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ON AN ARTICLE BY S. A. BARANNIKOV

FRANÇOIS LAUDENBACH

ABSTRACT. Given a Morse function f on a closed manifold M with distinct critical values, and given a field \mathbb{F} , there is a canonical complex, called the Morse-Barannikov complex, which is *equivalent* to any Morse complex associated with f and whose form is *simple*. In particular, the homology of M with coefficients in \mathbb{F} is immediately readable on this complex. The bifurcation theory of this complex in a generic one-parameter family of functions will be investigated. Applications to the boundary manifolds will be given.

Here is an expanded version of the lectures given in the Winter school organized in La Llagonne by the University Paul Sabatier in Toulouse (January 2013). There are three parts:

- I) The Morse-Barannikov complex (after C. Viterbo),
- II) Bifurcations,
- III) The non-empty boundary case.

1. THE MORSE-BARANNIKOV COMPLEX

We adopt a presentation which is a mix of the presentation given by S. Barannikov in [7] and a more abstract one given by C. Viterbo in his joint work with D. Le Peutrec and F. Nier [5] which we slightly simplify.

We are given a closed manifold M , a Morse function $f : M \rightarrow \mathbb{R}$ whose critical values are distinct, and a field \mathbb{F} . For each integer k the critical points of index k are numbered in the increasing order of the critical values: $f(p_1) < f(p_2) < \dots$ (the function is just generic and it is not assumed to be ordered). We shall often identify the set of critical points and the set of critical values.

For defining the *Morse complex* it is necessary to have two extra data:

- A (decreasing) *pseudo-gradient*, that is, a vector field on M which satisfies $X \cdot f < 0$ out of the critical points, and some non-degeneracy condition for the vanishing of X at each critical point; therefore, the zeroes of X are hyperbolic, so that, through each critical point p there are a stable manifold $W^s(p)$ and an unstable manifold $W^u(p)$.

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This pseudo-gradient is chosen Morse-Smale, indeed a generic property meaning that the stable manifolds are transverse to the unstable manifolds.

- An orientation of the unstable manifolds.

The Morse complex $C_*(f, X)$ is made as follows. In degree k the module $C_k(f, X)$ is the free \mathbb{Z} -module generated by the critical points of index k and the differential $\partial_k : C_k(f, X) \rightarrow C_{k-1}(f, X)$ counts the signed number of connecting orbits. Observe that, for (p, q) , a pair of critical points of respective index k and $k - 1$, $W^u(p) \cap W^s(q)$ is made of a finite number of connecting orbits. Since $W^s(q)$ is co-oriented by the orientation of $W^u(q)$, each orbit descending from p to q gets a sign. Define a_p^q to be the signed number of the connecting orbits and define the Morse differential $C_k(f, X)$ by

$$\partial_k(p) = \sum a_p^q q.$$

Theorem 1.1. (Milnor [3], Th. 7.2). *The Morse complex is a chain complex: $\partial \circ \partial = 0$. Moreover, $H(C_*(f, X)) \cong H_*(M; \mathbb{Z})$. A fortiori, $H(C_*(f, X); \mathbb{F}) \cong H_*(M; \mathbb{F})$.*

Definition 1.2. *A chain complex C_* with coefficients in \mathbb{F} is said to be \mathbb{F} -equivalent to $C_*(f, X)$ if it has the same generators and if its differential δ is made from ∂ by conjugating in each degree by an invertible upper triangular matrix T with coefficients in \mathbb{F} :*

$$\begin{array}{ccccc} C_{k+1}(f, X) \otimes \mathbb{F} & \xrightarrow{\partial} & C_k(f, X) \otimes \mathbb{F} & \xrightarrow{\partial} & C_{k-1}(f, X) \otimes \mathbb{F} \\ T \downarrow & & T \downarrow & & T \downarrow \\ C_{k+1} & \xrightarrow{\delta} & C_k & \xrightarrow{\delta} & C_{k-1} \end{array}$$

Here, $C_k(f, X) \otimes \mathbb{F}$ is a vector space equipped with its canonical ordered basis.

Remark 1.3. When changing the pseudo-gradient or the orientations of the unstable manifolds the Morse complex is changed by \mathbb{Z} -equivalence. Conversely, if $\dim M > 1$ and if the level sets of f are connected (or f has one local minimum and one local maximum only, *i.e.* f is *polar* in Morse's terminology) every \mathbb{Z} -equivalence is realizable by such changes. When the coefficients are in an field an \mathbb{F} -equivalence has no longer such a geometrical meaning in general. But, an \mathbb{F} -equivalence keeps the memory of the filtration by the sub-level sets of the function f . This fact will be used in the last step of the proof of Barannikov's theorem.

Theorem 1.4. (Barannikov [7]) *The Morse complex $C_*(f, X)$ is \mathbb{F} -equivalent to a simple complex (C_*, ∂_B) , that is, for every generator p , $\partial_B(p)$ is 0 or a generator and $\partial_B(p) \neq \partial_B(p')$ if $p \neq p'$ and $\partial_B(p) \neq 0$. Moreover, (C_*, ∂_B) is unique and depends only on $C_*(f, X) \otimes \mathbb{F}$ for any pseudo-gradient X .*

This complex is called the *Morse-Barannikov complex* associated with the Morse function f ; it depends on the field \mathbb{F} .

Corollary 1.5. *The homology $H_*(M; \mathbb{F})$ is graded isomorphic to the sub-space generated by the critical points having the homological type, in the sense given below: $\partial_B(p) = 0$ and $p \notin \text{Im } \partial_B$.*

One important point in the statement is the pairing of some critical points, the unpaired generators being "isolated" in the complex. This fact plays a deep rôle in the work by Y. Chekanov and P. Pushkar [2]. When the Morse complex is concentrated in two degrees, the

THE BARANNIKOV DIFFERENTIAL. Set $\partial_B(p) = \lambda(p)$ if p is of upper type and $\partial_B(p) = 0$ in the two other cases. According to the previous lemma, $\partial_B \circ \partial_B = 0$.

Lemma 1.7. *The map λ defines a bijection from the set of critical points of upper type onto the set of critical points of lower type.*

Proof. 1) INJECTIVITY. Let $q = \partial_B(p) = \partial_B(p')$ with $f(p) > f(p')$. We are using the same notation as in Lemma 1.6. We have $\Delta_q([p]) = \alpha I_q^p([q]) \neq 0$ in $H_k(f(p) - \varepsilon)$ and $\Delta_q([p']) = \alpha' I_q^{p'}([q]) \neq 0$ in $H_k(f(p') - \varepsilon)$. By construction $I_q^p \Delta_q([p']) = 0$. Thus, we have:

$$\begin{aligned} \Delta_q([p]) &= \Delta_q([p]) - \frac{\alpha}{\alpha'} I_q^p \Delta_q([p']) \\ &= \alpha I_q^p([q]) - \frac{\alpha}{\alpha'} \alpha' I_q^p([q]) = 0. \end{aligned}$$

Therefore, $f(q)$ is not the minimax value associated with p .

2) SURJECTIVITY. Let q be a critical point of lower type and index k . Set

$$\mu(q) = \inf_{\sigma} \max(f|\sigma)$$

where σ runs among the relative chains of $(+\infty, f(q) - \varepsilon)$ whose boundary is a relative cycle representing the class $[q]$. This $\mu(q)$ is a critical value of index $k + 1$ with $\mu(q) = f(p)$ for some critical point p . A chain σ approximating the infimum has a non vanishing class in $H_{k+1}(f(p) + \varepsilon, f(p) - \varepsilon)$; hence, $[\sigma] = \beta[p]$ with $\beta \in \mathbb{F}$, $\beta \neq 0$. We have $\Delta_q(\frac{1}{\beta}[p]) = I([q])$. Then, $\Delta_q([p]) \neq 0$. *A fortiori*, $\Delta([p]) \neq 0$ and p is of upper type. \square

At this point we have the uniqueness part in Barannikov's theorem.

Lemma 1.8. *The Morse-Barannikov complex is \mathbb{F} -equivalent to the Morse complex. In particular, its homology is isomorphic to $H_*(M, \mathbb{F})$.*

Proof. Suppose we have a chain complex (C_*, ∂) , \mathbb{F} -equivalent to the Morse complex and which is simple until the degree k . Then, $\partial(C_{k+1})$ is orthogonal to the critical points of upper type (with respect to the canonical scalar product of a based vector space). If not, $\partial \circ \partial \neq 0$.

Let p_1, \dots, p_m be the critical points of index $k + 1$, with $f(p_1) < \dots < f(p_m)$. Let q_1, \dots, q_r be the critical points of index k whose type is lower or homological; $f(q_1) < \dots < f(q_r)$. We assume that, for some $j \leq r$, we have $\partial p_i = 0$ or $\partial p_i = q_{k(i)}$ for every $i < j$, the map $i \mapsto k(i)$ being injective. We have

$$\partial p_j = \sum_{i < j, \partial p_i \neq 0} \alpha_i q_{k(i)} + \begin{cases} 0 \\ \text{or} \\ \beta_{k_0} q_{k_0} + \sum_{k < k_0, k \neq k(i)} \beta_k q_k, \quad \text{with } \beta_{k_0} \neq 0. \end{cases}$$

In the first case we use the following upper triangular matrix in degree $k + 1$:

$$\begin{aligned} T(p_j) &= p_j - \sum_{i < j, \partial p_i \neq 0} \alpha_i p_i \\ T(p_\ell) &= p_\ell \quad \text{if } \ell \neq j \end{aligned}$$

and we set $\bar{\partial} = \partial \circ T$. We get $\bar{\partial}(p_j) = 0$ and $\bar{\partial}(p_i) = \partial(p_i)$ for every $i < j$; so we have improved the simpleness of the differential.

In the second case, we use an upper triangular matrix in both degree $k + 1$ and k :

$$\begin{aligned} T(p_j) &= \frac{1}{\beta_{k_0}} \left(p_j - \sum_{i < j, \partial p_i \neq 0} \alpha_i p_i \right), \\ T(p_\ell) &= p_\ell \quad \text{if } \ell \neq j, \\ T(q_{k_0}) &= q_{k_0} + \sum_{k < k_0, k \neq k(i)} \frac{\beta_k}{\beta_{k_0}} q_k \\ T(q_k) &= q_k \quad \text{if } k \neq k_0. \end{aligned}$$

We set $\bar{\partial}_{k+1} = T^{-1} \circ \partial_{k+1} \circ T$ and $\bar{\partial}_k = \partial_k \circ T$. We observe that $\bar{\partial}_k = 0$ on the k -cycles as T keeps this set invariant. We have $\bar{\partial}(p_j) = q_{k_0}$ and $\bar{\partial}(p_i) = \partial(p_i)$ if $i < j$. Thus, $\bar{\partial}$ has the simple form for $1 \leq i \leq j$. Arguing this way recursively, we get a simple complex which is \mathbb{F} -equivalent to the Morse complex. \square

Since the equivalence relation involves upper triangular matrices only, $\partial(p)$ remains the class of $\Delta([p])$ in $H_*(f(p) - \varepsilon; \mathbb{F})$ as it is in the Morse complex. Therefore, when the \mathbb{F} -equivalent complex is simple, the type of each critical point can be easily derived and this complex is the Morse-Barannikov complex. The proof of Theorem 1.4 is completed. \square

2. BIFURCATIONS

Bifurcations occur in a path of functions. It follows from Thom's transversality theorems, as it is explained by J. Cerf in the beginning of [1], that the space \mathcal{F} of real smooth functions on M has as natural stratification whose strata of codimension ≤ 1 are the following:

- 0) The stratum \mathcal{F}_0 , an open dense set in \mathcal{F} , is formed by Morse functions whose critical values are all simple. The next strata are of codimension one.
- 1) The stratum \mathcal{F}_1 is formed by the functions whose critical values are simple and whose critical points are all non-degenerate but one the second derivative of which has a kernel of dimension one.
- 2) The stratum \mathcal{F}_2 is formed by the Morse functions whose critical values are all simple but one which has multiplicity 2.

A generic path has its end points in \mathcal{F}_0 and crosses $\mathcal{F}_1 \cup \mathcal{F}_2$ transversely in a finite number of points.

Since the Morse-Barannikov complex is well defined for functions in \mathcal{F}_0 only, it is necessary to study the bifurcation when crossing \mathcal{F}_1 and \mathcal{F}_2 . With the aim of this, it is convenient to introduce the Cerf diagram, which is defined as follows. Given a generic path $\gamma = (f_t)_{t \in [0,1]}$, its Cerf diagram is the union in $[0, 1] \times \mathbb{R}$ of $\{t\} \times f_t(\text{crit} f_t)$. It is made of finitely many smooth

arcs transverse to the verticals $\{t\} \times \mathbb{R}$, ending at cusp points or in $\{0, 1\} \times \mathbb{R}$, crossing one another transversely.

2.1. BIFURCATION AT BIRTH TIMES. This event is the crossing of the stratum \mathcal{F}_1 , which is co-oriented: the crossing in the positive direction corresponds to the birth of a pair of critical points of index k and $k + 1$ respectively. The crossing in the opposite direction corresponds to the cancellation of a pair of critical points.

The birth is modeled by the following formula:

$$f(x, y) = c + q(y) + x^3 - (t - t_0)x$$

where t_0 is the birth time, c is the critical value of the birth point, $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ are local coordinates at the birth point and q is a non-degenerate quadratic form of index k on \mathbb{R}^{n-1} . In the Cerf diagram, there is a cusp of coordinates (t_0, c) . For $0 < t - t_0$ small, there are two more critical points p_t, q_t of index $k + 1$ and k .

It follows from the model that p_t is of upper type and q_t is of lower type. The other critical points keep their type when crossing the birth time.

2.2. BIFURCATION AT DOUBLE CRITICAL VALUE TIME. This event is the crossing of the stratum \mathcal{F}_2 , say at time t_0 . Since the functions f_t are Morse for t close to t_0 , the critical points can be followed continuously on $(t_0 - \eta, t_0 + \eta)$. Denote by (p_t^1, p_t^2) the pair of critical points whose values cross at time t_0 ; say $f_t(p_t^1) < f_t(p_t^2)$ when $t < t_0$. The question is how the types of critical points are changing when t crosses t_0 .

First, one checks by hand that there is no change if p_t^1 and p_t^2 have distinct indices. Now, we are reduced to the case where both crossing critical values have the same index, say $k + 1$. This still leaves us with nine possibilities at time $t_0 - \eta$; some of them give rise to two distinct results at time $t_0 + \eta$ (in Barannikov's paper several cases are neglected corresponding to the case where one of the considered point is of homological type, since the considered manifold M is the sphere). There is a general rule which may be stated by considering the Cerf diagram. For f_t generic and for each coupled pair of critical points (p_t^i, q_t^i) of f_t , we join their critical values by a vertical segment I_t^i .

2.3. GENERAL RULE. *Let I_{\pm}^i denote the limit of I_t^i when t tends to $t_0 \pm 0$. Then, for each i there is some $j(i)$ such that $I_{-}^i = I_{+}^{j(i)}$.*

This statement can be checked by hand. All the possibilities allowed by this rule may appear. Of course, if the bifurcation of the Morse complex is known, the bifurcation of the associated Morse-Barannikov complex is so determined, by the uniqueness part in Theorem 1.4.

Example 2.4. Assume $f_t(p_t^1) < f_t(p_t^2)$ when $t < t_0$, p_t^1 of upper type and p_t^2 of homological type. Let c be the common critical value at time t_0 . At this time, the local unstable manifolds of the considered points are bounded by k -spheres $S^1(t_0)$ and $S^2(t_0)$ in the sub-level set $c - \varepsilon$. These spheres can be followed at time $t \in (t_0 - \eta, t_0)$ in the sub-level set $f_t(p_t^1) - \varepsilon$ if η is small with respect to ε . Certainly, $S^1(t)$ is not homologous to 0 in the sub-level set $f_t(p_t^1) - \varepsilon$. Concerning $S^2(t)$, there are two cases.

- (1) $S^2(t)$ is homologous to 0 in the sub-level set $f_t(p_t^1) - \varepsilon$.
- (2) $S^2(t)$ is homologous to $S^1(t)$ in the sub-level set $f_t(p_t^1) - \varepsilon$.

In case (1), $S^2(t)$ remains homologous to 0 in the sub-level set $f_t(p_t^2) - \varepsilon$ for $t > t_0$. It is easily seen that each critical point keeps his type.

In case (2), $S^2(t)$ is homologous to 0 only in a sub-level set whose boundary level is higher than $f_t(p_t^1)$. Of course, the two spheres remain homologous at every time close to t_0 and their class is not 0. This implies that, for $t = t_0 + \eta$, p_t^2 , which is the lower critical point, gets the upper type and it is coupled with the point q_t which was previously coupled with p_t^1 . The sphere $S^1(t)$ becomes homologous to 0 in a sub-level set whose boundary is at a level higher than $f_t(p_t^2)$. It is easy to check that p_t^1 becomes of homological type.

3. THE NON-EMPTY BOUNDARY CASE

Following S. Barannikov, we discuss in this section the problem of extending without critical points a germ of function given along the boundary M of a compact $(n+1)$ -dimensional manifold W , $M = \partial W$. This setting was already considered in 1934 by Morse-van Schaack [6] where Morse inequalities have been formulated and proven for manifolds with non-empty boundary. Notice that generically a function $F : W \rightarrow \mathbb{R}$ is a Morse function whose critical points lie in the interior of W and whose restriction to the boundary is Morse.

3.1. THE FRAMING. Here we use the terminology of Barannikov. Given a Morse function $f : M \rightarrow \mathbb{R}$ a framing is the data of one vertical arrow at each critical value of f . A germ $\bar{f} : M \times [0, \varepsilon) \rightarrow \mathbb{R}$ along the boundary without critical points along the boundary determines a framing according the following rule: for the critical p of f the arrow at $f(p)$ points up (resp. down) if $\langle df(p), \vec{n}(p) \rangle$ is positive (resp. negative), where $\vec{n}(p)$ is a tangent vector at p pointing inwards. The framing classifies the germ \bar{f} up to isotopy fixing the boundary.

This yields some information about the non-existence of an extension without critical points $F : W \rightarrow \mathbb{R}$ of the germ \bar{f} . For instance, if $M = S^n$ and the framing points up at the maximum, the maximum principle tells us that any extension must have at least one critical point. Barannikov's discussion will be of course more subtle. The framing may be attached to an \mathbb{F} -equivalence class of Morse complexes or to the Morse-Barannikov complex as well, since the generators are the same.

In what follows, we restrict ourselves to $M = S^n$ and $W = D^{n+1}$. Assume there is an extension without critical points F . Then, there is a one-parameter family of spheres S_t , $t \in [0, 1]$, such that $S_0 = M$, $S_{t'}$ lies inside S_t if $t' > t$ and S_1 is *standard*, that is, $F|_{S_1}$ has two critical points only, the framing pointing down at the maximum and pointing up at the minimum. Set $f_t := F|_{S_t}$; it is a function thought of as defined on M , equipped with a framing. We may assume that $t \mapsto f_t$ is a generic path of functions in the sense of Section 2. Therefore there is a sequence of bifurcations starting from a given germ and ending at the standard germ. At each generic time, we have a class of Morse complex and a framing. Revoir. By considering the sequence of bifurcations of the Morse-Barannikov framed complex, it is possible to derive the non-existence of such a sequence by applying a FINITE combinatorics.

REFERENCES

- [1] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. 39 (1970), 5-173.

- [2] Y. Chekanov, P. Pushkar, *Combinatorics of fronts of Legendrian links and the Arnol'd 4-conjectures*, Russian Math. Surveys 60:1 (2005), 95-149.
- [3] J. Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann & J. Sondow, Princeton Univ. Press, 1965.
- [4] F. Laudenbach, *A Morse complex on manifolds with boundary*, Geometriae Dedicata, Vol. 153 (1) (2011), 47-57.
- [5] D. Le Peutrec, F. Nier, C. Viterbo, *Precise Arrhenius law for p-forms, The Witten Laplacian and Morse-Barannikov complex*, Ann. Inst. Henri Poincaré 14 n°3 (2013), 567-610.
- [6] M.F. Morse, G.B. Van Schaack, *The critical point theory under general boundary conditions*, Annals of Math. 35 n° 3 (1934), 545-571.
- [7] S. Barannikov, *The framed Morse complex and its invariants*, Advances in Soviet Math. 21 (1994), 93-115.

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