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Abstract. Given a Morse function f on a closed manifold M with distinct critical values, and given a field $\mathbb{F}$, there is a canonical complex, called the Morse-Barannikov complex, which is *equivalent* to any Morse complex associated with f and whose form is *simple*. In particular, the homology of M with coefficients in $\mathbb{F}$ is immediately readable on this complex. The bifurcation theory of this complex in a generic one-parameter family of functions will be investigated. Applications to the boundary manifolds will be given.

Here is an expanded version of the lectures given in the Winter school organized in La Llagonne by the University Paul Sabatier in Toulouse (January 2013)[1]. There are three parts:

I) The Morse-Barannikov complex (after C. Viterbo),
II) Bifurcations,
III) The non-empty boundary case.

1. The Morse-Barannikov complex

We adopt a presentation which is a mix of the presentation given by S. Barannikov in [1] and a more abstract one given by C. Viterbo in his joint work with D. Le Peutrec and F. Nier [7] which we slightly simplify.

We are given a closed manifold M, a Morse function $f : M \to \mathbb{R}$ whose critical values are distinct, and a field $\mathbb{F}$. For each integer k the critical points of index k are numbered in the increasing order of the critical values: $f(p_1) < f(p_2) < \dots$ (the function is just generic and it is not assumed to be ordered). We shall often identify the set of critical points and the set of critical values.

For defining the *Morse complex* it is necessary to have two extra data:

- A (decreasing) *pseudo-gradient*, that is, a vector field on M which satifies $X \cdot f < 0$ out of the critical points, and some non-degeneracy condition for the vanishing of X at each critical point; therefore, the zeroes of X are hyperbolic. As a consequence, each critical point p has a stable manifold $W^s(p)$ and an unstable manifold $W^u(p)$. This pseudo-gradient is chosen Morse-Smale, a generic property meaning that the stable manifolds are transverse to the unstable manifolds.
- An orientation of the unstable manifolds.

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[1]This text was written in February 2013. It replaces the previous version posted by error.

The Morse complex $C_*(f,X)$ is made as follows. In degree k the module $C_k(f,X)$ is the free $\mathbb{Z}$-module generated by the critical points of index k and the differential $\partial_k : C_k(f,X) \to C_{k-1}(f,X)$ counts the signed number of connecting orbits. Observe that, for (p,q), a pair of critical points of respective index k and $k-1$, $W^u(p) \cap W^s(q)$ is made of a finite number of connecting orbits. Since $W^s(q)$ is co-oriented by the orientation of $W^u(q)$, each orbit descending from p to q gets a sign. Define a_p^q to be the signed number of the connecting orbits and define the Morse differential $C_k(f,X)$ by

$$\partial_k(p) = \sum a_p^q q\,.$$

Theorem 1.1. (Milnor [6]**, Th. 7.2)**. *The Morse complex is a chain complex: $\partial \circ \partial = 0$. Moreover, $H\left(C_*(f,X)\right) \cong H_*(M;\mathbb{Z})$. A fortiori, $H\left(C_*(f,X);\mathbb{F}\right) \cong H_*(M;\mathbb{F})$.*

Definition 1.2. *A chain complex C_* with coeffcients in $\mathbb{F}$ is said to be* $\mathbb{F}$-equivalent *to $C_*(f,X)$ if it has the same generators and if its differential δ is made from ∂ by conjugating in each degree by an invertible upper triangular matrix T with coefficients in $\mathbb{F}$:*

$$\begin{tikzcd}
C_{k+1}(f,X)\otimes\mathbb{F} \arrow[r, "\partial"] \arrow[d, "T"'] & C_k(f,X)\otimes\mathbb{F} \arrow[r, "\partial"] \arrow[d, "T"'] & C_{k-1}(f,X)\otimes\mathbb{F} \arrow[d, "T"'] \\
C_{k+1} \arrow[r, "\delta"] & C_k \arrow[r, "\delta"] & C_{k-1}
\end{tikzcd}$$

Here, $C_k(f,X)\otimes\mathbb{F}$ is a vector space equipped with its canonical ordered basis.

Remark 1.3. When changing the pseudo-gradient or the orientations of the unstable manifolds the Morse complex is changed by $\mathbb{Z}$-equivalence. Conversely, if $\dim M > 1$ and if the level sets of f are connected (or f has one local minimum and one local maximum only, *i.e.* f is *polar* in Morse's terminology) every $\mathbb{Z}$-equivalence is realizable by such changes. When the coefficients are in a field an $\mathbb{F}$-equivalence has no longer such a geometrical meaning in general. But, an $\mathbb{F}$-equivalence keeps the memory of the filtration by the sub-level sets of the function f. This fact will be used in the last step of the proof of Barannikov's theorem.

Theorem 1.4. (Barannikov [1]**)** *The Morse complex $C_*(f,X)$ is $\mathbb{F}$-equivalent to a* simple *complex (C_*,∂_B), that is, for every generator p, $\partial_B(p)$ is 0 or a generator and $\partial_B(p) \neq \partial_B(p')$ if $p \neq p'$ and $\partial_B(p) \neq 0$. Moreover, (C_*,∂_B) is unique and depends only on $C_*(f,X)\otimes\mathbb{F}$ for any pseudo-gradient X.*

This complex is called the *Morse-Barannikov complex* associated with the Morse function f; it depends on the field $\mathbb{F}$.

Corollary 1.5. *The homology $H_*(M;\mathbb{F})$ is graded isomorphic to the sub-space generated by the critical points having the homological type, in the sense given below: $\partial_B(p) = 0$ and $p \notin \operatorname{Im}\partial_B$.*

One important point in the statement is the coupling of some critical points, the unpaired generators being "isolated" in the complex. This fact plays a deep rôle in the work by Y. Chekanov & P. Pushkar [4]. When the Morse complex is concentrated in two degrees, the statement amounts to the fact that the double coset $GL(n,\mathbb{Z})/T(n)\times T(n)$ is isomophic to the symmetric group S_n, a fact which was important in Cerf's work on pseudo-isotopy (here, $T(n)$ denotes the sub-group of invertible upper triangular matrices) [3].

In Barannikov's paper the proof of existence follows more or less from Gauss' algorithm. The proof of uniqueness remains mysterious. It is clarified by C. Viterbo in [7].

Viterbo's important remark is that the critical points of the given Morse function f are divided in three types: *upper, lower* and *homological*, depending of the place of a zero map in the diagram below of $\mathbb{F}$-vector spaces and $\mathbb{F}$-linear maps, in which c denotes a critical value of index $k+1$ and, for brevity, $c+\varepsilon$ stands for the sub-level set $f^{c+\varepsilon} := f^{-1}\big((-\infty, c+\varepsilon]\big)$:

$$\begin{tikzcd}
 & & & \mathbb{F} & & & \\
0 \arrow[r] & H_{k+1}(c-\varepsilon) \arrow[r] & H_{k+1}(c+\varepsilon) \arrow[r, "J"] & H_{k+1}(c+\varepsilon, c-\varepsilon) \arrow[u, "\cong"'] \arrow[r, "\Delta"] \arrow[d, "I"] & H_k(c-\varepsilon) \arrow[r] & H_k(c+\varepsilon) \arrow[r] & 0 \\
 & & & H_{k+1}(+\infty, c-\varepsilon) \arrow[ur, "\Delta'"'] & & &
\end{tikzcd}$$

The horizontal line is an exact sequence. The critical point p such that $f(p) = c$ is said to be of *upper type* when $J = 0$, implying Δ injective. It is said to be of *lower type* when J is surjective and $I = 0$. It is said of *homological type* when I is injective and $\Delta = 0$. Clearly, since $\mathbb{F}$ is a field, all possibilities are covered, making a partition of the critical points.

The type of a critical point p of f is readable on the Morse complex with coefficients in $\mathbb{F}$, that is $C_*(f, X) \otimes \mathbb{F}$. For instance, p is of lower type if there is an $\mathbb{F}$-linear combination of critical points higher than p whose boundary is p; this property does not depend on the chosen pseudo-gradient X.

The coupling of critical points. If p is a critical point of index $k+1$, the local unstable manifold $W^u_{loc}(p)$ is unique up to isotopy and orientation. Set $[p] := [W^u_{loc}(p)] \in H_{k+1}(c+\varepsilon, c-\varepsilon)$ where $c = f(p)$. If p is of upper type, one defines

$$\lambda(p) := \inf_{\sigma} \max(f|\sigma)$$

where σ runs among the k-cycles of the sub-level set $f^{c-\varepsilon}$ representing $\Delta([p])$; it is a critical value $\lambda(p) = f(q)$ where q is a critical point of index k. By identifying critical point and critical value, we set $q := \lambda(p)$.

Lemma 1.6. *The critical point $q := \lambda(p)$ is of lower type.*

Proof. Denote $\Delta_q([p])$ the class of $\Delta([p])$ modulo the sub-level set $f(q) - \varepsilon$ in $H_k\big(f(p) - \varepsilon, f(q) - \varepsilon\big)$. By definition of the minimax, this class is not zero and we have

$$\Delta_q([p]) = \alpha I^p_q\big([q]\big), \alpha \in \mathbb{F}, \alpha \neq 0,$$

where $I^p_q : H_k(f(q)+\varepsilon, f(q)-\varepsilon) \to H_k(f(p)-\varepsilon, f(q)-\varepsilon)$ is induced by the inclusion. Thus, $W^u_{loc}(q)$ is the boundary of $\frac{1}{\alpha}W^u(p)$ in the pair $(+\infty, f(q)-\varepsilon)$. Hence $I([q]) = 0$. $\square$

The Barannikov differential. Set $\partial_B(p) = \lambda(p)$ if p is of upper type and $\partial_B(p) = 0$ in the two other cases. According to the previous lemma, $\partial_B \circ \partial_B = 0$.

Lemma 1.7. *The map λ defines a bijection from the set of critical points of upper type onto the set of critical points of lower type.*

Proof. 1) Injectivity. Let $q = \partial_B(p) = \partial_B(p')$ with $f(p) > f(p')$. We are using the same notation as in Lemma 1.6. We have $\Delta_q([p]) = \alpha I_q^p([q]) \neq 0$ in $H_k\big(f(p)-\varepsilon, f(q)-\varepsilon\big)$ and $\Delta_q([p']) = \alpha' I_q^{p'}([q]) \neq 0$ in $H_k\big(f(p')-\varepsilon, f(q)-\varepsilon\big)$. By construction $I_q^p \Delta_q([p']) = 0$. Thus, we have:

$$\begin{aligned}\Delta_q([p]) &= \Delta_q([p]) - \frac{\alpha}{\alpha'} I_q^p \Delta_q([p']) \\ &= \alpha I_q^p([q]) - \frac{\alpha}{\alpha'} \alpha' I_q^p([q]) = 0\,.\end{aligned}$$

Therefore, $f(q)$ is not the minimax value associated with p.

2) Surjectivity. Let q be a critical point of lower type and index k. Set

$$\mu(q) = \inf_{\sigma} \max(f|\sigma)$$

where σ runs among the relative chains of $(+\infty, f(q)-\varepsilon)$ whose boundary is a relative cycle representing the class $[q]$. This $\mu(q)$ is a critical value of index $k+1$ with $\mu(q) = f(p)$ for some critical point p. A chain σ approximating the infimum has a non vanishing class in $H_{k+1}\big(f(p)+\varepsilon, f(p)-\varepsilon\big)$; hence, $[\sigma] = \beta[p]$ in the pair $(f(p)+\varepsilon, f(p)-\varepsilon)$ with $\beta \in \mathbb{F},\ \beta \neq 0$. We have $\Delta_q\big([p]\big) = I_q^p(\frac{1}{\beta}[q])$. If this element is zero, this means that there is another relative chain bounded by $W^u(q)$ under the level of p, contradicting the definition of $\mu(q)$. Then, $\Delta_q([p]) \neq 0$. *A fortiori*, $\Delta([p]) \neq 0$ and p is of upper type.

We have also to show that $\lambda(p) = q$. By the above σ, $\Delta([p])$ is homologous to $[q]$ up a non zero scalar. If it is homologous to $[q']$ with $f(q') < f(q)$, then $I_q^p([q]) = 0$, and this is not the case. □

At this point we have the uniqueness part in Barannikov's theorem.

Lemma 1.8. *The Morse-Barannikov complex is $\mathbb{F}$-equivalent to the Morse complex. In particular, its homology is isomorphic to $H_*(M, \mathbb{F})$.*

Proof. Suppose we have a chain complex (C_*, ∂), $\mathbb{F}$-equivalent to the Morse complex and which is simple until the degree k. Then, $\partial\,(C_{k+1})$ is orthogonal to the critical points of upper type (with respect to the canonical scalar product of a based vector space). If not, $\partial \circ \partial \neq 0$.

Let $p_1, \ldots, p_m$ be the critical points of index $k+1$, with $f(p_1) < \ldots < f(p_m)$. Let $q_1, \ldots, q_r$ be the critical points of index k whose type is lower or homological; $f(q_1) < \ldots < f(q_r)$. We assume that, for some $j \leq r$, we have $\partial p_i = 0$ or $\partial p_i = q_{k(i)}$ for every $i < j$, the map $i \mapsto k(i)$ being injective. We have

$$\partial p_j = \sum_{i<j, \partial p_i \neq 0} \alpha_i\, q_{k(i)} + \begin{cases} 0 \\ \text{or} \\ \beta_{k_0}\, q_{k_0} + \sum\limits_{k<k_0, k\neq k(i)} \beta_k\, q_k, \quad \text{with} \quad \beta_{k_0} \neq 0\,. \end{cases}$$

In the first case we use the following upper triangular matrix in degree $k+1$:

$$\begin{aligned} T(p_j) &= p_j - \sum_{i<j, \partial p_i \neq 0} \alpha_i \, p_i \\ T(p_\ell) &= p_\ell \quad \text{if} \quad \ell \neq j \end{aligned}$$

and we set $\bar{\partial} = \partial \circ T$. We get $\bar{\partial}(p_j) = 0$ and $\bar{\partial}(p_i) = \partial(p_i)$ for every $i < j$; so we have improved the simpleness of the differential.

In the second case, we use an upper triangular matrix in both degree $k+1$ and k:

$$\begin{aligned} T(p_j) &= \tfrac{1}{\beta_{k_0}} \left(p_j - \sum_{i<j,\, \partial p_i \neq 0} \alpha_i \, p_i \right), \\ T(p_\ell) &= p_\ell \quad \text{if} \quad \ell \neq j, \\ T(q_{k_0}) &= q_{k_0} + \sum_{k<k_0, k \neq k(i)} \frac{\beta_k}{\beta_{k_0}} \, q_k \\ T(q_k) &= q_k \quad \text{if} \quad k \neq k_0 \,. \end{aligned}$$

We set $\bar{\partial}_{k+1} = T^{-1} \circ \partial_{k+1} \circ T$ and $\bar{\partial}_k = \partial_k \circ T$. We observe that $\bar{\partial}_k = 0$ on the k-cycles as T keeps this set invariant. We have $\bar{\partial}(p_j) = q_{k_0}$ and $\bar{\partial}(p_i) = \partial(p_i)$ if $i < j$. Thus, $\bar{\partial}$ has the simple form for $1 \leq i \leq j$. Arguing this way recursively, we get a simple complex which is $\mathbb{F}$-equivalent to the Morse complex. □

Since the equivalence relation involves upper triangular matrices only, $\partial(p)$ remains the class of $\Delta([p])$ in $H_*\big(f(p) - \varepsilon; \mathbb{F}\big)$ as it is in the Morse complex. Therefore, when the $\mathbb{F}$-equivalent complex is simple, the type of each critical point can be easily derived and this complex is the Morse-Barannikov complex. The proof of Theorem 1.4 is completed. □

2. Bifurcations

Bifurcations occur in a path of functions. It follows from Thom's transversality theorems, as it is explained by J. Cerf in the beginning of [3], that the space $\mathcal{F}$ of real smooth functions on M has a natural stratification whose strata of codimension ≤ 1 are the following:

0) The stratum $\mathcal{F}_0$, an open dense set in $\mathcal{F}$, is formed by Morse functions whose critical values are all simple. The next two strata are of codimension one.
1) The stratum $\mathcal{F}_1$ is formed by the functions whose critical values are simple and whose critical points are all non-degenerate but one where the Hessian has a kernel of dimension one.
2) The stratum $\mathcal{F}_2$ is formed by the Morse functions whose critical values are all simple but one which has multiplicity 2.
3) The complement $\mathcal{R}$ in $\mathcal{F}$ of the preceding strata.

A generic path has its end points in $\mathcal{F}_0$, avoids $\mathcal{R}$ which in turn is said to be of codimension greater than 1, and crosses $\mathcal{F}_1 \cup \mathcal{F}_2$ transversely in a finite number of points.

Since the Morse-Barannikov complex is well defined for functions in $\mathcal{F}_0$ only, it is necessary to study the bifurcation when crossing $\mathcal{F}_1$ and $\mathcal{F}_2$. In that aim, it is convenient to introduce the Barannikov diagram and the Cerf diagram, which are defined as follows.

The *Barannikov diagram* deals with a generic Morse function $f \in \mathcal{F}_0$. Let $n = \dim M$. The vertical lines $D_k, k = 0, 1, \ldots, n$, are drawn in the plane, the x-coordinate of D_k being $n - k$. The critical values of f of index k are marked on D_k. When the pair of critical points (p, q) are coupled in the Barannikov complex a segment is drawn from $f(p)$ to $f(q)$; clearly, the slope of this segment is negative (Fig. 1). The critical values, of homological type, in particular $\max f$ and $\min f$, remain non-connected to another one.

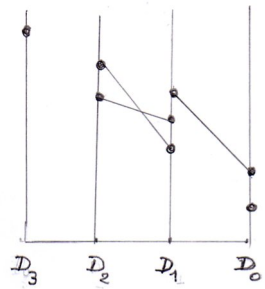


Figure 1

The *Cerf diagram* deals with a generic path $\gamma = (f_t)_{t\in[0,1]}$. Its Cerf diagram is the union in $[0, 1] \times \mathbb{R}$ of $\{t\} \times f_t(\mathrm{crit} f_t)$. It is made of finitely many smooth arcs transverse to the verticals $\{t\} \times \mathbb{R}$, ending at cusp points or in $\{0, 1\} \times \mathbb{R}$, crossing one another transversely.

2.1. Bifurcation at birth times. This event is the crossing of the stratum $\mathcal{F}_1$, which is co-oriented: the crossing in the positive direction corresponds to the birth of a pair of critical points of index k and $k + 1$ respectively. The crossing in the opposite direction corresponds to the cancellation of a pair of critical points.

The birth is modeled by the following formula:

$$f(x, y) = c + Q(y) + x^3 - (t - t_0)x,$$

where t_0 is the birth time, c is the critical value of the birth point, $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ are local coordinates at the birth point p_0 and Q is a non-degenerate quadratic form of index k on $\mathbb{R}^{n-1}$. In the Cerf diagram, there is a cusp of coordinates (t_0, c). For $0 < t_0 - t$ small, there are no critical points in the vicinity of p_0. For $0 < t - t_0$ small, there are two critical points p_t, q_t of index $k + 1$ and k close to p_0 (Fig. 2).

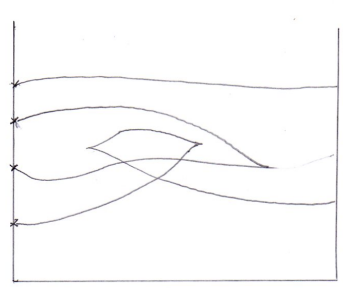


Figure 2

It follows from the model that p_t is of upper type and q_t is of lower type and this pair is coupled in the Barannikov diagram. The other critical points keep their type and coupling when crossing the birth time.

2.2. Bifurcation at double critical value time. This event is the crossing of the stratum $\mathcal{F}_2$, say at time t_0. Since the functions f_t are Morse for t close to t_0, the critical points can be followed continuously on $(t_0-\eta, t_0+\eta)$. Generically, the pseudo-gradient is Morse-Smale at the time t_0. Therefore, the Morse complex remains the same on a small interval. But, as the order of the critical values is modified the Barannikov complex could change.

Denote by (p_t^1, p_t^2) the pair of critical points whose values cross at time t_0; say $f_t(p_t^1) < f_t(p_t^2)$ when $t < t_0$; hence, $f_t(p_t^1) > f_t(p_t^2)$ when $t > t_0$. The question is how the types and coupling of critical points are changing when t crosses the time t_0.

With Barannikov we limit ourselves to the case when M is the n-sphere S^n and the crossing does not involve the extremal values, this latter question being left as an exercise to the reader. Since on a sphere the only critical points of homological type are the extrema, the crossing deals with critical values of upper/lower type.

We shall say that there is *no bifurcation* if the crossing keeps all critical points with their initial types and coupling (remember that, near a crossing time all the functions of the considered path are Morse and the critical points can be followed smoothly in time due to the implicit function theorem).

One checks by hand that there is no bifurcation if p_t^1 and p_t^2 have distinct indices. Now, we are reduced to the case where the two crossing critical values have the same index, say k.

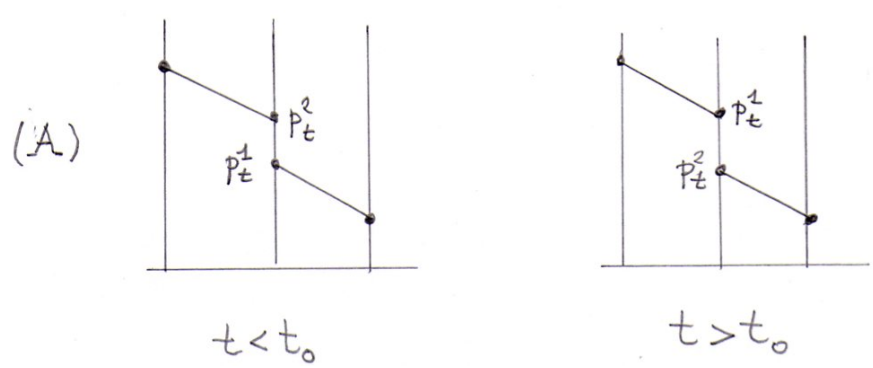
We are going to prove in the next three propositions that, in our restrictive setting, there are only three types of bifurcations which are shown with their Barannikov diagrams before and after crossing (Fig. 3).

2.3. Notation. Before stating the bifurcation propositions, it is useful to introduce some notation. Denote by c_0 the double critical value:

$$c_0 = f_{t_0}(p_{t_0}^1) = f_{t_0}(p_{t_0}^2).$$

We have two $(k-1)$-spheres Σ^1 and Σ^2 traced in the level set $f_{t_0} = c_0 - \varepsilon$ by the respective unstable manifolds of $p_{t_0}^1$ and $p_{t_0}^2$. When t runs in a small interval around t_0, these objects move by a small isotopy: $\Sigma_t^1, \Sigma_t^2 \subset (f_t = c_0 - \varepsilon)$.

Finally, if α is a non-zero $(k-1)$-homology class is the sub-level set $c_0 - \varepsilon$, we denote by $\lambda(\alpha)$ the critical value which is the infimum of c such that α vanishes in the relative homology $H_{k-1}(c_0 - \varepsilon, c; \mathbb{F})$ of the pair of the denoted sub-level sets.



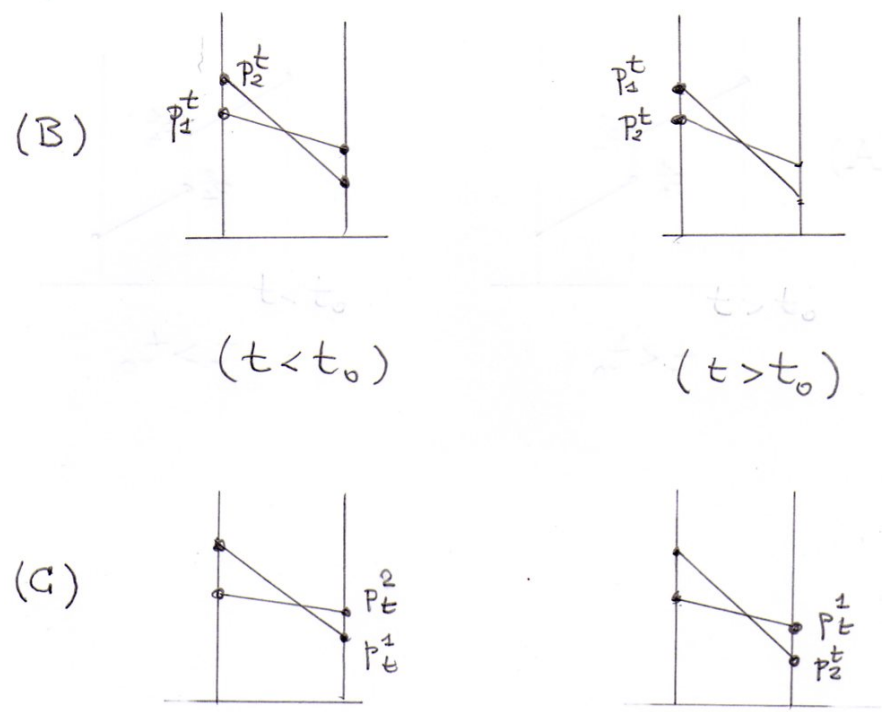


Figure 3

Proposition 2.4. *We assume that, for $t < t_0$, p_t^1 and p_t^2 have not the same type upper/lower. Then, the following holds true.*
1) *A bifurcation can occur only in case* (A left)*, that is: p_t^1 is of upper type and below p_t^2 which is of lower type.*
2) *Assuming the above necessary condition, a bifurcation occurs if and only if $[\Sigma^1] = [\Sigma^2]$ in $H_{k-1}(c_0; \mathbb{F})$ up to a scalar. In that case, for $t > t_0$, p_t^1 becomes of lower type and p_t^2 becomes of upper type; the new coupling is shown on* (A right)*.*

Proof. 1) Assume p_t^1 and p_t^2 are respectively of lower and upper type for $t < t_0$. When t is close to t_0, $t < t_0$, by assumption, Σ_t^1 is null-homologous in its sub-level set while Σ_t^2 is non-homologous to 0. Since these spheres change with t by an isotopy, this property persists up to $t = t_0$ and still a little further, up to some $t' > t_0$. Therefore the types are unchanged after crossing and it is easy to check that the coupling is also unchanged. Hence, no bifurcation.

2) Then, we assume (A left). Again, for $t < t_0$ and close to t_0, certainly $[\Sigma_t^1]$ is not homologous to 0 in $H_{k-1}((f_t = c_0 - \varepsilon); \mathbb{F})$. The unstable manifold $W^u(p_t^2)$ traces in $f_t = f_t(p_t^2) - \varepsilon'$, $\varepsilon' > 0$, a sphere which is homologous to 0 in its sub-level set if $f_t(p_t^2) - \varepsilon' > f_t(p_t^1)$. Thus, for Σ_t^2 in $f_t = c_0 - \varepsilon$, there are two possibilities: (i) $[\Sigma_t^2]$=0 or (ii) $[\Sigma_t^2]$ is a non-zero multiple of $[\Sigma_t^1]$ since it should vanish when passing in a sub-level set containing p_t^1. In both cases, this property persists up to t_0 and up to some $t' > t_0$. One checks easily that in case (i) there is no bifurcation.

Consider case (ii). The point $p_{t'}^2$ is of upper type, by the very definition. Now, with $\varepsilon' > 0$ small enough so that $f_{t'}(p_{t'}^2) < f_{t'}(p_{t'}^1) - \varepsilon'$, we see that the trace of $W^u(p_{t'}^1)$ in the level set

$f_{t'} = f_{t'}(p^1_{t'}) - \varepsilon'$ is homologous to 0 in its sub-level set. Therefore, $p^1_{t'}$ is of lower type and the types of the two critical points exchange. One checks that the coupling is as shown on (A right). □

Assume now p^1_t and p^2_t are both of upper type. By the same homological argument as before, the type cannot change. But the coupling could change. Denote q^1_t and q^2_t the points associated with p^1_t and p^2_t respectively before t_0.

Proposition 2.5. *In this situation, there is a bifurcation (exchange of coupling) if and only if* $\lambda([\Sigma^1]) = \lambda([\Sigma^2])$. *In that case, when* $t < t_0$, $f_t(q^1_t) > f_t(q^2_t)$ *as shown on* (B left).

Proof. Assume $\lambda([\Sigma^1]) < \lambda([\Sigma^2])$ (the opposite inequality is treated similarly). This implies that $[\Sigma^1]$ is 0 in the homology of the pair of sub-level sets $\big(c_0 - \varepsilon, \lambda([\Sigma^2])\big)$. But this vanishing holds true for every t close t_0. This proves that the coupling of p^1_t remains unchanged; hence, no bifurcation.

What happens in case of equality? First, we look at $t < t_0$. Since the function f_t, restricted to the sub-level set $c_0 - \varepsilon$, moves by isotopy when t runs in $(t_0 - \eta, t_0]$, one derives that $\lambda([\Sigma^1_t])$ varies continuously on this interval. But we know $\lambda([\Sigma^1_t]) = f_t(q^1_t)$; as a consequence we have

$$\lambda([\Sigma^1]) = \lambda([\Sigma^2]) = f_{t_0}(q^1_{t_0})\,.$$

For a small ε' so that $f_t(p^2_t) - \varepsilon' > f_t(p^1_t)$, the trace of $W^u(p^2_t)$ in $f_t = f_t(p^2_t) - \varepsilon'$ is homologous to 0 in the pair of sub-level sets $(f_t(p^2_t) - \varepsilon', f_t(q^1_t) - \varepsilon)$; indeed, the class of $W^u(q^1_t)$ is null in this pair which contains p^1_t. Therefore, $\lambda([\Sigma^2_t]) < \lambda([\Sigma^1_t])$ and, hence, $f_t(q^2_t) < f_t(q^1_t)$ as shown on (B left).

Second, we look at $t > t_0$. Now, $\lambda([\Sigma^2_t])$ varies continuously on $[t_0, t_0 + \eta)$; thus, this value is $f_t(q^1_t)$ due to the above equality at time t_0. Thus p^2_t is coupled with q^1_t. Therefore, p^1_t must be coupled with q^1_t; there is, indeed, no other free place! □

The last case to consider is when p^1_t and p^2_t are both of lower type. For homological reasons, there is no change of types. Let q^1_t and q^2_t be the points of index $k+1$ with which they are coupled respectively when $t < t_0$.

At time t_0, denotes by e_i, $i = 1, 2$, the k-cell traced by the unstable manifold of $p^i_{t_0}$ in the level set $f = c_0 - \varepsilon$. The map μ that we introduced in Lemma 1.7 is still defined: $\mu([e_i])$ is the infimum of c such that the class of e_i is 0 in the pair $(c, c_0 - \varepsilon)$ with coefficients in $\mathbb{F}$. Arguing similarly as in the previous proposition, one proves the following.

Proposition 2.6. *In this situation, there is a bifurcation (exchange of coupling) if and only if* $\mu([e_1]) = \mu([e_2])$. *In that case* $f_t(q^1_t) > f_t(q^2_t)$ *when* $t < t_0$ *as shown on* (C left).

3. The non-empty boundary case

Following S. Barannikov, we discuss in this section the problem of extending without critical points a germ of function given along the boundary M of a compact $(n+1)$-dimensional manifold W, $M = \partial W$. This setting was already considered in 1934 by Morse-van Schaack [8] where the

Morse inequalities have been formulated and proved for manifolds with non-empty boundary. Notice that generically a function $F : W \to \mathbb{R}$ is a Morse function whose critical points lie in the interior of W and whose restriction to the boundary is Morse.

Actually, the problem considered by Barannikov in [1] was more ambitious, that is, given a generic germ $\tilde{f}$ along the boundary M, to give a bound from below of the number of critical points of any generic extension $F : W \to \mathbb{R}$ of $\tilde{f}$, as acute as possible. We focus on $M = S^n$, the n-sphere, and $W = D^{n+1}$. Moreover, we limit ourselves to answer the question of knowing when this bound is positive. This problem was completely solved by S. Blank & F. Laudenbach [2] for $n = 1$ and by C. Curley [5] for $n = 2$.

3.1. The framing. Here we use Barannikov's terminology. Given a Morse function $f : M \to \mathbb{R}$ a *framing* of f is the data of one vertical arrow at each critical value of f. A generic germ $\tilde{f} : M \times [0, \varepsilon) \to \mathbb{R}$ along the boundary determines a framing according the following rule: for the critical p of f the arrow at $f(p)$ points up (resp. down) if $< d\tilde{f}(p), \vec{n}(p) >$ is positive (resp. negative), where $\vec{n}(p)$ is a tangent vector at p pointing inwards. Conversely, the framing classifies the germ $\tilde{f}$ up to isotopy fixing the boundary.

This yields some information about the non-existence of an extension without critical points $F : W \to \mathbb{R}$ of the germ $\tilde{f}$. For instance, if the framing points up at the maximum, the maximum principle tells us that any extension must have at least one critical point in the interior. Barannikov's discussion will be of course more subtle. The framing may be attached to a Morse function on M, an $\mathbb{F}$-equivalence class of Morse complexes or the associated Morse-Barannikov complex $C_B(f)$ as well. We will speak of the *framed Barannikov diagram.* We are going to look at the framed Barannikov complex and get some information in relation to the extension problem.

From now on, we restrict ourselves to $M = S^n$ and $W = D^{n+1}$. A framed function $f : S^n \to \mathbb{R}$ is said to be *standard* if f has two critical points only, the framing pointing down at the maximum and pointing up at the minimum. In this case, there is a standard extension to the $(n + 1)$-ball without critical points.

Now, start with a generic germ $\tilde{f}$ along S^n and assume there is an extension without critical points $F : D^{n+1} \to \mathbb{R}$. Then, there is a one-parameter family of spheres $S_t \subset D^{n+1},\ t \in [0, 1]$, such that $S_0 = M$, $S_{t'}$ lies inside S_t when $t' > t$ and the germ of F along S_1 is *standard.* Set $f_t := F|S_t$; it is a function thought of as defined on $M = S^n$, equipped with a framing due to the knowledge of $S_{t'}$ for $t' = t + \varepsilon,\ \varepsilon > 0$. The framing says the direction of moving of the spheres near a critical point p of f_t (up to isotopy, F may be locally thought of as the height function in $\mathbb{R}^3$).

Generically, for such a family of spheres, the map $t \mapsto f_t$ is a generic path of functions in the sense of Section 2. Therefore, there is a sequence of bifurcations starting from a given germ and ending at the standard germ.

There is no bifurcation of framing. At a birth time the pair (p_t, q_t) which has born with $f_t(p_t) > f_t(q_t)$ has the *standard* framing: the arrows are both up or both down; such a pair will be said to be a *standard pair.* At a crossing time t_0 involving the pair of critical values $f_t(p_t^1) < f_t(p_t^2)$, with the same indices, each critical point keeps its framing during the crossing. But, the rule of moving added to the rule of numbering implies that, for $t < t_0$, the arrow of p_t^1 is up and the arrow of p_t^2 is down (Compare Fig. 4). At each generic time, we have

a well-defined Morse-Barannikov framed complex and the bifurcations are those allowed by Propositions 2.4 to 2.6. Of course, in each case the bifurcation occurs only when the required homological condition is satisfied. If it is not, the crossing yields no bifurcation.

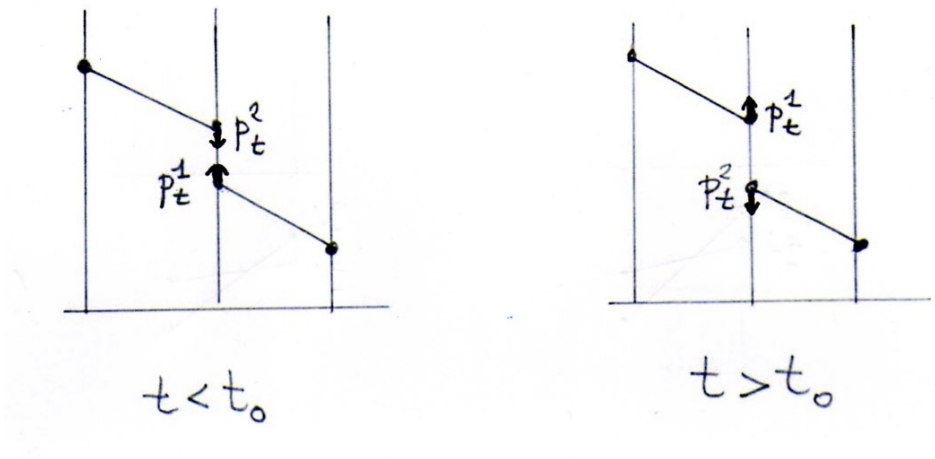


Figure 4

3.2. Barannikov idea's. Not any sequence of allowed bifurcations of the framed Barannikov's complex is realizable by a sequence of bifurcations of framed Morse complex; at the level of framed Barannikov diagram there is no longer homological condition. Therefore, there is a *formal problem* which is the following: given a framed Barannikov complex, does there exist a sequence of allowed bifurcations connecting it to the standard Barannikov complex (*i.e.* one maximum down and one minimum up)? If there is no solution to the formal problem, *a fortiori* there is no solution to the extension problem without critical point. Now, the question is whether it is possible to answer the formal problem in finite time. This question is solved by the last theorem in Barannikov's article.

Theorem 3.3. *Given a framed Barannikov complex C_0, if it is connected to the standard Barannikov complex C_{st}, then C_0 is connected to C_{st} without any birth.*

In particular, the formal problem reduces to a finite combinatorics.

Definition 3.4. *Let f be a framed Morse function. A coupled pair of critical points is said to be inverted when one of its two arrows points up and the other points down.*

There are two types of such inverted pairs: in type I (resp. type II) the upper point is equipped with an arrow pointing up (resp. down).

The index of a coupled pair (inverted or standard) will be the index of the upper point.

One checks on the list of bifurcations that such a pair could not disappear alone. At best, it is possible to shift the indices of the involved critical points (use the bifurcation of Fig. 4 one pair being inverted and the other being standard in the sense of the birth bifurcation). But, two inverted pairs involving of a bifurcation as shown on Fig.4 become both standard and then, each of them can be cancelled.

As a consequence, an obvious obstruction to extending without critical points is the parity of the number of inverted pairs in the initial framed Morse function. The obstruction which follows from Theorem 3.3 is more subtle as shown in the next example.

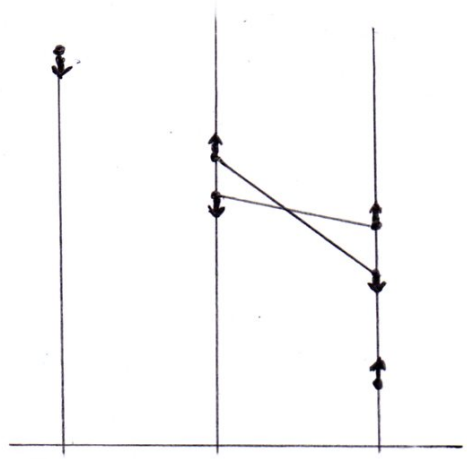


Figure 5

The proof of Theorem 3.3 is based on the fact that there is a very short list of bifurcations involving at least one inverted pair; moreover, the role of the two types, I and II, are completely different. Here is the list of these bifurcations:

(1) a pair of type I and a pair of type II of the same indices yield two standard pairs;
(2) two pairs of type I whose indices differ by 1 yield two standard pairs whose indices differ by 1;
(3) two standard pairs having the same index k yield a pair of type I and a pair of type II both having the index k;
(4) two standard pairs whose indices are k and $k+1$ yield two pairs of type II whose indices are still k and $k+1$;
(5) a pair of type I and index k and a standard pair of index $k \pm 1$ yield a pair of type II and index $k \pm 1$ and a standard pair of index k.

In particular, it is impossible to change the index of a pair of type II. For proving Theorem 3.3, without losing generality we may assume that the initial complex C_0 has no standard pairs. All the coupled pairs are inverted and we are facing the problem of canceling them, maybe with the help of introducing standard pairs (births) whose bifurcations could create new inverted pairs in good positions for a total cancellation. One checks that this event cannot happen.

According to the previous list, for canceling one pair A of type II it is required to have one pair B of type I and of the same index is the right position allowing the bifurcation (1). If B comes from births followed by the bifurcation (3), then B is born with another pair of type II still with the same index. So the price to pay the cancellation of A with B is the appearance of C which is almost in the same position as A in the Barannikov diagram, up to a shift of the heights in the direction of the arrows of the framing. So, nothing is gained. The complete proof follows the same line. We refer to Barannikov's article for the details.

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