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The Curious Reluctance to Define Prime Probability Statistically

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An elementary probability-based approach to estimating prime counting functions statistically

Bhupinder Singh Anand

Update of October 9, 2015

Abstract. All the known approximations of $\pi(n)$ for finite values of n are derived from real-valued functions that are asymptotic to $\pi(x)$, such as $\frac{x}{\log_e x}$, $Li(x)$ and Riemann's function $R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n)} li(x^{1/n})$. The degree of approximation for finite values of n is determined only heuristically, by conjecturing upon an error term in the asymptotic relation that can be seen to yield a closer approximation than others to the actual values of $\pi(n)$ for computable values of n . None of these can, however, claim to estimate $\pi(n)$ uniquely for all values of n . We show that statistically the probability of n being a prime is $\prod_{i=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i})$, and that statistically the expected value of the number $\pi(n)$ of primes less than or equal to n is given uniquely by $\sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i})$ for all values of n . We then demonstrate how this yields elementary probability-based proofs of the Prime Number Theorem, Dirichlet's Theorem, and the Twin-Prime Conjecture.

Keywords. prime counting function; prime probability function; Brocard's conjecture; Chebyshev's Theorem; complete system of incongruent residues; computational complexity; Dirichlet primes; Euler's constant γ ; expected value; factorising is polynomial time; Hardy-Littlewood conjecture; integer factorising algorithm; Law of Large Numbers; Mertens' theorem; mutually independent prime divisors; polynomial time algorithm; prime counting function $\pi(n)$; prime density; primes in an arithmetic progression; Prime Number Theorem; probability model; probabilistic number theory; twin primes.

2010 Mathematics Subject Classification. 11A07, 11A41, 11A51, 11N36, 11Y05, 11Y11, 11Y16

1. Introduction

“Prime numbers are the most basic objects in mathematics. They also are among the most mysterious, for after centuries of study, the structure of the set of prime numbers is still not well understood. Describing the distribution of primes is at the heart of much mathematics...”.¹

In the first half of this investigation we address the thesis that what makes the distribution of primes ‘mysterious’, and difficult to engage with for emerging scholars, is a curious—apparently implicit—reluctance to define prime probability statistically; an issue which may need to be addressed more extensively elsewhere.

In the second half, we explore the structure of divisibility (and, ipso facto, of primality)², and statistically define the probability of a number being prime.

We then show how this yields elementary (and unexpectedly related) probability-based proofs—derived from first principles—of fundamental prime properties such as the Prime Number Theorem, Dirichlet's Theorem, the Twin Prime Conjecture and the *PvNP* problem.³

¹Andrew Granville: from [this](#) AMS press release of 5 December 1997.

²Both structures become more transparent when displayed as in §5., Appendix II(A), Fig.6 and II(B), Fig.7.

³All of these have hitherto been considered as necessarily graduate or research level topics, but their probability-

1.A. The functions $\pi(x)$ and $\frac{x}{\log_e x}$: A historical perspective

To place this investigation in an appropriate historical perspective, we note that Adrien-Marie Legendre and Carl Friedrich Gauss are reported⁴ to have independently conjectured in 1796 that, if $\pi(x)$ denotes the number of primes less than x , then $\pi(x)$ is asymptotically equivalent to $\frac{x}{\log_e x}$.

Around 1848/1850, Pafnuty Lvovich Chebyshev proved that $\pi(x) \asymp \frac{x}{\log_e x}$, and confirmed that if $\pi(x)/\frac{x}{\log_e x}$ has a limit, then it must be 1⁵.

Fig.1: The asymptotic behaviour of the primes

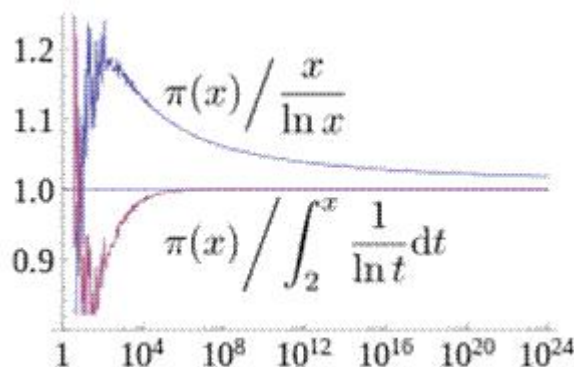


Fig.1: Graph showing ratio of the prime-counting function $\pi(x)$ to two of its approximations, $\frac{x}{\ln x}$ and $Li(x)$. As x increases (note x axis is logarithmic), both ratios tend towards 1. The ratio for $\frac{x}{\ln x}$ converges from above very slowly, while the ratio for $Li(x)$ converges more quickly from below.⁶

The question of whether $\pi(x)/\frac{x}{\log_e x}$ has a limit at all, or whether it oscillates, was answered—it has a limit—first by Jacques Hadamard and Charles Jean de la Vallée Poussin independently in 1896, using advanced argumentation involving functions of a complex variable⁷; and again independently by Paul Erdős and Atle Selberg⁸ in 1949/1950, using only elementary—but still abstruse—methods without involving functions of a complex variable.

1.B. A better approximation to $\pi(x)$: The integral $Li(x)$

We also note that, reportedly⁹:

“In a handwritten note on a reprint of his 1838 paper ‘Sur l’usage des séries infinies dans la théorie des nombres’, which he mailed to Carl Friedrich Gauss, Peter Gustav Lejeune Dirichlet conjectured (under a slightly different form appealing to a series rather than an integral) that an even better approximation to $\pi(x)$ is given by the offset logarithmic integral $Li(x)$ defined by:

based proofs are shown to be both simple and capable of being taught to, and reproduced by, any interested first-year undergraduate student of mathematics, or even a GCSE A level+ amateur enthusiast, with a spirit of enquiry. See §7., Appendix IV for the resources needed by a reader for following, and reproducing, the proofs of this paper.

⁴cf. Prime Number Theorem. (2014, June 10). In Wikipedia, The Free Encyclopedia. Retrieved 09:53, July 9, 2014, from http://en.wikipedia.org/w/index.php?title=Prime_number_theorem&oldid=612391868; see also [Gr95].

⁵[Di52], p.439; see also [HW60], p.9, Theorem 7 and p.345, §22.4 for a proof of Chebyshev’s Theorem.

⁶cf. Prime Number Theorem. (2014, June 10). In Wikipedia, The Free Encyclopedia. Retrieved 09:53, July 9, 2014, from http://en.wikipedia.org/w/index.php?title=Prime_number_theorem&oldid=612391868.

⁷[Di52], p.439; see also [Ti51], Chapter III, p.8 for details of Hadamard’s and de la Vallée Poussin’s proofs of the Prime Number Theorem.

⁸See [HW60], p.360, Theorem 433 for a proof of Selberg’s Theorem.

⁹cf. Prime Number Theorem. (2014, June 10). In Wikipedia, The Free Encyclopedia. Retrieved 09:53, July 9, 2014, from: http://en.wikipedia.org/w/index.php?title=Prime_number_theorem&oldid=612391868.

$$Li(x) = \int_2^x \frac{1}{\log_e t} \cdot dt = li(x) - li(2).”^{10}$$

We further note that in 1889 Jean de la Vallée Poussin proved¹¹ (cf. Fig.1):

“... that $Li(x)$ represents $\pi(x)$ more exactly than $\frac{x}{\log_e x}$ and its remaining approximations $\frac{x}{\log_e x} + \frac{x}{\log_e^2 x} + \dots + \frac{(m-1)!x}{\log_e^m x}$.”

1.C. Known approximations of $\pi(n)$ for finite values of n

We note that all the known approximations of $\pi(n)$ for finite values of n are derived from real-valued functions that are asymptotic to $\pi(x)$, such as $\frac{x}{\log_e x}$, $Li(x)$ and Riemann’s function $R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n)} li(x^{1/n})$.

The degree of approximation for finite values of n is determined only heuristically, by conjecturing upon an error term in the asymptotic relation that can be seen to yield the closest approximation upon comparison with the actual values of $\pi(n)$ for computable values of n (eg. Fig.2).

Fig.2: The distribution of the primes

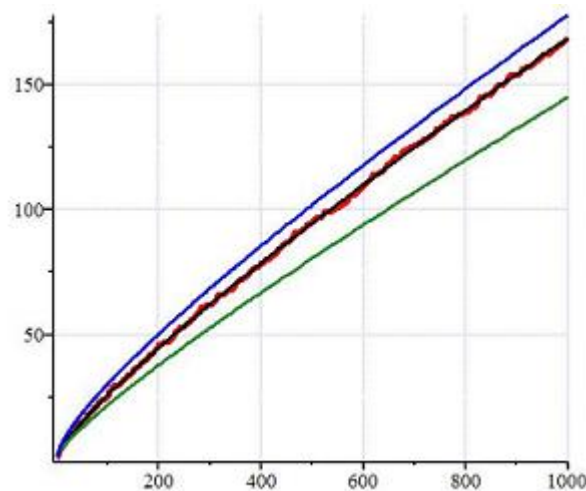


Fig.2: The above graph compares the actual number $\pi(x)$ (red) of primes $\leq x$ with the distribution of primes as estimated variously by the functions $Li(x)$ (blue), $R(x)$ (black), and $\frac{x}{\log_e x}$ (green), where $R(x)$ is Riemann’s function $\sum_{n=1}^{\infty} \frac{\mu(n)}{(n)} li(x^{1/n})$.¹²

The question remains:

- Is there a function which best approximates $\pi(n)$ for all values of n ?

1.D. Is there a unique function which *best* approximates $\pi(n)$ for *all* values of n ?

In this investigation we shall answer the above question affirmatively by:

¹⁰Where $li(x) = \int_0^x \frac{1}{\log_e t} \cdot dt$.

¹¹[Di52], p.440.

¹²cf. How Many Primes Are There? In *The Prime Pages*. Retrieved 10:29, September 27, 2015, from: <https://primes.utm.edu/howmany.html>.

- *first*, defining the statistical probability of an integer n being a prime; and
- *second*, showing that the statistically expected distribution of the primes—and hence the best approximation for $\pi(n)$ for all finite n —is given by the unique statistical prime counting function (cf. Fig.3):

$$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} \left(1 - \frac{1}{p_i}\right).$$

Fig.3: Statistically expected distribution of the primes

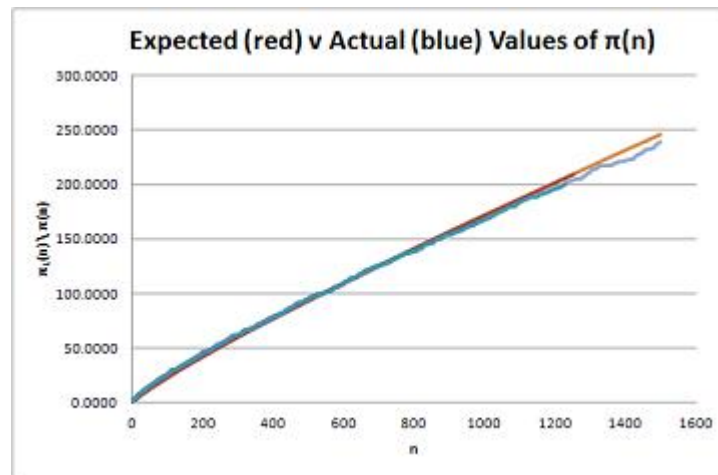


Fig.3: The above graph compares the statistically expected values (red) vs actual values (blue) of $\pi(n)$ for $4 \leq n \leq 1500$ ¹³, where the statistically expected value $\pi_L(n)$ of $\pi(n)$ is $\sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} \left(1 - \frac{1}{p_i}\right)$.

We shall then demonstrate how this yields elementary, probability-based, proofs of the Prime Number Theorem, Dirichlet’s Theorem, and the Twin-Prime Conjecture.

2. The curious reluctance to define prime probability statistically

2.A. Prime probability: conventional wisdom

Now, the explicit thesis of this investigation is that lack of recognition of $\pi_L(n)$ as *the* prime counting function for the number of primes $\leq n$ is, apparently, reflection of a curious—albeit implicit—reluctance to accept a statistical definition of prime probability as legitimate.

For instance, conventional number-theory wisdom appears to be that the distribution of primes suggested by the Prime Number Theorem¹⁴, $\pi(n) \sim \frac{n}{\log_e n}$, is such that the probability $\mathbb{P}(n \in \{p\})$ of an integer n being a prime p can *only* be *heuristically* estimated as $\frac{1}{\log_e n}$ ¹⁵; apparently reflecting an implicit faith in G. H. Hardy and J. E. Littlewood’s 1922 dictum that¹⁶:

“Probability is not a notion of pure mathematics, but of philosophy or physics”.

¹³See §6., Appendix III for the values of the above plot.

¹⁴[HW60], Theorem 6, p.9.

¹⁵“The chance of a random integer x being prime is about $1/\log x$ ” ... Chris K. Caldwell, *How Many Primes Are There?* In *The Prime Pages*. Retrieved 10:29, September 27, 2015, from: <https://primes.utm.edu/howmany.html>.

¹⁶[Gr95], p.19, fn.16 and p.20; see also [HL23], fn.4 on p.37, for the origin of the quote (courtesy Prof. Andrew Granville).

It is a dictum that can reasonably be taken by the laity to suggest, with some authority, that the statistical probability $\mathbb{P}(n \in \{p\})$ of an integer n being a prime p is also *not capable* of being well-defined statistically¹⁷ independently of the Theorem.

2.B. Statistical probability that a prime p divides n

However, such a conclusion would be misleading, since any lay investigation of such a probability from first principles:

(1) would begin naturally by considering if, and only if, conditions for i to be a divisor of n ;

(2) would move fairly straightforwardly to an elementary residue function such as $r_i(n)$ ¹⁸, defined (Definition 1) for all $n \geq 2$ and all $i \geq 2$ by:

$$n + r_i(n) \equiv 0 \pmod{i} \text{ where } i > r_i(n) \geq 0$$

since $r_i(n) = 0$ if, and only if, i is a divisor of n ;

(3) would then (Theorem 3.3) note for any $i \geq 2$ that:

$$\mathbb{M}_i = \{(0, 1, 2, \dots, i - 1), r_i(n), \frac{1}{i}\}$$

is a probability model¹⁹ for the values of $r_i(n)$ for $n \geq 2$;

(4) which would further imply:

(i) first (Corollary 3.4) that, by the standard definition of the statistical probability $\mathbb{P}(e)$ of an event e ²⁰, the probability $\mathbb{P}(p|n)$ that $r_p(n) = 0$ —whence the prime p divides n —is:

$$\mathbb{P}(p|n) = \frac{1}{p}$$

and that the probability $\mathbb{P}(p \nmid n)$ that $r_p(n) \neq 0$ —whence the prime p does not divide n —is:

$$\mathbb{P}(p \nmid n) = 1 - \frac{1}{p}$$

since the p numbers $0, 1, \dots, (p - 1)$ are all incongruent and form a complete system of residues²¹;

(ii) second (Lemma 3.5) that:

(a) the product of the individual probability that $r_{p_i}(n) = 0$ —whence the prime p_i divides the integer n —and the individual probability that $r_{p_j}(n) = 0$ —whence the prime $p_j \neq p_i$ divides n —is:

$$\mathbb{P}(p_i|n) \cdot \mathbb{P}(p_j|n) = \frac{1}{p_i} \cdot \frac{1}{p_j}$$

¹⁷See, for instance, [St02], Chapter 2, p.9, Theorem (*sic*) 2.1!

¹⁸Depicted graphically in §5., Appendix II(A), Fig.6.

¹⁹See §4., Appendix I.

²⁰See §4., Appendix I; also [Ko56], Chapter I, §1, Axiom III, p.2.

²¹[HW60], p.49.

(b) the joint probability $\mathbb{P}(p_i|n \cap p_j|n)$ that $r_{p_i}(n) = 0$ and $r_{p_j}(n) = 0$ —whence *both* the primes $p_i \neq p_j$ divide the integer n —is:

$$\mathbb{P}(p_i|n \cap p_j|n) = \frac{1}{p_i \cdot p_j}$$

since the $p_i \cdot p_j$ numbers $v \cdot p_i + u \cdot p_j$, where $p_i > u \geq 0$ and $p_j > v \geq 0$, are also all incongruent and form a complete system of residues²²;

(iii) and third (Theorem 3.8) that the prime divisors of any integer n are thus mutually independent by the standard definition of the ‘mutual independence’ of two events e_1 and e_2 ²³.

2.C. Statistical probability of n being a prime

Since it is easily shown that n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it would immediately then follow:

(i) first (Theorem 3.11) that the statistical probability of n being a prime p is given²⁴ by the prime probability function (cf. Fig.4):

$$\mathbb{P}(n \in \{p\}) = \prod_{i=1}^{\pi(\sqrt{n})} \left(1 - \frac{1}{p_i}\right) \sim \frac{2e^{-\gamma}}{\log_e n}, \quad 25$$

where $2e^{-\lambda} \approx 1.12292\dots$;²⁶

Fig.4: The graph of $y = \prod_{i=1}^{\pi(\sqrt{x})} \left(1 - \frac{1}{p_i}\right)$

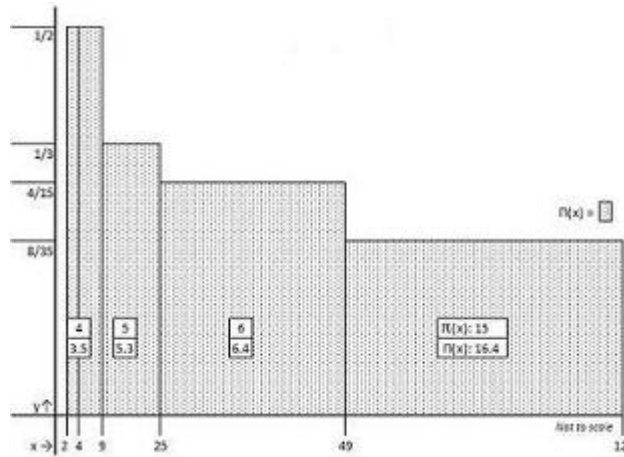


Fig.4: Graph of $y = \prod_{i=1}^{\pi(\sqrt{x})} \left(1 - \frac{1}{p_i}\right)$. The dotted rectangles represent $(p_{j+1}^2 - p_j^2) \prod_{i=1}^j \left(1 - \frac{1}{p_i}\right)$ for $j \geq 1$. Figures within boxes are values of the corresponding function within the interval (p_j^2, p_{j+1}^2) for $j \geq 2$. The area under the curve is $\Pi(x) = (x - p_n^2) \prod_{i=1}^n \left(1 - \frac{1}{p_i}\right) + \sum_{j=1}^{n-1} (p_{j+1}^2 - p_j^2) \prod_{i=1}^j \left(1 - \frac{1}{p_i}\right) + 2$ (see Fig.5).

²²Ibid., p.52, Theorem 59.

²³See §4, Appendix I; also [Ko56], Chapter VI, §1, Definition 1, p.57 and §2, p.58; see also [Ka59], p.54.

²⁴Compare [HL23], pp.36-37.

²⁵The asymptotic equivalence follows by Mertens's Theorem $\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log_e x}$, [HW60], Theorem 429, p.351.

²⁶[Gr95], p.13.

(ii) and second that (Theorem 3.13), by the Law of Large Numbers²⁷, the expected value²⁸ of the number $\pi(n)$ of primes less than or equal to n is (Definition 4) the prime counting function $\pi_L(n)$ (cf. Fig.5), such that:

$$\pi(n) \sim \pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i}).$$

Fig.5: The graph of $y = \sqcap(x) = \pi_L(x)$

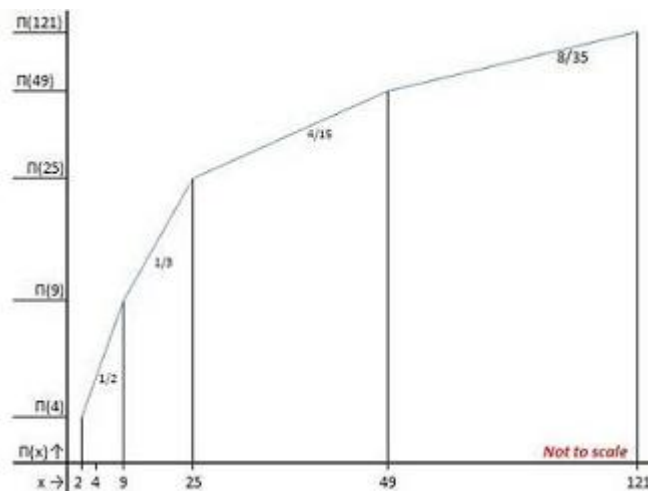


Fig.5: Graph of $y = \sqcap(x) = \pi_L(x) = (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i}) + \sum_{j=1}^{n-1} (p_{j+1}^2 - p_j^2) \prod_{i=1}^j (1 - \frac{1}{p_i}) + 2$ in the interval (p_n^2, p_{n+1}^2) . Note that the gradient in the interval (p_n^2, p_{n+1}^2) is $\prod_{i=1}^n (1 - \frac{1}{p_i})$.

2.D. The anomaly in approximating $\pi(n)$ heuristically: conventional wisdom

However conventional number theory wisdom—whilst reasonably conceding²⁹ that the *heuristic* probability of an integer n being prime *could* also be naïvely assumed as $\prod_{i=1}^{\sqrt{n}} (1 - \frac{1}{p_i})$ —seems to unreasonably argue against such naïvety, by concluding that the number $\pi(n)$ of primes less than or equal to n suggested by such probability would then be approximated by the *heuristic* prime counting function:

$$\pi_H(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i}) = n \cdot \prod_{i=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i}) \sim \frac{2 \cdot e^{-\gamma} n}{\log_e n}.$$

For instance, Hardy and Littlewood note that:

“In the first place we observe that any formula in the theory of primes, deduced from considerations of probability, is likely to be erroneous in just this way. Consider, for example, the problem ‘what is the chance that a large number n should be prime?’ We know that the answer is that the chance is approximately $\frac{1}{\log n}$.

Now the chance that n should not be divisible by any prime less than a *fixed* x is asymptotically equivalent to

²⁷See §4., Appendix I; also [Ko56], Chapter VI, §3, p.61.

²⁸See §4., Appendix II. Compare also [HL23], pp.36-37. See also §6., Appendix III for the expected values $\pi_L(n)$, and the actual values $\pi(n)$, for $4 \leq n \leq 1500$.

²⁹[Gr95], p.13.

$$\prod_{\varpi < x} \left(1 - \frac{1}{\varpi}\right)$$

and it would be natural to infer¹ that the chance required is asymptotically equivalent to

$$\prod_{\varpi < \sqrt{x}} \left(1 - \frac{1}{\varpi}\right)$$

But

$$\prod_{\varpi < \sqrt{x}} \left(1 - \frac{1}{\varpi}\right) \sim \frac{2e^{-C}}{\log n}$$

and our inference is incorrect, to the extent of a factor $2e^{-C}$.

¹ One might well replace $\varpi < \sqrt{x}$ by $\varpi < x$, in which case we should obtain a probability half as large. This remark is in itself enough to show the unsatisfactory character of the argument.”

... pp.36-37, G.H Hardy and J.E. Littlewood, *Some problems of ‘partitio numerorum:’ III: On the expression of a number as a sum of primes*, Acta Mathematica, December 1923, Volume 44, pp.1-70.

Now, even if we ignore the incongruity of treating x as ‘fixed’, the ‘character’ of the argument in Hardy and Littlewood’s footnoted remark can be considered ‘unsatisfactory’ only if we conflate necessity with sufficiency!

Otherwise, what we ought to reasonably conclude from the argument is that:

Lemma 2.1. *Whilst the statistical probability that n should not be divisible by any prime ϖ less than x is $\prod_{\varpi < x} (1 - \frac{1}{\varpi})$ if $x \leq \sqrt{n}$, it is defined by $\prod_{\varpi < \sqrt{n}} (1 - \frac{1}{\varpi})$ —and not by $\prod_{\varpi < x} (1 - \frac{1}{\varpi})$ —if $x > \sqrt{n}$.*

Proof: We shall show in §3.A. of this investigation that whilst—if $x > \sqrt{n}$ —the terms of the former product do, those of the latter product do not, define the statistical probabilities of the necessary and sufficient—mutually independent—conditions that *jointly* define the primality of n under the probability model (see §3.B.):

$$\mathbb{M}_i = \{(0, 1, 2, \dots, i - 1), r_i(n), \frac{1}{i}\}. \quad \square$$

Moreover, the argument that we may treat $\pi_H(n)$ as a *heuristic* approximation to $\pi(n)$ is ‘unreasonable’ since an apparent anomaly does, then, surface when we express $\pi(n)$ and the function $\pi_H(n)$ in terms of the number of primes determined by each function respectively in each interval (p_n^2, p_{n+1}^2) as follows:

$$\begin{aligned} \pi(p_{n+1}^2) &= \sum_{j=1}^n (\pi(p_{j+1}^2) - \pi(p_j^2)) + \pi(p_1^2) \\ \pi_H(p_{n+1}^2) &= p_{n+1}^2 \cdot \prod_{i=1}^{\pi(\sqrt{p_{n+1}^2})} \left(1 - \frac{1}{p_i}\right) \\ &= (\sum_{j=1}^n (p_{j+1}^2 - p_j^2) + p_1^2) \cdot \prod_{i=1}^n \left(1 - \frac{1}{p_i}\right) \\ &= \sum_{j=1}^n (p_{j+1}^2 \cdot \prod_{i=1}^n (1 - \frac{1}{p_i}) - p_j^2 \cdot \prod_{i=1}^n (1 - \frac{1}{p_i})) + p_1^2 \cdot \prod_{i=1}^n (1 - \frac{1}{p_i}) \end{aligned}$$

Reason: By Corollary 3.13, $\pi_L(n)$ is the expected value of $\pi(n)$, and, for any given $k > 1$:

$$\pi_L(p_{k+1}^2) - \pi_L(p_k^2) > 0 \text{ as } n \rightarrow \infty;$$

whilst, for any given $k > 1$ ³⁰:

$$p_{k+1}^2 \cdot \prod_{i=1}^n (1 - \frac{1}{p_i}) - p_k^2 \cdot \prod_{i=1}^n (1 - \frac{1}{p_i}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

More specifically, by Corollary 3.13 and Mertens' Theorem³¹, the expected value of the number of primes between the prime squares p_k^2 and p_{k+1}^2 (see Fig.4), for any $k > 1$, is given by:

$$\begin{aligned} \pi(p_{k+1}^2) - \pi(p_k^2) &\sim \pi_L(p_{k+1}^2) - \pi_L(p_k^2) \text{ as } k \rightarrow \infty \\ \pi_L(p_{k+1}^2) - \pi_L(p_k^2) &= (p_{k+1}^2 - p_k^2) \cdot \prod_{i=1}^k (1 - \frac{1}{p_i}) \\ &\geq ((p_k + 2)^2 - p_k^2) \cdot \prod_{i=1}^k (1 - \frac{1}{p_i}) \\ &\geq 4(p_k + 1) \cdot \prod_{i=1}^k (1 - \frac{1}{p_i}) \\ &\in O(\frac{p_k}{\log_e p_k}) \text{ as } k \rightarrow \infty \\ &\rightarrow \infty \text{ as } k \rightarrow \infty \end{aligned}$$

So, if we were to contrarily accept both $\pi_L(n)$ and $\pi_H(n)$ as prime counting functions, then the anomaly noted by Hardy and Littlewood would, indeed, follow from the Prime Number Theorem $\pi(n) \sim \frac{n}{\log_e n}$, since $\pi_H(n) \sim \frac{2 \cdot e^{-\gamma} n!}{\log_e n}$

Brocard's conjecture: We note without further comment that Brocard's conjecture:

$$\pi(p_{k+1}^2) - \pi(p_k^2) \geq 4$$

would follow if we could show that, for $k > 1$, the difference between $\pi(n)$ and $\pi_L(n)$ is always less than $4(p_k + 1) \cdot \prod_{i=1}^k (1 - \frac{1}{p_i}) + 1$.³²

2.E. The 'second' Hardy-Littlewood conjecture concerning prime density

We next note that the 'heuristic' definition of the probability of a number being prime, albeit discounted by Hardy and Littlewood as 'unsatisfactory', is not only justifiable statistically (as shown in §3.D.), but that Definition 4 immediately implies:

Theorem 2.2. $\pi_L(m+n) \leq \pi_L(m) + \pi_L(n)$ for all integers $m, n \geq 2$

Proof: The m terms of the summation $\pi_L(m) = \sum_{j=1}^m \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i})$ are identical to the first m terms of $\pi_L(m+n) = \sum_{j=1}^{m+n} \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i})$; whilst the k^{th} term $\prod_{i=1}^{\pi(\sqrt{k})} (1 - \frac{1}{p_i})$ of $\pi_L(n)$ is greater than the corresponding $(m+k)^{\text{th}}$ term $\prod_{i=1}^{\pi(\sqrt{m+k})} (1 - \frac{1}{p_i})$ of $\pi_L(m+n)$ for $m \geq 1, k \geq 1$ ³³. \square

We further have, by the Law of Large Numbers, that:

Corollary 2.3. $\pi(m+n) \leq \pi(m) + \pi(n)$ as $m \rightarrow \infty$ \square

³⁰Compare with what appears to be a similar argument in [St02], Chapter 2, p.9, Theorem (*sic*) 2.1.

³¹i.e., $\prod_{p \leq x} (1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log_e x}$, [HW60], Theorem 429, p.351.

³²cf. [Wikipedia: Brocard's conjecture](#).

³³As is graphically obvious from Fig.4.

The significance of Theorem 2.2 is seen if we compare:

(i) Theorem 2.2 with the definition of the ‘second’ Hardy-Littlewood 1923 conjecture in Richards³⁴ concerning the estimated density of primes as:

$$‘\pi(x + y) \leq \pi(x) + \pi(y) \text{ for all integers } x, y \geq 2’$$

where the author claims:

“We show that this assertion is probably false”;

(ii) and Corollary 2.3 with the original conjecture³⁵, where Hardy and Littlewood define:

$$“\varrho(x) = \overline{\lim}_{n \rightarrow \infty} (\pi(n + x) - \pi(n))”$$

and remark that:

“It is plain that the determination of a lower bound for $\varrho(x)$ is a problem of exceptional depth. . . . The problem of an upper bound has greater possibilities. . . . An examination of the primes less than 200 suggests forcibly that: $\varrho(x) \leq \pi(x)$ ($x \geq 2$)”.

3. An elementary probability-based approach to estimating prime counting functions statistically

In the rest of this investigation we demonstrate the far-reaching significance of defining the statistical probability of n being a prime by giving elementary probability-based proofs that:

(i) *The Prime Number Theorem*: First, by the Law of Large Numbers, we have $\pi(x) \sim \pi_L(x)$ since $p_{n+1}^2 - p_n^2 \rightarrow \infty$ (Corollary 3.13). Second, we note the function $\pi_L(x)/\frac{x}{\log_e x}$ is differentiable in the interval (p_n^2, p_{n+1}^2) with derivative $(\pi_L(x)/\frac{x}{\log_e x})' \in o(1)$ (Lemma 3.15). We conclude that both $\pi_L(x)/\frac{x}{\log_e x}$ and $\pi(x)/\frac{x}{\log_e x}$ do not oscillate as $x \rightarrow \infty$.

Chebyshev’s Theorem³⁶, $\pi(x) \asymp \frac{x}{\log_e x}$, then yields the Prime Number Theorem (Theorem 3.16): $\pi(x) \sim \frac{x}{\log_e x}$.

(ii) *Dirichlet’s Theorem*: By the Law of Large Numbers, the expected value of the number $\pi_{(a,d)}(n)$ of Dirichlet primes of the form $a + m.d$ which are less than or equal to n , where a, d are co-prime and $1 \leq a < d = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \dots q_k^{\alpha_k}$ (q_i prime), is given by the Dirichlet prime counting function $\pi_D(n)$ (Definition 6), such that:

$$\pi_{(a,d)}(n) \sim \pi_D(n) = \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k (1 - \frac{1}{q_i})^{-1} \cdot \pi_L(n) \rightarrow \infty.$$

(iii) *Twin Prime Theorem*: By the Law of Large Numbers, the expected value of the number $\pi_2(n)$ of twin primes $\leq n$ is given by the twin-prime counting function:

³⁴[Ri74], p.420.

³⁵In [HL23], pp.52-54.

³⁶[HW60], Theorem 7, p.9.

$$\pi_T(n) = \sum_{j=1}^n \mathbb{P}(j \in \{p\} \cap j + 2 \in \{p\}).$$

We conclude that there are infinitely many twin primes since we show that (Corollary 3.34):

$$\pi_2(n) \sim \pi_T(n) \sim e^{-2\gamma} \cdot \frac{n}{\log_e^2 n}.$$

3.A. The residues $r_i(n)$.

We begin by formally defining the residues $r_i(n)$ for all $n \geq 2$ and all $i \geq 2$ as below³⁷:

Definition 1. $n + r_i(n) \equiv 0 \pmod{i}$ where $i > r_i(n) \geq 0$.

Since each residue $r_i(n)$ cycles over the i values $(i - 1, i - 2, \dots, 0)$, these values are all incongruent and form a complete system of residues³⁸ mod i .

It immediately follows that:

Lemma 3.1. $r_i(n) = 0$ if, and only if, i is a divisor of n . □

3.B. The probability model $\mathbb{M}_i = \{(0, 1, 2, \dots, i - 1), r_i(n), \frac{1}{i}\}$

By the standard definition of the probability $\mathbb{P}(e)$ of an event e ³⁹, we have by Lemma 3.1 that:

Lemma 3.2. For any $n \geq 2$, $i \geq 2$ and any given integer $i > u \geq 0$:

- the probability $\mathbb{P}(r_i(n) = u)$ that $r_i(n) = u$ is $\frac{1}{i}$;
- $\sum_{u=0}^{i-1} \mathbb{P}(r_i(n) = u) = 1$;
- and the probability $\mathbb{P}(r_i(n) \neq u)$ that $r_i(n) \neq u$ is $1 - \frac{1}{i}$. □

By the standard definition of a probability model⁴⁰, we conclude that:

Theorem 3.3. For any $i \geq 2$, $\mathbb{M}_i = \{(0, 1, 2, \dots, i - 1), r_i(n), \frac{1}{i}\}$ is a probability model for the values of $r_i(n)$. □

Corollary 3.4. For any $n \geq 2$ and any prime $p \geq 2$, the probability $\mathbb{P}(r_p(n) = 0)$ that $r_p(n) = 0$, and that p divides n , is $\frac{1}{p}$; and the probability $\mathbb{P}(r_p(n) \neq 0)$ that $r_p(n) \neq 0$, and that p does not divide n , is $1 - \frac{1}{p}$. □

We also note the standard definition⁴¹:

Definition 2. Two events e_i and e_j are mutually independent for $i \neq j$ if, and only if, $\mathbb{P}(e_i \cap e_j) = \mathbb{P}(e_i) \cdot \mathbb{P}(e_j)$.

³⁷The residues $r_i(n)$ can also be graphically displayed variously as shown in the Appendix II in §5.

³⁸[HW60], p.49.

³⁹See §4., Appendix I; also [Ko56], Chapter I, §1, Axiom III, pg.2.

⁴⁰See §4., Appendix I.

⁴¹See §4., Appendix I; also [Ko56], Chapter VI, §1, Definition 1, pg.57 and §2, pg.58.

3.C. The prime divisors of any integer n are mutually independent

We then have that:

Lemma 3.5. *If p_i and p_j are two primes where $i \neq j$ then, for any $n \geq 2$, we have:*

$$\mathbb{P}((r_{p_i}(n) = u) \cap (r_{p_j}(n) = v)) = \mathbb{P}(r_{p_i}(n) = u) \cdot \mathbb{P}(r_{p_j}(n) = v)$$

where $p_i > u \geq 0$ and $p_j > v \geq 0$.

Proof: The $p_i \cdot p_j$ numbers $v \cdot p_i + u \cdot p_j$, where $p_i > u \geq 0$ and $p_j > v \geq 0$, are all incongruent and form a complete system of residues⁴² mod $(p_i \cdot p_j)$. Hence:

$$\mathbb{P}((r_{p_i}(n) = u) \cap (r_{p_j}(n) = v)) = \frac{1}{p_i \cdot p_j}$$

By Lemma 3.2:

$$\mathbb{P}(r_{p_i}(n) = u) \cdot \mathbb{P}(r_{p_j}(n) = v) = \left(\frac{1}{p_i}\right) \left(\frac{1}{p_j}\right).$$

The lemma follows. □

If $u = 0$ and $v = 0$ in Lemma 3.5, so that both p_i and p_j are prime divisors of n , we immediately conclude by Definition 2 that:

Corollary 3.6. $\mathbb{P}((r_{p_i}(n) = 0) \cap (r_{p_j}(n) = 0)) = \mathbb{P}(r_{p_i}(n) = 0) \cdot \mathbb{P}(r_{p_j}(n) = 0)$. □

We can also express this as:

Corollary 3.7. $\mathbb{P}(p_i | n \cap p_j | n) = \mathbb{P}(p_i | n) \cdot \mathbb{P}(p_j | n)$. □

We thus conclude that:

Theorem 3.8. *The prime divisors of any integer n are mutually independent.* □

3.C.a. Integer Factorising cannot be polynomial-time

We digress briefly from our investigation of prime counting functions to note that Theorem 3.8 immediately yields the actively pursued⁴³ (although prima facie unconnected) computational complexity consequence that no deterministic algorithm⁴⁴ can compute a factor of any randomly given integer n in polynomial time⁴⁵!

We note the standard definition⁴⁶:

Definition 3. *A deterministic algorithm computes a number-theoretical function $f(n)$ in polynomial-time if there exists k such that, for all inputs n , the algorithm computes $f(n)$ in $\leq (\log_e n)^k + k$ steps.*

⁴²[HW60], p.52, Theorem 59.

⁴³cf. [Cook].

⁴⁴A deterministic algorithm computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particular value as output.

⁴⁵cf. [Cook], p.1; also [Br00], p.1, fn.1.

⁴⁶cf. [Cook], p.1; also [Br00], p.1, fn.1: "For a polynomial-time algorithm the expected running time should be a polynomial in the length of the input, i.e. $O((\log N)^c)$ for some constant c ".

It then follows from Theorem 3.8 that:

Corollary 3.9. *Any deterministic algorithm that always computes a prime factor of n cannot be polynomial-time.*

Proof: Any computational process that successfully identifies a prime divisor of n must necessarily appeal to at least one logical operation for identifying such a factor.

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, and n may be the square of a prime, it follows from Theorem 3.8 that we necessarily require at least one logical operation for each prime $p \leq \sqrt{n}$ in order to logically determine whether p is a prime divisor of n .

Since the number of such primes is of the order $O(n/\log_e n)$, the number of computations required by any deterministic algorithm that always computes a prime factor of n cannot be polynomial-time—i.e. of order $O((\log_e n)^c)$ for any c —in the length of the input n . The corollary follows. \square

3.D. The statistical probability $\mathbb{P}(n \in \{p\})$ that n is a prime

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it follows immediately from Lemma 3.2 and Lemma 3.5 that:

Lemma 3.10. *For any $n \geq 2$, the probability $\mathbb{P}(n \in \{p\})$ of an integer n being a prime p is the probability that $r_{p_i}(n) \neq 0$ for any $1 \leq i \leq k$ if $p_k^2 \leq n < p_{k+1}^2$.* \square

By Corollary 3.4 we can express this by the statistical prime probability function (graphically illustrated in §2.C., Fig.4)⁴⁷:

Theorem 3.11. $\mathbb{P}(n \in \{p\}) = \prod_{i=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i}) \sim \frac{2e^{-\gamma}}{\log_e n}$. \square

It immediately follows that, for any $m > \pi(\sqrt{n})$:

Corollary 3.12. $\mathbb{P}(n \in \{p\}) > \prod_{i=1}^m (1 - \frac{1}{p_i})$. \square

3.E. The statistical prime counting function $\pi_L(n)$

It now follows from Theorem 3.11 that, since $p_{n+1}^2 - p_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, by the Law of Large Numbers⁴⁸, the expected value⁴⁹ of the number $\pi(n)$ of primes less than or equal to n is given by the prime counting function (graphically illustrated in §2.C., Fig.5):

Definition 4. $\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i})$.

Corollary 3.13. $\pi(n) \sim \pi_L(n)$. \square

⁴⁷We note that $Lt_{n \rightarrow \infty} \log_e n \cdot \prod_{i=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i}) = 2 \cdot e^{-\gamma} \approx 1.12292 \dots$ ([Gr95], p.13).

⁴⁸See §4., Appendix I; also [Ko56], Chapter VI, §3, p.61; [El79b], pp.52-57.

⁴⁹See §4., Appendix III. Compare also [HL23], pp.36-37.

3.F. The interval (p_n^2, p_{n+1}^2)

It also follows immediately from the definition of $\pi(x)$ as the number of primes less than or equal to x that:

Lemma 3.14. $\prod_{i=1}^{\pi(\sqrt{x})} (1 - \frac{1}{p_i}) = \prod_{i=1}^{\pi(\sqrt{x+1})} (1 - \frac{1}{p_i})$ for $p_n^2 \leq x < p_{n+1}^2$. □

We can also generalise the number-theoretic function of Definition 4 as the real-valued function:

Definition 5. $\pi_L(x) = \pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i})$ for $p_n^2 \leq x < p_{n+1}^2$. □

We note that the graph of $\pi_L(x)$ in the interval (p_n^2, p_{n+1}^2) for $n \geq 1$ is now a straight line with gradient $\prod_{i=1}^n (1 - \frac{1}{p_i})$, as illustrated in §2.C., Fig.5 where we defined $\pi_L(x)$ equivalently by:

$$\pi_L(x) = \pi(x) = (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i}) + \sum_{j=1}^{n-1} (p_{j+1}^2 - p_j^2) \prod_{i=1}^j (1 - \frac{1}{p_i}) + 2$$

3.G. The function $\pi_L(x)/\frac{x}{\log_e x}$

We consider next the function $\pi_L(x)/\frac{x}{\log_e x}$ in the interval (p_n^2, p_{n+1}^2) :

$$\pi_L(x)/\frac{x}{\log_e x} = (\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i}))/\frac{x}{\log_e x}$$

This now yields the derivative $(\pi_L(x) \cdot \frac{\log_e x}{x})'$ in the interval (p_n^2, p_{n+1}^2) as:

$$\begin{aligned} & \pi_L(x) \cdot (\frac{\log_e x}{x})' + (\pi_L(x))' \cdot \frac{\log_e x}{x} \\ & (\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i})) \cdot (\frac{\log_e x}{x})' + (\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i}))' \cdot \frac{\log_e x}{x} \\ & (\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^n (1 - \frac{1}{p_i})) \cdot (\frac{1}{x^2} - \frac{\log_e x}{x^2}) + (\prod_{i=1}^n (1 - \frac{1}{p_i})) \cdot \frac{\log_e x}{x} \end{aligned}$$

Since $p_n^2 \leq x < p_{n+1}^2$ and $\pi_L(x) \sim \pi(x)$ by the Law of Large Numbers, by Mertens⁵⁰ and Chebyshev's Theorems we can express the above as:

$$\begin{aligned} & \sim (\pi_L(p_n^2) + \frac{e^{-\gamma}(x-p_n^2)}{\log_e n}) \cdot (\frac{1}{x^2} - \frac{\log_e x}{x^2}) + \frac{e^{-\gamma} \cdot \log_e x}{x \cdot \log_e n} \\ & \sim (\frac{\pi_L(p_n^2)}{x} + \frac{e^{-\gamma}}{\log_e n} (1 - \frac{p_n^2}{x})) \cdot \frac{(1 - \log_e x)}{x} + \frac{e^{-\gamma} \cdot \log_e x}{x \cdot \log_e n} \\ & \sim (\frac{\pi_L(p_n^2)}{p_n^2} \cdot \frac{p_n^2}{x} + \frac{e^{-\gamma}}{\log_e n} (1 - \frac{p_n^2}{x})) \cdot \frac{(1 - 2 \cdot \log_e p_n)}{p_n^2} + \frac{2 \cdot e^{-\gamma} \cdot \log_e p_n}{p_n^2 \cdot \log_e n} \end{aligned}$$

Since each term $\rightarrow 0$ as $n \rightarrow \infty$, we conclude that the function $\pi_L(x)/\frac{x}{\log_e x}$ does not oscillate but tends to a limit as $x \rightarrow \infty$ since:

Lemma 3.15. $(\pi_L(x)/\frac{x}{\log_e x})' \in o(1)$. □

3.H. An elementary probability-based proof of the Prime Number Theorem

The above now yields an elementary probability-based proof that:

Theorem 3.16. $\pi(x) \sim x/\log_e x$

⁵⁰[HW60], Theorem 429, p.351.

Proof: By Lemma 3.15 $(\pi_L(x)/\frac{x}{\log_e x})' \in o(1)$; whence the function $\pi_L(x)/\frac{x}{\log_e x}$ does not oscillate but tends to a limit as $x \rightarrow \infty$.

Since $p_{n+1}^2 - p_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, and $\pi(x) \sim \pi_L(x)$ by the Law of Large Numbers, the theorem follows from Chebyshev's Theorem that $\pi(x) \asymp x/\log_e x$. \square

3.I. Primes in an arithmetic progression

We consider next Dirichlet's Theorem, which is the assertion that if a and d are co-prime and $1 \leq a < d$, then the arithmetic progression $a + m.d$, where $m \geq 1$, contains an infinitude of (Dirichlet) primes.

We first note that Lemma 3.5 can be extended to prime powers in general⁵¹:

Lemma 3.17. *If p_i and p_j are two primes where $i \neq j$ then, for any $n \geq 2$, $\alpha, \beta \geq 1$, we have:*

$$\mathbb{P}((r_{p_i^\alpha}(n) = u) \cap (r_{p_j^\beta}(n) = v)) = \mathbb{P}(r_{p_i^\alpha}(n) = u) \cdot \mathbb{P}(r_{p_j^\beta}(n) = v)$$

where $p_i^\alpha > u \geq 0$ and $p_j^\beta > v \geq 0$.

Proof: The $p_i^\alpha \cdot p_j^\beta$ numbers $v \cdot p_i^\alpha + u \cdot p_j^\beta$, where $p_i^\alpha > u \geq 0$ and $p_j^\beta > v \geq 0$, are all incongruent and form a complete system of residues⁵² mod $(p_i^\alpha \cdot p_j^\beta)$. Hence:

$$\mathbb{P}((r_{p_i^\alpha}(n) = u) \cap (r_{p_j^\beta}(n) = v)) = \frac{1}{p_i^\alpha \cdot p_j^\beta}$$

By Lemma 3.2:

$$\mathbb{P}(r_{p_i^\alpha}(n) = u) \cdot \mathbb{P}(r_{p_j^\beta}(n) = v) = \left(\frac{1}{p_i^\alpha}\right) \left(\frac{1}{p_j^\beta}\right).$$

The lemma follows. \square

If $u = 0$ and $v = 0$ in Lemma 3.17, so that both p_i and p_j are prime divisors of n , we immediately conclude by Definition 2 that:

Corollary 3.18. $\mathbb{P}((r_{p_i^\alpha}(n) = 0) \cap (r_{p_j^\beta}(n) = 0)) = \mathbb{P}(r_{p_i^\alpha}(n) = 0) \cdot \mathbb{P}(r_{p_j^\beta}(n) = 0)$. \square

We can also express this as:

Corollary 3.19. $\mathbb{P}(p_i^\alpha | n \cap p_j^\beta | n) = \mathbb{P}(p_i^\alpha | n) \cdot \mathbb{P}(p_j^\beta | n)$. \square

We thus conclude that:

Theorem 3.20. *For any two primes $p \neq q$ and natural numbers $n, \alpha, \beta \geq 1$, whether or not p^α divides n is independent of whether or not q^β divides n .* \square

⁵¹Hint: The following arguments may be easier to follow if we visualise the residues $r_{p_i^\alpha}(n)$ and $r_{p_j^\beta}(n)$ as they would occur in §5., Fig.6 and Fig.7.

⁵²[HW60], p.52, Theorem 59.

3.I.a. The probability that n is a prime of the form $a + m.d$

We note next that:

Lemma 3.21. *For any co-prime natural numbers $1 \leq a < d = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \dots q_k^{\alpha_k}$ where:*

$$q_1 < q_2 < \dots < q_k \text{ are primes and } \alpha_1, \alpha_2 \dots \alpha_k \geq 1 \text{ are natural numbers;}$$

the natural number n is of the form $a + m.d$ for some natural number $m \geq 1$ if, and only if:

$$a + r_{q_i^{\alpha_i}}(n) \equiv 0 \pmod{q_i^{\alpha_i}} \text{ for all } 1 \leq i \leq k$$

where $0 \leq r_i(n) < i$ is defined for all $i > 1$ by:

$$n + r_i(n) \equiv 0 \pmod{i} .$$

Proof: First, if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \dots q_k^{\alpha_k}$, then:

$$\begin{aligned} n &\equiv a \pmod{d} \\ \text{and: } n + r_{q_i^{\alpha_i}}(n) &\equiv 0 \pmod{q_i^{\alpha_i}} \text{ for all } 1 \leq i \leq k \\ \text{whence: } a + r_{q_i^{\alpha_i}}(n) &\equiv 0 \pmod{q_i^{\alpha_i}} \text{ for all } 1 \leq i \leq k \end{aligned}$$

Second:

$$\begin{aligned} \text{If: } a + r_{q_i^{\alpha_i}}(n) &\equiv 0 \pmod{q_i^{\alpha_i}} \text{ for all } 1 \leq i \leq k \\ \text{and: } n + r_{q_i^{\alpha_i}}(n) &\equiv 0 \pmod{q_i^{\alpha_i}} \text{ for all } 1 \leq i \leq k \\ \text{then: } n - a &\equiv 0 \pmod{q_i^{\alpha_i}} \text{ for all } 1 \leq i \leq k \\ \text{whence: } n &\equiv a \pmod{d} \end{aligned}$$

The Lemma follows. □

By Lemma 3.2, it follows that:

Corollary 3.22. *The probability that $a + r_{q_i^{\alpha_i}}(n) \equiv 0 \pmod{q_i^{\alpha_i}}$ for any $1 \leq i \leq k$ is $\frac{1}{q_i^{\alpha_i}}$.* □

By Theorem 3.20, it further follows that:

Corollary 3.23. *The joint probability that $a + r_{q_i^{\alpha_i}}(n) \equiv 0 \pmod{q_i^{\alpha_i}}$ for all $1 \leq i \leq k$ is $\prod_{i=1}^k \frac{1}{q_i^{\alpha_i}}$.* □

We conclude by Lemma 3.21 that:

Corollary 3.24. *The probability that n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \dots q_k^{\alpha_k}$ is $\prod_{i=1}^k \frac{1}{q_i^{\alpha_i}}$.* □

It follows that:

Corollary 3.25. *The probability $\mathbb{P}(n \in \{p\} \cap n \in \{a + m.d\})$ that n is a Dirichlect prime of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1} . q_2^{\alpha_2} \dots q_k^{\alpha_k}$ is:*

$$\prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k \left(1 - \frac{1}{q_i}\right)^{-1} \cdot \mathbb{P}(n \in \{p\}).$$

Proof: Since a, d are co-prime, we have by Lemma 3.21 that if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1} . q_2^{\alpha_2} \dots q_k^{\alpha_k}$, we have that:

$$\begin{aligned} n &\equiv a \pmod{q_i} && \text{for all } 1 \leq i \leq k \\ \text{whilst : } n + r_i(n) &\equiv 0 \pmod{i} && \text{for all } 1 \leq i \\ \text{whence : } a + r_{q_i}(n) &\equiv 0 \pmod{q_i} && \text{for all } 1 \leq i \leq k \\ r_{q_i}(n) &\neq 0 && \text{for all } 1 \leq i \leq k \\ \text{and : } q_i &\nmid n && \text{for all } 1 \leq i \leq k \end{aligned}$$

Hence, if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1} . q_2^{\alpha_2} \dots q_k^{\alpha_k}$ and $(a, d) = 1$, the probability that $q_i \nmid n$ for all $1 \leq i \leq k$ is 1.

By Lemma 3.10, Theorem 3.11 and Theorem 3.20, the probability that any $n \geq q_k^2$ is a Dirichlect prime of the form $a + m.d$ is thus:

$$\begin{aligned} &\prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{\substack{2 \leq p \leq \sqrt{n} \\ p \neq q_i \text{ for } 1 \leq i \leq k}} \left(1 - \frac{1}{p}\right) \\ &= \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k \left(1 - \frac{1}{q_i}\right)^{-1} \cdot \prod_{2 \leq p \leq \sqrt{n}} \left(1 - \frac{1}{p}\right) \\ &= \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k \left(1 - \frac{1}{q_i}\right)^{-1} \cdot \prod_{j=1}^{\pi(\sqrt{n})} \left(1 - \frac{1}{p_j}\right) \\ &= \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k \left(1 - \frac{1}{q_i}\right)^{-1} \cdot \mathbb{P}(n \in \{p\}) \end{aligned}$$

The Corollary follows. □

3.I.b. An elementary probability-based proof of Dirichlect's Theorem

It further follows from Theorem 3.11 that, since $p_{n+1}^2 - p_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, by the Law of Large Numbers⁵³ the expected value of the number $\pi_{(a,d)}(n)$ of Dirichlect primes, of the form $a + m.d$ for some natural number $m \geq 1$ and $1 \leq a < d = q_1^{\alpha_1} . q_2^{\alpha_2} \dots q_k^{\alpha_k}$, that are less than or equal to any $n \geq q_k^2$ is given by the Dirichlect prime counting function:

$$\text{Definition 6. } \pi_D(n) = \sum_{l=1}^n \left(\prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k \left(1 - \frac{1}{q_i}\right)^{-1} \cdot \mathbb{P}(l \in \{p\}) \right).$$

We conclude that:

Lemma 3.26. $\pi_{(a,d)}(n) \sim \pi_D(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Proof: If a, d are co-prime and $1 \leq a < d = q_1^{\alpha_1} . q_2^{\alpha_2} \dots q_k^{\alpha_k}$, we have for any $n \geq q_k^2$:

⁵³See §4., Appendix I; also [Ko56], Chapter VI, §3, pg. 61.

$$\begin{aligned}
\pi_D(n) &= \sum_{l=1}^n (\prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k (1 - \frac{1}{q_i})^{-1} \cdot \mathbb{P}(l \in \{p\})) \\
&= \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k (1 - \frac{1}{q_i})^{-1} \cdot \pi_L(n) \\
&= \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k (1 - \frac{1}{q_i})^{-1} \cdot \sum_{l=1}^n \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j}) \\
&\geq \prod_{i=1}^k \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^k (1 - \frac{1}{q_i})^{-1} \cdot n \cdot \prod_{j=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_j})
\end{aligned}$$

The lemma follows since, by Mertens' Theorem, $\prod_{p \leq x} (1 - \frac{1}{p}) \sim \frac{e^{-\lambda}}{\log_e x}$, we have that:

$$n \cdot \prod_{j=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_j}) \sim \frac{2e^{-\gamma} n}{\log_e(n)} \rightarrow \infty \text{ as } n \rightarrow \infty. \quad \square$$

We conclude by the Law of Large Numbers when applied to the interval, $p_{n+1}^2 - p_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, that:

Theorem 3.27. *There are an infinity of primes in any arithmetic progression $a + m \cdot d$ where $(a, d) = 1$ ⁵⁴.* □

3.J. An elementary probability-based proof that there are infinitely many twin-primes

We next note that, by Theorem 3.11, we can define the twin-prime counting function $\pi_T(n)$, which gives the expected value of the number $\pi_2(n)$ of twin primes $(p_i, p_{i+1} = p_i + 2)$ for $3 \leq p_i \leq n$ as:

$$\text{Definition 7. } \pi_T(n) = \sum_{j=1}^n \mathbb{P}(j \in \{p\} \cap j + 2 \in \{p\})$$

In order to estimate $\pi_T(n)$, we first define:

Definition 8. *An integer n is a $\mathbb{T}\mathbb{W}$ integer if, and only if, $r_{p_i}(n) \neq 0$ and $r_{p_i}(n) \neq 2$ for all $1 \leq i \leq \pi(\sqrt{n})$.*

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, we then have that:

Lemma 3.28. *If n is a $\mathbb{T}\mathbb{W}$ integer, then n is a prime.*

Proof: The lemma follows immediately from Definition 8, Definition 1 and Lemma 3.1. □

Lemma 3.29. *If n is a $\mathbb{T}\mathbb{W}$ integer, then $n + 2$ is either a prime or $p_{\pi(\sqrt{n})+1}^2$.*

Proof: By Definition 8 and Definition 1:

$$\begin{aligned}
r_{p_i}(n) &\neq 2 \text{ for all } 1 \leq i \leq \pi(\sqrt{n}) \\
n + 2 &\neq \lambda \cdot i \text{ for all } 2 \leq i \leq p_{\pi(\sqrt{n})}, \lambda \geq 1
\end{aligned}$$

Hence, if $n + 2$ is divisible by $p_{\pi(\sqrt{n})+1}$, then $n + 2 = p_{\pi(\sqrt{n})+1}^2$; else it is a prime. □

Since each residue $r_i(n)$ cycles over the i values $(i - 1, i - 2, \dots, 0)$, these values are all incongruent and form a complete system of residues *mod* i . It thus follows from Definition 8 and Section 3.B. that the probability of $n \geq 9$ being a $\mathbb{T}\mathbb{W}$ integer is:

⁵⁴Compare [HW60], p.13, Theorem 15*.

Lemma 3.30. $\mathbb{P}(n \in \{\text{TW}\}) = \prod_{i=2}^{\pi(\sqrt{n})} (1 - \frac{2}{p_i})$. □

The number $\pi_{\text{TW}}(n)$ of TW integers ≥ 9 but $\leq n$ is thus:

Lemma 3.31. $\pi_{\text{TW}}(n) = \sum_{j=9}^n \prod_{i=2}^{\pi(\sqrt{j})} (1 - \frac{2}{p_i})$. □

Since the number of TW integers such that $n + 2 = p_{\pi(\sqrt{n})+1}^2$ is not more than $\pi(\sqrt{n})$, it also follows that, for $n \geq 9$:

Lemma 3.32. $\pi_T(n) \geq \sum_{j=9}^n \prod_{i=2}^{\pi(\sqrt{j})} (1 - \frac{2}{p_i}) - \pi(\sqrt{n})$. □

We further note that:

Theorem 3.33. $\pi_T(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Proof: We have by Lemma 3.32 that, for $n \geq 9$:

$$\begin{aligned} \pi_T(n) &\geq (n - 9) \cdot \prod_{i=2}^{\pi(\sqrt{n})} (1 - \frac{2}{p_i}) - \pi(\sqrt{n}) \\ &\geq (n - 9) \cdot \prod_{i=2}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i})(1 - \frac{1}{(p_i-1)}) - \pi(\sqrt{n}) \\ &\geq (n - 9) \cdot \prod_{i=2}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i})(1 - \frac{1}{p_{i-1}}) - \pi(\sqrt{n}) \\ &\geq (n - 9) \cdot \prod_{i=2}^{\pi(\sqrt{n})} (1 - \frac{1}{p_{i-1}})^2 - \pi(\sqrt{n}) \\ &\geq (n - 9) \cdot \prod_{i=1}^n (1 - \frac{1}{p_i})^2 - \pi(\sqrt{n}) \end{aligned}$$

Now, by Chebyshev's and Mertens' Theorems, we have that:

$$\begin{aligned} (n - 9) \cdot \prod_{i=1}^n (1 - \frac{1}{p_i})^2 - \pi(\sqrt{n}) &\sim (n - 9) \cdot (\frac{e^{-\gamma}}{\log_e n})^2 - \pi(\sqrt{n}) \\ &\sim e^{-2\gamma} \cdot \frac{n}{\log_e^2 n} - \frac{9e^{-2\gamma}}{\log_e^2 n} - O(\frac{\sqrt{n}}{\log_e n}) \\ &\rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

The theorem follows. □

Since $p_{n+1}^2 - p_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, it follows by the Law of Large Numbers that $\pi_2(n) \sim \pi_T(n) \sim \pi_{\text{TW}}(n)$. We conclude that there are infinitely many twin primes, and that⁵⁵:

Corollary 3.34. $\pi_2(n) \sim e^{-2\gamma} \cdot \frac{n}{\log_e^2 n}$. □

3.K. The Generalised Prime Counting Function: $\sum_{j=1}^n \prod_{i=a}^{\pi(\sqrt{j})} (1 - \frac{b}{p_i})$

We note that the argument of Theorem 3.33 in §3.J. is a special case of the limiting behaviour of the Generalised Prime Counting Function $\sum_{j=1}^n \prod_{i=a}^{\pi(\sqrt{j})} (1 - \frac{b}{p_i})$, which estimates the number of integers $\leq n$ such that there are b values that cannot occur amongst the residues $r_{p_i}(n)$ for $a \leq i \leq \pi(\sqrt{j})$ ⁵⁶:

⁵⁵Where $e^{-2\gamma} = 0.3152373316\dots$; compare [HW60], p.371, §22.20: $\pi_2(n) \sim 2C_2 \cdot \frac{n}{\log_e^2 n}$, where $C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.66016181584\dots$

⁵⁶Thus $b = 1$ yields an estimate for the number of primes $\leq n$, and $b = 2$ an estimate for the number of TW primes (Definition 8) $\leq n$.

Theorem 3.35. $\sum_{j=1}^n \prod_{i=a}^{\pi(\sqrt{j})} (1 - \frac{b}{p_i}) \rightarrow \infty$ as $n \rightarrow \infty$ if $p_a > b \geq 1$.

Proof: For $p_a > b \geq 1$, we have that:

$$\begin{aligned} \sum_{j=1}^n \prod_{i=a}^{\pi(\sqrt{j})} (1 - \frac{b}{p_i}) &\geq \sum_{j=p_a^2}^n \prod_{i=a}^{\pi(\sqrt{j})} (1 - \frac{b}{p_i}) \\ &\geq \sum_{j=p_a^2}^n \prod_{i=a}^{\pi(\sqrt{n})} (1 - \frac{b}{p_i}) \\ &\geq (n - p_a^2) \cdot \prod_{i=a}^{\pi(\sqrt{n})} (1 - \frac{b}{p_i}) \\ &\geq (n - p_a^2) \cdot \prod_{i=a}^n (1 - \frac{b}{p_i}) \end{aligned}$$

The theorem follows if:

$$\log_e(n - p_a^2) + \sum_{i=a}^n \log_e(1 - \frac{b}{p_i}) \rightarrow \infty$$

(i) We note first the standard result for $|x| < 1$ that:

$$\log_e(1 - x) = - \sum_{m=1}^{\infty} \frac{x^m}{m}$$

For any $p_i > b \geq 1$, we thus have:

$$\log_e(1 - \frac{b}{p_i}) = - \sum_{m=1}^{\infty} \frac{(b/p_i)^m}{m} = -\frac{b}{p_i} - \sum_{m=2}^{\infty} \frac{(b/p_i)^m}{m}$$

Hence:

$$\sum_{i=a}^n \log_e(1 - \frac{b}{p_i}) = - \sum_{i=a}^n (\frac{b}{p_i}) - \sum_{i=a}^n (\sum_{m=2}^{\infty} \frac{(b/p_i)^m}{m})$$

(ii) We note next that, for all $i \geq a$:

$$c < (1 - \frac{b}{p_a}) \rightarrow c < (1 - \frac{b}{p_i})$$

It follows for any such c that:

$$\sum_{m=2}^{\infty} \frac{(b/p_i)^m}{m} \leq \sum_{m=2}^{\infty} (\frac{b}{p_i})^m = \frac{(b/p_i)^2}{1 - b/p_i} \leq \frac{b^2}{c \cdot p_i^2}$$

Since:

$$\sum_{i=1}^{\infty} \frac{1}{p_i^2} = O(1)$$

it further follows that:

$$\sum_{i=a}^n (\sum_{m=2}^{\infty} \frac{(b/p_i)^m}{m}) \leq \sum_{i=a}^n (\frac{b^2}{c \cdot p_i^2}) = O(1)$$

(iii) From the standard result⁵⁷:

$$\sum_{p \leq x} \frac{1}{p} = \log_e \log_e x + O(1) + o(1)$$

it then follows that:

$$\begin{aligned} \sum_{i=a}^n \log_e \left(1 - \frac{b}{p_i}\right) &\geq -\sum_{i=a}^n \left(\frac{b}{p_i}\right) - O(1) \\ &\geq -b \cdot (\log_e \log_e n + O(1) + o(1)) - O(1) \end{aligned}$$

The theorem follows since:

$$\log_e(n - p_a^2) - b \cdot (\log_e \log_e n + O(1) + o(1)) - O(1) \rightarrow \infty$$

and so:

$$\log_e(n - p_a^2) + \sum_{i=a}^n \log_e \left(1 - \frac{b}{p_i}\right) \rightarrow \infty \quad \square$$

4. Appendix I: Definitions of some terms and concepts of Probability Theory

Probability model⁵⁸: A *probability model* is a mathematical representation of a random phenomenon. It is defined by its sample space, events within the sample space, and probabilities associated with each event.

- *The sample space* S for a probability model is the set of all possible outcomes.
- *An event* A is a subset of the sample space S .
- *A probability* is a numerical value assigned to a given event A .

Distribution Function⁵⁹: Let X be a random variable which denotes the value of the outcome of a certain experiment, and assume that this experiment has only finitely many possible outcomes. Let Ω be the sample space of the experiment (i.e., the set of all possible values of X , or equivalently, the set of all possible outcomes of the experiment). A *distribution function* for X is a real-valued function m whose domain is Ω and which satisfies:

1. $m(\omega) \geq 0$, for all $\omega \in \Omega$, and
2. $\sum_{\omega \in \Omega} m(\omega) = 1$.

For any subset E of Ω , we define the *probability* of E to be the number $P(E)$ given by

$$P(E) = \sum_{\omega \in E} m(\omega)$$

Some notations⁶⁰: Let A and B be two sets. Then the union of A and B is the set

⁵⁷[HW60], p.351, Theorem 427.

⁵⁸cf. <http://www.stat.yale.edu/Courses/1997-98/101/probint.htm>.

⁵⁹Excerpted from [GS97], Chapter 1, §1.2, p.19.

⁶⁰Excerpted from [GS97], Chapter 1, §1.2, p.21.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The intersection of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The difference of A and B is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

The set A is a subset of B , written $A \subset B$, if every element of A is also an element of B . Finally, the complement of A is the set

$$\bar{A} = \{x \mid x \in \Omega \text{ and } x \notin A\}.$$

Mutual Independence⁶¹: A set of events $\{A_1, A_2, \dots, A_n\}$ is said to be *mutually independent* if for any subset $\{A_i, A_j, \dots, A_m\}$ of these events we have

$$P(A_i \cap A_j \cap \dots \cap A_m) = P(A_i)P(A_j) \dots P(A_m),$$

or equivalently, if for any sequence $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_n$ with $\bar{A}_j = A_j$ or \bar{A}_j ,

$$P(\bar{A}_i \cap \bar{A}_j \cap \dots \cap \bar{A}_m) = P(\bar{A}_i)P(\bar{A}_j) \dots P(\bar{A}_m).$$

Expected Value⁶²: Let X be a numerically-valued discrete random variable with sample space Ω and distribution function $m(x)$. The *expected value* $E(X)$ is defined by:

$$E(X) = \sum_{x \in \Omega} xm(x),$$

provided this sum converges absolutely. We often refer to the expected value as the mean, and denote $E(X)$ by μ for short. If the above sum does not converge absolutely, then we say that X does not have an expected value.

Law of Large Numbers⁶³: Let X_1, X_2, \dots, X_n be an independent trials process, with finite expected value $\mu = E(X_j)$ and finite variance $\sigma^2 = V(X_j)$. Let $S_n = X_1 + X_2 + \dots + X_n$. Then for any $\epsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \rightarrow 0$$

as $n \rightarrow \infty$. Equivalently,

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \epsilon\right) \rightarrow 1$$

as $n \rightarrow \infty$.

⁶¹Excerpted from [GS97], Chapter 4, §4.1, Definition 4.2, p.141.

⁶²Excerpted from [GS97], Chapter 5, §5.1, p.183.

⁶³Excerpted from [GS97], Chapter 8, §8.1, p.307, Theorem 8.2.

5. Appendix II: The residue function $r_i(n)$

We graphically illustrate how the residues $r_i(n)$ occur naturally as values of:

A: The natural-number based residue functions $R_i(n)$;

B: The natural-number based residue sequences $E(n)$;

and as the output of:

C: The natural-number based algorithm $E_{\mathbb{N}}$;

D: The prime-number based algorithm $E_{\mathbb{P}}$;

E: The prime-number based algorithm $E_{\mathbb{Q}}$.

A: The natural-number based residue functions $R_i(n)$

The residues $r_i(n)$ can be defined for all $n \geq 1$ as the values of the natural-number based residue functions $R_i(n)$, defined for all $i \geq 1$ as below in Fig.6. We note that each function $R_i(n)$ cycles through the values $(i - 1, i - 2, \dots, 0)$ with period i .

Fig.6: The natural-number based residue functions $R_i(n)$

Function: $R_1n \ R_2n \ R_3n \ R_4n \ R_5n \ R_6n \ R_7n \ R_8n \ R_9n \ R_{10}n \ R_{11}n \ \dots \ R_n n$

$n = 1$	0	1	2	3	4	5	6	7	8	9	10	... n-1
$n = 2$	0	0	1	2	3	4	5	6	7	8	9	... n-2
$n = 3$	0	1	0	1	2	3	4	5	6	7	8	... n-3
$n = 4$	0	0	2	0	1	2	3	4	5	6	7	... n-4
$n = 5$	0	1	1	3	0	1	2	3	4	5	6	... n-5
$n = 6$	0	0	0	2	4	0	1	2	3	4	5	... n-6
$n = 7$	0	1	2	1	3	5	0	1	2	3	4	... n-7
$n = 8$	0	0	1	0	2	4	6	0	1	2	3	... n-8
$n = 9$	0	1	0	3	1	3	5	7	0	1	2	... n-9
$n = 10$	0	0	2	2	0	2	4	6	8	0	1	... n-10
$n = 11$	0	1	1	1	4	1	3	5	7	9	0	... n-11

$n \quad r_1n \ r_2n \ r_3n \ r_4n \ r_5n \ r_6n \ r_7n \ r_8n \ r_9n \ r_{10}n \ r_{11}n \ \dots 0$

Fig.6: The natural-number based residue functions $R_i(n)$

B: The natural-number based residue sequences $E(n)$

The above residues $r_i(n)$ can also be viewed alternatively as values of the associated residue sequences, $E(n) = \{r_i(n) : i \geq 1\}$, defined for all $n \geq 1$, as illustrated below in Fig.7.

We note that:

- The sequences highlighted in red identify a prime⁶⁴ p (since $r_i(p) \neq 0$ for $1 < i < p$);
- The ‘boundary’ residues $r_1(n) = 0$ and $r_n(n) = 0$ are identified in cyan.

⁶⁴Conventionally defined as integers that are not divisible by any smaller integer other than 1.

Fig.7: The natural-number based residue sequences $E(n)$

Function: $R_1n R_2n R_3n R_4n R_5n R_6n R_7n R_8n R_9n R_{10}n R_{11}n \dots R_n n$

$E(1):$	0	1	2	3	4	5	6	7	8	9	10	...	n-1
$E(2):$	0	0	1	2	3	4	5	6	7	8	9	...	n-2
$E(3):$	0	1	0	1	2	3	4	5	6	7	8	...	n-3
$E(4):$	0	0	2	0	1	2	3	4	5	6	7	...	n-4
$E(5):$	0	1	1	3	0	1	2	3	4	5	6	...	n-5
$E(6):$	0	0	0	2	4	0	1	2	3	4	5	...	n-6
$E(7):$	0	1	2	1	3	5	0	1	2	3	4	...	n-7
$E(8):$	0	0	1	0	2	4	6	0	1	2	3	...	n-8
$E(9):$	0	1	0	3	1	3	5	7	0	1	2	...	n-9
$E(10):$	0	0	2	2	0	2	4	6	8	0	1	...	n-10
$E(11):$	0	1	1	1	4	1	3	5	7	9	0	...	n-11
...													
$E(n):$	r_1n	r_2n	r_3n	r_4n	r_5n	r_6n	r_7n	r_8n	r_9n	$r_{10}n$	$r_{11}n$...	0
...													

Fig.7: The natural-number based residue sequences $E(n)$ **C: The output of a natural-number based algorithm $E_{\mathbb{N}}$**

We give below in Fig.8 the output for $1 \leq n \leq 11$ of a natural-number based algorithm $E_{\mathbb{N}}$ that computes the values $r_i(n)$ of the sequence $E_{\mathbb{N}}(n)$ for only $1 \leq i \leq n$ for any given n .

Fig.8: The output of the natural-number based algorithm $E_{\mathbb{N}}$

Divisors:	1	2	3	4	5	6	7	8	9	10	11	...	n	...
$E_{\mathbb{N}}(1):$	0													
$E_{\mathbb{N}}(2):$	0	0												
$E_{\mathbb{N}}(3):$	0	1	0											
$E_{\mathbb{N}}(4):$	0	0	2	0										
$E_{\mathbb{N}}(5):$	0	1	1	3	0									
$E_{\mathbb{N}}(6):$	0	0	0	2	4	0								
$E_{\mathbb{N}}(7):$	0	1	2	1	3	5	0							
$E_{\mathbb{N}}(8):$	0	0	1	0	2	4	6	0						
$E_{\mathbb{N}}(9):$	0	1	0	3	1	3	5	7	0					
$E_{\mathbb{N}}(10):$	0	0	2	2	0	2	4	6	8	0				
$E_{\mathbb{N}}(11):$	0	1	1	1	4	1	3	5	7	9	0			
...														
$E_{\mathbb{N}}(n):$	r_1n	r_2n	r_3n	r_4n	r_5n	r_6n	r_7n	r_8n	r_9n	$r_{10}n$	$r_{11}n$...	0	
...														

Fig.8: The output of the natural-number based algorithm $E_{\mathbb{N}}$

D: The output of the prime-number based algorithm $E_{\mathbb{P}}$

We give below in Fig.9 the output for $2 \leq n \leq 31$ of a prime-number based algorithm $E_{\mathbb{Q}}$ that computes the values $q_i(n) = r_{p_i}(n)$ of the sequence $E_{\mathbb{P}}(n)$ for only each prime $2 \leq p_i \leq n$ for any given n .

Fig.9: The output of the prime-number based algorithm $E_{\mathbb{P}}$

Prime:	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	$\dots p_n \dots$
Divisor:	2	3	5	7	11	13	17	19	23	29	31	$\dots p_n \dots$

$E_{\mathbb{P}}(2):$	0											
$E_{\mathbb{P}}(3):$	1	0										
$E_{\mathbb{P}}(4):$	0	2										
$E_{\mathbb{P}}(5):$	1	1	0									
$E_{\mathbb{P}}(6):$	0	0	4									
$E_{\mathbb{P}}(7):$	1	2	3	0								
$E_{\mathbb{P}}(8):$	0	1	2	6								
$E_{\mathbb{P}}(9):$	1	0	1	5								
$E_{\mathbb{P}}(10):$	0	2	0	4								
$E_{\mathbb{P}}(11):$	1	1	4	3	0							
$E_{\mathbb{P}}(12):$	0	0	3	2	10							
$E_{\mathbb{P}}(13):$	1	2	2	1	9	0						
$E_{\mathbb{P}}(14):$	0	1	1	0	8	12						
$E_{\mathbb{P}}(15):$	1	0	0	6	7	11						
$E_{\mathbb{P}}(16):$	0	2	4	5	6	10						
$E_{\mathbb{P}}(17):$	1	1	3	4	5	9	0					
$E_{\mathbb{P}}(18):$	0	0	2	3	4	8	16					
$E_{\mathbb{P}}(19):$	1	2	1	2	3	7	15	0				
$E_{\mathbb{P}}(20):$	0	1	0	1	2	6	14	18				
$E_{\mathbb{P}}(21):$	1	0	4	0	1	5	13	17				
$E_{\mathbb{P}}(22):$	0	2	3	6	0	4	12	16				
$E_{\mathbb{P}}(23):$	1	1	2	5	10	3	11	15	0			
$E_{\mathbb{P}}(24):$	0	0	1	4	9	2	10	14	22			
$E_{\mathbb{P}}(25):$	1	2	0	3	8	1	9	13	21			
$E_{\mathbb{P}}(26):$	0	1	4	2	7	0	8	12	20			
$E_{\mathbb{P}}(27):$	1	0	3	1	6	12	7	11	19			
$E_{\mathbb{P}}(28):$	0	2	2	0	5	11	6	10	18			
$E_{\mathbb{P}}(29):$	1	1	1	6	4	10	5	9	17	0		
$E_{\mathbb{P}}(30):$	0	0	0	5	3	9	4	8	16	28		
$E_{\mathbb{P}}(31):$	1	2	4	4	2	8	3	7	15	27	0	
\dots												
$E_{\mathbb{P}}(n):$	$q_1 n$	$q_2 n$	$q_3 n$	$q_4 n$	$q_5 n$	$q_6 n$	$q_7 n$	$q_8 n$	$q_9 n$	$q_{10} n$	$q_{11} n$	$\dots 0$
\dots												

Fig.9: The output of the prime-number based algorithm $E_{\mathbb{P}}$

E: The output of the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$

We give below in Fig.10 the output for $2 \leq n \leq 121$ of the two prime-number based algorithms $E_{\mathbb{P}}$ (whose output $\{q_i(n) = r_{p_i}(n) : 1 \leq i \leq \pi(n)\}$ is shown only partially, partly in cyan) and $E_{\mathbb{Q}}$ (whose

output $q_i(n) = \{r_{p_i}(n) : 1 \leq i \leq \pi(\sqrt{n})\}$ is highlighted in black and red, the latter indicating the generation of a prime sequence and, ipso facto, definition of the corresponding prime⁶⁵.

Fig.10: The output of the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$

Prime: p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8 p_9 p_{10} p_{11} $\dots p_n \dots$
 Divisor: 2 3 5 7 11 13 17 19 23 29 31 $\dots p_n \dots$
 Function: $Q_1 n$ $Q_2 n$ $Q_3 n$ $Q_4 n$ $Q_5 n$ $Q_6 n$ $Q_7 n$ $Q_8 n$ $Q_9 n$ $Q_{10} n$ $Q_{11} n$ \dots

$E_{\mathbb{Q}}(2)$: 0 (Prime by definition)
 $E_{\mathbb{Q}}(3)$: 1 0
 $E_{\mathbb{Q}}(4)$: 0 2
 $E_{\mathbb{Q}}(5)$: 1 1 0
 $E_{\mathbb{Q}}(6)$: 0 0 4
 $E_{\mathbb{Q}}(7)$: 1 2 3 0
 $E_{\mathbb{Q}}(8)$: 0 1 2 6
 $E_{\mathbb{Q}}(9)$: 1 0 1 5
 $E_{\mathbb{Q}}(10)$: 0 2 0 4
 $E_{\mathbb{Q}}(11)$: 1 1 4 3 0
 $E_{\mathbb{Q}}(12)$: 0 0 3 2 10
 $E_{\mathbb{Q}}(13)$: 1 2 2 1 9 0
 $E_{\mathbb{Q}}(14)$: 0 1 1 0 8 12
 $E_{\mathbb{Q}}(15)$: 1 0 0 6 7 11
 $E_{\mathbb{Q}}(16)$: 0 2 4 5 6 10
 $E_{\mathbb{Q}}(17)$: 1 1 3 4 5 9 0
 $E_{\mathbb{Q}}(18)$: 0 0 2 3 4 8 16
 $E_{\mathbb{Q}}(19)$: 1 2 1 2 3 7 15 0
 $E_{\mathbb{Q}}(20)$: 0 1 0 1 2 6 14 18
 $E_{\mathbb{Q}}(21)$: 1 0 4 0 1 5 13 17
 $E_{\mathbb{Q}}(22)$: 0 2 3 6 0 4 12 16
 $E_{\mathbb{Q}}(23)$: 1 1 2 5 10 3 11 15 0
 $E_{\mathbb{Q}}(24)$: 0 0 1 4 9 2 10 14 22
 $E_{\mathbb{Q}}(25)$: 1 2 0 3 8 1 9 13 21
 $E_{\mathbb{Q}}(26)$: 0 1 4 2 7 0 8 12 20
 $E_{\mathbb{Q}}(27)$: 1 0 3 1 6 12 7 11 19
 $E_{\mathbb{Q}}(28)$: 0 2 2 0 5 11 6 10 18
 $E_{\mathbb{Q}}(29)$: 1 1 1 6 4 10 5 9 17 0
 $E_{\mathbb{Q}}(30)$: 0 0 0 5 3 9 4 8 16 28
 $E_{\mathbb{Q}}(31)$: 1 2 4 4 2 8 3 7 15 27 0
 $E_{\mathbb{Q}}(32)$: 0 1 3 3 1 7 2 6 14 26 30
 $E_{\mathbb{Q}}(33)$: 1 0 2 2 0 6 1 5 13 25 29
 $E_{\mathbb{Q}}(34)$: 0 2 1 1 10 5 0 4 12 24 28
 $E_{\mathbb{Q}}(35)$: 1 1 0 0 9 4 16 3 11 23 27
 $E_{\mathbb{Q}}(36)$: 0 0 4 6 8 3 15 2 10 22 26
 $E_{\mathbb{Q}}(37)$: 1 2 3 5 7 2 14 1 9 21 25
 $E_{\mathbb{Q}}(38)$: 0 1 2 4 6 1 13 0 8 20 24
 $E_{\mathbb{Q}}(39)$: 1 0 1 3 5 0 12 18 7 19 23

⁶⁵For informal reference and perspective, formal definitions of both the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$ are given in this work in progress *Factorising all $m \leq n$ is of order $\Theta(\sum_{i=2}^n \pi(\sqrt{i}))$.*

$E_{\mathbb{Q}}(40)$:	0	2	0	2	4	12	11	17	6	18	22
$E_{\mathbb{Q}}(41)$:	1	1	4	1	3	11	10	16	5	17	21
$E_{\mathbb{Q}}(42)$:	0	0	3	0	2	10	9	15	4	16	20
$E_{\mathbb{Q}}(43)$:	1	2	2	6	1	9	8	14	3	15	19
$E_{\mathbb{Q}}(44)$:	0	1	1	5	0	8	7	13	2	14	18
$E_{\mathbb{Q}}(45)$:	1	0	0	4	10	7	6	12	1	13	17
$E_{\mathbb{Q}}(46)$:	0	2	4	3	9	6	5	11	0	12	16
$E_{\mathbb{Q}}(47)$:	1	1	3	2	8	5	4	10	22	11	15
$E_{\mathbb{Q}}(48)$:	0	0	2	1	7	4	3	9	21	10	14
$E_{\mathbb{Q}}(49)$:	1	2	1	0	6	3	2	8	20	9	13
$E_{\mathbb{Q}}(50)$:	0	1	0	6	5	2	1	7	19	8	12
$E_{\mathbb{Q}}(51)$:	1	0	4	5	4	1	0	6	18	7	11
$E_{\mathbb{Q}}(52)$:	0	2	3	4	3	0	16	5	17	6	10
$E_{\mathbb{Q}}(53)$:	1	1	2	3	2	12	15	4	16	5	9
$E_{\mathbb{Q}}(54)$:	0	0	1	2	1	11	14	3	15	4	8
$E_{\mathbb{Q}}(55)$:	1	2	0	1	0	10	13	2	14	3	7
$E_{\mathbb{Q}}(56)$:	0	1	4	0	10	9	12	1	13	2	6
$E_{\mathbb{Q}}(57)$:	1	0	3	6	9	8	11	0	12	1	5
$E_{\mathbb{Q}}(58)$:	0	2	2	5	8	7	10	18	11	0	4
$E_{\mathbb{Q}}(59)$:	1	1	1	4	7	6	9	17	10	28	3
$E_{\mathbb{Q}}(60)$:	0	0	0	3	6	5	8	16	9	27	2
$E_{\mathbb{Q}}(61)$:	1	2	4	2	5	4	7	15	8	26	1
$E_{\mathbb{Q}}(62)$:	0	1	3	1	4	3	6	14	7	25	0
$E_{\mathbb{Q}}(63)$:	1	0	2	0	3	2	5	13	6	24	30
$E_{\mathbb{Q}}(64)$:	0	2	1	6	2	1	4	12	5	23	29
$E_{\mathbb{Q}}(65)$:	1	1	0	5	1	0	3	11	4	22	28
$E_{\mathbb{Q}}(66)$:	0	0	4	4	0	12	2	10	3	21	27
$E_{\mathbb{Q}}(67)$:	1	2	3	3	10	11	1	9	2	20	26
$E_{\mathbb{Q}}(68)$:	0	1	2	2	9	10	0	8	1	19	25
$E_{\mathbb{Q}}(69)$:	1	0	1	1	8	9	16	7	0	18	24
$E_{\mathbb{Q}}(70)$:	0	2	0	0	7	8	15	6	22	17	23
$E_{\mathbb{Q}}(71)$:	1	1	4	6	6	7	14	5	21	16	22
$E_{\mathbb{Q}}(72)$:	0	0	3	5	5	6	13	4	20	15	21
$E_{\mathbb{Q}}(73)$:	1	2	2	4	4	5	12	3	19	14	20
$E_{\mathbb{Q}}(74)$:	0	1	1	3	3	4	11	2	18	13	19
$E_{\mathbb{Q}}(75)$:	1	0	0	2	2	3	10	1	17	12	18
$E_{\mathbb{Q}}(76)$:	0	2	4	1	1	2	9	0	16	11	17
$E_{\mathbb{Q}}(77)$:	1	1	3	0	0	1	8	18	15	10	16
$E_{\mathbb{Q}}(78)$:	0	0	2	6	10	0	7	17	14	9	15
$E_{\mathbb{Q}}(79)$:	1	2	1	5	9	12	6	16	13	8	14
$E_{\mathbb{Q}}(80)$:	0	1	0	4	8	11	5	15	12	7	13
$E_{\mathbb{Q}}(81)$:	1	0	4	3	7	10	4	14	11	6	12
$E_{\mathbb{Q}}(82)$:	0	2	3	2	6	9	3	13	10	5	11
$E_{\mathbb{Q}}(83)$:	1	1	2	1	5	8	2	12	9	4	10
$E_{\mathbb{Q}}(84)$:	0	0	1	0	4	7	1	11	8	3	9
$E_{\mathbb{Q}}(85)$:	1	2	0	6	3	6	0	10	7	2	8
$E_{\mathbb{Q}}(86)$:	0	1	4	5	2	5	16	9	6	1	7
$E_{\mathbb{Q}}(87)$:	1	0	3	4	1	4	15	8	5	0	6
$E_{\mathbb{Q}}(88)$:	0	2	2	3	0	3	14	7	4	28	5
$E_{\mathbb{Q}}(89)$:	1	1	1	2	10	2	13	6	3	27	4

$E_{\mathbb{Q}}(90):$	0	0	0	1	9	1	12	5	2	26	3	
$E_{\mathbb{Q}}(91):$	1	2	4	0	8	0	11	4	1	25	2	
$E_{\mathbb{Q}}(92):$	0	1	3	6	7	12	10	3	0	24	1	
$E_{\mathbb{Q}}(93):$	1	0	2	5	6	11	9	2	22	23	0	
$E_{\mathbb{Q}}(94):$	0	2	1	4	5	10	8	1	21	22	30	
$E_{\mathbb{Q}}(95):$	1	1	0	3	4	9	7	0	20	21	29	
$E_{\mathbb{Q}}(96):$	0	0	4	2	3	8	6	18	19	20	28	
$E_{\mathbb{Q}}(97):$	1	2	3	1	2	7	5	17	18	19	27	
$E_{\mathbb{Q}}(98):$	0	1	2	0	1	6	4	16	17	18	26	
$E_{\mathbb{Q}}(99):$	1	0	1	6	0	5	3	15	16	17	25	
$E_{\mathbb{Q}}(100):$	0	2	0	5	10	4	2	14	15	16	24	
$E_{\mathbb{Q}}(101):$	1	1	4	4	9	3	1	13	14	15	23	
$E_{\mathbb{Q}}(102):$	0	0	3	3	8	2	0	12	13	14	22	
$E_{\mathbb{Q}}(103):$	1	2	2	2	7	1	16	11	12	13	21	
$E_{\mathbb{Q}}(104):$	0	1	1	1	6	0	15	10	11	12	20	
$E_{\mathbb{Q}}(105):$	1	0	0	0	5	12	14	9	10	11	19	
$E_{\mathbb{Q}}(106):$	0	2	4	6	4	11	13	8	9	10	18	
$E_{\mathbb{Q}}(107):$	1	1	3	5	3	10	12	7	8	9	17	
$E_{\mathbb{Q}}(108):$	0	0	2	4	2	9	11	6	7	8	16	
$E_{\mathbb{Q}}(109):$	1	2	1	3	1	8	10	5	6	7	15	
$E_{\mathbb{Q}}(110):$	0	1	0	2	0	7	9	4	5	6	14	
$E_{\mathbb{Q}}(111):$	1	0	4	1	10	6	8	3	4	5	13	
$E_{\mathbb{Q}}(112):$	0	2	3	0	9	5	7	2	3	4	12	
$E_{\mathbb{Q}}(113):$	1	1	2	6	8	4	6	1	2	3	11	
$E_{\mathbb{Q}}(114):$	0	0	1	5	7	3	5	0	1	2	10	
$E_{\mathbb{Q}}(115):$	1	2	0	4	6	2	4	18	0	1	9	
$E_{\mathbb{Q}}(116):$	0	1	4	3	5	1	3	17	22	0	8	
$E_{\mathbb{Q}}(117):$	1	0	3	2	4	0	2	16	21	28	7	
$E_{\mathbb{Q}}(118):$	0	2	2	1	3	12	1	15	20	27	6	
$E_{\mathbb{Q}}(119):$	1	1	1	0	2	11	0	14	19	26	5	
$E_{\mathbb{Q}}(120):$	0	0	0	6	1	10	16	13	18	25	4	
$E_{\mathbb{Q}}(121):$	1	2	4	5	0	9	15	12	17	24	3	
...												
$E_{\mathbb{Q}}(n):$	$q_1 n$	$q_2 n$	$q_3 n$	$q_4 n$	$q_5 n$	$q_6 n$	$q_7 n$	$q_8 n$	$q_9 n$	$q_{10} n$	$q_{11} n$...
...												

Prime:	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	... p_n ...
Divisor:	2	3	5	7	11	13	17	19	23	29	31	... p_n ...

Fig.10: The output of the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$

6. Appendix III: Distribution of primes ≤ 1500

The following table gives the statistically expected values $\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$, and the actual values $\pi(n)$, of the primes less than or equal to n for $4 \leq n \leq 1500$.

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
1	2	0	1/2	1		0.0000	0
2	3	0	1/3	1	1/2	0.5000	1
3	5	0	4/15	1	1/2	1.0000	2
4	7	1	8/35	2	1/3	1.3333	2
5	11	1	16/77	2	1/3	1.6667	3
6	13	1	89/464	2	1/3	2.0000	3
7	17	1	165/914	2	1/3	2.3333	4
8	19	1	157/918	2	1/3	2.6667	4
9	23	2	62/379	3	4/15	2.9333	4
10	29	2	157/994	3	4/15	3.2000	4
11	31	2	142/929	3	4/15	3.4667	5
12	37	2	29/195	3	4/15	3.7333	5
13	41	2	139/958	3	4/15	4.0000	6
14	43	2	89/628	3	4/15	4.2667	6
15	47	2	62/447	3	4/15	4.5333	6
16	53	2	112/823	4	4/15	4.8000	6
17	59	2	40/299	4	4/15	5.0667	7
18	61	2	5/38	4	4/15	5.3333	7
19	67	2	7/54	4	4/15	5.6000	8
20	71	2	40/313	4	4/15	5.8667	8
21	73	2	15/119	4	4/15	6.1333	8
22	79	2	85/683	4	4/15	6.4000	8
23	83	2	15/122	4	4/15	6.6667	9
24	89	2	31/255	4	4/15	6.9333	9
25	97	3	106/881	5	8/35	7.1619	9
26	101	3	109/915	5	8/35	7.3905	9
27	103	3	86/729	5	8/35	7.6190	9
28	107	3	97/830	5	8/35	7.8476	9
29	109	3	11/95	5	8/35	8.0762	10
30	113	3	7/61	5	8/35	8.3048	10
31	127	3	101/887	5	8/35	8.5333	11
32	131	3	20/177	5	8/35	8.7619	11
33	137	3	47/419	5	8/35	8.9905	11
34	139	3	49/440	5	8/35	9.2190	11
35	149	3	25/226	5	8/35	9.4476	11
36	151	3	10/91	6	8/35	9.6762	11
37	157	3	63/577	6	8/35	9.9048	12
38	163	3	79/728	6	8/35	10.1333	12
39	167	3	48/445	6	8/35	10.3619	12
40	173	3	77/718	6	8/35	10.5905	12
41	179	3	61/572	6	8/35	10.8190	13
42	181	3	7/66	6	8/35	11.0476	13
43	191	3	94/891	6	8/35	11.2762	14
44	193	3	89/848	6	8/35	11.5048	14
45	197	3	26/249	6	8/35	11.7333	14
46	199	3	8/77	6	8/35	11.9619	14
47	211	3	76/735	6	8/35	12.1905	15
48	223	3	7/68	6	8/35	12.4190	15
49	227	4	33/322	7	16/77	12.6268	15
50	229	4	5/49	7	16/77	12.8346	15
51	233	4	19/187	7	16/77	13.0424	15
52	239	4	43/425	7	16/77	13.2502	15
53	241	4	40/397	7	16/77	13.4580	16
54	251	4	57/568	7	16/77	13.6658	16
55	257	4	1/10	7	16/77	13.8736	16
56	263	4	24/241	7	16/77	14.0814	16
57	269	4	63/635	7	16/77	14.2892	16
58	271	4	60/607	7	16/77	14.4970	16
59	277	4	13/132	7	16/77	14.7048	17
60	281	4	58/591	7	16/77	14.9126	17
61	283	4	31/317	7	16/77	15.1203	18
62	293	4	23/236	7	16/77	15.3281	18
63	307	4	17/175	7	16/77	15.5359	18
64	311	4	58/599	8	16/77	15.7437	18
65	313	4	61/632	8	16/77	15.9515	18
66	317	4	61/634	8	16/77	16.1593	18
67	331	4	40/417	8	16/77	16.3671	19
68	337	4	68/711	8	16/77	16.5749	19
69	347	4	72/755	8	16/77	16.7827	19
70	349	4	31/326	8	16/77	16.9905	19
71	353	4	11/116	8	16/77	17.1983	20
72	359	4	33/349	8	16/77	17.4061	20
73	367	4	43/456	8	16/77	17.6139	21
74	373	4	79/840	8	16/77	17.8216	21
75	379	4	59/629	8	16/77	18.0294	21
76	383	4	45/481	8	16/77	18.2372	21
77	389	4	60/643	8	16/77	18.4450	21
78	397	4	39/419	8	16/77	18.6528	21
79	401	4	61/657	8	16/77	18.8606	22
80	409	4	64/691	8	16/77	19.0684	22
81	419	4	79/855	9	16/77	19.2762	22
82	421	4	33/358	9	16/77	19.4840	22
83	431	4	8/87	9	16/77	19.6918	23
84	433	4	89/970	9	16/77	19.8996	23
85	439	4	13/142	9	16/77	20.1074	23
86	443	4	39/427	9	16/77	20.3152	23
87	449	4	37/406	9	16/77	20.5229	23
88	457	4	1/11	9	16/77	20.7307	23
89	461	4	48/529	9	16/77	20.9385	24
90	463	4	67/740	9	16/77	21.1463	24
91	467	4	73/808	9	16/77	21.3541	24
92	479	4	11/122	9	16/77	21.5619	24
93	487	4	35/389	9	16/77	21.7697	24
94	491	4	51/568	9	16/77	21.9775	24
95	499	4	69/770	9	16/77	22.1853	24
96	503	4	11/123	9	16/77	22.3931	24
97	509	4	54/605	9	16/77	22.6009	25
98	521	4	71/797	9	16/77	22.8087	25
99	523	4	77/866	9	16/77	23.0165	25
100	541	4	71/800	10	16/77	23.2242	25
101	547	4	59/666	10	16/77	23.4320	26
102	557	4	81/916	10	16/77	23.6398	26
103	563	4	73/827	10	16/77	23.8476	27
104	569	4	43/488	10	16/77	24.0554	27
105	571	4	19/216	10	16/77	24.2632	27

n	P_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
106	577	4	85/968	10	16/77	24.4710	27
107	587	4	27/308	10	16/77	24.6788	28
108	593	4	7/80	10	16/77	24.8866	28
109	599	4	65/744	10	16/77	25.0944	29
110	601	4	43/493	10	16/77	25.3022	29
111	607	4	31/356	10	16/77	25.5100	29
112	613	4	2/23	10	16/77	25.7177	29
113	617	4	23/265	10	16/77	25.9255	30
114	619	4	50/577	10	16/77	26.1333	30
115	631	4	77/890	10	16/77	26.3411	30
116	641	4	85/984	10	16/77	26.5489	30
117	643	4	37/429	10	16/77	26.7567	30
118	647	4	80/929	10	16/77	26.9645	30
119	653	4	46/535	10	16/77	27.1723	30
120	659	4	54/629	10	16/77	27.3801	30
121	661	5	3/35	11	89/464	27.5719	30
122	673	5	41/479	11	89/464	27.7637	30
123	677	5	10/117	11	89/464	27.9555	30
124	683	5	46/539	11	89/464	28.1473	30
125	691	5	64/751	11	89/464	28.3391	30
126	701	5	4/47	11	89/464	28.5309	30
127	709	5	56/659	11	89/464	28.7227	31
128	719	5	51/601	11	89/464	28.9146	31
129	727	5	5/59	11	89/464	29.1064	31
130	733	5	60/709	11	89/464	29.2982	31
131	739	5	6/71	11	89/464	29.4900	32
132	743	5	33/391	11	89/464	29.6818	32
133	751	5	59/700	11	89/464	29.8736	32
134	757	5	25/297	11	89/464	30.0654	32
135	761	5	67/797	11	89/464	30.2572	32
136	769	5	45/536	11	89/464	30.4490	32
137	773	5	68/811	11	89/464	30.6408	33
138	787	5	17/203	11	89/464	30.8326	33
139	797	5	23/275	11	89/464	31.0244	34
140	809	5	35/419	11	89/464	31.2163	34
141	811	5	73/875	11	89/464	31.4081	34
142	821	5	1/12	11	89/464	31.5999	34
143	823	5	64/769	11	89/464	31.7917	34
144	827	5	33/397	12	89/464	31.9835	34
145	829	5	45/542	12	89/464	32.1753	34
146	839	5	17/205	12	89/464	32.3671	34
147	853	5	41/495	12	89/464	32.5589	34
148	857	5	23/278	12	89/464	32.7507	34
149	859	5	79/956	12	89/464	32.9425	35
150	863	5	26/315	12	89/464	33.1343	35
151	877	5	31/376	12	89/464	33.3261	36
152	881	5	7/85	12	89/464	33.5179	36
153	883	5	51/620	12	89/464	33.7098	36
154	887	5	47/572	12	89/464	33.9016	36
155	907	5	49/597	12	89/464	34.0934	36
156	911	5	71/866	12	89/464	34.2852	36
157	919	5	19/232	12	89/464	34.4770	37
158	929	5	76/929	12	89/464	34.6688	37
159	937	5	38/465	12	89/464	34.8606	37
160	941	5	4/49	12	89/464	35.0524	37
161	947	5	19/233	12	89/464	35.2442	37
162	953	5	29/356	12	89/464	35.4360	37
163	967	5	52/639	12	89/464	35.6278	38
164	971	5	10/123	12	89/464	35.8196	38
165	977	5	51/628	12	89/464	36.0115	38
166	983	5	46/567	12	89/464	36.2033	38
167	991	5	62/765	12	89/464	36.3951	39
168	997	5	57/704	12	89/464	36.5869	39
169	1009	6	11/136	13	165/914	36.7674	39
170	1013	6	8/99	13	165/914	36.9479	39
171	1019	6	49/607	13	165/914	37.1285	39
172	1021	6	5/62	13	165/914	37.3090	39
173	1031	6	17/211	13	165/914	37.4895	40
174	1033	6	59/733	13	165/914	37.6700	40
175	1039	6	39/485	13	165/914	37.8506	40
176	1049	6	67/834	13	165/914	38.0311	40
177	1051	6	37/461	13	165/914	38.2116	40
178	1061	6	17/212	13	165/914	38.3921	40
179	1063	6	29/362	13	165/914	38.5727	41
180	1069	6	2/25	13	165/914	38.7532	41
181	1087	6	79/988	13	165/914	38.9337	42
182	1091	6	56/701	13	165/914	39.1142	42
183	1093	6	17/213	13	165/914	39.2948	42
184	1097	6	37/464	13	165/914	39.4753	42
185	1103	6	29/364	13	165/914	39.6558	42
186	1109	6	71/892	13	165/914	39.8363	42
187	1117	6	47/591	13	165/914	40.0169	42
188	1123	6	70/881	13	165/914	40.1974	42
189	1129	6	62/781	13	165/914	40.3779	42
190	1151	6	51/643	13	165/914	40.5584	42
191	1153	6	21/265	13	165/914	40.7390	43
192	1163	6	27/341	13	165/914	40.9195	43
193	1171	6	25/316	13	165/914	41.1000	44
194	1181	6	43/544	13	165/914	41.2805	44
195	1187	6	68/861	13	165/914	41.4611	44
196	1193	6	58/735	14	165/914	41.6416	44
197	1201	6	41/520	14	165/914	41.8221	45
198	1213	6	62/787	14	165/914	42.0026	45
199	1217	6	27/343	14	165/914	42.1832	46
200	1223	6	7/89	14	165/914	42.3637	46
201	1229	6	69/878	14	165/914	42.5442	46
202	1231	6	34/433	14	165/914	42.7247	46
203	1237	6	55/701	14	165/914	42.9053	46
204	1249	6	45/574	14	165/914	43.0858	46
205	1259	6	64/817	14	165/914	43.2663	46
206	1277	6	49/626	14	165/914	43.4468	46
207	1279	6	14/179	14	165/914	43.6274	46
208	1283	6	44/563	14	165/914	43.8079	46
209	1289	6	36/461	14	165/914	43.9884	46
210	1291	6	65/833	14	165/914	44.1689	46

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
211	1297	6	63/808	14	165/914	44.3495	47
212	1301	6	6/77	14	165/914	44.5300	47
213	1303	6	71/912	14	165/914	44.7105	47
214	1307	6	62/797	14	165/914	44.8910	47
215	1319	6	37/476	14	165/914	45.0716	47
216	1321	6	8/103	14	165/914	45.2521	47
217	1327	6	69/889	14	165/914	45.4326	47
218	1361	6	47/606	14	165/914	45.6131	47
219	1367	6	31/400	14	165/914	45.7937	47
220	1373	6	57/736	14	165/914	45.9742	47
221	1381	6	64/827	14	165/914	46.1547	47
222	1399	6	29/375	14	165/914	46.3352	47
223	1409	6	67/867	14	165/914	46.5158	48
224	1423	6	59/764	14	165/914	46.6963	48
225	1427	6	24/311	15	165/914	46.8768	48
226	1429	6	31/402	15	165/914	47.0574	48
227	1433	6	43/558	15	165/914	47.2379	49
228	1439	6	69/896	15	165/914	47.4184	49
229	1447	6	1/13	15	165/914	47.5989	50
230	1451	6	1/13	15	165/914	47.7795	50
231	1453	6	1/13	15	165/914	47.9600	50
232	1459	6	47/612	15	165/914	48.1405	50
233	1471	6	33/430	15	165/914	48.3210	51
234	1481	6	51/665	15	165/914	48.5016	51
235	1483	6	21/274	15	165/914	48.6821	51
236	1487	6	53/692	15	165/914	48.8626	51
237	1489	6	61/797	15	165/914	49.0431	51
238	1493	6	27/353	15	165/914	49.2237	51
239	1499	6	12/157	15	165/914	49.4042	52
240	1511	6	11/144	15	165/914	49.5847	52
241	1523	6	10/131	15	165/914	49.7652	53
242	1531	6	46/603	15	165/914	49.9458	53
243	1543	6	17/223	15	165/914	50.1263	53
244	1549	6	8/105	15	165/914	50.3068	53
245	1553	6	67/880	15	165/914	50.4873	53
246	1559	6	7/92	15	165/914	50.6679	53
247	1567	6	53/697	15	165/914	50.8484	53
248	1571	6	69/908	15	165/914	51.0289	53
249	1579	6	6/79	15	165/914	51.2094	53
250	1583	6	17/224	15	165/914	51.3900	53
251	1597	6	38/501	15	165/914	51.5705	54
252	1601	6	26/343	15	165/914	51.7510	54
253	1607	6	5/66	15	165/914	51.9315	54
254	1609	6	67/885	15	165/914	52.1121	54
255	1613	6	23/304	15	165/914	52.2926	54
256	1619	6	71/939	16	165/914	52.4731	54
257	1621	6	30/397	16	165/914	52.6536	55
258	1627	6	29/384	16	165/914	52.8342	55
259	1637	6	4/53	16	165/914	53.0147	55
260	1657	6	66/875	16	165/914	53.1952	55
261	1663	6	64/849	16	165/914	53.3757	55
262	1667	6	11/146	16	165/914	53.5563	55
263	1669	6	71/943	16	165/914	53.7368	56
264	1693	6	38/505	16	165/914	53.9173	56
265	1697	6	37/492	16	165/914	54.0978	56
266	1699	6	59/785	16	165/914	54.2784	56
267	1709	6	16/213	16	165/914	54.4589	56
268	1721	6	53/706	16	165/914	54.6394	56
269	1723	6	67/893	16	165/914	54.8199	57
270	1733	6	3/40	16	165/914	55.0005	57
271	1741	6	32/427	16	165/914	55.1810	58
272	1747	6	37/494	16	165/914	55.3615	58
273	1753	6	39/521	16	165/914	55.5420	58
274	1759	6	30/401	16	165/914	55.7226	58
275	1777	6	8/107	16	165/914	55.9031	58
276	1783	6	55/736	16	165/914	56.0836	58
277	1787	6	18/241	16	165/914	56.2641	59
278	1789	6	58/777	16	165/914	56.4447	59
279	1801	6	47/630	16	165/914	56.6252	59
280	1811	6	17/228	16	165/914	56.8057	59
281	1823	6	74/993	16	165/914	56.9862	60
282	1831	6	61/819	16	165/914	57.1668	60
283	1847	6	30/403	16	165/914	57.3473	61
284	1861	6	59/793	16	165/914	57.5278	61
285	1867	6	67/901	16	165/914	57.7083	61
286	1871	6	11/148	16	165/914	57.8889	61
287	1873	6	13/175	16	165/914	58.0694	61
288	1877	6	49/660	16	165/914	58.2499	61
289	1879	7	21/283	17	157/918	58.4209	61
290	1889	7	31/418	17	157/918	58.5920	61
291	1901	7	55/742	17	157/918	58.7630	61
292	1907	7	2/27	17	157/918	58.9340	61
293	1913	7	2/27	17	157/918	59.1050	62
294	1931	7	41/554	17	157/918	59.2761	62
295	1933	7	27/365	17	157/918	59.4471	62
296	1949	7	19/257	17	157/918	59.6181	62
297	1951	7	15/203	17	157/918	59.7891	62
298	1973	7	63/853	17	157/918	59.9602	62
299	1979	7	11/149	17	157/918	60.1312	62
300	1987	7	47/637	17	157/918	60.3022	62
301	1993	7	25/339	17	157/918	60.4732	62
302	1997	7	30/407	17	157/918	60.6443	62
303	1999	7	61/828	17	157/918	60.8153	62
304	2003	7	31/421	17	157/918	60.9863	62
305	2011	7	63/856	17	157/918	61.1573	62
306	2017	7	32/435	17	157/918	61.3284	62
307	2027	7	5/68	17	157/918	61.4994	63
308	2029	7	28/381	17	157/918	61.6704	63
309	2039	7	57/776	17	157/918	61.8414	63
310	2053	7	29/395	17	157/918	62.0125	63
311	2063	7	51/695	17	157/918	62.1835	64
312	2069	7	41/559	17	157/918	62.3545	64
313	2081	7	64/873	17	157/918	62.5255	65
314	2083	7	17/232	17	157/918	62.6965	65
315	2087	7	26/355	17	157/918	62.8676	65

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
316	2089	7	50/683	17	157/918	63.0386	65
317	2099	7	3/41	17	157/918	63.2096	66
318	2111	7	52/711	17	157/918	63.3806	66
319	2113	7	25/342	17	157/918	63.5517	66
320	2129	7	35/479	17	157/918	63.7227	66
321	2131	7	13/178	17	157/918	63.8937	66
322	2137	7	10/137	17	157/918	64.0647	66
323	2141	7	17/233	17	157/918	64.2358	66
324	2143	7	52/713	18	157/918	64.4068	66
325	2153	7	39/535	18	157/918	64.5778	66
326	2161	7	29/398	18	157/918	64.7488	66
327	2179	7	26/357	18	157/918	64.9199	66
328	2203	7	19/261	18	157/918	65.0909	66
329	2207	7	35/481	18	157/918	65.2619	66
330	2213	7	4/55	18	157/918	65.4329	66
331	2221	7	45/619	18	157/918	65.6040	67
332	2237	7	21/289	18	157/918	65.7750	67
333	2239	7	56/771	18	157/918	65.9460	67
334	2243	7	31/427	18	157/918	66.1170	67
335	2251	7	50/689	18	157/918	66.2881	67
336	2267	7	14/193	18	157/918	66.4591	67
337	2269	7	53/731	18	157/918	66.6301	68
338	2273	7	5/69	18	157/918	66.8011	68
339	2281	7	46/635	18	157/918	66.9721	68
340	2287	7	58/801	18	157/918	67.1432	68
341	2293	7	49/677	18	157/918	67.3142	68
342	2297	7	62/857	18	157/918	67.4852	68
343	2309	7	35/484	18	157/918	67.6562	68
344	2311	7	6/83	18	157/918	67.8273	68
345	2333	7	25/346	18	157/918	67.9983	68
346	2339	7	13/180	18	157/918	68.1693	68
347	2341	7	27/374	18	157/918	68.3403	69
348	2347	7	7/97	18	157/918	68.5114	69
349	2351	7	22/305	18	157/918	68.6824	70
350	2357	7	23/319	18	157/918	68.8534	70
351	2371	7	8/111	18	157/918	69.0244	70
352	2377	7	17/236	18	157/918	69.1955	70
353	2381	7	71/986	18	157/918	69.3665	71
354	2383	7	28/389	18	157/918	69.5375	71
355	2389	7	10/139	18	157/918	69.7085	71
356	2393	7	21/292	18	157/918	69.8796	71
357	2399	7	56/779	18	157/918	70.0506	71
358	2411	7	12/167	18	157/918	70.2216	71
359	2417	7	13/181	18	157/918	70.3926	72
360	2423	7	14/195	18	157/918	70.5637	72
361	2437	8	15/209	19	62/379	70.7272	72
362	2441	8	33/460	19	62/379	70.8908	72
363	2447	8	18/251	19	62/379	71.0544	72
364	2459	8	61/851	19	62/379	71.2180	72
365	2467	8	23/321	19	62/379	71.3816	72
366	2473	8	53/740	19	62/379	71.5452	72
367	2477	8	31/433	19	62/379	71.7088	73
368	2503	8	37/517	19	62/379	71.8724	73
369	2521	8	47/657	19	62/379	72.0359	73
370	2531	8	64/895	19	62/379	72.1995	73
371	2539	8	1/14	19	62/379	72.3631	73
372	2543	8	1/14	19	62/379	72.5267	73
373	2549	8	1/14	19	62/379	72.6903	74
374	2551	8	1/14	19	62/379	72.8539	74
375	2557	8	1/14	19	62/379	73.0175	74
376	2579	8	58/813	19	62/379	73.1811	74
377	2591	8	44/617	19	62/379	73.3447	74
378	2593	8	71/996	19	62/379	73.5082	74
379	2609	8	30/421	19	62/379	73.6718	75
380	2617	8	26/365	19	62/379	73.8354	75
381	2621	8	68/955	19	62/379	73.9990	75
382	2633	8	20/281	19	62/379	74.1626	75
383	2647	8	18/253	19	62/379	74.3262	76
384	2657	8	50/703	19	62/379	74.4898	76
385	2659	8	61/858	19	62/379	74.6534	76
386	2663	8	14/197	19	62/379	74.8169	76
387	2671	8	13/183	19	62/379	74.9805	76
388	2677	8	37/521	19	62/379	75.1441	76
389	2683	8	23/324	19	62/379	75.3077	77
390	2687	8	11/155	19	62/379	75.4713	77
391	2689	8	31/437	19	62/379	75.6349	77
392	2693	8	49/691	19	62/379	75.7985	77
393	2699	8	28/395	19	62/379	75.9621	77
394	2707	8	9/127	19	62/379	76.1257	77
395	2711	8	17/240	19	62/379	76.2892	77
396	2713	8	65/918	19	62/379	76.4528	77
397	2719	8	39/551	19	62/379	76.6164	78
398	2729	8	15/212	19	62/379	76.7800	78
399	2731	8	36/509	19	62/379	76.9436	78
400	2741	8	7/99	20	62/379	77.1072	78
401	2749	8	47/665	20	62/379	77.2708	79
402	2753	8	13/184	20	62/379	77.4344	79
403	2767	8	44/623	20	62/379	77.5979	79
404	2777	8	67/949	20	62/379	77.7615	79
405	2789	8	59/836	20	62/379	77.9251	79
406	2791	8	63/893	20	62/379	78.0887	79
407	2797	8	39/553	20	62/379	78.2523	79
408	2801	8	65/922	20	62/379	78.4159	79
409	2803	8	58/823	20	62/379	78.5795	80
410	2819	8	36/511	20	62/379	78.7431	80
411	2833	8	5/71	20	62/379	78.9067	80
412	2837	8	44/625	20	62/379	79.0702	80
413	2843	8	62/881	20	62/379	79.2338	80
414	2851	8	14/199	20	62/379	79.3974	80
415	2857	8	41/583	20	62/379	79.5610	80
416	2861	8	49/697	20	62/379	79.7246	80
417	2879	8	61/868	20	62/379	79.8882	80
418	2887	8	64/911	20	62/379	80.0518	80
419	2897	8	46/655	20	62/379	80.2154	81
420	2903	8	45/641	20	62/379	80.3789	81

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
421	2909	8	4/57	20	62/379	80.5425	82
422	2917	8	59/841	20	62/379	80.7061	82
423	2927	8	27/385	20	62/379	80.8697	82
424	2939	8	19/271	20	62/379	81.0333	82
425	2953	8	67/956	20	62/379	81.1969	82
426	2957	8	11/157	20	62/379	81.3605	82
427	2963	8	18/257	20	62/379	81.5241	82
428	2969	8	53/757	20	62/379	81.6876	82
429	2971	8	66/943	20	62/379	81.8512	82
430	2999	8	41/586	20	62/379	82.0148	82
431	3001	8	37/529	20	62/379	82.1784	83
432	3011	8	43/615	20	62/379	82.3420	83
433	3019	8	13/186	20	62/379	82.5056	84
434	3023	8	16/229	20	62/379	82.6692	84
435	3037	8	19/272	20	62/379	82.8328	84
436	3041	8	28/401	20	62/379	82.9964	84
437	3049	8	43/616	20	62/379	83.1599	84
438	3061	8	3/43	20	62/379	83.3235	84
439	3067	8	3/43	20	62/379	83.4871	85
440	3079	8	50/717	20	62/379	83.6507	85
441	3083	8	29/416	21	62/379	83.8143	85
442	3089	8	63/904	21	62/379	83.9779	85
443	3109	8	65/933	21	62/379	84.1415	86
444	3119	8	53/761	21	62/379	84.3051	86
445	3121	8	11/158	21	62/379	84.4686	86
446	3137	8	68/977	21	62/379	84.6322	86
447	3163	8	43/618	21	62/379	84.7958	86
448	3167	8	8/115	21	62/379	84.9594	86
449	3169	8	21/302	21	62/379	85.1230	87
450	3181	8	13/187	21	62/379	85.2866	87
451	3187	8	41/590	21	62/379	85.4502	87
452	3191	8	38/547	21	62/379	85.6138	87
453	3203	8	5/72	21	62/379	85.7774	87
454	3209	8	52/749	21	62/379	85.9409	87
455	3217	8	49/706	21	62/379	86.1045	87
456	3221	8	63/908	21	62/379	86.2681	87
457	3229	8	12/173	21	62/379	86.4317	88
458	3251	8	19/274	21	62/379	86.5953	88
459	3253	8	54/779	21	62/379	86.7589	88
460	3257	8	7/101	21	62/379	86.9225	88
461	3259	8	23/332	21	62/379	87.0861	89
462	3271	8	41/592	21	62/379	87.2496	89
463	3299	8	9/130	21	62/379	87.4132	90
464	3301	8	29/419	21	62/379	87.5768	90
465	3307	8	42/607	21	62/379	87.7404	90
466	3313	8	46/665	21	62/379	87.9040	90
467	3319	8	13/188	21	62/379	88.0676	91
468	3323	8	58/839	21	62/379	88.2312	91
469	3329	8	66/955	21	62/379	88.3948	91
470	3331	8	19/275	21	62/379	88.5584	91
471	3343	8	23/333	21	62/379	88.7219	91
472	3347	8	29/420	21	62/379	88.8855	91
473	3359	8	39/565	21	62/379	89.0491	91
474	3361	8	59/855	21	62/379	89.2127	91
475	3371	8	2/29	21	62/379	89.3763	91
476	3373	8	2/29	21	62/379	89.5399	91
477	3389	8	2/29	21	62/379	89.7035	91
478	3391	8	59/856	21	62/379	89.8671	91
479	3407	8	39/566	21	62/379	90.0306	92
480	3413	8	29/421	21	62/379	90.1942	92
481	3433	8	23/334	21	62/379	90.3578	92
482	3449	8	59/857	21	62/379	90.5214	92
483	3457	8	17/247	21	62/379	90.6850	92
484	3461	8	15/218	22	62/379	90.8486	92
485	3463	8	13/189	22	62/379	91.0122	92
486	3467	8	59/858	22	62/379	91.1758	92
487	3469	8	11/160	22	62/379	91.3394	93
488	3491	8	20/291	22	62/379	91.5029	93
489	3499	8	9/131	22	62/379	91.6665	93
490	3511	8	34/495	22	62/379	91.8301	93
491	3517	8	16/233	22	62/379	91.9937	94
492	3527	8	67/976	22	62/379	92.1573	94
493	3529	8	7/102	22	62/379	92.3209	94
494	3533	8	33/481	22	62/379	92.4845	94
495	3539	8	19/277	22	62/379	92.6481	94
496	3541	8	12/175	22	62/379	92.8116	94
497	3547	8	17/248	22	62/379	92.9752	94
498	3557	8	49/715	22	62/379	93.1388	94
499	3559	8	52/759	22	62/379	93.3024	95
500	3571	8	5/73	22	62/379	93.4660	95
501	3581	8	48/701	22	62/379	93.6296	95
502	3583	8	23/336	22	62/379	93.7932	95
503	3593	8	49/716	22	62/379	93.9568	96
504	3607	8	13/190	22	62/379	94.1204	96
505	3613	8	29/424	22	62/379	94.2839	96
506	3617	8	8/117	22	62/379	94.4475	96
507	3623	8	35/512	22	62/379	94.6111	96
508	3631	8	49/717	22	62/379	94.7747	96
509	3637	8	11/161	22	62/379	94.9383	97
510	3643	8	64/937	22	62/379	95.1019	97
511	3659	8	45/659	22	62/379	95.2655	97
512	3671	8	37/542	22	62/379	95.4291	97
513	3673	8	23/337	22	62/379	95.5926	97
514	3677	8	67/982	22	62/379	95.7562	97
515	3691	8	56/821	22	62/379	95.9198	97
516	3697	8	3/44	22	62/379	96.0834	97
517	3701	8	3/44	22	62/379	96.2470	97
518	3709	8	58/851	22	62/379	96.4106	97
519	3719	8	34/499	22	62/379	96.5742	97
520	3727	8	25/367	22	62/379	96.7378	97
521	3733	8	19/279	22	62/379	96.9014	98
522	3739	8	16/235	22	62/379	97.0649	98
523	3761	8	13/191	22	62/379	97.2285	99
524	3767	8	23/338	22	62/379	97.3921	99
525	3769	8	10/147	22	62/379	97.5557	99

n	P_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
526	3779	8	27/397	22	62/379	97.7193	99
527	3793	8	65/956	22	62/379	97.8829	99
528	3797	8	52/765	22	62/379	98.0465	99
529	3803	9	7/103	23	157/994	98.2044	99
530	3821	9	57/839	23	157/994	98.3624	99
531	3823	9	65/957	23	157/994	98.5203	99
532	3833	9	11/162	23	157/994	98.6783	99
533	3847	9	26/383	23	157/994	98.8362	99
534	3851	9	49/722	23	157/994	98.9942	99
535	3853	9	65/958	23	157/994	99.1521	99
536	3863	9	66/973	23	157/994	99.3100	99
537	3877	9	63/929	23	157/994	99.4680	99
538	3881	9	4/59	23	157/994	99.6259	99
539	3889	9	65/959	23	157/994	99.7839	99
540	3907	9	33/487	23	157/994	99.9418	99
541	3911	9	67/989	23	157/994	100.0998	100
542	3917	9	17/251	23	157/994	100.2577	100
543	3919	9	13/192	23	157/994	100.4157	100
544	3923	9	22/325	23	157/994	100.5736	100
545	3929	9	67/990	23	157/994	100.7316	100
546	3931	9	50/739	23	157/994	100.8895	100
547	3943	9	37/547	23	157/994	101.0475	101
548	3947	9	33/488	23	157/994	101.2054	101
549	3967	9	24/355	23	157/994	101.3634	101
550	3989	9	39/577	23	157/994	101.5213	101
551	4001	9	5/74	23	157/994	101.6793	101
552	4003	9	5/74	23	157/994	101.8372	101
553	4007	9	67/992	23	157/994	101.9951	101
554	4013	9	21/311	23	157/994	102.1531	101
555	4019	9	59/874	23	157/994	102.3110	101
556	4021	9	11/163	23	157/994	102.4690	101
557	4027	9	67/993	23	157/994	102.6269	102
558	4049	9	57/845	23	157/994	102.7849	102
559	4051	9	64/949	23	157/994	102.9428	102
560	4057	9	6/89	23	157/994	103.1008	102
561	4073	9	6/89	23	157/994	103.2587	102
562	4079	9	31/460	23	157/994	103.4167	102
563	4091	9	19/282	23	157/994	103.5746	103
564	4093	9	13/193	23	157/994	103.7326	103
565	4099	9	20/297	23	157/994	103.8905	103
566	4111	9	41/609	23	157/994	104.0485	103
567	4127	9	7/104	23	157/994	104.2064	103
568	4129	9	36/535	23	157/994	104.3644	103
569	4133	9	59/877	23	157/994	104.5223	104
570	4139	9	53/788	23	157/994	104.6803	104
571	4153	9	39/580	23	157/994	104.8382	105
572	4157	9	8/119	23	157/994	104.9961	105
573	4159	9	33/491	23	157/994	105.1541	105
574	4177	9	17/253	23	157/994	105.3120	105
575	4201	9	35/521	23	157/994	105.4700	105
576	4211	9	9/134	24	157/994	105.6279	105
577	4217	9	28/417	24	157/994	105.7859	106
578	4219	9	29/432	24	157/994	105.9438	106
579	4229	9	10/149	24	157/994	106.1018	106
580	4231	9	31/462	24	157/994	106.2597	106
581	4241	9	43/641	24	157/994	106.4177	106
582	4243	9	67/999	24	157/994	106.5756	106
583	4253	9	58/865	24	157/994	106.7336	106
584	4259	9	12/179	24	157/994	106.8915	106
585	4261	9	38/567	24	157/994	107.0495	106
586	4271	9	53/791	24	157/994	107.2074	106
587	4273	9	14/209	24	157/994	107.3654	107
588	4283	9	29/433	24	157/994	107.5233	107
589	4289	9	46/687	24	157/994	107.6812	107
590	4297	9	65/971	24	157/994	107.8392	107
591	4327	9	17/254	24	157/994	107.9971	107
592	4337	9	55/822	24	157/994	108.1551	107
593	4339	9	39/583	24	157/994	108.3130	108
594	4349	9	21/314	24	157/994	108.4710	108
595	4357	9	45/673	24	157/994	108.6289	108
596	4363	9	49/733	24	157/994	108.7869	108
597	4373	9	53/793	24	157/994	108.9448	108
598	4391	9	29/434	24	157/994	109.1028	108
599	4397	9	65/973	24	157/994	109.2607	109
600	4409	9	37/554	24	157/994	109.4187	109
601	4421	9	42/629	24	157/994	109.5766	110
602	4423	9	49/734	24	157/994	109.7346	110
603	4441	9	58/869	24	157/994	109.8925	110
604	4447	9	1/15	24	157/994	110.0505	110
605	4451	9	1/15	24	157/994	110.2084	110
606	4457	9	1/15	24	157/994	110.3664	110
607	4463	9	1/15	24	157/994	110.5243	111
608	4481	9	1/15	24	157/994	110.6822	111
609	4483	9	1/15	24	157/994	110.8402	111
610	4493	9	1/15	24	157/994	110.9981	111
611	4507	9	1/15	24	157/994	111.1561	111
612	4513	9	1/15	24	157/994	111.3140	111
613	4517	9	61/916	24	157/994	111.4720	112
614	4519	9	51/766	24	157/994	111.6299	112
615	4523	9	44/661	24	157/994	111.7879	112
616	4547	9	38/571	24	157/994	111.9458	112
617	4549	9	34/511	24	157/994	112.1038	113
618	4561	9	61/917	24	157/994	112.2617	113
619	4567	9	55/827	24	157/994	112.4197	114
620	4583	9	51/767	24	157/994	112.5776	114
621	4591	9	47/707	24	157/994	112.7356	114
622	4597	9	65/978	24	157/994	112.8935	114
623	4603	9	61/918	24	157/994	113.0515	114
624	4621	9	19/286	24	157/994	113.2094	114
625	4637	9	18/271	25	157/994	113.3673	114
626	4639	9	17/256	25	157/994	113.5253	114
627	4643	9	16/241	25	157/994	113.6832	114
628	4649	9	61/919	25	157/994	113.8412	114
629	4651	9	29/437	25	157/994	113.9991	114
630	4657	9	14/211	25	157/994	114.1571	114

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
631	4663	9	40/603	25	157/994	114.3150	115
632	4673	9	51/769	25	157/994	114.4730	115
633	4679	9	49/739	25	157/994	114.6309	115
634	4691	9	59/890	25	157/994	114.7889	115
635	4703	9	34/513	25	157/994	114.9468	115
636	4721	9	11/166	25	157/994	115.1048	115
637	4723	9	53/800	25	157/994	115.2627	115
638	4729	9	41/619	25	157/994	115.4207	115
639	4733	9	10/151	25	157/994	115.5786	115
640	4751	9	48/725	25	157/994	115.7366	115
641	4759	9	28/423	25	157/994	115.8945	116
642	4783	9	9/136	25	157/994	116.0525	116
643	4787	9	44/665	25	157/994	116.2104	117
644	4789	9	60/907	25	157/994	116.3683	117
645	4793	9	25/378	25	157/994	116.5263	117
646	4799	9	65/983	25	157/994	116.6842	117
647	4801	9	8/121	25	157/994	116.8422	118
648	4813	9	31/469	25	157/994	117.0001	118
649	4817	9	15/227	25	157/994	117.1581	118
650	4831	9	59/893	25	157/994	117.3160	118
651	4861	9	36/545	25	157/994	117.4740	118
652	4871	9	7/106	25	157/994	117.6319	118
653	4877	9	7/106	25	157/994	117.7899	119
654	4889	9	27/409	25	157/994	117.9478	119
655	4903	9	53/803	25	157/994	118.1058	119
656	4909	9	13/197	25	157/994	118.2637	119
657	4919	9	19/288	25	157/994	118.4217	119
658	4931	9	25/379	25	157/994	118.5796	119
659	4933	9	49/743	25	157/994	118.7376	120
660	4937	9	6/91	25	157/994	118.8955	120
661	4943	9	59/895	25	157/994	119.0534	121
662	4951	9	29/440	25	157/994	119.2114	121
663	4957	9	57/865	25	157/994	119.3693	121
664	4967	9	28/425	25	157/994	119.5273	121
665	4969	9	11/167	25	157/994	119.6852	121
666	4973	9	27/410	25	157/994	119.8432	121
667	4987	9	16/243	25	157/994	120.0011	121
668	4993	9	21/319	25	157/994	120.1591	121
669	4999	9	31/471	25	157/994	120.3170	121
670	5003	9	5/76	25	157/994	120.4750	121
671	5009	9	5/76	25	157/994	120.6329	121
672	5011	9	64/973	25	157/994	120.7909	121
673	5021	9	34/517	25	157/994	120.9488	122
674	5023	9	43/654	25	157/994	121.1068	122
675	5039	9	33/502	25	157/994	121.2647	122
676	5051	9	14/213	26	157/994	121.4227	122
677	5059	9	55/837	26	157/994	121.5806	123
678	5077	9	9/137	26	157/994	121.7386	123
679	5081	9	58/883	26	157/994	121.8965	123
680	5087	9	22/335	26	157/994	122.0544	123
681	5099	9	13/198	26	157/994	122.2124	123
682	5101	9	30/457	26	157/994	122.3703	123
683	5107	9	17/259	26	157/994	122.5283	124
684	5113	9	46/701	26	157/994	122.6862	124
685	5119	9	62/945	26	157/994	122.8442	124
686	5147	9	49/747	26	157/994	123.0021	124
687	5153	9	4/61	26	157/994	123.1601	124
688	5167	9	4/61	26	157/994	123.3180	124
689	5171	9	63/961	26	157/994	123.4760	124
690	5179	9	35/534	26	157/994	123.6339	124
691	5189	9	50/763	26	157/994	123.7919	125
692	5197	9	19/290	26	157/994	123.9498	125
693	5209	9	64/977	26	157/994	124.1078	125
694	5227	9	41/626	26	157/994	124.2657	125
695	5231	9	11/168	26	157/994	124.4237	125
696	5233	9	51/779	26	157/994	124.5816	125
697	5237	9	18/275	26	157/994	124.7395	125
698	5261	9	25/382	26	157/994	124.8975	125
699	5273	9	53/810	26	157/994	125.0554	125
700	5279	9	7/107	26	157/994	125.2134	125
701	5281	9	45/688	26	157/994	125.3713	126
702	5297	9	24/367	26	157/994	125.5293	126
703	5303	9	17/260	26	157/994	125.6872	126
704	5309	9	37/566	26	157/994	125.8452	126
705	5323	9	10/153	26	157/994	126.0031	126
706	5333	9	33/505	26	157/994	126.1611	126
707	5347	9	49/750	26	157/994	126.3190	126
708	5351	9	55/842	26	157/994	126.4770	126
709	5381	9	16/245	26	157/994	126.6349	127
710	5387	9	54/827	26	157/994	126.7929	127
711	5393	9	63/965	26	157/994	126.9508	127
712	5399	9	25/383	26	157/994	127.1088	127
713	5407	9	65/996	26	157/994	127.2667	127
714	5413	9	46/705	26	157/994	127.4247	127
715	5417	9	3/46	26	157/994	127.5826	127
716	5419	9	3/46	26	157/994	127.7405	127
717	5431	9	3/46	26	157/994	127.8985	127
718	5437	9	3/46	26	157/994	128.0564	127
719	5441	9	50/767	26	157/994	128.2144	128
720	5443	9	35/537	26	157/994	128.3723	128
721	5449	9	26/399	26	157/994	128.5303	128
722	5471	9	43/660	26	157/994	128.6882	128
723	5477	9	37/568	26	157/994	128.8462	128
724	5479	9	48/737	26	157/994	129.0041	128
725	5483	9	14/215	26	157/994	129.1621	128
726	5501	9	25/384	26	157/994	129.3200	128
727	5503	9	11/169	26	157/994	129.4780	129
728	5507	9	52/799	26	157/994	129.6359	129
729	5519	9	19/292	27	157/994	129.7939	129
730	5521	9	62/953	27	157/994	129.9518	129
731	5527	9	8/123	27	157/994	130.1098	129
732	5531	9	8/123	27	157/994	130.2677	129
733	5557	9	29/446	27	157/994	130.4256	130
734	5563	9	34/523	27	157/994	130.5836	130
735	5569	9	13/200	27	157/994	130.7415	130

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_j)$	$\pi(n)$
736	5573	9	49/754	27	157/994	130.8995	130
737	5581	9	41/631	27	157/994	131.0574	130
738	5591	9	28/431	27	157/994	131.2154	130
739	5623	9	48/739	27	157/994	131.3733	131
740	5639	9	5/77	27	157/994	131.5313	131
741	5641	9	5/77	27	157/994	131.6892	131
742	5647	9	52/801	27	157/994	131.8472	131
743	5651	9	59/909	27	157/994	132.0051	132
744	5653	9	22/339	27	157/994	132.1631	132
745	5657	9	17/262	27	157/994	132.3210	132
746	5659	9	53/817	27	157/994	132.4790	132
747	5669	9	12/185	27	157/994	132.6369	132
748	5683	9	19/293	27	157/994	132.7949	132
749	5689	9	26/401	27	157/994	132.9528	132
750	5693	9	54/833	27	157/994	133.1108	132
751	5701	9	7/108	27	157/994	133.2687	133
752	5711	9	58/895	27	157/994	133.4266	133
753	5717	9	53/818	27	157/994	133.5846	133
754	5737	9	55/849	27	157/994	133.7425	133
755	5741	9	41/633	27	157/994	133.9005	133
756	5743	9	43/664	27	157/994	134.0584	133
757	5749	9	9/139	27	157/994	134.2164	134
758	5779	9	38/587	27	157/994	134.3743	134
759	5783	9	20/309	27	157/994	134.5323	134
760	5791	9	53/819	27	157/994	134.6902	134
761	5801	9	11/170	27	157/994	134.8482	135
762	5807	9	24/371	27	157/994	135.0061	135
763	5813	9	13/201	27	157/994	135.1641	135
764	5821	9	41/634	27	157/994	135.3220	135
765	5827	9	15/232	27	157/994	135.4800	135
766	5839	9	32/495	27	157/994	135.6379	135
767	5843	9	17/263	27	157/994	135.7959	135
768	5849	9	19/294	27	157/994	135.9538	135
769	5851	9	21/325	27	157/994	136.1117	136
770	5857	9	48/743	27	157/994	136.2697	136
771	5861	9	27/418	27	157/994	136.4276	136
772	5867	9	64/991	27	157/994	136.5856	136
773	5869	9	39/604	27	157/994	136.7435	137
774	5879	9	49/759	27	157/994	136.9015	137
775	5881	9	2/31	27	157/994	137.0594	137
776	5897	9	2/31	27	157/994	137.2174	137
777	5903	9	2/31	27	157/994	137.3753	137
778	5923	9	2/31	27	157/994	137.5333	137
779	5927	9	2/31	27	157/994	137.6912	137
780	5939	9	2/31	27	157/994	137.8492	137
781	5953	9	63/977	27	157/994	138.0071	137
782	5981	9	47/729	27	157/994	138.1651	137
783	5987	9	37/574	27	157/994	138.3230	137
784	6007	9	31/481	28	157/994	138.4810	137
785	6011	9	27/419	28	157/994	138.6389	137
786	6029	9	48/745	28	157/994	138.7969	137
787	6037	9	21/326	28	157/994	138.9548	138
788	6043	9	19/295	28	157/994	139.1127	138
789	6047	9	17/264	28	157/994	139.2707	138
790	6053	9	32/497	28	157/994	139.4286	138
791	6067	9	15/233	28	157/994	139.5866	138
792	6073	9	41/637	28	157/994	139.7445	138
793	6079	9	13/202	28	157/994	139.9025	138
794	6089	9	24/373	28	157/994	140.0604	138
795	6091	9	57/886	28	157/994	140.2184	138
796	6101	9	11/171	28	157/994	140.3763	138
797	6113	9	51/793	28	157/994	140.5343	139
798	6121	9	29/451	28	157/994	140.6922	139
799	6131	9	9/140	28	157/994	140.8502	139
800	6133	9	9/140	28	157/994	141.0081	139
801	6143	9	59/918	28	157/994	141.1661	139
802	6151	9	16/249	28	157/994	141.3240	139
803	6163	9	39/607	28	157/994	141.4820	139
804	6173	9	30/467	28	157/994	141.6399	139
805	6197	9	7/109	28	157/994	141.7978	139
806	6199	9	7/109	28	157/994	141.9558	139
807	6203	9	47/732	28	157/994	142.1137	139
808	6211	9	26/405	28	157/994	142.2717	139
809	6217	9	19/296	28	157/994	142.4296	140
810	6221	9	55/857	28	157/994	142.5876	140
811	6229	9	12/187	28	157/994	142.7455	141
812	6247	9	46/717	28	157/994	142.9035	141
813	6257	9	56/873	28	157/994	143.0614	141
814	6263	9	49/764	28	157/994	143.2194	141
815	6269	9	37/577	28	157/994	143.3773	141
816	6271	9	62/967	28	157/994	143.5353	141
817	6277	9	5/78	28	157/994	143.6932	141
818	6287	9	5/78	28	157/994	143.8512	141
819	6299	9	48/749	28	157/994	144.0091	141
820	6301	9	28/437	28	157/994	144.1671	141
821	6311	9	64/999	28	157/994	144.3250	142
822	6317	9	18/281	28	157/994	144.4830	142
823	6323	9	57/890	28	157/994	144.6409	143
824	6329	9	60/937	28	157/994	144.7988	143
825	6337	9	21/328	28	157/994	144.9568	143
826	6343	9	37/578	28	157/994	145.1147	143
827	6353	9	8/125	28	157/994	145.2727	144
828	6359	9	8/125	28	157/994	145.4306	144
829	6361	9	62/969	28	157/994	145.5886	145
830	6367	9	19/297	28	157/994	145.7465	145
831	6373	9	41/641	28	157/994	145.9045	145
832	6379	9	11/172	28	157/994	146.0624	145
833	6389	9	36/563	28	157/994	146.2204	145
834	6397	9	39/610	28	157/994	146.3783	145
835	6421	9	14/219	28	157/994	146.5363	145
836	6427	9	48/751	28	157/994	146.6942	145
837	6449	9	37/579	28	157/994	146.8522	145
838	6451	9	63/986	28	157/994	147.0101	145
839	6469	9	26/407	28	157/994	147.1681	146
840	6473	9	61/955	28	157/994	147.3260	146

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
841	6481	10	38/595	29	142/929	147.4789	146
842	6491	10	56/877	29	142/929	147.6317	146
843	6521	10	3/47	29	142/929	147.7846	146
844	6529	10	3/47	29	142/929	147.9374	146
845	6547	10	3/47	29	142/929	148.0903	146
846	6551	10	3/47	29	142/929	148.2431	146
847	6553	10	55/862	29	142/929	148.3960	146
848	6563	10	40/627	29	142/929	148.5488	146
849	6569	10	31/486	29	142/929	148.7017	146
850	6571	10	25/392	29	142/929	148.8545	146
851	6577	10	22/345	29	142/929	149.0074	146
852	6581	10	19/298	29	142/929	149.1602	146
853	6599	10	16/251	29	142/929	149.3131	147
854	6607	10	29/455	29	142/929	149.4659	147
855	6619	10	13/204	29	142/929	149.6188	147
856	6637	10	49/769	29	142/929	149.7716	147
857	6653	10	56/879	29	142/929	149.9245	148
858	6659	10	10/157	29	142/929	150.0773	148
859	6661	10	10/157	29	142/929	150.2302	149
860	6673	10	27/424	29	142/929	150.3830	149
861	6679	10	17/267	29	142/929	150.5359	149
862	6689	10	24/377	29	142/929	150.6887	149
863	6691	10	38/597	29	142/929	150.8416	150
864	6701	10	7/110	29	142/929	150.9945	150
865	6703	10	7/110	29	142/929	151.1473	150
866	6709	10	46/723	29	142/929	151.3002	150
867	6719	10	25/393	29	142/929	151.4530	150
868	6733	10	18/283	29	142/929	151.6059	150
869	6737	10	29/456	29	142/929	151.7587	150
870	6761	10	11/173	29	142/929	151.9116	150
871	6763	10	48/755	29	142/929	152.0644	150
872	6779	10	41/645	29	142/929	152.2173	150
873	6781	10	15/236	29	142/929	152.3701	150
874	6791	10	19/299	29	142/929	152.5230	150
875	6793	10	42/661	29	142/929	152.6758	150
876	6803	10	27/425	29	142/929	152.8287	150
877	6823	10	35/551	29	142/929	152.9815	151
878	6827	10	55/866	29	142/929	153.1344	151
879	6829	10	4/63	29	142/929	153.2872	151
880	6833	10	4/63	29	142/929	153.4401	151
881	6841	10	4/63	29	142/929	153.5929	152
882	6857	10	53/835	29	142/929	153.7458	152
883	6863	10	37/583	29	142/929	153.8986	153
884	6869	10	54/851	29	142/929	154.0515	153
885	6871	10	21/331	29	142/929	154.2043	153
886	6883	10	55/867	29	142/929	154.3572	153
887	6899	10	47/741	29	142/929	154.5101	154
888	6907	10	13/205	29	142/929	154.6629	154
889	6911	10	61/962	29	142/929	154.8158	154
890	6917	10	22/347	29	142/929	154.9686	154
891	6947	10	40/631	29	142/929	155.1215	154
892	6949	10	9/142	29	142/929	155.2743	154
893	6959	10	59/931	29	142/929	155.4272	154
894	6961	10	55/868	29	142/929	155.5800	154
895	6967	10	51/805	29	142/929	155.7329	154
896	6971	10	14/221	29	142/929	155.8857	154
897	6977	10	19/300	29	142/929	156.0386	154
898	6983	10	43/679	29	142/929	156.1914	154
899	6991	10	29/458	29	142/929	156.3443	154
900	6997	10	44/695	30	142/929	156.4971	154
901	7001	10	5/79	30	142/929	156.6500	154
902	7013	10	5/79	30	142/929	156.8028	154
903	7019	10	5/79	30	142/929	156.9557	154
904	7027	10	46/727	30	142/929	157.1085	154
905	7039	10	31/490	30	142/929	157.2614	154
906	7043	10	21/332	30	142/929	157.4142	154
907	7057	10	53/838	30	142/929	157.5671	155
908	7069	10	59/933	30	142/929	157.7199	155
909	7079	10	38/601	30	142/929	157.8728	155
910	7103	10	11/174	30	142/929	158.0257	155
911	7109	10	50/791	30	142/929	158.1785	156
912	7121	10	45/712	30	142/929	158.3314	156
913	7127	10	57/902	30	142/929	158.4842	156
914	7129	10	52/823	30	142/929	158.6371	156
915	7151	10	41/649	30	142/929	158.7899	156
916	7159	10	6/95	30	142/929	158.9428	156
917	7177	10	6/95	30	142/929	159.0956	156
918	7187	10	6/95	30	142/929	159.2485	156
919	7193	10	37/586	30	142/929	159.4013	157
920	7207	10	25/396	30	142/929	159.5542	157
921	7211	10	19/301	30	142/929	159.7070	157
922	7213	10	45/713	30	142/929	159.8599	157
923	7219	10	13/206	30	142/929	160.0127	157
924	7229	10	53/840	30	142/929	160.1656	157
925	7237	10	47/745	30	142/929	160.3184	157
926	7243	10	34/539	30	142/929	160.4713	157
927	7247	10	7/111	30	142/929	160.6241	157
928	7253	10	7/111	30	142/929	160.7770	157
929	7283	10	57/904	30	142/929	160.9298	158
930	7297	10	29/460	30	142/929	161.0827	158
931	7307	10	22/349	30	142/929	161.2355	158
932	7309	10	15/238	30	142/929	161.3884	158
933	7321	10	38/603	30	142/929	161.5413	158
934	7331	10	31/492	30	142/929	161.6941	158
935	7333	10	55/873	30	142/929	161.8470	158
936	7349	10	8/127	30	142/929	161.9998	158
937	7351	10	57/905	30	142/929	162.1527	159
938	7369	10	58/921	30	142/929	162.3055	159
939	7393	10	59/937	30	142/929	162.4584	159
940	7411	10	43/683	30	142/929	162.6112	159
941	7417	10	35/556	30	142/929	162.7641	160
942	7433	10	9/143	30	142/929	162.9169	160
943	7451	10	9/143	30	142/929	163.0698	160
944	7457	10	37/588	30	142/929	163.2226	160
945	7459	10	19/302	30	142/929	163.3755	160

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
946	7477	10	29/461	30	142/929	163.5283	160
947	7481	10	59/938	30	142/929	163.6812	161
948	7487	10	10/159	30	142/929	163.8340	161
949	7489	10	41/652	30	142/929	163.9869	161
950	7499	10	21/334	30	142/929	164.1397	161
951	7507	10	43/684	30	142/929	164.2926	161
952	7517	10	11/175	30	142/929	164.4454	161
953	7523	10	45/716	30	142/929	164.5983	162
954	7529	10	23/366	30	142/929	164.7511	162
955	7537	10	59/939	30	142/929	164.9040	162
956	7541	10	12/191	30	142/929	165.0569	162
957	7547	10	25/398	30	142/929	165.2097	162
958	7549	10	51/812	30	142/929	165.3626	162
959	7559	10	13/207	30	142/929	165.5154	162
960	7561	10	27/430	30	142/929	165.6683	162
961	7573	11	14/223	31	29/195	165.8170	162
962	7577	11	43/685	31	29/195	165.9657	162
963	7583	11	59/940	31	29/195	166.1144	162
964	7589	11	61/972	31	29/195	166.2631	162
965	7591	11	47/749	31	29/195	166.4119	162
966	7603	11	49/781	31	29/195	166.5606	162
967	7607	11	17/271	31	29/195	166.7093	163
968	7621	11	35/558	31	29/195	166.8580	163
969	7639	11	18/287	31	29/195	167.0067	163
970	7643	11	19/303	31	29/195	167.1555	163
971	7649	11	59/941	31	29/195	167.3042	164
972	7669	11	41/654	31	29/195	167.4529	164
973	7673	11	43/686	31	29/195	167.6016	164
974	7681	11	45/718	31	29/195	167.7504	164
975	7687	11	47/750	31	29/195	167.8991	164
976	7691	11	49/782	31	29/195	168.0478	164
977	7699	11	26/415	31	29/195	168.1965	165
978	7703	11	55/878	31	29/195	168.3452	165
979	7717	11	29/463	31	29/195	168.4940	165
980	7723	11	31/495	31	29/195	168.6427	165
981	7727	11	33/527	31	29/195	168.7914	165
982	7741	11	36/575	31	29/195	168.9401	165
983	7753	11	38/607	31	29/195	169.0888	166
984	7757	11	42/671	31	29/195	169.2376	166
985	7759	11	45/719	31	29/195	169.3863	166
986	7789	11	50/799	31	29/195	169.5350	166
987	7793	11	56/895	31	29/195	169.6837	166
988	7817	11	1/16	31	29/195	169.8324	166
989	7823	11	1/16	31	29/195	169.9812	166
990	7829	11	1/16	31	29/195	170.1299	166
991	7841	11	1/16	31	29/195	170.2786	167
992	7853	11	1/16	31	29/195	170.4273	167
993	7867	11	1/16	31	29/195	170.5761	167
994	7873	11	1/16	31	29/195	170.7248	167
995	7877	11	1/16	31	29/195	170.8735	167
996	7879	11	1/16	31	29/195	171.0222	167
997	7883	11	1/16	31	29/195	171.1709	168
998	7901	11	1/16	31	29/195	171.3197	168
999	7907	11	1/16	31	29/195	171.4684	168
1000	7919	11	1/16	31	29/195	171.6171	168
1001	7927	11	1/16	31	29/195	171.7658	168
1002	7933	11	1/16	31	29/195	171.9145	169
1003	7937	11	1/16	31	29/195	172.0633	169
1004	7949	11	60/961	31	29/195	172.2120	169
1005	7951	11	54/865	31	29/195	172.3607	169
1006	7963	11	48/769	31	29/195	172.5094	169
1007	7993	11	44/705	31	29/195	172.6581	169
1008	8009	11	41/657	31	29/195	172.8069	169
1009	8011	11	38/609	31	29/195	172.9556	169
1010	8017	11	35/561	31	29/195	173.1043	169
1011	8039	11	33/529	31	29/195	173.2530	169
1012	8053	11	61/978	31	29/195	173.4018	169
1013	8059	11	29/465	31	29/195	173.5505	170
1014	8069	11	27/433	31	29/195	173.6992	170
1015	8081	11	26/417	31	29/195	173.8479	170
1016	8087	11	49/786	31	29/195	173.9966	170
1017	8089	11	47/754	31	29/195	174.1454	170
1018	8093	11	45/722	31	29/195	174.2941	170
1019	8101	11	43/690	31	29/195	174.4428	171
1020	8111	11	62/995	31	29/195	174.5915	171
1021	8117	11	20/321	31	29/195	174.7402	172
1022	8123	11	19/305	31	29/195	174.8890	172
1023	8147	11	55/883	31	29/195	175.0377	172
1024	8161	11	53/851	32	29/195	175.1864	172
1025	8167	11	17/273	32	29/195	175.3351	172
1026	8171	11	33/530	32	29/195	175.4838	172
1027	8179	11	16/257	32	29/195	175.6326	172
1028	8191	11	31/498	32	29/195	175.7813	172
1029	8209	11	15/241	32	29/195	175.9300	172
1030	8219	11	44/707	32	29/195	176.0787	172
1031	8221	11	57/916	32	29/195	176.2275	173
1032	8231	11	14/225	32	29/195	176.3762	173
1033	8233	11	27/434	32	29/195	176.5249	174
1034	8237	11	13/209	32	29/195	176.6736	174
1035	8243	11	13/209	32	29/195	176.8223	174
1036	8263	11	25/402	32	29/195	176.9711	174
1037	8269	11	61/981	32	29/195	177.1198	174
1038	8273	11	12/193	32	29/195	177.2685	174
1039	8287	11	35/563	32	29/195	177.4172	175
1040	8291	11	57/917	32	29/195	177.5659	175
1041	8293	11	11/177	32	29/195	177.7147	175
1042	8297	11	11/177	32	29/195	177.8634	175
1043	8311	11	32/515	32	29/195	178.0121	175
1044	8317	11	21/338	32	29/195	178.1608	175
1045	8329	11	41/660	32	29/195	178.3095	175
1046	8353	11	10/161	32	29/195	178.4583	175
1047	8363	11	10/161	32	29/195	178.6070	175
1048	8369	11	29/467	32	29/195	178.7557	175
1049	8377	11	19/306	32	29/195	178.9044	176
1050	8387	11	28/451	32	29/195	179.0532	176

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
1051	8389	11	46/741	32	29/195	179.2019	177
1052	8419	11	9/145	32	29/195	179.3506	177
1053	8423	11	62/999	32	29/195	179.4993	177
1054	8429	11	61/983	32	29/195	179.6480	177
1055	8431	11	60/967	32	29/195	179.7968	177
1056	8443	11	59/951	32	29/195	179.9455	177
1057	8447	11	25/403	32	29/195	180.0942	177
1058	8461	11	49/790	32	29/195	180.2429	177
1059	8467	11	8/129	32	29/195	180.3916	177
1060	8501	11	8/129	32	29/195	180.5404	177
1061	8513	11	39/629	32	29/195	180.6891	178
1062	8521	11	54/871	32	29/195	180.8378	178
1063	8527	11	38/613	32	29/195	180.9865	179
1064	8537	11	15/242	32	29/195	181.1352	179
1065	8539	11	59/952	32	29/195	181.2840	179
1066	8543	11	51/823	32	29/195	181.4327	179
1067	8563	11	43/694	32	29/195	181.5814	179
1068	8573	11	7/113	32	29/195	181.7301	179
1069	8581	11	7/113	32	29/195	181.8789	180
1070	8597	11	7/113	32	29/195	182.0276	180
1071	8599	11	34/549	32	29/195	182.1763	180
1072	8609	11	47/759	32	29/195	182.3250	180
1073	8623	11	20/323	32	29/195	182.4737	180
1074	8627	11	59/953	32	29/195	182.6225	180
1075	8629	11	13/210	32	29/195	182.7712	180
1076	8641	11	32/517	32	29/195	182.9199	180
1077	8647	11	19/307	32	29/195	183.0686	180
1078	8663	11	25/404	32	29/195	183.2173	180
1079	8669	11	37/598	32	29/195	183.3661	180
1080	8677	11	55/889	32	29/195	183.5148	180
1081	8681	11	6/97	32	29/195	183.6635	180
1082	8689	11	6/97	32	29/195	183.8122	180
1083	8693	11	6/97	32	29/195	183.9609	180
1084	8699	11	35/566	32	29/195	184.1097	180
1085	8707	11	52/841	32	29/195	184.2584	180
1086	8713	11	40/647	32	29/195	184.4071	180
1087	8719	11	17/275	32	29/195	184.5558	181
1088	8731	11	28/453	32	29/195	184.7046	181
1089	8737	11	61/987	33	29/195	184.8533	181
1090	8741	11	11/178	33	29/195	185.0020	181
1091	8747	11	38/615	33	29/195	185.1507	182
1092	8753	11	43/696	33	29/195	185.2994	182
1093	8761	11	16/259	33	29/195	185.4482	183
1094	8779	11	58/939	33	29/195	185.5969	183
1095	8783	11	47/761	33	29/195	185.7456	183
1096	8803	11	31/502	33	29/195	185.8943	183
1097	8807	11	41/664	33	29/195	186.0430	184
1098	8819	11	61/988	33	29/195	186.1918	184
1099	8821	11	5/81	33	29/195	186.3405	184
1100	8831	11	5/81	33	29/195	186.4892	184
1101	8837	11	5/81	33	29/195	186.6379	184
1102	8839	11	49/794	33	29/195	186.7866	184
1103	8849	11	34/551	33	29/195	186.9354	185
1104	8861	11	24/389	33	29/195	187.0841	185
1105	8863	11	19/308	33	29/195	187.2328	185
1106	8867	11	52/843	33	29/195	187.3815	185
1107	8887	11	14/227	33	29/195	187.5302	185
1108	8893	11	14/227	33	29/195	187.6790	185
1109	8923	11	23/373	33	29/195	187.8277	186
1110	8929	11	32/519	33	29/195	187.9764	186
1111	8933	11	9/146	33	29/195	188.1251	186
1112	8941	11	9/146	33	29/195	188.2739	186
1113	8951	11	49/795	33	29/195	188.4226	186
1114	8963	11	31/503	33	29/195	188.5713	186
1115	8969	11	57/925	33	29/195	188.7200	186
1116	8971	11	13/211	33	29/195	188.8687	186
1117	8999	11	13/211	33	29/195	189.0175	187
1118	9001	11	30/487	33	29/195	189.1662	187
1119	9007	11	17/276	33	29/195	189.3149	187
1120	9011	11	38/617	33	29/195	189.4636	187
1121	9013	11	21/341	33	29/195	189.6123	187
1122	9029	11	54/877	33	29/195	189.7611	187
1123	9041	11	33/536	33	29/195	189.9098	188
1124	9043	11	41/666	33	29/195	190.0585	188
1125	9049	11	61/991	33	29/195	190.2072	188
1126	9059	11	4/65	33	29/195	190.3559	188
1127	9067	11	4/65	33	29/195	190.5047	188
1128	9091	11	4/65	33	29/195	190.6534	188
1129	9103	11	4/65	33	29/195	190.8021	189
1130	9109	11	51/829	33	29/195	190.9508	189
1131	9127	11	39/634	33	29/195	191.0996	189
1132	9133	11	58/943	33	29/195	191.2483	189
1133	9137	11	50/813	33	29/195	191.3970	189
1134	9151	11	42/683	33	29/195	191.5457	189
1135	9157	11	19/309	33	29/195	191.6944	189
1136	9161	11	49/797	33	29/195	191.8432	189
1137	9173	11	15/244	33	29/195	191.9919	189
1138	9181	11	26/423	33	29/195	192.1406	189
1139	9187	11	48/781	33	29/195	192.2893	189
1140	9199	11	11/179	33	29/195	192.4380	189
1141	9203	11	51/830	33	29/195	192.5868	189
1142	9209	11	29/472	33	29/195	192.7355	189
1143	9221	11	18/293	33	29/195	192.8842	189
1144	9227	11	25/407	33	29/195	193.0329	189
1145	9239	11	32/521	33	29/195	193.1816	189
1146	9241	11	60/977	33	29/195	193.3304	189
1147	9257	11	7/114	33	29/195	193.4791	189
1148	9277	11	7/114	33	29/195	193.6278	189
1149	9281	11	52/847	33	29/195	193.7765	189
1150	9283	11	31/505	33	29/195	193.9253	189
1151	9293	11	24/391	33	29/195	194.0740	190
1152	9311	11	17/277	33	29/195	194.2227	190
1153	9319	11	44/717	33	29/195	194.3714	191
1154	9323	11	37/603	33	29/195	194.5201	191
1155	9337	11	10/163	33	29/195	194.6689	191

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
1156	9341	11	10/163	34	29/195	194.8176	191
1157	9343	11	43/701	34	29/195	194.9663	191
1158	9349	11	23/375	34	29/195	195.1150	191
1159	9371	11	49/799	34	29/195	195.2637	191
1160	9377	11	13/212	34	29/195	195.4125	191
1161	9391	11	42/685	34	29/195	195.5612	191
1162	9397	11	45/734	34	29/195	195.7099	191
1163	9403	11	16/261	34	29/195	195.8586	192
1164	9413	11	54/881	34	29/195	196.0073	192
1165	9419	11	19/310	34	29/195	196.1561	192
1166	9421	11	22/359	34	29/195	196.3048	192
1167	9431	11	25/408	34	29/195	196.4535	192
1168	9433	11	28/457	34	29/195	196.6022	192
1169	9437	11	34/555	34	29/195	196.7510	192
1170	9439	11	40/653	34	29/195	196.8997	192
1171	9461	11	52/849	34	29/195	197.0484	193
1172	9463	11	3/49	34	29/195	197.1971	193
1173	9467	11	3/49	34	29/195	197.3458	193
1174	9473	11	3/49	34	29/195	197.4946	193
1175	9479	11	3/49	34	29/195	197.6433	193
1176	9491	11	3/49	34	29/195	197.7920	193
1177	9497	11	3/49	34	29/195	197.9407	193
1178	9511	11	59/964	34	29/195	198.0894	193
1179	9521	11	47/768	34	29/195	198.2382	193
1180	9533	11	38/621	34	29/195	198.3869	193
1181	9539	11	32/523	34	29/195	198.5356	194
1182	9547	11	26/425	34	29/195	198.6843	194
1183	9551	11	23/376	34	29/195	198.8330	194
1184	9587	11	43/703	34	29/195	198.9818	194
1185	9601	11	57/932	34	29/195	199.1305	194
1186	9613	11	17/278	34	29/195	199.2792	194
1187	9619	11	48/785	34	29/195	199.4279	195
1188	9623	11	59/965	34	29/195	199.5767	195
1189	9629	11	14/229	34	29/195	199.7254	195
1190	9631	11	39/638	34	29/195	199.8741	195
1191	9643	11	36/589	34	29/195	200.0228	195
1192	9649	11	11/180	34	29/195	200.1715	195
1193	9661	11	11/180	34	29/195	200.3203	196
1194	9677	11	41/671	34	29/195	200.4690	196
1195	9679	11	19/311	34	29/195	200.6177	196
1196	9689	11	46/753	34	29/195	200.7664	196
1197	9697	11	35/573	34	29/195	200.9151	196
1198	9719	11	59/966	34	29/195	201.0639	196
1199	9721	11	8/131	34	29/195	201.2126	196
1200	9733	11	8/131	34	29/195	201.3613	196
1201	9739	11	45/737	34	29/195	201.5100	197
1202	9743	11	29/475	34	29/195	201.6587	197
1203	9749	11	21/344	34	29/195	201.8075	197
1204	9767	11	47/770	34	29/195	201.9562	197
1205	9769	11	13/213	34	29/195	202.1049	197
1206	9781	11	44/721	34	29/195	202.2536	197
1207	9787	11	49/803	34	29/195	202.4024	197
1208	9791	11	59/967	34	29/195	202.5511	197
1209	9803	11	23/377	34	29/195	202.6998	197
1210	9811	11	28/459	34	29/195	202.8485	197
1211	9817	11	38/623	34	29/195	202.9972	197
1212	9829	11	53/869	34	29/195	203.1460	197
1213	9833	11	5/82	34	29/195	203.2947	198
1214	9839	11	5/82	34	29/195	203.4434	198
1215	9851	11	5/82	34	29/195	203.5921	198
1216	9857	11	5/82	34	29/195	203.7408	198
1217	9859	11	42/689	34	29/195	203.8896	199
1218	9871	11	32/525	34	29/195	204.0383	199
1219	9883	11	49/804	34	29/195	204.1870	199
1220	9887	11	22/361	34	29/195	204.3357	199
1221	9901	11	17/279	34	29/195	204.4844	199
1222	9907	11	46/755	34	29/195	204.6332	199
1223	9923	11	41/673	34	29/195	204.7819	200
1224	9929	11	12/197	34	29/195	204.9306	200
1225	9931	11	12/197	35	29/195	205.0793	200
1226	9941	11	31/509	35	29/195	205.2281	200
1227	9949	11	19/312	35	29/195	205.3768	200
1228	9967	11	26/427	35	29/195	205.5255	200
1229	9973	11	26/427	35	29/195	205.6742	201
1230	10007	11	33/542	35	29/195	205.8229	201
1231	10009	11	33/542	35	29/195	205.9717	202
1232	10037	11	54/887	35	29/195	206.1204	202
1233	10039	11	54/887	35	29/195	206.2691	202
1234	10061	11	7/115	35	29/195	206.4178	202
1235	10067	11	7/115	35	29/195	206.5665	202
1236	10069	11	7/115	35	29/195	206.7153	202
1237	10079	11	7/115	35	29/195	206.8640	203
1238	10091	11	58/953	35	29/195	207.0127	203
1239	10093	11	58/953	35	29/195	207.1614	203
1240	10099	11	37/608	35	29/195	207.3101	203
1241	10103	11	37/608	35	29/195	207.4589	203
1242	10111	11	23/378	35	29/195	207.6076	203
1243	10133	11	23/378	35	29/195	207.7563	203
1244	10139	11	39/641	35	29/195	207.9050	203
1245	10141	11	39/641	35	29/195	208.0537	203
1246	10151	11	16/263	35	29/195	208.2025	203
1247	10159	11	16/263	35	29/195	208.3512	203
1248	10163	11	41/674	35	29/195	208.4999	203
1249	10169	11	41/674	35	29/195	208.6486	204
1250	10177	11	59/970	35	29/195	208.7974	204
1251	10181	11	59/970	35	29/195	208.9461	204
1252	10193	11	52/855	35	29/195	209.0948	204
1253	10211	11	52/855	35	29/195	209.2435	204
1254	10223	11	9/148	35	29/195	209.3922	204
1255	10243	11	9/148	35	29/195	209.5410	204
1256	10247	11	9/148	35	29/195	209.6897	204
1257	10253	11	9/148	35	29/195	209.8384	204
1258	10259	11	38/625	35	29/195	209.9871	204
1259	10267	11	38/625	35	29/195	210.1358	205
1260	10271	11	49/806	35	29/195	210.2846	205

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
1261	10273	11	49/806	35	29/195	210.4333	205
1262	10289	11	20/329	35	29/195	210.5820	205
1263	10301	11	20/329	35	29/195	210.7307	205
1264	10303	11	31/510	35	29/195	210.8794	205
1265	10313	11	31/510	35	29/195	211.0282	205
1266	10321	11	11/181	35	29/195	211.1769	205
1267	10331	11	11/181	35	29/195	211.3256	205
1268	10333	11	11/181	35	29/195	211.4743	205
1269	10337	11	11/181	35	29/195	211.6231	205
1270	10343	11	46/757	35	29/195	211.7718	205
1271	10357	11	46/757	35	29/195	211.9205	205
1272	10369	11	24/395	35	29/195	212.0692	205
1273	10391	11	24/395	35	29/195	212.2179	205
1274	10399	11	50/823	35	29/195	212.3667	205
1275	10427	11	50/823	35	29/195	212.5154	205
1276	10429	11	13/214	35	29/195	212.6641	205
1277	10433	11	13/214	35	29/195	212.8128	206
1278	10453	11	54/889	35	29/195	212.9615	206
1279	10457	11	54/889	35	29/195	213.1103	207
1280	10459	11	28/461	35	29/195	213.2590	207
1281	10463	11	28/461	35	29/195	213.4077	207
1282	10477	11	15/247	35	29/195	213.5564	207
1283	10487	11	15/247	35	29/195	213.7051	208
1284	10499	11	47/774	35	29/195	213.8539	208
1285	10501	11	47/774	35	29/195	214.0026	208
1286	10513	11	49/807	35	29/195	214.1513	208
1287	10529	11	49/807	35	29/195	214.3000	208
1288	10531	11	17/280	35	29/195	214.4488	208
1289	10559	11	17/280	35	29/195	214.5975	209
1290	10567	11	36/593	35	29/195	214.7462	209
1291	10589	11	36/593	35	29/195	214.8949	210
1292	10597	11	19/313	35	29/195	215.0436	210
1293	10601	11	19/313	35	29/195	215.1924	210
1294	10607	11	21/346	35	29/195	215.3411	210
1295	10613	11	21/346	35	29/195	215.4898	210
1296	10627	11	44/725	36	29/195	215.6385	210
1297	10631	11	44/725	36	29/195	215.7872	211
1298	10639	11	23/379	36	29/195	215.9360	211
1299	10651	11	23/379	36	29/195	216.0847	211
1300	10657	11	25/412	36	29/195	216.2334	211
1301	10663	11	25/412	36	29/195	216.3821	212
1302	10667	11	27/445	36	29/195	216.5308	212
1303	10687	11	27/445	36	29/195	216.6796	213
1304	10691	11	60/989	36	29/195	216.8283	213
1305	10709	11	60/989	36	29/195	216.9770	213
1306	10711	11	33/544	36	29/195	217.1257	213
1307	10723	11	33/544	36	29/195	217.2745	214
1308	10729	11	37/610	36	29/195	217.4232	214
1309	10733	11	37/610	36	29/195	217.5719	214
1310	10739	11	41/676	36	29/195	217.7206	214
1311	10753	11	41/676	36	29/195	217.8693	214
1312	10771	11	47/775	36	29/195	218.0181	214
1313	10781	11	47/775	36	29/195	218.1668	214
1314	10789	11	55/907	36	29/195	218.3155	214
1315	10799	11	55/907	36	29/195	218.4642	214
1316	10831	11	2/33	36	29/195	218.6129	214
1317	10837	11	2/33	36	29/195	218.7617	214
1318	10847	11	2/33	36	29/195	218.9104	214
1319	10853	11	2/33	36	29/195	219.0591	215
1320	10859	11	2/33	36	29/195	219.2078	215
1321	10861	11	2/33	36	29/195	219.3565	216
1322	10867	11	2/33	36	29/195	219.5053	216
1323	10883	11	2/33	36	29/195	219.6540	216
1324	10889	11	2/33	36	29/195	219.8027	216
1325	10891	11	2/33	36	29/195	219.9514	216
1326	10903	11	2/33	36	29/195	220.1002	216
1327	10909	11	2/33	36	29/195	220.2489	217
1328	10937	11	2/33	36	29/195	220.3976	217
1329	10939	11	2/33	36	29/195	220.5463	217
1330	10949	11	2/33	36	29/195	220.6950	217
1331	10957	11	2/33	36	29/195	220.8438	217
1332	10973	11	2/33	36	29/195	220.9925	217
1333	10979	11	2/33	36	29/195	221.1412	217
1334	10987	11	2/33	36	29/195	221.2899	217
1335	10993	11	2/33	36	29/195	221.4386	217
1336	11003	11	2/33	36	29/195	221.5874	217
1337	11027	11	2/33	36	29/195	221.7361	217
1338	11047	11	55/908	36	29/195	221.8848	217
1339	11057	11	55/908	36	29/195	222.0335	217
1340	11059	11	47/776	36	29/195	222.1822	217
1341	11069	11	47/776	36	29/195	222.3310	217
1342	11071	11	41/677	36	29/195	222.4797	217
1343	11083	11	41/677	36	29/195	222.6284	217
1344	11087	11	37/611	36	29/195	222.7771	217
1345	11093	11	37/611	36	29/195	222.9259	217
1346	11113	11	33/545	36	29/195	223.0746	217
1347	11117	11	33/545	36	29/195	223.2233	217
1348	11119	11	60/991	36	29/195	223.3720	217
1349	11131	11	31/512	36	29/195	223.5207	217
1350	11149	11	56/925	36	29/195	223.6695	217
1351	11159	11	56/925	36	29/195	223.8182	217
1352	11161	11	25/413	36	29/195	223.9669	217
1353	11171	11	52/859	36	29/195	224.1156	217
1354	11173	11	48/793	36	29/195	224.2643	217
1355	11177	11	48/793	36	29/195	224.4131	217
1356	11197	11	44/727	36	29/195	224.5618	217
1357	11213	11	23/380	36	29/195	224.7105	217
1358	11239	11	21/347	36	29/195	224.8592	217
1359	11243	11	21/347	36	29/195	225.0079	217
1360	11251	11	59/975	36	29/195	225.1567	217
1361	11257	11	59/975	36	29/195	225.3054	218
1362	11261	11	19/314	36	29/195	225.4541	218
1363	11273	11	19/314	36	29/195	225.6028	218
1364	11279	11	53/876	36	29/195	225.7515	218
1365	11287	11	53/876	36	29/195	225.9003	218

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n (1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})} (1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})} (1 - 1/p_i)$	$\pi(n)$
1366	11299	11	17/281	36	29/195	226.0490	218
1367	11311	11	17/281	36	29/195	226.1977	219
1368	11317	11	32/529	36	29/195	226.3464	219
1369	11321	12	32/529	37	139/958	226.4915	219
1370	11329	12	15/248	37	139/958	226.6366	219
1371	11351	12	15/248	37	139/958	226.7817	219
1372	11353	12	15/248	37	139/958	226.9268	219
1373	11369	12	15/248	37	139/958	227.0719	220
1374	11383	12	28/463	37	139/958	227.2170	220
1375	11393	12	28/463	37	139/958	227.3621	220
1376	11399	12	54/893	37	139/958	227.5072	220
1377	11411	12	54/893	37	139/958	227.6523	220
1378	11423	12	13/215	37	139/958	227.7974	220
1379	11437	12	13/215	37	139/958	227.9425	220
1380	11443	12	50/827	37	139/958	228.0876	220
1381	11447	12	50/827	37	139/958	228.2327	221
1382	11467	12	24/397	37	139/958	228.3777	221
1383	11471	12	24/397	37	139/958	228.5228	221
1384	11483	12	35/579	37	139/958	228.6679	221
1385	11489	12	35/579	37	139/958	228.8130	221
1386	11491	12	11/182	37	139/958	228.9581	221
1387	11497	12	11/182	37	139/958	229.1032	221
1388	11503	12	11/182	37	139/958	229.2483	221
1389	11519	12	11/182	37	139/958	229.3934	221
1390	11527	12	53/877	37	139/958	229.5385	221
1391	11549	12	53/877	37	139/958	229.6836	221
1392	11551	12	31/513	37	139/958	229.8287	221
1393	11579	12	31/513	37	139/958	229.9738	221
1394	11587	12	20/331	37	139/958	230.1189	221
1395	11593	12	20/331	37	139/958	230.2640	221
1396	11597	12	29/480	37	139/958	230.4091	221
1397	11617	12	29/480	37	139/958	230.5542	221
1398	11621	12	38/629	37	139/958	230.6992	221
1399	11633	12	38/629	37	139/958	230.8443	222
1400	11657	12	9/149	37	139/958	230.9894	222
1401	11677	12	9/149	37	139/958	231.1345	222
1402	11681	12	9/149	37	139/958	231.2796	222
1403	11689	12	9/149	37	139/958	231.4247	222
1404	11699	12	9/149	37	139/958	231.5698	222
1405	11701	12	9/149	37	139/958	231.7149	222
1406	11717	12	34/563	37	139/958	231.8600	222
1407	11719	12	43/712	37	139/958	232.0051	222
1408	11731	12	25/414	37	139/958	232.1502	222
1409	11743	12	25/414	37	139/958	232.2953	223
1410	11777	12	57/944	37	139/958	232.4404	223
1411	11779	12	57/944	37	139/958	232.5855	223
1412	11783	12	16/265	37	139/958	232.7306	223
1413	11789	12	16/265	37	139/958	232.8756	223
1414	11801	12	39/646	37	139/958	233.0207	223
1415	11807	12	39/646	37	139/958	233.1658	223
1416	11813	12	23/381	37	139/958	233.3109	223
1417	11821	12	23/381	37	139/958	233.4560	223
1418	11827	12	30/497	37	139/958	233.6011	223
1419	11831	12	30/497	37	139/958	233.7462	223
1420	11833	12	44/729	37	139/958	233.8913	223
1421	11839	12	44/729	37	139/958	234.0364	223
1422	11863	12	7/116	37	139/958	234.1815	223
1423	11867	12	7/116	37	139/958	234.3266	224
1424	11887	12	7/116	37	139/958	234.4717	224
1425	11897	12	7/116	37	139/958	234.6168	224
1426	11903	12	7/116	37	139/958	234.7619	224
1427	11909	12	7/116	37	139/958	234.9070	225
1428	11923	12	54/895	37	139/958	235.0521	225
1429	11927	12	54/895	37	139/958	235.1971	226
1430	11933	12	33/547	37	139/958	235.3422	226
1431	11939	12	40/663	37	139/958	235.4873	226
1432	11941	12	26/431	37	139/958	235.6324	226
1433	11953	12	26/431	37	139/958	235.7775	227
1434	11959	12	45/746	37	139/958	235.9226	227
1435	11969	12	45/746	37	139/958	236.0677	227
1436	11971	12	19/315	37	139/958	236.2128	227
1437	11981	12	19/315	37	139/958	236.3579	227
1438	11987	12	31/514	37	139/958	236.5030	227
1439	12007	12	31/514	37	139/958	236.6481	228
1440	12011	12	12/199	37	139/958	236.7932	228
1441	12037	12	12/199	37	139/958	236.9383	228
1442	12041	12	12/199	37	139/958	237.0834	228
1443	12043	12	12/199	37	139/958	237.2285	228
1444	12049	12	53/879	38	139/958	237.3736	228
1445	12071	12	53/879	38	139/958	237.5186	228
1446	12073	12	29/481	38	139/958	237.6637	228
1447	12097	12	29/481	38	139/958	237.8088	229
1448	12101	12	17/282	38	139/958	237.9539	229
1449	12107	12	17/282	38	139/958	238.0990	229
1450	12109	12	56/929	38	139/958	238.2441	229
1451	12113	12	17/282	38	139/958	238.3892	230
1452	12119	12	22/365	38	139/958	238.5343	230
1453	12143	12	22/365	38	139/958	238.6794	231
1454	12149	12	49/813	38	139/958	238.8245	231
1455	12157	12	49/813	38	139/958	238.9696	231
1456	12161	12	59/979	38	139/958	239.1147	231
1457	12163	12	59/979	38	139/958	239.2598	231
1458	12197	12	37/614	38	139/958	239.4049	231
1459	12203	12	37/614	38	139/958	239.5500	232
1460	12211	12	47/780	38	139/958	239.6951	232
1461	12227	12	47/780	38	139/958	239.8401	232
1462	12239	12	5/83	38	139/958	239.9852	232
1463	12241	12	5/83	38	139/958	240.1303	232
1464	12251	12	5/83	38	139/958	240.2754	232
1465	12253	12	5/83	38	139/958	240.4205	232
1466	12263	12	5/83	38	139/958	240.5656	232
1467	12269	12	5/83	38	139/958	240.7107	232
1468	12277	12	5/83	38	139/958	240.8558	232
1469	12281	12	5/83	38	139/958	241.0009	232
1470	12289	12	5/83	38	139/958	241.1460	232

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^n(1 - 1/p_j)$	$[(\sqrt{n})]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1 - 1/p_j)$	$\pi_L(n) = \sum_{j=1}^n \prod_{i=1}^{\pi(\sqrt{j})}(1 - 1/p_i)$	$\pi(n)$
1471	12301	12	5/83	38	139/958	241.2911	233
1472	12323	12	48/797	38	139/958	241.4362	233
1473	12329	12	53/880	38	139/958	241.5813	233
1474	12343	12	38/631	38	139/958	241.7264	233
1475	12347	12	38/631	38	139/958	241.8715	233
1476	12373	12	28/465	38	139/958	242.0166	233
1477	12377	12	28/465	38	139/958	242.1616	233
1478	12379	12	51/847	38	139/958	242.3067	233
1479	12391	12	51/847	38	139/958	242.4518	233
1480	12401	12	41/681	38	139/958	242.5969	233
1481	12409	12	23/382	38	139/958	242.7420	234
1482	12413	12	18/299	38	139/958	242.8871	234
1483	12421	12	18/299	38	139/958	243.0322	235
1484	12433	12	49/814	38	139/958	243.1773	235
1485	12437	12	49/814	38	139/958	243.3224	235
1486	12451	12	44/731	38	139/958	243.4675	235
1487	12457	12	31/515	38	139/958	243.6126	236
1488	12473	12	13/216	38	139/958	243.7577	236
1489	12479	12	13/216	38	139/958	243.9028	237
1490	12487	12	13/216	38	139/958	244.0479	237
1491	12491	12	13/216	38	139/958	244.1930	237
1492	12497	12	34/565	38	139/958	244.3380	237
1493	12503	12	47/781	38	139/958	244.4831	238
1494	12511	12	21/349	38	139/958	244.6282	238
1495	12517	12	21/349	38	139/958	244.7733	238
1496	12527	12	50/831	38	139/958	244.9184	238
1497	12539	12	50/831	38	139/958	245.0635	238
1498	12541	12	37/615	38	139/958	245.2086	238
1499	12547	12	37/615	38	139/958	245.3537	239
1500	12553	12	53/881	38	139/958	245.4988	239

7. Appendix IV: Resources

The only resources needed by a reader for following, and reproducing, the proofs of this paper are knowledge of:

1. Chapters I and II (The series of primes) of Hardy and Wright: Introduction to the Theory of Numbers.
2. Chapter V (Congruences and residues) of Hardy and Wright: Introduction to the Theory of Numbers.
3. The statement of Mertens Theorem (Theorem 429, p.351) in Hardy and Wright: Introduction to the Theory of Numbers.
4. The standard definition of a Probability Model (reproduced in §4., Appendix I).
5. The following definitions (reproduced in §4., Appendix I) from Charles Grinstead and Laurie Snell: Introduction to Probability ;
 - (i) Distribution Function (Chapter 1)
 - (ii) Mutual independence (Chapter 4)
 - (iii) Expected value (Chapter 5)
 - (iv) Law of Large Numbers (Chapter 8)
6. The definition of a polynomial-time algorithm in Cook (p.1): The P versus NP Problem.

No familiarity is needed of any advanced topics covered in the above texts or in P. D. T. A. Elliott: Probabilistic Number Theory.

Ideally, all the lemmas, theorems and corollaries of the paper could be given as exercises to students in any introductory course or textbook on the Theory of Numbers.

The thesis of the paper for a student of Number Theory is that all the known functions such as $x/\log_e x$, $Li(x)$, and Riemann's $R(x)$ are only proven to be asymptotically equivalent to $\pi(x)$ (Fig.2). There can be many such functions. However, none of them can claim to yield a mathematically unique estimate of $\pi(n)$ for finite values of n .

The function $\pi_L(n)$ defined in Section 1.D. is *the* mathematically unique function that gives the statistically expected value of $\pi(n)$ for any finite n (Fig.3). The uniqueness follows from Corollary 2.13.

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