}^m$. All this implies that the sublevel set $\mathcal{E} = \{z \in \mathbb{R}^m : \mathcal{V}(z) < \alpha(\bar{d})\}$ is contained in $\{z \in \mathbb{R}^m : d(z, E) \in [0, \bar{d}]\}$ and that $\text{cl}(E)$ is contained in \mathcal{E} . Besides, \mathcal{E} verifies condition \mathbb{B} with the vector field χ and the function $\kappa = \mathcal{V} - \alpha(\bar{d})$. \square

Suppose we have a diffeomorphism $\tau_e^* : \mathcal{O}_e \rightarrow \tau_e^*(\mathcal{O}_e)$ where $\tau_e^*(\mathcal{O}_e)$ is a strict subset of \mathbb{R}^m . To satisfy property P2, we need to modify this diffeomorphism. With Corollary 3.3, Lemma 3.4 and Lemma 3.7, this is possible if $\tau_e^*(\mathcal{O}_e)$ verifies condition \mathbb{B} or \mathcal{O}_e is C^2 -diffeomorphic to \mathbb{R}^m and τ_e^* is C^2 . But for this, the given diffeomorphism τ_e^* needs to be defined on an outer approximation of \mathcal{O}_e . If this is not the case, we can take advantage of the following lemma :

LEMMA 3.8. *Let a diffeomorphism ψ be defined on an open set E in \mathbb{R}^p . Consider a bounded open set E' such that $\text{cl}(E') \subset E$. There exists a strictly positive real number d_0 such that for all x in E' ,*

$$d(\psi(x), \mathbb{R}^m \setminus \psi(E)) > d_0.$$

Proof. Suppose that, for each integer k , there exists x_k in E' such that

$$d(\psi(x_k), \mathbb{R}^m \setminus \psi(E)) \leq \frac{1}{k}.$$

By compactness, there exists x^* in $\text{cl}(E') \subset E$ such that $d(\psi(x^*), \mathbb{R}^m \setminus \psi(E)) = 0$. Thus, $\psi(x^*)$ is in $\text{cl}(\mathbb{R}^m \setminus \psi(E)) \cap \psi(E)$. But since E is open, $\psi(E)$ is also open by Brouwer's invariance theorem. Thus, $\text{cl}(\mathbb{R}^m \setminus \psi(E)) \cap \psi(E)$ is the empty set and we have a contradiction. \square

To round the difficulty mentioned before this Lemma, we consider a bounded open set \mathcal{O}'_e satisfying $\mathcal{A} \times \{0\} \subset \mathbf{c1}(\mathcal{O}'_e) \subset \mathcal{O}_e$. According to Lemma 3.8, there exists d_0 such that,

$$\begin{aligned} d((x, w), \mathbb{R}^m \setminus (\mathcal{O}_e)) &> d_0, & \forall (x, w) \in \mathcal{O}'_e, \\ d(\tau_e^*(x, w), (\tau_e^*(\mathbb{R}^m \setminus (\mathcal{O}_e)))) &> d_0, & \forall (x, w) \in \mathcal{O}'_e. \end{aligned}$$

Thus, if $\tau_e^*(\mathcal{O}_e)$ verifies condition \mathbb{B} or \mathcal{O}_e is C^2 -diffeomorphic to \mathbb{R}^m and τ_e^* is C^2 , by applying Lemma 3.7 with $\bar{d} \leq d_0$ to $\tau_e^*(\mathcal{O}_e)$ or to \mathcal{O}_e , we get the existence of an open bounded set \mathcal{O}_e^1 verifying $\mathcal{A} \times \{0\} \subset \mathbf{c1}(\mathcal{O}_e^1) \subset \mathcal{O}_e^1 \subset \mathcal{O}_e$ and such that $\tau_e^*(\mathcal{O}_e^1)$ verifies condition \mathbb{B} or \mathcal{O}_e^1 is C^2 -diffeomorphic to \mathbb{R}^m . We are now able to apply Corollary 3.3, and obtain a diffeomorphism $\overline{\tau}_e^* : \mathcal{O}_e^1 \rightarrow \mathbb{R}^m$ the image of which is \mathbb{R}^m .

3.4. Diffeomorphism extension from a submersion. As in Section 2.4, we consider now the particular case where there exists a full-rank C^1 function $F : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$ such that

$$F(\tau^*(x)) = 0 \quad \forall x \in \mathcal{S}_x.$$

Thanks to Propositions 2.1 and 2.2, for ϵ sufficiently small, the function

$$(x, w) \mapsto \tau_e^*(x, w) = \tau^*(x) + \frac{\partial F}{\partial \xi}(\tau^*(x))^T w \quad (3.4)$$

is a diffeomorphism on $\mathcal{S}_x \times \mathcal{B}_\epsilon(0)$ but the set $\tau_e^*(\mathcal{S}_x \times \mathcal{B}_\epsilon(0))$ may not be the entire \mathbb{R}^m as required for P2 to hold.

PROPOSITION 3.9. *Assume the existence of a full-rank C^1 function $F : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$ such that the set*

$$\{\xi \in \mathbb{R}^m : |F(\xi)| \in [0, 1]\} \quad (3.5)$$

is compact and we have

$$\tau^*(\mathcal{S}_x) = \{\xi \in \mathbb{R}^m, F(\xi) = 0\}. \quad (3.6)$$

Let also ϵ be such that the function defined in (3.4) is a diffeomorphism on $\mathcal{S}_x \times \mathcal{B}_\epsilon(0)$. Under these conditions, there exists a strictly positive real number δ such that the set

$$E^\delta = \{\xi \in \mathbb{R}^m, |F(\xi)| < \delta\}$$

satisfies

$$\tau^*(\mathcal{S}_x) \subseteq E^\delta \subseteq \tau_e^*(\mathcal{S}_x \times \mathcal{B}_\epsilon(0)). \quad (3.7)$$

Moreover if F is C^2 on \mathbb{R}^m , then E^δ verifies condition \mathbb{B} .

Proof. We start by showing the existence of a strictly positive real number r such that

$$\{\xi \in \mathbb{R}^m : d(\xi, \tau^*(\mathcal{S}_x)) \leq r\} \subset \tau_e^*(\mathcal{S}_x \times \mathcal{B}_\epsilon(0)). \quad (3.8)$$

We know that for all ξ_0 in $\tau^*(\mathcal{S}_x)$, there exists x_0 in \mathcal{S}_x satisfying $\xi_0 = \tau_e^*(x_0, 0)$. Moreover, the Jacobian of τ_e^* at $(x_0, 0)$ is invertible. It follows from the Implicit function Theorem that, for each ξ_0 in $\tau^*(\mathcal{S}_x)$, there exists a strictly positive real number r_ξ such that, for all ξ in $\tau^*(\mathcal{S}_x)$, satisfying

$|\xi - \xi_0| \leq r_\xi$, there exists (x, w) in $\mathcal{S}_x \times \mathcal{B}_\varepsilon(0)$ satisfying $\tau_e^*(x, w) = \xi$. Existence of r follows from compactness of $\text{cl}(\mathcal{S}_x)$ and therefore of $\text{cl}(\tau^*(\mathcal{S}_x))$.

Secondly, there exists $\delta > 0$ such that E^δ is contained in $\{\xi \in \mathbb{R}^m : d(\xi, \tau^*(\mathcal{S}_x)) \leq r\}$. Indeed, assume this is not the case. Then, for each integer k , there exists ξ_k satisfying

$$|F(\xi_k)| \leq \frac{1}{k} \quad , \quad d(\xi_k, \tau^*(\mathcal{S}_x)) > r.$$

The sequence being in the compact set defined in (3.5), there is a subsequence which converges to ξ^* satisfying

$$F(\xi^*) = 0 \quad , \quad d(\xi^*, \tau^*(\mathcal{S}_x)) \geq r.$$

This contradicts (3.6). Then (3.7) follows from (3.6) and (3.8).

Finally, for the last point, we note that, with χ the C^1 vector field defined by

$$\chi(\xi) = -\frac{\partial F}{\partial \xi}(\xi)^T \left(\frac{\partial F}{\partial \xi}(\xi) \frac{\partial F}{\partial \xi}(\xi)^T \right)^{-1} F(\xi),$$

we have, for all ξ in \mathbb{R}^m , $L_\chi(F^\top F)(\xi) = -F(\xi)^\top F(\xi)$. This implies that the closed set $\tau^*(\mathcal{S}_x)$ is globally attractive for χ and that the transversality assumption is verified. Thus, E^δ satisfies \mathbb{B} . \square

The example below illustrates the fact that, if we have a submersion F giving an (outer) approximation only of $\tau_e^*(\mathcal{S}_x \times \mathcal{S}_w)$ we can still apply the diffeomorphism image extension of Theorem 3.2.

EXAMPLE 6 (Continuation of Example 2). In Example 2, we have introduced the function

$$F(\xi) = \xi_2 \xi_3 - \xi_1 \xi_4 \triangleq \frac{1}{2} \xi^T M \xi$$

as a submersion on $\mathbb{R}^4 \setminus \{0\}$ satisfying

$$F(\tau^*(x)) = 0. \tag{3.9}$$

With it we have ‘‘augmented’’ the injective immersion τ^* given in (1.3) as

$$\tau_e^*(x, w) = \tau^*(x) + \frac{\partial F^T}{\partial \xi}(\tau^*(x)) w = \tau^*(x) + M \tau^*(x) w$$

which is a diffeomorphism on $\mathcal{O}_\varepsilon = \mathcal{S}_x \times]-\varepsilon_\diamond, -\varepsilon_\diamond[$ for some strictly positive real number $-\varepsilon_\diamond$. To satisfy $P2$ we need an image extension.

It is difficult to obtain an exact representation of the image $\tau_e^*(\mathcal{O}_\varepsilon)$, but we can use F to obtain an approximation. Indeed, with (3.9), we obtain $F(\tau_e^*(x, w)) = |\tau^*(x)|^2 w$ or equivalently $w = \frac{F(\tau_e^*(x, w))}{|\tau^*(x)|^2}$, for $|\tau^*(x)|$ non zero, i.e. away from the observability singularity $x_1 = x_2 = 0$. Hence ensuring that $\xi = \tau_e^*(x, w)$ remains in

$$E^\delta = \{\xi \in \mathbb{R}^m, F(\xi)^2 < \delta\}$$

ensures that w remains small unless x is close to the observability singularity.

In Example 5, we have obtained in (3.2) an expression of a diffeomorphism $\phi : \mathbb{R}^4 \rightarrow E^\delta$ which is the identity on

$$E_\varepsilon^\delta = \left\{ \xi \in \mathbb{R}^4, \left(\frac{1}{2} \xi^T M \xi \right)^2 \leq e^{-4\varepsilon} \delta \right\}.$$

With it, we can replace $\tau_e^* : \mathcal{O}_e \rightarrow \tau_e^*(\mathcal{O}_e)$ by $\phi^{-1} \circ \tau_e^*$, ensuring that, for any (\hat{x}, \hat{w}) in \mathbb{R}^4 , $\tau_e^*(\hat{x}, \hat{w})$ is in E^δ for δ sufficiently small. In doing so, the domain of non invertibility of the Jacobian has been reduced. But we still have a problem for (\hat{x}_1, \hat{x}_2) close to the origin where the observability singularity occurs. \triangle

4. Modifying A.II and A.C to allow P1 and P2 to be satisfied. Throughout Sections 2 and 3, we have studied tools which enable us to fulfill conditions P1 and P2 of Proposition 1.1. However, they need conditions on the dimensions or on the domain of injectivity \mathcal{O} which are not always satisfied : contractibility for Jacobian extension and diffeomorphism extension, limited number of pairs (m, n) for the $P[m, n]$ problem, etc. Expressed in terms of our initial problem, these conditions are limitations on the data f, h and τ^* we considered. In the following, we show by means of examples that, in some cases, these data can be modified in such a way that our various tools apply and give a satisfactory solution. Such modifications are possible since we restrict our attention to system solutions which remain in \mathcal{A} . Therefore we can modify arbitrarily the data f, h and τ^* outside this set. For example we can add arbitrary “fictitious” components to the measured output y as long as they are constant on \mathcal{A} .

4.1. For contractibility. It may happen that the set \mathcal{O} attached to τ^* is not contractible, for example due to an observability singularity. We have seen Jacobian completion and diffeomorphism image extension may be prevented by this. A possible approach to round this problem when we know the system solutions stay away from singularities is to add a fictitious output and to modify Assumptions A.II and A.C.

EXAMPLE 7 (Continuation of Example 3). The observer we have obtained at the end of Example 3 for the harmonic oscillator with unknown frequency is not satisfactory in particular because of the singularity at $\hat{x}_1 = \hat{x}_2 = 0$. To round this difficulty we add, to the given measurement $y = x_1$, the following “fictitious” one :

$$y_2 = h_2(x) = \wp(x_1, x_2) [x_3 - \varsigma]$$

with ς in $]0, r[$ and $\wp(x_1, x_2) = \max \left\{ 0, \frac{1}{r^2} - (x_1^2 + x_2^2) \right\}^4$. By construction this measurement is zero on \mathcal{A} meaning that we do not change the data on this set. The interest of y_2 is to give us access to x_3 directly. Indeed, consider the new function τ^* defined as

$$\tau^*(x) = (x_1, x_2, -x_1 x_3, -x_2 x_3, \wp(x_1, x_2) [x_3 - \varsigma]) . \quad (4.1)$$

τ^* is C^1 on \mathbb{R}^3 and its Jacobian is :

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \\ \frac{\partial \wp}{\partial x_1} [x_3 - \varsigma] & \frac{\partial \wp}{\partial x_2} [x_3 - \varsigma] & \wp \end{pmatrix}, \quad (4.2)$$

which has full rank 3 on \mathbb{R}^3 , since $\wp(x_1, x_2) \neq 0$ when $x_1 = x_2 = 0$. It follows that the singularity has disappeared and this new τ^* is an injective immersion on the entire \mathbb{R}^3 which is contractible.

We have shown in Example 4 how Wazewski's algorithm allows us to get in this case a C^2 function $\gamma : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ satisfying :

$$\det \left(\frac{\partial \tau^*}{\partial x}(x) \gamma(x) \right) = 0 \quad \forall x \in \mathbb{R}^3 .$$

This gives us $\tau_e^*(x, w) = \tau^*(x) + \gamma(x)w$ which is a C^2 -diffeomorphism on $\mathbb{R}^3 \times \mathcal{B}_\varepsilon(0)$, with ε sufficiently small.

Furthermore, $\mathcal{O}_e = \mathbb{R}^3 \times \mathcal{B}_\varepsilon(0)$ being diffeomorphic to \mathbb{R}^5 , Corollary 3.3 applies and provides a modification $\overline{\tau}_e^*$ of τ_e^* satisfying P1 and P2. \triangle

4.2. For a solvable $P[m, n]$ problem. As already seen in Example 3, it is appropriate to exploit the fact that some rows of $\frac{\partial \tau^*}{\partial x}$, say p of them, are independent for all x in \mathcal{A} . Indeed this allows to transform the Jacobian completion problem into a $P[m - p, n - p]$ problem. If the new pair $(m - p, n - p)$ is not in table (2.7) but $(m - p + k, n - p)$, we can still add k arbitrary rows to $\frac{\partial \tau^*}{\partial x}$.

EXAMPLE 8 (Continuation of Example 7). In Example 7, by adding the fictitious measured output $y_2 = h_2(x)$, we have obtained another function τ^* for the harmonic oscillator with unknown frequency which is an injective immersion on \mathbb{R}^3 . In this case, we have $n = 3$ and $m = 5$ which gives a pair not in table (2.7). But, as already exploited in Example 3, the first 2 rows of the Jacobian $\frac{\partial \tau^*}{\partial x}$ in (4.2) are independent for all x in \mathbb{R}^3 . It follows that our Jacobian completion problem reduces to complement the vector

$$(-x_1, -x_2, \wp(x_1, x_2))$$

This is a $P[3, 1]$ problem which is not in the table (2.7). Instead, the pair (4, 1) is in the table, meaning that the following vector can be completed via a universal formula

$$(-x_1, -x_2, \wp(x_1, x_2), 0) .$$

We have added a zero component, without changing the full rank property. Actually this vector is extracted from the Jacobian of

$$\tau^*(x) = (x_1, x_2, -x_1x_3, -x_2x_3, \wp(x_1, x_2)[x_3 - \varsigma], 0) \quad (4.3)$$

where the added last component at 0 is consistent with the high gain observer paradigm, when we add another fictitious measured output

$$y_3 = 0 .$$

A complement is

$$\begin{pmatrix} x_2 & -\wp & 0 \\ -x_1 & 0 & -\wp \\ 0 & -x_1 & -x_2 \\ \wp & x_2 & -x_1 \end{pmatrix}$$

It gives the diffeomorphism

$$\tau_e^*(x, w) = \left(x_1, x_2, [-x_1x_3 + x_2w_1 - \wp(x_1, x_2)w_2], [-x_2x_3 - x_1w_1 - \wp(x_1, x_2)w_3], \right. \\ \left. [\wp(x_1, x_2)[x_3 - \varsigma] - x_1w_2 - x_2w_3], [\wp(x_1, x_2)w_1 + x_2w_2 - x_1w_3] \right).$$

the Jacobian of which is :

$$\frac{\partial \tau_e^*}{\partial x}(x, w) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -x_3 - \frac{\partial \wp}{\partial x_1} w_2 & w_1 - \frac{\partial \wp}{\partial x_2} w_2 & -x_1 & x_2 & -\wp & 0 \\ -w_1 - \frac{\partial \wp}{\partial x_1} w_3 & -x_3 - \frac{\partial \wp}{\partial x_2} w_3 & -x_2 & -x_1 & 0 & -\wp \\ \frac{\partial \wp}{\partial x_1} [x_3 - \varsigma] - w_2 & \frac{\partial \wp}{\partial x_2} [x_3 - \varsigma] - w_3 & \wp & 0 & -x_1 & -x_2 \\ \frac{\partial \wp}{\partial x_1} w_1 - w_3 & \frac{\partial \wp}{\partial x_2} w_1 + w_2 & 0 & \wp & x_2 & -x_1 \end{pmatrix}$$

with determinant

$$(x_1^2 + x_2^2 + \wp(x_1, x_2)^2)^2$$

which is nowhere 0 on \mathbb{R}^6 . So τ_e^* is a diffeomorphism defined on \mathbb{R}^6 which is also surjective since it is proper (Hadamard-Levy Theorem) thanks to

$$x_1 = \hat{\xi}_1, \quad x_2 = \hat{\xi}_2,$$

$$\begin{pmatrix} \hat{\xi}_1 & \hat{\xi}_2 & -\wp(\hat{\xi}_1, \hat{\xi}_2) & 0 \\ -\hat{\xi}_2 & -\hat{\xi}_1 & 0 & -\wp(\hat{\xi}_1, \hat{\xi}_2) \\ \wp(\hat{\xi}_1, \hat{\xi}_2) & 0 & -\hat{\xi}_1 & -\hat{\xi}_2 \\ 0 & \wp(\hat{\xi}_1, \hat{\xi}_2) & \hat{\xi}_2 & -\hat{\xi}_1 \end{pmatrix} \begin{pmatrix} x_3 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \hat{\xi}_3 \\ \hat{\xi}_4 \\ \hat{\xi}_5 \\ \hat{\xi}_6 \end{pmatrix}.$$

So, with the addition of the fictitious measured outputs $y_2 = h_2(x)$ and $y_3 = 0$, we have obtained the new function τ^* , given in (4.3), to be used, in place of (1.3), as another starting point for the construction of an observer and in particular for the construction of a function τ_e^* satisfying the properties P1 and P2 of Proposition 1.1. Also to obtain a set $\varphi\mathcal{T}$ satisfying the property P3, we can follow the high gain observer paradigm and complete the expression of φ in (1.2) taking advantage of the fact that, for x in \mathcal{A} , we have

$$\dot{y}_2 = \overbrace{\wp(x_1, x_2)[x_3 - \varsigma]} = 0, \quad \dot{y}_3 = 0$$

This motivates the new function φ

$$\varphi(\hat{\xi}, \hat{x}, y) = \begin{pmatrix} \hat{\xi}_2 + \ell k_1(y - \hat{x}_1) \\ \hat{\xi}_3 + \ell^2 k_2(y - \hat{x}_1) \\ \hat{\xi}_4 + \ell^3 k_3(y - \hat{x}_1) \\ \mathbf{sat}(\hat{x}_1 \hat{x}_3^2) + \ell^4 k_4(y - \hat{x}_1) \\ -a \hat{\xi}_5 \\ -b \hat{\xi}_6 \end{pmatrix}.$$

where the function \mathbf{sat} is defined in (1.9) and a and b are arbitrary strictly positive real numbers. With picking ℓ large enough, it can be paired with any function $\tau : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ which is locally

Lipschitz. With all this, we have obtained the following observer for the harmonic oscillator with unknown frequency

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{w}}_1 \\ \dot{\hat{w}}_2 \\ \dot{\hat{w}}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\hat{x}_3 - \frac{\partial \varphi}{\partial \hat{x}_1} \hat{w}_2 & \hat{w}_1 - \frac{\partial \varphi}{\partial x_2} \hat{w}_2 & -\hat{x}_1 & \hat{x}_2 & -\varphi & 0 \\ -\hat{w}_1 - \frac{\partial \varphi}{\partial \hat{x}_1} \hat{w}_3 & -\hat{x}_3 - \frac{\partial \varphi}{\partial x_2} \hat{w}_3 & -\hat{x}_2 & -\hat{x}_1 & 0 & -\varphi \\ \frac{\partial \varphi}{\partial x_1} [\hat{x}_3 - \varsigma] - \hat{w}_2 & \frac{\partial \varphi}{\partial x_2} [\hat{x}_3 - \varsigma] - \hat{w}_3 & \varphi & 0 & -\hat{x}_1 & -\hat{x}_2 \\ \frac{\partial \varphi}{\partial x_1} \hat{w}_1 - \hat{w}_3 & \frac{\partial \varphi}{\partial x_2} \hat{w}_1 + \hat{w}_2 & 0 & \varphi & \hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1} \times \begin{pmatrix} \hat{x}_2 + \ell k_1 (y - \hat{x}_1) \\ [-\hat{x}_1 \hat{x}_3 + \hat{x}_2 \hat{w}_1 - \varphi(\hat{x}_1, \hat{x}_2) \hat{w}_2] + \ell^2 k_2 (y - \hat{x}_1) \\ [-\hat{x}_2 \hat{x}_3 - \hat{x}_1 \hat{w}_1 - \varphi(\hat{x}_1, \hat{x}_2) \hat{w}_3] + \ell^3 k_3 (y - \hat{x}_1) \\ \mathbf{sat}(\hat{x}_1 \hat{x}_3^2) + \ell^4 k_4 (y - \hat{x}_1) \\ -a [\varphi(\hat{x}_1, \hat{x}_2) [\hat{x}_3 - \varsigma] - \hat{x}_1 \hat{w}_2 - \hat{x}_2 \hat{w}_3] \\ -b [\varphi(\hat{x}_1, \hat{x}_2) \hat{w}_1 + \hat{x}_2 \hat{w}_2 - \hat{x}_1 \hat{w}_3] \end{pmatrix}. \quad (4.4)$$

It is globally defined and globally convergent for any solution of the oscillator initialized in

$$\mathcal{A} = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in]\frac{1}{r}, r[, x_3 \in]0, r[\} .$$

As a final remark, we observe that the manifold $\hat{\xi}_5 = \hat{\xi}_6 = 0$ is invariant. This implies the existence of an observer with order reduced to 4. Unfortunately it cannot be expressed with coordinates $\mathbf{c} = (x_1, x_2, x_3, \bar{w})$ since the following Jacobian is singular for $\xi_1 = \xi_2 = 0$:

$$\frac{\partial \mathbf{c}}{\partial \xi_1, \dots, \xi_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & -\frac{\xi_1}{\xi_1^2 + \xi_2^2 + \varphi(\xi_1, \xi_2)^2} & -\frac{\xi_2}{\xi_1^2 + \xi_2^2 + \varphi(\xi_1, \xi_2)^2} \\ * & * & * & * \end{pmatrix}$$

where * denotes an unimportant term.

The observer (4.4) is just an illustration of what can be obtained by using in a very nominal way our tools. We do not claim any property for it. For example, by using another design, an observer of dimension 2, globally convergent on \mathcal{A} , can be obtained. \triangle

5. Example of the bioreactor. As another illustration we consider the model of bioreactor presented in [8] and modeled by the dynamics :

$$\dot{x}_1 = \frac{a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_1 , \quad \dot{x}_2 = -\frac{a_3 a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_2 + u a_4 , \quad y = x_1$$

where the a_i 's are strictly positive real numbers and the control u verifies : $0 < u_{min} < u(t) < u_{max} < a_1$. This system evolves in the set $\mathcal{O} = \{x \in \mathbb{R}^2 : x_1 > \varepsilon_1 , x_2 > -a_2 x_1\}$ which is forward invariant.

A high gain observer design leads us to consider the function $\tau^* : \mathcal{O} \rightarrow \mathbb{R}^2$ defined as :

$$\tau^*(x_1, x_2) = \left(x_1, \frac{a_1 x_1 x_2}{a_2 x_1 + x_2} \right) .$$

It is a diffeomorphism onto

$$\tau^*(\mathcal{O}) = \{\xi \in \mathbb{R}^2 : \xi_1 > 0, a_1\xi_1 > \xi_2\} .$$

The image by τ^* of the bioreactor dynamics is of the form

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 + g_1(\xi_1)u \\ \dot{\xi}_2 &= \varphi_m(\xi_1, \xi_2) + g_2(\xi_1, \xi_2)u\end{aligned}$$

for which the following high gain observer can be built (see [8]):

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, u) + S_\infty^{-1}C^T(y - C\hat{\xi}) \quad (5.1)$$

where $C = (1, 0)^T$ and S_∞ is solution of $A^T S_\infty + S_\infty A - C^T C = -\ell S_\infty$ for ℓ sufficiently large. Since τ^* is a diffeomorphism, the expression of this observer in the x -coordinates is :

$$\dot{\hat{x}} = \begin{pmatrix} \frac{a_1 \hat{x}_1 \hat{x}_2}{a_2 \hat{x}_1 + \hat{x}_2} - u \hat{x}_1 \\ -\frac{a_3 a_1 \hat{x}_1 \hat{x}_2}{a_2 \hat{x}_1 + \hat{x}_2} - u \hat{x}_2 + u a_4 \end{pmatrix} + \begin{pmatrix} a_1 a_2 \hat{x}_1^2 & 0 \\ -a_1 a_2 \hat{x}_1^2 & (a_2 \hat{x}_1 + \hat{x}_2)^2 \end{pmatrix} \frac{S_\infty^{-1} C^T (y - \hat{x}_1)}{a_1 a_2 \hat{x}_1^2} . \quad (5.2)$$

Unfortunately the right hand side has singularities at $\hat{x}_1 = 0$ and $\hat{x}_2 = -a_1 \hat{x}_1$ and nothing guarantees that \mathcal{O} is still invariant for the observer dynamics (5.2). To round this difficulty, we can modify τ^* to make its image equal to \mathbb{R}^2 , while keeping it unchanged inside a subset of \mathcal{O} , arbitrarily close to \mathcal{O} .

To construct this modification, we view the image $\tau^*(\mathcal{O})$ as the intersection $\tau^*(\mathcal{O}) = E_1 \cap E_2$ with :

$$E_1 = \{(\xi_1, \xi_2) \in \mathbb{R}^2, \xi_1 > \varepsilon_1\} \quad , \quad E_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2, a_1 \xi_1 > \xi_2\} .$$

This exhibits the fact that $\tau^*(\mathcal{O})$ does not satisfy the condition \mathbb{B} since its boundary is not C^1 . We could smoothen this boundary to remove its ‘‘corner’’. But we prefer to exploit its particular ‘‘shape’’ and proceed as follows :

- first, we build a diffeomorphism $\phi_1 : \mathbb{R}^2 \rightarrow E_1$ which acts on ξ_1 without changing ξ_2 .
- then, we build a diffeomorphism $\phi_2 : \mathbb{R}^2 \rightarrow E_2$ which acts on ξ_2 without changing ξ_1 .
- finally, we take $\bar{\tau}^* = \phi_1^{-1} \circ \phi_2^{-1} \circ \tau^* : \mathcal{O} \rightarrow \mathbb{R}^2$.

To build ϕ_1 and ϕ_2 , we follow Lemma 3.4 since E_1 and E_2 satisfy condition \mathbb{B} with :

$$\kappa_1(\xi) = \varepsilon_1 - \xi_1 \quad , \quad \kappa_2(\xi) = \xi_2 - a_1 \xi_1 \quad , \quad \chi_1(\xi) = \begin{pmatrix} -(\xi_1 - 1) \\ 0 \end{pmatrix} \quad , \quad \chi_2(\xi) = \begin{pmatrix} 0 \\ -(\xi_2 + 1) \end{pmatrix} .$$

By following the same steps as in Example 6, with ε an arbitrary small strictly positive real number and ν defined in (3.3), we obtain :

$$\begin{aligned}t_{\xi,1} &= \ln \frac{1-\xi_1}{1-\varepsilon} \\ E_{\varepsilon,1} &= \{(\xi_1, \xi_2) \in \mathbb{R}^2, \xi_1 > 1 - (1 - \varepsilon)e^{-\varepsilon}\} \\ \phi_1(\xi) &= \begin{cases} \xi & , \text{ if } \xi \in E_{\varepsilon,1} \\ e^{-t_{\xi,1} - \nu(t_{\xi,1})}(\xi_1 - 1) + 1 & , \text{ otherwise} \end{cases} \\ \hline t_{\xi,2} &= \ln \frac{\xi_2 + 1}{a_1 \xi_1 + 1} \quad , \\ E_{\varepsilon,2} &= \{(\xi_1, \xi_2) \in \mathbb{R}^2, \xi_2 \leq e^{-\varepsilon}(a_1 \xi_1 + 1) - 1\} \\ \phi_2(\xi) &= \begin{cases} \xi & , \text{ if } \xi \in E_{\varepsilon,2} \\ e^{-t_{\xi,2} - \nu(t_{\xi,2})}(\xi_2 + 1) - 1 & , \text{ otherwise} \end{cases}\end{aligned} \quad (5.3)$$

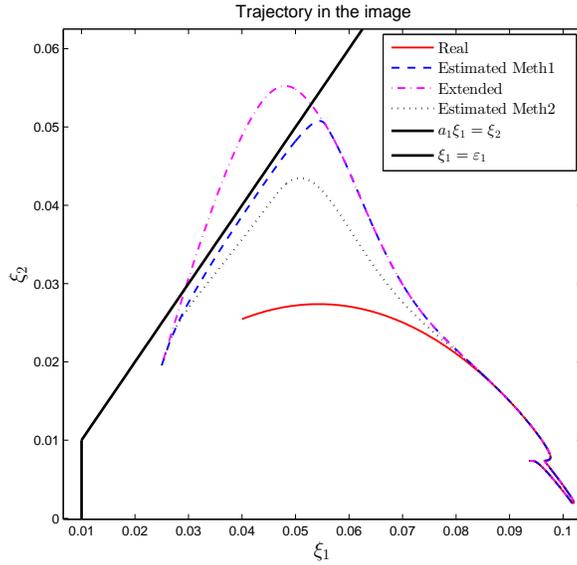


FIG. 5.1. *Bioreactor and observers solutions in the $\hat{\xi}$ -coordinates*

We remind the reader that, in the $\hat{\xi}$ -coordinates, the observer dynamics are the same let it be built with τ^* or with $\overline{\tau^*}$. The difference is only in the way \hat{x} is related to $\hat{\xi}$. Moreover this “way” is such that it has no effect on the system solution we have

$$\tau^*(x) = \overline{\tau^*}(x) \quad \forall x \in \mathcal{O} \text{ “} - \varepsilon \text{”} .$$

As a consequence the difference between τ^* and $\overline{\tau^*}$ acts only during the transient, making sure that \hat{x} never reaches a singularity of the Jacobian of $\overline{\tau^*}$.

We present in Figure 5.1 the results in the $\hat{\xi}$ coordinates (to allow us to see the effects of both τ^* and $\overline{\tau^*}$) of a simulation with (similar to [8]) :

$$\begin{aligned} a_1 = a_2 = a_3 = 1, \quad a_4 = 0.1 \\ u(t) = 0.08 \text{ for } t \leq 10, \quad = 0.02 \text{ for } 10 \leq t \leq 20, \quad = 0.08 \text{ for } t \geq 20 \\ x(0) = (0.04, 0.07), \quad \hat{x}(0) = (0.03, 0.09), \quad \ell = 5. \end{aligned}$$

The solid black curves are the singularities locus. The red (= solid dark) curve represents the bioreactor solution. The magenta (= light grey dashdot) curve represents the solution of the observer built with $\overline{\tau^*}$. It evolves freely in \mathbb{R}^2 according to the dynamics (5.1), not worried by any constraints. The blue (= dark dashed) curve represents its image by ϕ^{-1} which brings it back inside the constrained domain where $\tau = \tau^{*-1}$ can be used. This means these two curves represent the same object but viewed via different coordinates.

The solution of the observer built with τ^* coincides with the magenta (= light grey dashdot) curve up to the point it reaches one solid black curve of a singularity locus. At that point it leaves $\tau^*(\mathcal{O})$ and consequently stops to exist in the \hat{x} -coordinates.

As proposed in [17, 3], instead of keeping the raw dynamics (5.1) untouched as above, we can modify them to force $\hat{\xi}$ to remain in the set $\tau^*(\mathcal{O})$. Taking advantage of the fact that this set is convex, the modification proposed in [3] consists in adding to (5.1) the term

$$\mathcal{M}(\hat{\xi}) = -\gamma S_\infty^{-1} \frac{\partial \mathfrak{h}}{\partial \hat{\xi}}(\hat{\xi})^T \mathfrak{h}(\hat{\xi}) \quad (5.4)$$

where $\mathfrak{h}(\hat{\xi}) = \begin{pmatrix} \max\{\kappa_1(\hat{\xi}) + \delta, 0\}^2 \\ \max\{\kappa_2(\hat{\xi}) + \delta, 0\}^2 \end{pmatrix}$ with δ an arbitrary small real number and γ a sufficiently large real number. The solution corresponding to this modified observer dynamics is shown in Figure 5.1 with the dotted black curve. As expected it stays away from the singularities locus in a very efficient way. But for this method to apply we have the restriction that $\tau^*(\mathcal{O})$ should be convex, instead of satisfying the less restrictive condition \mathbb{B} . Moreover, to guarantee that $\hat{\xi}$ is in $\tau^*(\mathcal{O})$, γ has to be large enough and even larger when the measurement noise is larger. On the contrary, when the observer is built with $\overline{\tau^*}$, there is no need to tune properly any parameter to obtain convergence, at least theoretically. Nevertheless there maybe some numerical problems when $\hat{\xi}$ becomes too large or equivalently $\phi^{-1}(\hat{\xi})$ is too close to the boundary of $\tau^*(\mathcal{O})$. To round this problem we can select the "thickness" of the layer (parameter ε in (5.3)) sufficiently large. Actually instead of "opposing" the two methods, we suggest to combine them. The modification (5.4) makes sure $\hat{\xi}$ does not go too far, and $\overline{\tau^*}$ makes sure that \hat{x} stays away from the singularities locus.

6. Conclusion. We have presented a method to express the dynamics of an observer in the given system original coordinates enlarging its domain of validity and possibly avoiding the difficult left-inversion of an injective immersion.

It relies on the assumption that we know an injective immersion and a converging observer for the immersed system. The idea is not to modify this observer dynamics but to map it back to the original coordinates in a different way. Our construction involves two tools : the extension of an injective immersion into a diffeomorphism through Jacobian completion and the extension of the image of the obtained diffeomorphism to enlarge the domain of validity.

For Jacobian completion, we are relying on the results of Wazewski [21] and Eckmann [7]. It allows us to build a diffeomorphism by complementing the given coordinates with new ones and to write the given observer dynamics in these augmented coordinates.

For diffeomorphism extension, we have proposed our own method widely inspired from [12, Chapter 8] and [18, pages 2, 7 to 14 and 16 to 18].

In our presentation we have assumed the system is time-invariant and autonomous. Adding time-variations is not a problem but dealing with exogenous inputs is more complex. This is in part due to the fact that, as far as we know, the theory of observers, in presence of such inputs, relying on immersion into a space of larger dimension, as high gain observers or nonlinear Luenberger observers, is not available yet. Progress on this topic has to be made before trying to extend our results.

Appendix A. Construction of a diffeomorphism from an open set to \mathbb{R}^m . In this section we prove a result slightly more general than Lemma 3.4. For this we use the following notations:

The complementary, closure and boundary of a set S are denoted S^c , $\text{cl}(S)$ and ∂S , respectively. The Hausdorff distance d_H between two sets A and B is defined by :

$$d_H(A, B) = \max \left\{ \sup_{z_A \in A} \inf_{z_B \in B} |z_A - z_B|, \sup_{z_B \in B} \inf_{z_A \in A} |z_A - z_B| \right\}$$

With $Z(z, t)$ we denote the (unique) solution, at time t , to $\dot{z} = \chi(z)$ going through x at time 0.

LEMMA A.1. *Let E be an open strict subset of \mathbb{R}^m verifying \mathbb{B} , with a C^s vector field χ . Then, for any strictly positive real number ε , there exists a C^s -diffeomorphism $\phi: \mathbb{R}^m \rightarrow E$, such that, with*

$$\Sigma = \bigcup_{t \in [0, \varepsilon]} Z(\partial E, t) ,$$

we have $\phi(z) = z$ for all $z \in E_\varepsilon = E \cap \Sigma^c$ and $d_H(\partial E_\varepsilon, \partial E) \leq \varepsilon \sup_z |\chi(z)|$.

Proof. We start by establishing some properties.

- E is forward invariant by χ . This is a direct consequence of points 1 and 3 of the condition \mathbb{B} .
- Σ is closed. Take a sequence (z_k) of points in Σ converging to z^* . By definition, there exists a sequence (t_k) , such that :

$$t_k \in [0, \varepsilon] \quad \text{and} \quad Z(z_k, -t_k) \in \partial E \quad \forall k \in \mathbb{N} .$$

Since $[0, \varepsilon]$ is compact, one can extract a subsequence $(t_{\sigma(k)})$ converging to t^* in $[0, \varepsilon]$, and by continuity of the function $(z, t) \mapsto Z(z, -t)$, $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$ tends to $Z(z^*, -t^*)$ which is in ∂E , since ∂E is closed. Finally, because t^* is in $[0, \varepsilon]$, z^* is in Σ by definition.

- Σ is contained in $\text{cl}(E)$. Since, E is forward invariant by χ , and so is $\text{cl}(E)$ (see [11, Theorem 16.3]). This implies

$$\partial E \subset \Sigma = \bigcup_{t \in [0, \varepsilon]} Z(\partial E, t) \subset \text{cl}(E) = E \cup \partial E .$$

At this point, it is useful to note that, because Σ is a closed subset of the open set E , we have $\Sigma \cap E = \Sigma \setminus \partial E$. This implies :

$$E \setminus E_\varepsilon = (E_\varepsilon)^c \cap E = (E^c \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \setminus \partial E, \quad (\text{A.1})$$

and $E = E_\varepsilon \cup (\Sigma \setminus \partial E)$.

With all these properties at hand, we define now two functions t_z and θ_z . The assumptions of global attractiveness of the closed set K_0 contained in E open, of transversality of χ to ∂E , and the property of forward-invariance of E , imply that, for all z in E^c , there exists a unique non negative real number t_z satisfying:

$$\kappa(Z(z, t_z)) = 0 \quad \iff \quad Z(z, t_z) \in \partial E.$$

The same arguments in reverse time allow us to see that, for all z in Σ , t_z exists, is unique and in $[-\varepsilon, 0]$. This way, the function $z \rightarrow t_z$ is defined on $(E_\varepsilon)^c$. Next, for all z in $(E_\varepsilon)^c$, we define :

$$\theta_z = Z(z, t_z).$$

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions $z \mapsto t_z$ and $z \mapsto \theta_z$ are C^s on $(E_\varepsilon)^c$.

REMARK 5. κ having constant rank 1 in a neighborhood of ∂E , this set is a closed, regular submanifold of \mathbb{R}^m . The arguments above show that $z \mapsto (\theta_z, t_z)$ is a diffeomorphism between E^c and $\partial E \times [0, +\infty[$. Since ∂E is a deformation retract of E^c and the open unit ball is diffeomorphic

to \mathbb{R}^m [10], if E were bounded, E^c could be seen as a h -cobordism between ∂E and the unit sphere \mathbb{S}^{m-1} and t_z as a Morse function with no critical point in E^c . See [18] for instance.

Now we evaluate t_z for z in $\partial\Sigma$. Let z be arbitrary in $\partial\Sigma$ and therefore in Σ which is closed. Assume its corresponding t_z is in $] -\varepsilon, 0[$. The Implicit Function Theorem shows that $z \mapsto t_z$ and $z \mapsto \theta_z$ are defined and continuous on a neighborhood of z . Therefore, there exists a strictly positive real number r satisfying

$$\forall y \in \mathcal{B}_r(z), \exists t_y \in] -\varepsilon, 0[: Z(y, t_y) \in \partial E .$$

This implies that the neighborhood $\mathcal{B}_r(z)$ of z is contained in Σ , in contradiction with the fact that z is on the boundary of Σ .

This shows that, for all z in $\partial\Sigma$, t_z is either 0 or $-\varepsilon$. We write this as $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$, with the notation $(\partial\Sigma)_i = \{z \in \Sigma : t_z = -\varepsilon\}$.

Now we want to prove $\partial E_\varepsilon \subset (\partial\Sigma)_i$. To obtain this result, we start by showing :

$$\partial E_\varepsilon \cap \partial E = \emptyset \quad \text{and} \quad \partial E_\varepsilon \subset \partial\Sigma . \quad (\text{A.2})$$

Suppose the existence of z in $\partial E_\varepsilon \cap \partial E$. z being in ∂E , its corresponding t_z is 0. By the Implicit Function Theorem, there exists a strictly positive real number r such that,

$$\forall y \in \mathcal{B}_r(z), \exists t_y \in] -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[: Z(y, t_y) \in \partial E .$$

But, by definition, any y , for which there exists t_y in $] -\frac{\varepsilon}{2}, 0[$, is in Σ . If instead t_y is strictly positive, then necessarily y is in E^c , because E is forward-invariant by χ and a solution starting in E cannot reach ∂E in positive finite time. We have obtained : $\mathcal{B}_r(z) \subset \Sigma \cup E^c = (E_\varepsilon)^c$. $\mathcal{B}_r(z)$ being a neighborhood of z , this contradicts the fact that z is in the boundary of E_ε .

At this point, we have proved that $\partial E_\varepsilon \cap \partial E = \emptyset$, and, because E_ε is contained in E , this implies $\partial E_\varepsilon \subset E$. With this, (A.2) will be established by proving that we have $\partial E_\varepsilon \subset \partial\Sigma$. Let z be arbitrary in ∂E_ε and therefore in E which is open. There exists a strictly positive real number r such that we have :

$$\mathcal{B}_r(z) \cap E_\varepsilon = \mathcal{B}_r(z) \cap (E \cap \Sigma^c) \neq \emptyset, \quad \mathcal{B}_r(z) \cap E_\varepsilon^c = \mathcal{B}_r(z) \cap (E^c \cup \Sigma) \neq \emptyset, \quad \mathcal{B}_r(z) \subset E .$$

This implies $\mathcal{B}_r(z) \cap \Sigma^c \neq \emptyset$ and $\mathcal{B}_r(z) \cap \Sigma \neq \emptyset$ and therefore that z is in $\partial\Sigma$.

We have established $\partial E_\varepsilon \cap \partial E = \emptyset$, $\partial E_\varepsilon \subset \partial\Sigma$ and $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$. This does imply :

$$\partial E_\varepsilon \subset (\partial\Sigma)_i = \{z \in E : t_z = -\varepsilon\} . \quad (\text{A.3})$$

This allows us to extend by continuity the definition of t_z to \mathbb{R}^m by letting $t_z = -\varepsilon$ for all $z \in E_\varepsilon$.

Thanks to all these preparatory steps, we are finally ready to define a function $\phi : \mathbb{R}^m \rightarrow E$ as:

$$\phi(z) = \begin{cases} Z(z, t_z + \nu(t_z)), & \text{if } z \in (E_\varepsilon)^c, \\ z, & \text{if } z \in E_\varepsilon, \end{cases} \quad (\text{A.4})$$

where ν is an arbitrary C^s and strictly decreasing function defined on \mathbb{R} satisfying:

$$\nu(t) = -t \quad \forall t \leq -\varepsilon, \quad \lim_{t \rightarrow +\infty} \nu(t) = 0.$$

The image of ϕ is contained in E since we have $E_\varepsilon \subset E$ and :

$$\begin{aligned} t_z + \nu(t_z) &> t_z & \forall z \in E_\varepsilon^c, \\ Z(z, t_z) &\in \partial E, \\ Z(z, t) &\in E & \forall (z, t) \in \partial E \times \mathbb{R}_{>0}. \end{aligned}$$

The continuity of the functions $(z, t) \in \mathbb{R}^m \times \mathbb{R} \mapsto Z(z, t) \in \mathbb{R}$ and $z \in E_\varepsilon^c \mapsto t_z \in [-\varepsilon, +\infty[$ implies that this function ϕ is continuous at least on $\mathbb{R}^m \setminus \partial E_\varepsilon$. Also, for any z in ∂E_ε , t_z is defined and equal to $-\varepsilon$ (see (A.3)). So, for any strictly positive real number η , there exists a real number r such that :

$$\begin{aligned} |t_y + \varepsilon| &\leq \eta & \forall y \in \mathcal{B}_r(z), \\ \nu(t_y) + \varepsilon &\leq \eta & \forall y \in \mathcal{B}_r(z), \\ \phi(y) &= y & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon, \\ \phi(y) &= Z(y, t_y + \nu(t_y)) & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon^c. \end{aligned}$$

Since we have :

$$\phi(z) = Z(z, t_z + \nu(t_z)) = Z(z, -\varepsilon + \nu(-\varepsilon)) = z,$$

we conclude that ϕ is also continuous at z .

By differentiating, we obtain :

– at any interior point z of $(E_\varepsilon)^c$

$$\frac{\partial \phi}{\partial z}(z) = \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z));$$

– at any z in E_ε (which is open) $\frac{\partial \phi}{\partial z}(z) = I$. Also, for any z in ∂E_ε , we have :

$$\begin{aligned} \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) &= \frac{\partial Z}{\partial z}(z, 0) + \chi(Z(z, 0)) \frac{\partial t_z}{\partial z}(z)(1 - 1), \\ &= I. \end{aligned}$$

This implies that ϕ is C^1 on \mathbb{R}^m .

We now show that ϕ is invertible. Let y be arbitrary in $E \cap E_\varepsilon^c = E \cap \Sigma$. There exists t_y in $[-\varepsilon, 0[$. The function ν being strictly monotonic, $\nu^{-1}(t_y)$ exists and is in $[-\varepsilon, +\infty[$. This allows us to define properly ϕ^{-1} as :

$$\phi^{-1}(y) = \begin{cases} Z(y, t_y - \nu^{-1}(-t_y)), & \text{if } y \in E \setminus E_\varepsilon \\ y, & \text{if } y \in E_\varepsilon \end{cases} \quad (\text{A.5})$$

This function is an inverse of ϕ as can be seen by reverting the flow induced by χ when needed. Also, with the same arguments as before, we can prove that it is C^1 .

This implies that ϕ is a diffeomorphism from \mathbb{R}^m to E .

Besides, the functions $z \mapsto Z(z, t)$ for $t > 0$, $z \mapsto t_z$ and ν being C^s , ϕ is C^s at any interior point of $(E_\varepsilon)^c$. By continuity of $\nu^{(r)}$ for $r \leq s$, it can be verified that ϕ is also C^s on the boundary ∂E_ε . So, ϕ is a C^s -diffeomorphism from \mathbb{R}^m to E .

Finally, we note that, for any point z_ε in ∂E_ε , there exists a point z in ∂E satisfying :

$$|z_\varepsilon - z| = \left| \int_0^\varepsilon \chi(Z(z, s)) ds \right| \leq \varepsilon \sup_\zeta |\chi(\zeta)| .$$

And conversely, for any z in ∂E , there exist z_ε in ∂E_ε satisfying :

$$|z_\varepsilon - z| = \left| \int_0^\varepsilon \chi(Z(z, s)) ds \right| \leq \varepsilon \sup_\zeta |\chi(\zeta)| .$$

It follows that

$$d_H(\partial E_\varepsilon, \partial E) \leq \varepsilon \sup_\zeta |\chi(\zeta)| \tag{A.6}$$

and ε may be chosen as small as needed. \square

Appendix B. Diffeomorphism extension from a ball.

LEMMA B.1 (3.5). *Let R be a strictly positive real number and consider a C^2 diffeomorphism $\psi : \mathcal{B}_R(0) \rightarrow \psi(\mathcal{B}_R(0)) \subset \mathbb{R}^m$. For any strictly positive real number ε in $]0, 1[$, there exists a diffeomorphism $\psi_\varepsilon : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $\psi_\varepsilon(z) = \psi(z)$ for all z in $\text{cl}(\mathcal{B}_{\frac{R}{1+\varepsilon}}(0))$.*

Proof. It sufficient to prove that our assumptions imply [12, Theorem 8.1.4] applies.

We let

$$U = \mathcal{B}_{\frac{R}{1+\frac{\varepsilon}{2}}}(0) \quad , \quad A = \text{cl}(\mathcal{B}_{\frac{R}{1+\varepsilon}}(0)) \quad , \quad I = \left] -\frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2} \right[,$$

and, without loss of generality we may assume that $\psi(0) = 0$.

Then, consider the function $F : U \times I \rightarrow \mathbb{R}^m$ defined as

$$F(z, t) = \left(\frac{\partial \psi}{\partial z}(0) \right)^{-1} \frac{\psi(zt)}{t} \quad , \quad \forall t \in I \setminus \{0\} \quad , \quad \varphi(z, 0) = z .$$

We start by showing that F is an isotopy of U .

- For any t in I , the function $z \mapsto F_t(z) = F(z, t)$ is an embedding from U onto $F_t(U) \subset \mathbb{R}^m$. Indeed, for any pair (z_a, z_b) in U^2 satisfying $F(z_a, t) = F(z_b, t)$, we obtain $\psi(z_a t) = \psi(z_b t)$ where $(z_a t, z_b t)$ is in U^2 . The function ψ being injective on this set, we have $z_a = z_b$ which establishes F_t is injective. Moreover, we have:

$$\frac{\partial F_t}{\partial z}(z) = \left(\frac{\partial \psi}{\partial z}(0) \right)^{-1} \frac{\partial \psi}{\partial z}(zt) \quad \forall t \in I \setminus \{0\} \quad , \quad \frac{\partial F_0}{\partial z}(z) = Id .$$

Hence, F_t is full rank on U and therefore an embedding.

- For all z in U , the function $t \mapsto F(z, t)$ is C^1 . This follows directly from the fact that, the function ψ being C^2 , and $\psi(0) = 0$, we have

$$\frac{\psi(zt)}{t} = \frac{\partial \psi}{\partial z}(0)z + z' \left(\frac{\partial^2 \psi}{\partial z \partial z}(0) \right) z \frac{t}{2} + o(t) .$$

In particular, we obtain $\frac{\partial F}{\partial t}(z, t) = \left(\frac{\partial \psi}{\partial z}(0) \right)^{-1} \rho(z, t)$ where

$$\rho(z, t) = \frac{1}{t^2} \left[\frac{\partial \psi}{\partial z}(zt)zt - \psi(zt) \right] \quad \forall t \in I \setminus \{0\} \quad , \quad \rho(z, 0) = \frac{1}{2} z' \left(\frac{\partial^2 \psi}{\partial z \partial z}(0) \right) z .$$

Moreover, for all t in I , the function $z \mapsto \frac{\partial F}{\partial t}(z, t)$ is locally Lipschitz and therefore gives rise to an ordinary differential equation with unique solutions.

Also the set $\bigcup_{(z,t) \in U \times I} \{(F(z, t), t)\}$ is open. This follows from Brouwer's Invariance theorem since the function $(z, t) \mapsto (F(z, t), t)$ is a diffeomorphism on the open set $U \times I$. With [12, Theorem 8.1.4], we know there exists a diffeotopy F_e from $\mathbb{R}^m \times I$ onto \mathbb{R}^m which satisfies $F_e = F$ on $A \times [0, 1]$. Thus, the diffeomorphism $\psi_e = F_e(\cdot, 1)$ defined on \mathbb{R}^m onto \mathbb{R}^m verifies $\psi_e(z) = F_e(z, 1) = F(z, 1) = \psi(z)$ for all $z \in A$. \square

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