# Abelianization of Subgroups of Reflection Group and their Braid Group; an Application to Cohomology

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# Abelianization of Subgroups of Reflection Groups and their Braid Groups; an Application to Cohomology

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Friday 28th August, 2015

**Abstract**

The final result of this article gives the order of the extension

$$1 \longrightarrow P/[P,P] \xrightarrow{j} B/[P,P] \xrightarrow{p} W \longrightarrow 1$$

as an element of the cohomology group $H^2(W, P/[P,P])$ (where $B$ and $P$ stands for the braid group and the pure braid group associated to the complex reflection group $W$). To obtain this result, we first refine Stanley-Springer's theorem on the abelianization of a reflection group to describe the abelianization of the stabilizer $N_H$ of a hyperplane $H$. The second step is to describe the abelianization of big subgroups of the braid group $B$ of $W$. More precisely, we just need a group homomorphism from the inverse image of $N_H$ by $p$ (where $p : B \to W$ is the canonical morphism) but a slight enhancement gives a complete description of the abelianization of $p^{-1}(W')$ where $W'$ is a reflection subgroup of $W$ or the stabilizer of a hyperplane. We also suggest a lifting construction for every element of the centralizer of a reflection in $W$.

## 1 Introduction

Let us start with setting the framework. Let $V$ be a finite dimensional complex vector space; a *reflection* is a non trivial finite order element $s$ of $\mathrm{GL}(V)$ which pointwise fixes a hyperplane of $V$, called *the hyperplane of* $s$. The *line of* $s$ is the one dimensional eigenspace of $s$ associated to the non trivial eigenvalue of $s$. Let $W \subset \mathrm{GL}(V)$ be a *(complex) reflection group* that is to say a finite group generated by reflections. We denote by $\mathscr{S}$ the set of reflections of $W$ and $\mathscr{H}$ the set of hyperplanes of $W$ :

$$\mathscr{S} = \{s \in W, \quad \mathrm{codimKer}\,(s - \mathrm{id}) = 1\} \quad \text{and} \quad \mathscr{H} = \{\mathrm{Ker}\,(s - \mathrm{id}), \quad s \in \mathscr{S}\}\,.$$

For a reflection group, we denote by $V^{\mathrm{reg}} = V \setminus \cup_{H \in \mathscr{H}} H$ the set of regular vectors. According to a classical result of Steinberg [12, Corollary 1.6], $V^{\mathrm{reg}}$ is precisely the set of vectors that no non trivial element of $W$ fixes. Thus, the canonical map $\pi : V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ is a Galois covering. So let us fix a base point $x_0 \in V^{\mathrm{reg}}$ and denote by $P = \pi_1(V^{\mathrm{reg}}, x_0)$ and $B = \pi_1(V^{\mathrm{reg}}/W, \pi(x_0))$ the fundamental groups of $V^{\mathrm{reg}}$ and its quotient $V^{\mathrm{reg}}/W$, we obtain the short exact sequence

$$1 \longrightarrow P \longrightarrow B \xrightarrow{p} W \longrightarrow 1 \tag{1}$$

The groups $B$ and $P$ are respectively called the *braid group* and the *pure braid group* of $W$.

The final result of this article (Corollary 27) gives the order of the extension

$$1 \longrightarrow P/[P,P] \xrightarrow{j} B/[P,P] \xrightarrow{p} W \longrightarrow 1$$

as an element of the cohomology group $H^2(W, P/[P,P])$. This order turns out to be the integer $\kappa(W)$ defined by Marin in [9]. As explained in [9], this integer is linked to the periodicity of the monodromy representation of $B$ associated to the action of $W$ on its set of hyperplanes.

To obtain this result, we first describe in section 2 the abelianization of some subgroups of complex reflection groups. Specifically, we study the stabilizer of a hyperplane $H$ which is the same as the centralizer of a reflection of hyperplane $H$. Contrary to the case of Coxeter groups, this is not a reflection subgroup of the complex reflection group $W$ in general. The first step is to refine Stanley-Springer's theorem [13][14] on the abelianization of a reflection group (see Proposition 6 in Section 2).

The rationale also relies on a good description (Section 3) of abelianization of various types of big subgroups of the braid group $B$ of $W$ (here "big" stands for "containing the pure braid group $P$"). Though we just need to construct a group homomorphism from the inverse image of the stabilizer of $H$ by $p$ with values in $\mathbb{Q}$ (see Definition 19 and Proposition 23), we give in fact a complete description of the abelianization of $p^{-1}(W')$ with $W'$ a reflection subgroup of $W$ or the stabilizer of a hyperplane (Proposition 15 and Proposition 24). We also suggest a lifting construction for every element of the centralizer of a reflection in $W$ generalizing the construction of the generator of monodromy of [4, p.14] (see Remark 21).

Orbits of hyperplanes and ramification index are gathered in tables in the last section.

We finish the introduction with some notations. We fix a $W$-invariant hermitian product on $V$ denoted by $\langle \cdot, \cdot \rangle$ : orthogonality will always be relative to this particular hermitian product. We denote by $S(V^*)$ the symmetric algebra of the dual of $V$ which is also the polynomial functions on $V$.

**Notation 1 – Around a Hyperplane of $W$.** For $H \in \mathscr{H}$,

- one chooses $\alpha_H \in V^*$ a linear form with kernel $H$;
- one sets $W_H = \mathrm{Fix}_W(H) = \{g \in W, \quad \forall x \in H, \quad gx = x\}$. This is a cyclic subgroup of $W$. We denote by $e_H$ its order and by $s_H$ its generator with determinant $\zeta_H = \exp(2i\pi/e_H)$. Except for identity, the elements of $W_H$ are precisely the reflections of $W$ whose hyperplane is $H$. The reflection $s_H$ is called the *distinguished reflection for $H$* in $W$.

For $n$ a positive integer, we denote by $\mathbb{U}_n$ the group of the $n^{\text{th}}$ root of unity in $\mathbb{C}$ and by $\mathbb{U}$ the group of unit complex numbers. For a group $G$, we denote by $[G, G]$ the commutator subgroup of $G$ and by $G^{\mathrm{ab}} = G/[G, G]$ the abelianization of $G$.

# 2 Abelianization of Subgroups of Reflection Groups

Stanley-Springer's Theorem (see [13, Theorem 4.3.4][14, Theorem 3.1]) gives an explicit description of the group of linear characters of a reflection group using the conjugacy classes of hyperplanes. Naturally, it applies to all reflection subgroups of a reflection group and in particular to parabolic subgroups thanks to Steinberg's theorem [12, Theorem 1.5]. But since $P/[P, P]$ is the $\mathbb{Z}W$ permutation module defined by the hyperplanes of $W$, we are interested in the stabilizer of a hyperplane which is not in general a reflection subgroup of $W$. So we have to go deeper in the study of the stabilizer of a hyperplane.

Before starting our study of the stabilizer of a hyperplane, we write down Stanley-Springer's Theorem because we will use it many times.

**Theorem 2 – Stanley-Springer's Theorem.** For every map $n : \mathscr{H} \to \mathbb{N}$ constant on the $W$-orbits of $\mathscr{H}$, there exists a linear character $\chi : W \to \mathbb{C}^\times$ such that $\chi(s_H) = \det(s_H)^{-n_H}$.

Moreover, the $\chi$-isotypic component of $S(V^*)$ is a free $S(V^*)^W$-module of rank 1 generated by

$$Q_\chi = \prod_{H \in \mathscr{H}} {\alpha_H}^{n_H}$$

where the $n_H$ are related to $\chi$ by the formula above and satisfy the relations $0 \leqslant n_H \leqslant e_H - 1$.

For $H \in \mathscr{H}$, we set

$$N_H = \{w \in W, \quad wH = H\} = \{w \in W, \quad ws_H = s_H w\}$$

the stabilizer of $H$ which is also the centralizer of $s_H$. We denote by $D = H^\perp$ the line of $s_H$ (or of every reflection of $W$ with hyperplane $H$) it is the unique $N_H$-stable line of $V$ such that $H \oplus D = V$ and $N_H$ is also the stabilizer of $D$ (see [3, Proposition 1.19]). We denote the parabolic subgroup associated to $D$ by $C_H = \{w \in W, \quad \forall x \in D, \ wx = x\}$.

Since $D$ is a line stable by every element of $N_H$ and $W_H \subset N_H$, there exists an integer $f_H$ such that $e_H \mid f_H$ and the following sequence is exact

$$1 \longrightarrow C_H \xrightarrow{i} N_H \xrightarrow{r} \mathbb{U}_{f_H} \longrightarrow 1 \tag{2}$$

where $i$ is the natural inclusion and $r$ denote the restriction to $D$. We define $r$ to be the *natural linear character of $N_H$*.

Before stating our main result on the abelianization of the stabilizer of a hyperplane, let us start with a straightforward lemma.

**Lemma 3 – Abelianization of an exact sequence.** Let us consider the following exact sequence of groups where $M$ is an abelian group.

$$1 \longrightarrow C \xrightarrow{i} N \xrightarrow{r} M \longrightarrow 1$$

Then the following sequence is exact

$$C^{\mathrm{ab}} \xrightarrow{i^{\mathrm{ab}}} N^{\mathrm{ab}} \xrightarrow{r^{\mathrm{ab}}} M \longrightarrow 1$$

Moreover the map $i^{\mathrm{ab}}$ is injective if and only if $[N, N] = [C, C]$. When $C^{\mathrm{ab}}$ and $M$ are finite, the injectivity of $i^{\mathrm{ab}}$ is equivalent to $|C^{\mathrm{ab}}||M| = |N^{\mathrm{ab}}|$.

We also have the following criterion : the map $i^{\mathrm{ab}}$ is injective if and only if the canonical restriction map $\mathrm{Hom}_{\mathrm{gr.}}(N, \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{gr.}}(C, \mathbb{C}^\times)$ is surjective.

*Proof.* The first injectivity criterion is an easy diagram chasing computation. The second one is trivial. Let us focus on the third one. Since $\mathbb{C}^\times$ is a commutative group, we have the following commutative square whose vertical arrows are isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{gr.}}(N, \mathbb{C}^\times) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr.}}(C, \mathbb{C}^\times) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{gr.}}(N^{\mathrm{ab}}, \mathbb{C}^\times) & \xrightarrow{\circ i^{\mathrm{ab}}} & \mathrm{Hom}_{\mathrm{gr.}}(C^{\mathrm{ab}}, \mathbb{C}^\times) \end{array}$$

Moreover, since $\mathbb{C}^\times$ is a divisible abelian group, $i^{\mathrm{ab}}$ is injective if and only if $\circ i^{\mathrm{ab}}$ is surjective. □

Before applying the preceding lemma to the stabilizer of a hyperplane, let us introduce a definition and a classical linear algebra lemma.

**Definition 4 – Commuting hyperplanes.** Let $H, H' \in \mathscr{H}$. We say that $H$ *and* $H'$ *commute* if $s_H$ and $s_{H'}$ commute.

We denote by $\mathscr{H}_H$ the set of hyperplanes which commute with $H$ and $\mathscr{H}'_H = \mathscr{H}_H \smallsetminus \{H\}$.

The next lemma on commuting hyperplanes is stated in [3, Lemma 1.7].

**Lemma 5** We have the following equivalent characterizations :

($i$) the hyperplanes $H$ and $H'$ commute
($ii$) $H = H'$ or $D = H^\perp \subset H'$
($iii$) every reflection of $W$ with hyperplane $H$ commutes with every reflection of $W$ with hyperplane $H'$
($iv$) there exists a reflection of $W$ with hyperplane $H$ which commutes with a reflection of $W$ with hyperplane $H'$

As a consequence of Lemma 3, we are now able to formulate the following proposition.

**Proposition 6 – Abelianization of the stabilizer of a hyperplane.** For a hyperplane $H \in \mathscr{H}$, the following sequence is exact

$$C_H^{\mathrm{ab}} \xrightarrow{i^{\mathrm{ab}}} N_H^{\mathrm{ab}} \xrightarrow{r^{\mathrm{ab}}} \mathbb{U}_{f_H} \longrightarrow 1$$

Moreover, we have the following geometric characterization of the injectivity of $i^{\mathrm{ab}}$ : the map $i^{\mathrm{ab}}$ is injective if and only if the orbits of the hyperplanes commuting with $H$ under $N_H$ and $C_H$ are the same.

*Proof.* Steinberg's theorem and Lemma 5 ($ii$) ensure us that $C_H$ is the reflection subgroup of $W$ generated by the $s_{H'}$ for $H' \neq H$ commuting with $H$. Thanks to Theorem 2, we are able to describe the linear characters of $C_H$. For every linear character $\delta$ of $C_H$, there exists integers $e_{\mathcal{O}}$ for $\mathcal{O} \in \mathscr{H}'_H/C_H$ such that the polynomial

$$Q_\delta = \prod_{\mathcal{O} \in \mathscr{H}'_H/C_H} \prod_{H' \in \mathcal{O}} \alpha_{H'}{}^{e_{\mathcal{O}}} \in S(V^*)$$

verifies $gQ_\delta = \delta(g)Q_\delta$ for every $g \in C_H$.

Let us assume that the orbits of the hyperplanes commuting with $H$ under $N_H$ and $C_H$ are the same. For every $g \in N_H$ and every $\mathcal{O} \in \mathscr{H}'_H/C_H$, there exists $\lambda_g \in \mathbb{C}^\times$ such that

$$g \prod_{H' \in \mathcal{O}} \alpha_{H'} = \lambda_g \prod_{H' \in \mathcal{O}} \alpha_{H'}$$

So we obtain that, for every $g \in N_H$, there exists $\mu_g \in \mathbb{C}^\times$ such that $gQ_\delta = \mu_g Q_\delta$. Thus every linear character of $C_H$ extends to $N_H$ and Lemma 3 tells us that the map $i^{\mathrm{ab}}$ is injective.

Let us assume now that every linear character of $C_H$ extends to $N_H$. The orbit of $H$ under $N_H$ and $C_H$ is $\{H\}$. So let us consider an orbit $\mathcal{O} \in \mathscr{H}'_H/C_H$. We define

$$Q = \prod_{H' \in \mathcal{O}} \alpha_{H'} \in S(V^*)\,.$$

Thanks to Theorem 2, $Q$ define a linear character $\chi$ of $C_H$: for every $c \in C_H$, there exists $\chi(c) \in \mathbb{C}^\times$ such that $cQ = \chi(c)Q$ for every $c \in C_H$. We then consider the $N_H$-submodule $M$ of $S(V^*)$ generated by $Q$. As a vector space, $M$ is generated by the family $(nQ)_{n \in N_H}$. But, since $C_H$ is normal in $N_H$, we obtain for $c \in C_H$,

$$cnQ = nn^{-1}cnQ = n\chi(n^{-1}cn)Q\,.$$

Since $\chi$ extends to $N_H$, we have $\chi(n^{-1}cn) = \chi(c)$ and then $cnQ = \chi(c)nQ$. Theorem 2 allows us to conclude that $nQ = \lambda_n Q$ for some $\lambda_n \in \mathbb{C}^\times$. Since $S(V^*)$ is a UFD, we obtain that $\mathcal{O}$ is still an orbit under $N_H$. □

**Remark 7 – Commuting Orbits.** The orbits of the hyperplanes commuting with $H$ under $N_H$ and $C_H$ are the same for every hyperplane $H$ of every complex reflection group except the hyperplanes of the exceptional group $G_{25}$ and the hyperplanes $H_i$ $(1 \leqslant i \leqslant r)$ of the group $G(de, e, r)$ when $r = 3$ and $e$ is even (see section 5 for the notations).

In Section 5, we give tables for the various orbits of hyperplanes for the infinite series $G(de, e, r)$. For the exceptional complex reflection groups, we check the injectivity or non-injectivity of $i^{\mathrm{ab}}$ using the package CHEVIE of GAP [6][8].

For a hyperplane $H \in \mathscr{H}$, the comparison of $e_H$ and $f_H$ leads to the following definition.

**Definition 8 – Ramification at a hyperplane.** We define $d_H = f_H/e_H$ to be *the index of ramification of $W$ at the hyperplane $H$*. We say that $W$ is *unramified at $H$* if $d_H = 1$.

We say that an element $w \in N_H$ such that $r(w) = \exp(2i\pi/f_H)$ *realizes the ramification.*

**Remark 9 – The Coxeter Case.** When $H$ is an unramified hyperplane, we have $N_H = C_H \times W_H$ which is generated by reflections thanks to Steinberg's theorem [12, Theorem 1.5] and $s_H$ realizes the ramification. Moreover $i^{\mathrm{ab}}$ is trivially injective.

In a Coxeter group, every hyperplane is unramified. Indeed, the eigenvalue on the line $D$ of an element of $N_H$ is a finite order element of the field of the real numbers.

**Remark 10 – The 2-dimensional Case.** When $W$ is a 2-dimensional reflection group, $N_H$ is an abelian group and $i = i^{\mathrm{ab}}$ is injective.

In section 5, we give tables for the values of $e_H, f_H$ and $d_H$ for every hyperplane of every complex reflection groups. From these tables, we obtain the following remarks.

**Remark 11 – Unramified $G(de, e, r)$.** All the hyperplanes of $G(de, e, r)$ are unramified only when $r = 1$ or when $G(de, e, r)$ is a Coxeter group (that is to say if $d = 2$ and $e = 1$ and $r \geqslant 2$ (Coxeter group of type $B_r$) or if $d = 1$ and $e = 2$ and $r \geqslant 3$ (Coxeter group of type $D_r$) or if $d = 1$ and $e = 1$ and $r \geqslant 3$ (Coxeter group of type $A_{r-1}$) of if $d = 1$ and $r = 2$ (Coxeter group of type $I_2(e)$).

**Remark 12 – Unramified exceptional groups.** The only non Coxeter groups for which every hyperplane is unramified are $G_8, G_{12}$ and $G_{24}$.

**Remark 13 – Generating Set.** Since $C_H$ is the parabolic subgroup associated to $D$, it is generated by the reflections it contains (this is Steinberg's theorem). Moreover, if $w_H \in N_H$ realizes the ramification. Then, the exact sequence (2) tells us that $N_H$ is generated by $w_H$ and the family of $s_{H'}$ such that $H' \in \mathscr{H}'_H$.

# 3 Abelianization of Subgroups of Braid Groups

In this section, we describe abelianizations of subgroups of $B$ containing $P$ that is to say of inverse images of subgroups $W'$ of $W$. Explicitly, we are able to give a complete description of $p^{-1}(W')^{\mathrm{ab}}$ if $W'$ is a reflection subgroup (Proposition 15) or if $W'$ is the stabilizer of a hyperplane under geometrical assumptions on the hyperplane (Proposition 24). We also construct a particular linear character of $p^{-1}(N_H)$ lifting the natural linear character $r$ of $N_H$ which is of importance for the next section (Definition 19).

Our method is similar to the method of [4] for the description of $B^{\mathrm{ab}}$ : we integrate along paths invariants polynomial functions. So, we have first to construct invariant polynomial functions and then verify that we have constructed enough of them.

## 3.1 Subgroup Generated by Reflections

In this subsection, we fix $C$ a subgroup of $W$ generated by reflections. We denote by $\mathscr{H}_C \subset \mathscr{H}$ the set of hyperplanes of $C$. For $H \in \mathscr{H}_C$, then $C_H = \{c \in C, \quad \forall\, x \in H,\ cx = x\}$ is a subgroup of $W_H$ and so generated by ${s_H}^{a_H}$ with $a_H \mid e_H$. For $H \in \mathscr{H} \smallsetminus \mathscr{H}_C$, we set $a_H = e_H$. We then obtain $C = \langle {s_H}^{a_H},\ H \in \mathscr{H} \rangle$. For $\mathcal{C} \in \mathscr{H}/C$ a $C$-class of hyperplanes of $W$, we denote by $a_{\mathcal{C}}$ the common value of $a_H$ for $H \in \mathcal{C}$.

The aim of this subsection is to give a description of the abelianization of $p^{-1}(C) \subset B$. For this, we follow the method of [4] and we start to exhibit invariants which will be useful to show the freeness of our generating set of $p^{-1}(C)^{\mathrm{ab}}$.

**Lemma 14 – An invariant.** We define, for $\mathcal{C} \in \mathscr{H}/C$,

$$\alpha_{\mathcal{C}} = \prod_{H \in \mathcal{C}} {\alpha_H}^{e_H/a_H} \in S(V^*)\,.$$

Then $\alpha_{\mathcal{C}}$ is invariant under the action of $C$.

*Proof.* If $\mathcal{C}$ is a class of hyperplanes of $\mathscr{H}_C$ then this is an easy consequence of Theorem 2.

Assume that $\mathcal{C}$ is not a class of hyperplanes of $\mathscr{H}_C$. Let us choose a reflection $s$ of $C$ and let $n_s$ be the order of $s$. Since $\mathcal{C}$ is not a class in $\mathscr{H}_C$, the hyperplane of $s$ does not belong to $\mathcal{C}$. We then deduce that the orbits of $\mathcal{C}$ under the action of $\langle s\rangle$ are of two types.

First type : the orbits of $H \in \mathcal{C}$ such that $s_H$ and $s$ commute. Since $ss_Hs^{-1} = s_{s(H)}$, we then deduce that $s(H) = H$. And so the orbit of $H$ under $\langle s\rangle$ is reduced to $H$. We denote by $H_s$ the hyperplane of $s$. Since $H \neq H_s$, Lemma 5 tells us that $D = H^\perp \subset H_s$ and so $s$ acts trivially on $D$ which is identified to the line spanned by $\alpha_H$ through the inner product.

Second type : the orbits of $H \in \mathcal{C}$ such that $s_Hs \neq ss_H$. If $s^iH = H$ then $s^i$ and $s_H$ commute and thus, Lemma 5 ensures us that $s^i$ is trivial. We then obtain that the orbit of $H$ under $\langle s\rangle$ has cardinality $n_s$. So if we denote by $Q$ the following product $\alpha_H\alpha_{sH}\cdots\alpha_{s^{n_s-1}H} = \lambda\alpha_H s\alpha_H\cdots s^{n_s-1}\alpha_H$ with $\lambda \in \mathbb{C}^\times$, we have $sQ = Q$.

We then easily obtain $s\alpha_{\mathcal{C}} = \alpha_{\mathcal{C}}$ for every $s \in C$ and so $\alpha_{\mathcal{C}}$ is invariant under the action of $C$. □

Before stating the main result of the subsection, we recall the notion of "generator of the monodromy around a hyperplane" as defined in [4, p.14]. For $H \in \mathscr{H}$, we define a generator of the monodromy around $H$ to be a path $s_{H,\gamma}$ in $V^{\mathrm{reg}}$ which is the composition of three paths. The first path is a path $\gamma$ going from $x_0$ to a point $x_H$ which is near $H$ and far from other hyperplanes. To describe the second path, we write $x_H = h + d$ with $h \in H$ and $d \in D = H^\perp$, and the second path is $t \in [0,1] \mapsto h + \exp(2i\pi t/e_H)d$ going from $x_H$ to $s_H(x_H)$. The third path is $s_H(\gamma^{-1})$ going from $s_H(x_H)$ to $s_H(x_0)$. We can now state our abelianization result.

**Proposition 15 – Abelianization of subgroups of the braid group.** Let $C$ be a subgroup of $W$ generated by reflections. Then $p^{-1}(C)^{\mathrm{ab}}$ is the free abelian group over $\mathscr{H}/C$ the $C$-classes of hyperplanes of $W$.

Explicitly, we have $p^{-1}(C) = \langle {s_{H,\gamma}}^{a_H},\ (H,\gamma)\rangle$ (see [4, Theorem 2.18]). For $\mathcal{C} \in \mathscr{H}/C$, we denote by $(s_{\mathcal{C}}^{a_C})^{\mathrm{ab}}$ the common value in $p^{-1}(C)^{\mathrm{ab}}$ of the ${s_{H,\gamma}}^{a_H}$ for $H \in \mathcal{C}$. Then $p^{-1}(C)^{\mathrm{ab}} = \langle (s_{\mathcal{C}}^{a_C})^{\mathrm{ab}},\quad \mathcal{C} \in \mathscr{H}/C\rangle$. Moreover, for $\mathcal{C} \in \mathscr{H}/C$, there exists a group homomorphism $\varphi_{\mathcal{C}} : p^{-1}(C) \to \mathbb{Z}$ such that $\varphi_{\mathcal{C}}((s_{\mathcal{C}}^{a_C})^{\mathrm{ab}}) = 1$ and $\varphi_{\mathcal{C}}((s_{\mathcal{C}'}^{a_{C'}})^{\mathrm{ab}}) = 0$ for $\mathcal{C}' \neq \mathcal{C}$.

*Proof.* First of all, Lemma 2.14.(2) of [4] shows that ${s_{H,\gamma}}^{a_H} = {s_{H,\gamma'}}^{a_H}$ in $p^{-1}(C)^{\mathrm{ab}}$. Now, for $c \in C$, we choose $x \in p^{-1}(C)$ such that $p(x) = c$. We have $xs_{H,\gamma}x^{-1} = s_{cH,x(c\gamma)}$. So ${s_{cH,x(c\gamma)}}^{a_{cH}}$ and ${s_{H,\gamma}}^{a_H}$ are conjugate by an element of $p^{-1}(C)$. So, we have

$$ {s_{cH,x(c\gamma)}}^{a_{cH}} = {s_{H,\gamma}}^{a_H} \in p^{-1}(C)^{\mathrm{ab}}\,. $$

And then $p^{-1}(C)^{\mathrm{ab}} = \langle (s_{\mathcal{C}}^{a_C})^{\mathrm{ab}},\quad \mathcal{C} \in \mathscr{H}/C\rangle$.

Let us now show that the family $((s_{\mathcal{C}}^{a_C})^{\mathrm{ab}})_{\mathcal{C}\in\mathscr{H}/C}$ is free over $\mathbb{Z}$. We identify $p^{-1}(C)$ with

$$ p^{-1}(C) = \left( \bigsqcup_{c,c'\in C} \pi_1(V^{\mathrm{reg}}, cx_0, c'x_0) \right) /C $$

where $\pi_1(V^{\mathrm{reg}}, cx_0, c'x_0)$ denotes the homotopy classes of paths from $c(x_0)$ to $c'(x_0)$ and the action of $C$ on paths is simply the composition.

Since $\alpha_{\mathcal{C}} : V^{\mathrm{reg}} \to \mathbb{C}^\times$ is $C$-invariant (Lemma 14), the functoriality of $\pi_1$ defines a group homomorphism $\pi_1(\alpha_{\mathcal{C}})$ from $p^{-1}(C)$ to $\pi_1(\mathbb{C}^\times, \alpha_{\mathcal{C}}(x_0))$. Moreover, the map

$$ I : \gamma \longmapsto \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{d}z}{z} $$

realizes a group isomorphism between $\pi_1(\mathbb{C}^\times, \alpha_{\mathcal{C}}(x_0))$ and $\mathbb{Z}$. The composition of these two maps defines a group homomorphism. We denote it by $\varphi_{\mathcal{C}}$ and we now want to show that $\varphi_{\mathcal{C}}$ verifies the condition stated in the Proposition.

For $H \in \mathcal{C}$ and $\mathcal{C}' \in \mathscr{H}/C$, let us compute $\varphi_{\mathcal{C}'}({s_{H,\gamma}}^{a_H})$. The path ${s_{H,\gamma}}^{a_H}$ is the composition of three paths. The first one is $\gamma$, the third one is ${s_H}^{a_H}(\gamma^{-1})$ and the second one is $\eta : t \in [0,1] \mapsto h + \exp(2i\pi a_H t/e_H)d$.

Since $\alpha_{\mathcal{C}'}$ is $C$-invariant, when we apply $\pi_1(\alpha_{\mathcal{C}'})$, the first part of the path and the third one are inverse from each other. So when applying $I$, they do not appear. We thus obtain

$$ \varphi_{\mathcal{C}'}({s_{H,\gamma}}^{a_H}) = \frac{1}{2i\pi}\int_{\alpha_{\mathcal{C}'}\circ\eta} \frac{\mathrm{d}z}{z}\,. $$

Using the logarithmic derivative, we obtain

$$ \varphi_{\mathcal{C}'}({s_{H,\gamma}}^{a_H}) = \frac{1}{2i\pi}\sum_{H'\in\mathcal{C}'} \frac{e_{H'}}{a_{H'}}\int_0^1 2i\pi\frac{a_H}{e_H}\frac{\exp(2i\pi a_H t/e_H)\alpha_{H'}(d)}{\alpha_{H'}(h+\exp(2i\pi a_H t/e_H)d)}\,\mathrm{d}t $$

To compute this sum, we regroup the terms according to the orbit of $H'$ under $\langle {s_H}^{a_H}\rangle$.

Lemma 5 shows that there are three types of orbits : two types of orbits reduced to one single hyperplane and one other type of orbits corresponding to reflections that do not commute with $s_H$.

Let us first study the orbits reduced to one single hyperplane. The first type corresponds to the hyperplane $H$ whose term of the sum is 1 and this term appears if and only if $H \in \mathcal{C}'$. The second type corresponds to hyperplanes $H'$ such that $D = H^\perp \subset H'$. The corresponding term of the sum is 0 since $\alpha_{H'}(d) = 0$

Let us now study the non trivial orbits. The orbits of $H'$ under ${s_H}^{a_H}$ is $\{H', \dots, {s_H}^{a_H(e_H/a_H-1)}(H')\}$. Moreover, since a quotient of the form $\alpha_{H'}(x)/\alpha_{H'}(y)$ does not depend of the linear form with kernel $H'$, we can replace $\alpha_{{s_H}^k H'}$ by $s_H^k \alpha_{H'}$ to obtain

$$\begin{aligned}\frac{\exp(2i\pi a_H t/e_H)\alpha_{{s_H}^{-ka_H}H'}(d)}{\alpha_{{s_H}^{-ka_H}H'}(h)+\exp(2i\pi a_H t/e_H)\alpha_{{s_H}^{-ka_H}H'}(d)}&\\ = \frac{\exp(2i\pi a_H t/e_H){s_H}^{-a_H k}\alpha_{H'}(d)}{{s_H}^{-a_H k}\alpha_{H'}(h)+\exp(2i\pi a_H t/e_H){s_H}^{-a_H k}\alpha_{H'}(d)}&\\ = \frac{\exp(2i\pi a_H (t+k)/e_H)\alpha_{H'}(d)}{\alpha_{H'}(h)+\exp(2i\pi a_H (t+k)/e_H)\alpha_{H'}(d)}\,.&\end{aligned}$$

Considering the sum over the orbit under $\langle {s_H}^{a_H}\rangle$ of $H'$, we obtain

$$\sum_{k=0}^{e_H/a_H-1}\int_0^1 \frac{\exp(2i\pi a_H(t+k)/e_H)\alpha_{H'}(d)}{\alpha_{H'}(h)+\exp(2i\pi a_H(t+k)/e_H)\alpha_{H'}(d)}\,\mathrm{d}t = \frac{e_H}{a_H}\int_0^1 \frac{\exp(2i\pi t)\alpha_{H'}(d)}{\alpha_{H'}(h)+\exp(2i\pi t)\alpha_{H'}(d)}\,\mathrm{d}t\,.$$

Since $x_H$ is chosen such that $\alpha_{H'}(h)\neq 0$ for $H'\neq H$ and $d$ is small, the last term is 0 as the index of the circle of center 0 and radius $|\alpha_{H'}(d)|$ relatively to the point $-\alpha_{H'}(h)$. □

**Remark 16 – Extreme cases.** The two extreme cases where $C=1$ and $C=W$ may be found in [4, Prop. 2.2.(2)] and [4, Theorem 2.17.(2)]. In the first case, $p^{-1}(C)=P$ is the pure braid group whose abelianization is the free abelian group over $\mathscr{H}$. In the second case $p^{-1}(C)=B$ is the braid group whose abelianization is the free abelian group over $\mathscr{H}/W$.

**Remark 17** The logarithmic derivative shows that for every $\gamma\in p^{-1}(C)$ and $n\in\mathbb{Z}$, we have

$$\int_{\alpha_{\mathcal{C}}{}^n\circ\gamma}\frac{\mathrm{d}z}{z} = n\varphi_{\mathcal{C}}(\gamma)\,.$$

## 3.2 Stabilizer of a hyperplane

Let us recall the notation of Section 2; we consider $H\in\mathscr{H}$ a hyperplane of the reflection group $W$. We denote by $N_H$ the stabilizer of $H$ in $W$ and $C_H$ the parabolic subgroup of $W$ associated to the line $D=H^\perp$. The set of hyperplanes commuting with $H$ is $\mathscr{H}_H$ (see Definition 4).

### A GROUP HOMOMORPHISM

The aim of this paragraph is to construct an "extension" of the natural character of $N_H$ to the group $p^{-1}(N_H)$ which will be useful for the third section. We still follow the method of [4] : we construct an invariant function with values in $\mathbb{C}^\times$ (Lemma 18) and integrate it (Definition 19). To obtain the "extension" properties of the linear character of $p^{-1}(N_H)$ (Proposition 23), we construct a lifting in the braid group of the elements of $N_H$ (Remark 21). This lifting is inspired from the construction of the generator of the monodromy.

**Lemma 18 – An invariant function.** The function $\alpha_{N_H}={\alpha_H}^{f_H}\in S(V^*)$ is invariant under $N_H$.

*Proof.* This is clear since the line spanned by $\alpha_H$ is identified to $D$ through the inner product. □

**Definition 19 – The group homomorphism.** As in the proof of Proposition 15, we write

$$p^{-1}(N_H)=\left(\bigsqcup_{n,n'\in N_H}\pi_1(V^{\mathrm{reg}},nx_0,n'x_0)\right)/N_H\,.$$

Since $\alpha_{N_H}: V^{\mathrm{reg}}\to\mathbb{C}^\times$ is $N_H$-invariant (Lemma 18), the functoriality of $\pi_1$ allows us to define a group homomorphism $\pi_1(\alpha_{N_H})$ from $p^{-1}(N_H)$ to $\pi_1(\mathbb{C}^\times,\alpha_{N_H}(x_0))$. Moreover, the map

$$I:\gamma\longmapsto\frac{1}{2i\pi}\int_\gamma\frac{\mathrm{d}z}{z}$$

realizes a group isomorphism between $\pi_1(\mathbb{C}^\times,\alpha_{N_H}(x_0))$ and $\mathbb{Z}$. The composition of this two maps defines a group homomorphism $\rho':p^{-1}(N_H)\to\mathbb{Z}$. We also define $\rho={f_H}^{-1}\rho':p^{-1}(N_H)\to\mathbb{Q}$.

**Remark 20 – Center of the braid group of $G_{31}$.** In [4, Theorem 2.24], it is shown that the center of the braid group $B$ of an irreducible reflection group $W$ is an infinite cyclic group generated by $\beta : t \mapsto \exp(2i\pi t/|Z(W)|)x_0$ (where $x_0 \in V^{\mathrm{reg}}$ is a base point) for all but six exceptional reflection groups. In his articles [1][2], Bessis proves that the result holds for all reflection groups but the exceptional one $G_{31}$.

This remark is a first step toward the case of $G_{31}$ : we show that if $ZB$ is an infinite cyclic group, it is generated by $\beta$. For this, let us consider $H \in \mathscr{H}$ a hyperplane of $G_{31}$ and $\rho'$ the group homomorphism defined above. Since $ZB \subset p^{-1}(N_H)$, $\rho'$ restricts to a group homomorphism from $ZB$ to $\mathbb{Z}$ such that $\rho'(\beta) = 1$. So if $ZB$ is an infinite cyclic group, it is generated by $\beta$.

**Remark 21 – The lifting construction.** Let us consider $w \in N_H$. We now construct a path $\widetilde{w}$ in $V^{\mathrm{reg}}$ starting from $x_0$ and ending at $w(x_0)$ : $p(\widetilde{w}) = w$. We use the notations of the description of the generators of the monodromy around $H$ : we write $x_H = h + d$ with $h \in H$ and $\alpha_{H'}(h) \neq 0$ for $H' \neq H$ and $d \in D = H^{\perp}$. Since $w \in N_H$, we have $w(x_H) = h' + \exp(2ik\pi/f_H)d$ with $h' \in H$ and $0 \leqslant k < f_H$.

The path $\widetilde{w}$ consists into four parts. As in the case of the generators of the monodromy, the first part is a path $\gamma$ from $x_0$ to $x_H$ and the fourth path is $w(\gamma^{-1})$ from $w(x_H)$ to $w(x_0)$. Let us now describe the second part and the third part. The second part of $\widetilde{w}$ is the path

$$t \in [0,1] \mapsto h + \exp(2ik\pi t/f_H)d \in V^{\mathrm{reg}}\,.$$

The third part is of the form $t \in [0,1] \mapsto \theta(t) + \exp(2ik\pi/f_H)d$ where $\theta(t)$ is a path in the complex affine line $\mathscr{D}$ generated by $h'$ and $h$. It is easy to force the third part of $\widetilde{w}$ to stay in $V^{\mathrm{reg}}$ since its image is contained in the affine line $\exp(2ik\pi/f_H)d + \mathscr{D}$ which is parallel to the hyperplane $H$ and meets each of the other hyperplanes in a single point : so we just have to avoid a finite number of points in $\mathbb{C}$.

**Remark 22 – Generating set.** We have seen in Remark 13 that

$$N_H = \langle w_H, s_{H'},\ H' \in \mathscr{H}_H \rangle$$

where $w_H \in N_H$ is a once and for all fixed element realizing the ramification. It is now an easy consequence of Theorem 2.18 of [4] that

$$p^{-1}(N_H) = \langle \widetilde{w_H}, s_{H',\gamma}, {s_{H'',\gamma'}}^{e_{H''}},\quad H' \in \mathscr{H}_H,\ H'' \in \mathscr{H} \smallsetminus \mathscr{H}_H,\ \gamma,\ \gamma' \rangle\,.$$

It remains to show that the constructed group homomorphism $\rho$ is an "extension" of the natural character of $N_H$. More precisely, we have the following proposition.

**Proposition 23 – The "extension" property.** We have the following commutative square

$$\begin{array}{ccc} p^{-1}(N_H) & \xrightarrow{\ \rho\ } & \mathbb{Q} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi'} \\ N_H & \xrightarrow{\ r\ } & \mathbb{U}_{f_H} \end{array}$$

where $\pi' : x \in \mathbb{Q} \mapsto \exp(2i\pi x)$.

*Proof.* Using the generating set of $N_H$ given in Remark 22, we only need to show that

(*i*) $\rho'(\widetilde{w_H}) = 1$
(*ii*) $\rho'(s_{H,\gamma}) = f_H/e_H$
(*iii*) $\rho'(s_{H',\gamma}) = 0$ for $H' \in \mathscr{H}'_H = \mathscr{H}_H \smallsetminus \{H\}$
(*iv*) $\rho'({s_{H',\gamma}}^{e_{H'}}) = 0$ for $H' \in \mathscr{H} \smallsetminus \mathscr{H}_H$.

As in the proof of Proposition 15, the $\gamma$-part of $s_{H,\gamma}$ (resp. $s_{H',\gamma}$ for $H' \in \mathscr{H}'_H$ and ${s_{H',\gamma}}^{e_{H'}}$ for $H' \in \mathscr{H} \smallsetminus \mathscr{H}_H$) does not appear in the computation of $\rho'$. We thus obtain

$$\rho'(s_{H,\gamma}) = \frac{1}{2i\pi}\int_0^1 f_H \frac{2i\pi}{e_H}\frac{\alpha_H(\exp(2i\pi t/e_H)d)}{\alpha_H(h + d\exp(2i\pi t/e_H))}\mathrm{d}t = f_H/e_H\,.$$

For $H' \in \mathscr{H}'_H$, we set $x_{H'} = h' + d'$ with $h' \in H'$ and $d' \in D' = H'^{\perp}$. We then obtain

$$\rho'(s_{H',\gamma}) = \frac{1}{2i\pi}\int_0^1 f_H \frac{2i\pi}{e_{H'}}\frac{\alpha_H(\exp(2i\pi t/e_{H'})d')}{\alpha_H(h' + d'\exp(2i\pi t/e_{H'}))}\mathrm{d}t = 0$$

since $\alpha_H(d') = 0$ for $H \in \mathscr{H}'_H$. With the same arguments, we obtain for $H' \in \mathscr{H} \smallsetminus \mathscr{H}_H$

$$\rho'({s_{H',\gamma}}^{e_H}) = \frac{1}{2i\pi}\int_0^1 2i\pi f_H \frac{\exp(2i\pi t)\alpha_H(d')}{\alpha_H(h') + \exp(2i\pi t)\alpha_H(d')}\mathrm{d}t = 0$$

since $d'$ is small and $\alpha_H(h') \neq 0$.

For $\widetilde{w_H}$, neither the first and fourth part are involved in the computation nor the third one. Moreover, as in the computation of $\rho'(s_{H,\gamma})$ the second part of $\widetilde{w_H}$ gives 1. □

### The Stabilizer Case

In this paragraph, we extend the results of Section 2 to the braid group. Namely, since $p : B/[P,P] \to W$ is a surjective homomorphism, the classical isomorphism theorems give the following short exact sequence

$$1 \longrightarrow p^{-1}(C_H) \xrightarrow{j} p^{-1}(N_H) \xrightarrow{rp} \mathbb{U}_{f_H} \longrightarrow 1$$

which gives rise to the following exact sequence (Lemma 3)

$$p^{-1}(C_H)^{\mathrm{ab}} \xrightarrow{j^{\mathrm{ab}}} p^{-1}(N_H)^{\mathrm{ab}} \xrightarrow{(rp)^{\mathrm{ab}}} \mathbb{U}_{f_H} \longrightarrow 1$$

and Proposition 6 extends to the braid group in the following way.

**Proposition 24 – Abelianization in the braid group.** If the orbits of the hyperplanes of $\mathscr{H}$ under $N_H$ and $C_H$ are the same, the map $j^{\mathrm{ab}}$ is injective.

Moreover under this hypothesis, $p^{-1}(N_H)^{\mathrm{ab}}$ is the free abelian group with basis $\widetilde{w_H}$, $(s_{\mathcal{C}})^{\mathrm{ab}}$ for $\mathcal{C} \in \mathscr{H}'_H/C_H$ and $(s_{\mathcal{C}}^{e_{\mathcal{C}}})^{\mathrm{ab}}$ for $\mathcal{C} \in (\mathscr{H} \smallsetminus \mathscr{H}_H)/C_H$.

*Proof.* From Lemma 3, it is enough to show that every linear character of $p^{-1}(C_H)$ with values in $\mathbb{C}^\times$ extends to $p^{-1}(N_H)$. But the group of linear characters of $p^{-1}(C_H)$ is generated by the $\exp(z\varphi_{\mathcal{C}})$ for $z \in \mathbb{C}$ and $\mathcal{C}$ an orbit of $\mathscr{H}$ under $C_H$. So it suffices to show that $\varphi_{\mathcal{C}}$ extends to $p^{-1}(N_H)$.

Since the orbits of $\mathscr{H}$ under $C_H$ and $N_H$ are the same, then for every $\mathcal{C} \in \mathscr{H}/C_H$, there exists $n \in \mathbb{N}^*$ such that ${\alpha_{\mathcal{C}}}^n$ is invariant under $N_H$ (see Lemma 14 for the definition of $\alpha_{\mathcal{C}}$). Then Remark 17 shows that

$$\psi_{\mathcal{C}} : \gamma \in p^{-1}(N_H) \longmapsto \frac{1}{n} \int_{{\alpha_{\mathcal{C}}}^n \circ \gamma} \frac{\mathrm{d}z}{z} \in \mathbb{Q}$$

is a well defined linear character of $p^{-1}(N_H)$ extending $\varphi_{\mathcal{C}}$.

Proposition 15 applied to $C_H$ ensures us that $p^{-1}(C_H)^{\mathrm{ab}}$ is the free abelian group generated by $(s_{\{H\}}^{e_{\{H\}}})^{\mathrm{ab}}$, $(s_{\mathcal{C}})^{\mathrm{ab}}$ for $\mathcal{C} \in \mathscr{H}'_H/C_H$ and $(s_{\mathcal{C}}^{e_{\mathcal{C}}})^{\mathrm{ab}}$ for $\mathcal{C} \in (\mathscr{H} \smallsetminus \mathscr{H}_H)/C_H$. Moreover, we have $\widetilde{w_H}^{f_H} \in p^{-1}(C_H)$ and, thanks to Remark 17,

$$\varphi_{\{H\}}(\widetilde{w_H}^{f_H}) = \frac{1}{f_H}\rho'(\widetilde{w_H}^{f_H}) = 1$$

We then deduce that the family $\widetilde{w_H}^{f_H}$, $(s_{\mathcal{C}})^{\mathrm{ab}}$ for $\mathcal{C} \in \mathscr{H}'_H/C_H$ and $(s_{\mathcal{C}}^{e_{\mathcal{C}}})^{\mathrm{ab}}$ for $\mathcal{C} \in (\mathscr{H} \smallsetminus \mathscr{H}_H)/C_H$ is a basis for $p^{-1}(C_H)^{\mathrm{ab}}$. The short exact sequence

$$1 \longrightarrow p^{-1}(C_H)^{\mathrm{ab}} \xrightarrow{j^{\mathrm{ab}}} p^{-1}(N_H)^{\mathrm{ab}} \xrightarrow{(rp)^{\mathrm{ab}}} \mathbb{U}_{f_H} \longrightarrow 1$$

gives the result. □

**Remark 25 – Comparison of orbits.** In this remark, we give a list of the hyperplanes for which the orbits of hyperplanes under $N_H$ and $C_H$ are not the same. Of course, we find again in this list the hyperplanes of Remark 7 but we have to add some others.

Let us consider the infinite series (see Section 5 for notations). When $H = H_i$, the orbits under $N_H$ and $C_H$ are always the same except when $r = 3$ and $e$ is even and when $r = 2$ and $e \geqslant 3$. If $H = H_{i,j,\zeta}$, the orbits under $N_H$ and $C_H$ are the same when $de$ is even and $r \neq 3$ or when $r = 3$ and $e \in \{1, 3\}$ or when $r = 2$ and $d = e = 1$.

For the exceptional types, $G_{25}$ is the only case where the commuting orbits under $N_H$ and $C_H$ are not the same. The only exceptional types where the non commuting orbits under $N_H$ and $C_H$ are not the same are $G_4$, the second (named after GAP) class of hyperplanes of $G_6$, the first (named after GAP) class of hyperplanes of $G_{13}$ and the third (named after GAP) class of hyperplanes of $G_{15}$.

## 4 An Application to Cohomology

In this section, we apply the preceding constructions and results to obtain a group cohomology result. Specifically, the derived subgroup of $P$ is normal in $B$, so we obtain the following short exact sequence

$$1 \longrightarrow P/[P,P] \xrightarrow{j} B/[P,P] \xrightarrow{p} W \longrightarrow 1 \tag{3}$$

which induces a structure of $W$-module on $P^{\mathrm{ab}}$. By a classical result on hyperplanes arrangements (see [10] for example), the $W$-module $P^{\mathrm{ab}}$ is nothing else that the permutation module $\mathbb{Z}\mathscr{H}$ and this section describes the extension (3) as an element of $H^2(W, \mathbb{Z}\mathscr{H})$ using methods of low-dimensional cohomology.

The rationale breaks down into three steps and each step consists of a translation of a standard isomorphism between cohomology groups in terms of group extensions.

(*i*) We decompose $\mathscr{H}$ into orbits under $W$ : $\mathscr{H} = \sqcup \mathcal{C}$ and uses the isomorphism

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} H^2(W, \mathbb{Z}\mathcal{C})$$

(*ii*) In each orbit, we set a hyperplane $H_\mathcal{C}$ and then $\mathbb{Z}\mathcal{C} = \mathrm{Ind}_{N_\mathcal{C}}^W(\mathbb{Z})$ where $N_\mathcal{C}$ is the stabilizer of $H_\mathcal{C}$. Shapiro's lemma (see [5, Proposition III.6.2]) then gives us

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} H^2(N_\mathcal{C}, \mathbb{Z})$$

(*iii*) The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $N_\mathcal{C}$-modules gives a long exact sequence in cohomology. Since $|N_\mathcal{C}|$ is invertible in $\mathbb{Q}$, we have $H^1(N_\mathcal{C}, \mathbb{Q}) = H^2(N_\mathcal{C}, \mathbb{Q}) = 0$ and so we obtain the isomorphism $H^2(N_\mathcal{C}, \mathbb{Z}) = H^1(N_\mathcal{C}, \mathbb{Q}/\mathbb{Z})$ and

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} H^1(N_\mathcal{C}, \mathbb{Q}/\mathbb{Z}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} \mathrm{Hom}_{\mathrm{gr.}}(N_\mathcal{C}, \mathbb{Q}/\mathbb{Z})\,.$$

The results of this section are the following proposition and corollary.

**Proposition 26 – Description.** Under the isomorphism

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} \mathrm{Hom}_{\mathrm{gr.}}(N_\mathcal{C}, \mathbb{Q}/\mathbb{Z})$$

the extension (3) corresponds to the family $(r_\mathcal{C} : N_\mathcal{C} \to \mathbb{Q}/\mathbb{Z})_{\mathcal{C} \in \mathscr{H}/W}$ where $r_\mathcal{C}$ is the natural linear character of $N_\mathcal{C}$ (we identify $\mathbb{U}_{f_{H_\mathcal{C}}}$ with a subgroup of $\mathbb{Q}/\mathbb{Z}$ via the exponential map).

The next corollary is a trivial consequence of Proposition 26 and generalizes a result of Digne [7, 5.1] for the case of Coxeter groups.

**Corollary 27 – Order in $H^2(W, \mathbb{Z}\mathscr{H})$.** Since the order of $r_\mathcal{C}$ is $f_{H_\mathcal{C}}$, we deduce that the order of the extension (3) is $\kappa(W) = \mathrm{lcm}(f_{H_\mathcal{C}}, \mathcal{C} \in \mathscr{H}/W)$ (this integer $\kappa(W)$ was first introduced in [9]).

The rest of the section is devoted to the proof of Proposition 26 : one subsection for each of the three steps.

## 4.1 First step : splitting into orbits

The isomorphism

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} H^2(W, \mathbb{Z}\mathcal{C})$$

is simply given by applying the various projections $p_\mathcal{C} : \mathbb{Z}\mathscr{H} \to \mathbb{Z}\mathcal{C}$ to a 2-cocycle with values in $\mathbb{Z}\mathscr{H}$ where

$$p_\mathcal{C} : \sum_{H \in \mathscr{H}} \lambda_H H \longmapsto \sum_{H \in \mathcal{C}} \lambda_H H\,.$$

To give a nice expression of the corresponding extensions, we need the following lemma.

**Lemma 28 – Extension and direct sum.** Let $G$ be a group, $X = Y \oplus Z$ a direct sum of $G$-modules and

$$0 \longrightarrow X \xrightarrow{u} E \xrightarrow{v} G \longrightarrow 1$$

an extension of $G$ by $X$. We denote by $q : X \to Y$ the first projection and $\varphi$ the class of the extension $E$ in $H^2(G, X)$. The extension associated to $q(\varphi)$ is

$$0 \longrightarrow Y \longrightarrow E/Z \longrightarrow G \longrightarrow 1$$

*Proof.* Let us denote by $\theta : E \to E/Z$ the natural surjection and $i : Y \to Y \oplus Z$ the natural map. Let us first remark that $Z$ is normal in $E$ since $Z$ is stable by the action of $G$. Since $v$ is trivial on $Z$, then it induces a group homomorphism $\widetilde{v} : E/Z \to G$ whose kernel is $X/Z = Y$. Thus the sequence

$$0 \longrightarrow Y \xrightarrow{\theta u i} E/Z \xrightarrow{\widetilde{v}} G \longrightarrow 1 \tag{4}$$

is an exact one.

If $s : G \to E$ is a set-theoretic section of $v$, then $\theta s$ is a set-theoretic section of $\widetilde{v}$. The expression of a 2-cocycle associated to an extension in terms of a set-theoretic section gives the result. □

For $\mathcal{C} \in \mathscr{H}/W$, we denote by $B_\mathcal{C}$ the quotient group

$$B_\mathcal{C} = B/\langle [P, P], {s_{H,\gamma}}^{e_H}, \quad H \notin \mathcal{C} \rangle$$

Lemma 28 tells us that the extension (3) is equivalent to the family of extensions

$$0 \longrightarrow \mathbb{Z}\mathcal{C} \xrightarrow{jc} B_\mathcal{C} \xrightarrow{pc} W \longrightarrow 1 \tag{5}$$

for $\mathcal{C} \in \mathscr{H}/W$.

### 4.2 Second step : the induction argument

In each orbit $\mathcal{C} \in \mathscr{H}/W$, we choose a hyperplane $H_{\mathcal{C}} \in \mathcal{C}$ and write $\mathbb{Z}\mathcal{C} = \mathrm{Ind}_{N_{\mathcal{C}}}^{W}(\mathbb{Z})$ where $N_{\mathcal{C}} \subset W$ is the stabilizer of $H_{\mathcal{C}}$. Shapiro's isomorphism lemma [5, Proposition III.6.2] shows that $H^2(W, \mathbb{Z}\mathcal{C}) = H^2(N_{\mathcal{C}}, \mathbb{Z})$. Exercise III.8.2 of [5] tells us that in term of 2-cocycles Shapiro's isomorphism is described as follow

$$S : (\varphi : G^2 \to \mathbb{Z}\mathcal{C}) \longmapsto (f_{\mathcal{C}} \circ \varphi : {N_{\mathcal{C}}}^2 \to \mathbb{Z})$$

where $f_{\mathcal{C}} : \mathbb{Z}\mathcal{C} \to \mathbb{Z}$ is the projection onto the $H_{\mathcal{C}}$-component.

Decomposing Shapiro's isomorphism into the following two steps

$$(\varphi : G^2 \to \mathbb{Z}\mathcal{C}) \longmapsto (\varphi : {N_{\mathcal{C}}}^2 \to \mathbb{Z}\mathcal{C}) \longmapsto (f_{\mathcal{C}} \circ \varphi : {N_{\mathcal{C}}}^2 \to \mathbb{Z}),$$

allows us to interpret it in terms of group extensions. Exercice IV.3.1.(a) of [5] gives a description of the first step : the corresponding extension is given by

$$0 \longrightarrow \mathbb{Z}\mathcal{C} \longrightarrow {p_{\mathcal{C}}}^{-1}(N_{\mathcal{C}}) \longrightarrow N_{\mathcal{C}} \longrightarrow 1$$

since ${p_{\mathcal{C}}}^{-1}(N_{\mathcal{C}})$ is the fiber product of $B_{\mathcal{C}}$ and $N_{\mathcal{C}}$ over $W$. Moreover, since $f_{\mathcal{C}}$ is a split surjection as a $N_{\mathcal{C}}$-module map, Lemma 28 gives us the following extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow {p_{\mathcal{C}}}^{-1}(N_{\mathcal{C}})/\langle {s_{H,\gamma}}^{e_H},\ H \in \mathcal{C} \smallsetminus \{H_{\mathcal{C}}\}\rangle \longrightarrow N_{\mathcal{C}} \longrightarrow 1$$

Finally, the extension (3) is equivalent to the family of extensions

$$0 \longrightarrow \mathbb{Z} \longrightarrow B'_{\mathcal{C}} \xrightarrow{p_{\mathcal{C}}} N_{\mathcal{C}} \longrightarrow 1 \tag{6}$$

where $B'_{\mathcal{C}} = p^{-1}(N_{\mathcal{C}})/\langle [P,P], {s_{H,\gamma}}^{e_H},\ H \neq H_{\mathcal{C}}\rangle$ and $\mathcal{C} \in \mathscr{H}/W$.

### 4.3 Third step : linear character

For the third step, we use results and notations of Section 2 and Section 3. Let us consider the group homomorphism $\rho_{\mathcal{C}} : p^{-1}(N_{\mathcal{C}}) \to \mathbb{Q}$ of Definition 19. Since it is trivial on $\langle [P,P], {s_{H,\gamma}}^{e_H},\ H \neq H_{\mathcal{C}}\rangle$, it induces a group homomorphism from $B'_{\mathcal{C}}$ to $\mathbb{Q}$ still denoted by $\rho_{\mathcal{C}}$. Moreover, since $\rho_{\mathcal{C}}({s_{H_{\mathcal{C}},\gamma}}^{e_{H_{\mathcal{C}}}}) = 1$, Proposition 23 gives the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \rho_{\mathcal{C}}} & & \uparrow{\scriptstyle r_{\mathcal{C}}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & B'_{\mathcal{C}} & \xrightarrow{p_{\mathcal{C}}} & N_{\mathcal{C}} & \longrightarrow & 1
\end{array}$$

Exercises IV.3.2 and IV.3.3 of [5] tell us precisely that the group homomorphism corresponding to (6) is $r_{\mathcal{C}}$. So the extension (3) is equivalent to the family $(r_{\mathcal{C}})_{\mathcal{C} \in \mathscr{H}/W}$. This concludes the proof of Proposition 26.

## 5 Tables

### 5.1 The infinite series

In this subsection, we bring together tables for the orbits of the hyperplanes of $G(de, e, r)$ under the centralizer of a reflection and under the parabolic subgroup associated to the line of the reflection and tables for the values of $f_H$ and the index of ramification. So let us consider the complex reflection group $G(de, e, r)$ acting on $\mathbb{C}^r$ with canonical basis $(e_1, \ldots, e_r)$. The standard point of $\mathbb{C}^r$ is denoted by $(z_1, \ldots, z_r)$.

The hyperplanes of $G(de, e, r)$ are $H_i = \{z_i = 0\}$ for $i \in \{1, \ldots, r\}$ (when $d > 1$) and $H_{i,j,\zeta} = \{z_i = \zeta z_j\}$ for $i < j$ and $\zeta \in \mathbb{U}_{de}$ (when $r \geqslant 2$). They split in general into two conjugacy classes under $G(de, e, r)$ whose representant may be chosen as follow $H_1$ and $H_{1,2,1}$.

Let us continue with more notations. For every triple of integers $d, e, r$, we denote by $\pi : G(de, e, r) \to \mathbb{U}$ the following group morphism : for $g \in G(de, e, r)$, $\pi(g)$ is the product of the nonzero coefficients of the monomial matrix $g$. When $e$ is even, we denote by $e' = e/2$. We denote by $e'' = e/\gcd(e, 3)$ and by $P$ the set of elements of $\mathbb{U}_{de}$ with strictly positive imaginary part.

#### The case of the hyperplane $H_1 = \{z_1 = 0\}$

We then have $d > 1$. The stabilizer $N$ of $H_1$ is described by

$$N = \{(\alpha, g), \quad g \in G(de, 1, r-1),\ \alpha \in \mathbb{U}_{de},\ (\pi(g)\alpha)^d = 1\}$$

and the pointwise stabilizer $C$ of $D_1 = {H_1}^{\perp} = \mathbb{C}e_1$ is $C = G(de, e, r-1)$. Table 1 gives the orbits of the hyperplanes under $N$ and $C$. In Table 1, C.O. stands commuting orbits and N.C.O. stand for non commuting orbits.

### The hyperplane $H_{1,2,\exp(2i\pi/de)}$ with $r = 2$

We set $\zeta = \exp(2i\pi/de)$. The reflection of $G(de, e, 2)$ with hyperplane $H_{1,2,\zeta}$ is

$$s = \begin{bmatrix} & \zeta \\ \zeta^{-1} & \end{bmatrix}$$

The line of $s$ is $D = \mathbb{C}(e_2 - \zeta e_1)$. The centralizer of $s$ is given by

$$N = \left\{ d_\lambda = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}, t_\lambda = \begin{bmatrix} & \lambda\zeta \\ \lambda\zeta^{-1} & \end{bmatrix},\ \lambda^{2d} = 1 \right\}.$$

The eigenvalue of $t_\lambda$ on $D$ is $\lambda$ whereas the eigenvalue of $d_\lambda$ on $D$ is $-\lambda$. So the parabolic subgroup $C$ associated to $D$ is $C = \{\mathrm{id}, -s\}$. The orbits of hyperplane under $C$ and $N$ are the same. The commuting ones are $\{H_{1,2,\zeta}\}$ and $\{H_{1,2,-\zeta}\}$. The non commuting ones are $\{H_1, H_2\}$ and $\{H_{1,2,\mu}, H_{1,2,\zeta^2\mu^{-1}}\}$ for $\mu \in \mathbb{U}_{de} \setminus \{\pm\zeta\}$.

### The hyperplane $H_{1,2,1} = \{z_1 = z_2\}$

We then have $r \geqslant 2$. The reflection of $G(de, e, r)$ with hyperplane $H_{1,2,1}$ is the transposition $\tau_{12}$ swapping 1 and 2. Since the elements of $G(de, e, r)$ are monomial matrices, an element of $G(de, e, r)$ commuting with $\tau_{12}$ stabilizes the subspace spanned by $e_1$ and $e_2$. Thus the stabilizer $N$ of $H_{1,2,1}$ is given by

$$N = \left\{ d_{\lambda,g} = \left[\begin{array}{c|c} \lambda & \\ \hline & g \end{array}\right], \quad t_{\lambda,g} = \left[\begin{array}{c|c} & \lambda \\ \hline \lambda & \\ \hline & g \end{array}\right], \right.$$
$$\left. \lambda \in \mathbb{U}_{de},\ g \in G(de, 1, r-2),\ (\pi(g)\lambda^2)^d = 1 \right\}$$

The line of $\tau_{12}$ is $\mathbb{C}(e_1 - e_2)$. So the eigenvalue of $d_{\lambda,g}$ on $\mathbb{C}(e_1 - e_2)$ is $\lambda$ whereas the eigenvalue of $t_{\lambda,g}$ on $\mathbb{C}(e_1 - e_2)$ is $-\lambda$. Thus, when $de$ is odd, the parabolic subgroup associated to the line $\mathbb{C}(e_1 - e_2)$ is given by

$$C = \left\{ \left[\begin{array}{c|c} 1 & \\ \hline & g \end{array}\right], \quad g \in G(de, e, r-2) \right\}$$

and when $de$ is even, the parabolic subgroup associated to the line $\mathbb{C}(e_1 - e_2)$ is given by

$$C = \left\{ \left[\begin{array}{c|c} 1 & \\ \hline & g \end{array}\right], \left[\begin{array}{c|c} -1 & \\ \hline & g \end{array}\right], \quad g \in G(de, e, r-2) \right\}$$

Table 2 gives the orbits the hyperplanes of $G(de, e, r)$ under $N$ and $C$. In Table 2, C.O. stands commuting orbits and N.C.O. stand for non commuting orbits.

### Value for $f_H$ and the index of ramification

The computations of the preceding paragraphs also lead to Table 3 which brings together the values of $e_H, f_H$ and $d_H$ for every class of hyperplanes. In Table 3, we set $\zeta = \exp(2i\pi/de)$.

We obtain the following errata for the proposition 6.1 of [9]. Let us consider $r \geqslant 2$. For $W = G(de, e, r)$, we have $\kappa(W) = 2de$ if $de$ is odd and $r \geqslant 3$. We have $\kappa(W) = de$ if ($d \neq 1$ and $r = 2$ and $de$ even) or ($r \geqslant 3$ and $de$ even). We have $\kappa(W) = 2de$ if ($d \neq 1$ and $r = 2$ and $de$ odd). We have $\kappa(W) = 2$ if $d = 1$ and $r = 2$ [1].

## 5.2 Exceptional types

With the package CHEVIE of GAP [8][6], we obtain Table 4 for the values of $e_H, f_H$ and $f_H/e_H$ for the hyperplanes of the exceptional reflection groups. In particular, the only non Coxeter groups with only unramified hyperplanes are $G_8, G_{12}$ and $G_{24}$. Table 4 can also easily be obtained from the table of [9] for the value of $\kappa(W)$. The first and fifth columns stand for the number of the group in the Shephard and Todd classification. We also write instructions which determines, for a given hyperplane $H$, the orbits of commuting and non commuting hyperplanes under $N_H$ and $C_H$ which is used to obtain the results of Remark 7 and Remark 25.

[1] Comparing to the preceding versions, we correct here the value of $\kappa(W)$ for $de$ odd and $r = 2$. This was pointed out by Ivan Marin.

| | | $N_{H_1}$ | $C_{H_1}$ |
|---|---|---|---|
| $r \geqslant 4$ | C.O. | $H_1$<br>$\{H_i,\ i \geqslant 2\}$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 2,\ \zeta \in \mathbb{U}_{de}\}$ | $H_1$<br>$\{H_i,\ i \geqslant 2\}$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 2,\ \zeta \in \mathbb{U}_{de}\}$ |
| | N.C.O. | $\{H_{1,j,\zeta},\ j \geqslant 2,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,j,\zeta},\ j \geqslant 2,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 3$ and $e$ odd | C.O. | $H_1$<br>$\{H_2, H_3\}$<br>$\{H_{2,3,\zeta},\ \zeta \in \mathbb{U}_{de}\}$ | $H_1$<br>$\{H_2, H_3\}$<br>$\{H_{2,3,\zeta},\ \zeta \in \mathbb{U}_{de}\}$ |
| | N.C.O. | $\{H_{1,i,\zeta},\ i = 2, 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,i,\zeta},\ i = 2, 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 3$ and $e$ even | C.O. | $H_1$<br>$\{H_2, H_3\}$<br>$\{H_{2,3,\zeta},\ \zeta \in \mathbb{U}_{de}\}$ | $H_1$<br>$\{H_2, H_3\}$<br>$\{H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de'}$ |
| | N.C.O. | $\{H_{1,i,\zeta},\ i = 2, 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,i,\zeta},\ i = 2, 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 2$ and $e$ odd | C.O. | $H_1$<br>$H_2$ | $H_1$<br>$H_2$ |
| | N.C.O | $\{H_{1,2,\zeta},\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,2,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{d}$ |
| $r = 2$ and $e$ even | C.O. | $H_1$<br>$H_2$ | $H_1$<br>$H_2$ |
| | N.C.O. | $\{H_{1,2,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de'}$ | $\{H_{1,2,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{d}$ |
| $r = 1$ | C.O. | $H_1$ | $H_1$ |

Table 1: Orbits of hyperplanes under $N_{H_1}$ and $C_{H_1}$

| | | $N_{H_{1,2,1}}$ | $C_{H_{1,2,1}}$ |
|---|---|---|---|
| $r \geqslant 5$ and $de$ odd | C.O. | $H_{1,2,1}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $H_{1,2,1}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_1\}, \{H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$<br>$\{H_{1,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$<br>$\{H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r \geqslant 5$ and $de$ even | C.O. | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 4$ and $de$ odd | C.O. | $\{H_{1,2,1}\}; \{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,2,1}\}; \{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_1\}, \{H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$<br>$\{H_{1,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$<br>$\{H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 4$ and $e$ odd and $d$ even | C.O. | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 4$ and $e$ even | C.O. | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de'}\}$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de} \setminus \mathbb{U}_{de'}\}$ | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$<br>$\{H_i,\ i \geqslant 3\}$ if $d > 1$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de'}\}$<br>$\{H_{i,j,\zeta},\ i \neq j \geqslant 3,\ \zeta \in \mathbb{U}_{de} \setminus \mathbb{U}_{de'}\}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geqslant 3,\ \zeta \in \mathbb{U}_{de}\}$ |
| $r = 3$ and $de$ odd | C.O. | $\{H_{1,2,1}\}; \{H_3\}$ if $d > 1$ | $\{H_{1,2,1}\}; \{H_3\}$ if $d > 1$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,3,\zeta}, H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de''}$ | $\{H_1\}, \{H_2\}$ if $d > 1$<br>$\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$<br>$U_c = \{H_{1,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$<br>$U'_c = \{H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$ |
| | C.O. | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ |

Orbits of hyperplanes under the stabiliser of $H_{1,2,1}$ and parabolic subgroup associated to $\mathbb{C}(e_1 - e_2)$

$r = 3$ and $de$ even

| | | $N_{H_{1,2,1}}$ | $C_{H_{1,2,1}}$ |
|---|---|---|---|
| | | $H_3$ if $d>1$ | $H_3$ if $d>1$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d>1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,3,\zeta}, H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de''}$ | $\{H_1, H_2\}$ if $d>1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$<br>$\{H_{1,3,\zeta}, H_{2,3,-\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{d}$ |
| $r=2$ and $de$ odd | C.O. | $H_{1,2,1}$ | $H_{1,2,1}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d>1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ | $\{H_1\}, \{H_2\}$ if $d>1$<br>$\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$ |
| $r=2$ and $de$ even | C.O. | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ | $\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ |
| | N.C.O. | $\{H_1, H_2\}$ if $d>1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ | $\{H_1, H_2\}$ if $d>1$<br>$\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ |

Table 2: Orbits of hyperplanes under $N_{H_{1,2,1}}$ and $C_{H_{1,2,1}}$

| $H$ | | | $f_H$ | $e_H$ | $d_H$ |
|---|---|---|---|---|---|
| $z_1 = 0$ | $r = 1$ | | $d$ | $d$ | 1 |
| $z_1 = 0$ | $r \geqslant 2$ | $d \neq 1$ | $de$ | $d$ | $e$ |
| $z_2 = z_1$ | $r \geqslant 3$ | $de$ odd | $2de$ | 2 | $de$ |
| $z_2 = z_1$ | $r \geqslant 3$ | $de$ even | $de$ | 2 | $de/2$ |
| $z_2 = z_1$ | $r = 2$ | $e$ odd and $d$ even | $d$ | 2 | $d/2$ |
| $z_2 = z_1$ | $r = 2$ | $e$ even or $d$ odd | $2d$ | 2 | $d$ |
| $z_2 = \zeta z_1$ | $r = 2$ | $e$ even | $2d$ | 2 | $d$ |

Table 3: Values for the ramification index for $G(de, e, r)$

| ST | $e_H$ | $f_H$ | $d_H$ | ST | $e_H$ | $f_H$ | $d_H$ |
|---|---|---|---|---|---|---|---|
| 4 | 3 | 6 | 2 | 21 | 2,3 | 12,12 | 6,4 |
| 5 | 3,3 | 6,6 | 2,2 | 22 | 2 | 4 | 2 |
| 6 | 2,3 | 4,12 | 2,4 | 23 | 2 | 2 | 1 |
| 7 | 2,3,3 | 12,12,12 | 6,4,4 | 24 | 2 | 2 | 1 |
| 8 | 4 | 4 | 1 | 25 | 3 | 6 | 2 |
| 9 | 2,4 | 8,8 | 4,2 | 26 | 3,2 | 6,6 | 2,3 |
| 10 | 3,4 | 12,12 | 4,3 | 27 | 2 | 6 | 3 |
| 11 | 2,3,4 | 24,24,24 | 12,8,6 | 28 | 2,2 | 2,2 | 1,1 |
| 12 | 2 | 2 | 1 | 29 | 2 | 4 | 2 |
| 13 | 2,2 | 8,4 | 4,2 | 30 | 2 | 2 | 1 |
| 14 | 2,3 | 6,6 | 3,2 | 31 | 2 | 4 | 2 |
| 15 | 2,3,2 | 12,12,24 | 6,4,12 | 32 | 3 | 6 | 2 |
| 16 | 5 | 10 | 2 | 33 | 2 | 6 | 3 |
| 17 | 2,5 | 20,20 | 10,4 | 34 | 2 | 6 | 3 |
| 18 | 3,5 | 30,30 | 10,6 | 35 | 2 | 2 | 1 |
| 19 | 2,3,5 | 60,60,60 | 30,20,12 | 36 | 2 | 2 | 1 |
| 20 | 3 | 6 | 2 | 37 | 2 | 2 | 1 |

Table 4: Values for the ramification index for the exceptional groups

**Acknowledgement.** I wish to thank Ivan Marin for fruitful discussions and supports.

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