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TAME EXPANSIONS OF ω -STABLE THEORIES AND DEFINABLE GROUPS

HAYDAR GÖRAL

ABSTRACT. We study groups definable in tame expansions of ω -stable theories. Assuming several tameness conditions, we obtain structural theorems for groups definable and groups interpretable in these expansions. As our main example, by characterizing independence in the pair (K, G) where K is an algebraically closed field and G is a multiplicative subgroup of K^\times with the Mann property, we show the pair (K, G) satisfies the assumptions. In particular, this provides a characterization of definable and interpretable groups in (K, G) in terms of algebraic groups in K and interpretable groups in G . Furthermore, Morley rank and U-rank are computed in (K, G) and both ranks agree.

1. INTRODUCTION

The model theory of pairs has been studied extensively for some time and attracted considerable attention. In particular, stable theories with a predicate were studied in the paper of E. Casanovas and M. Ziegler [4], which gives the criterion for a pair to be stable. If the base model is strongly minimal, this criterion coincides with the induced structure on the predicate to be stable. Their result in [4] involves that of B. Zilber [22] and also B. Poizat's result on the ω -stability of the theory of pairs of algebraically closed fields [18]. Definable groups in stable theories constitute a recurrent topic in model theory and play a significant role in the classification theory pioneered by B. Zilber in the 1970's. In this article, we also work with definable and interpretable groups in expansions of ω -stable theories by a predicate. We impose several tameness conditions to characterize groups definable and interpretable in the expansions in terms of definable groups in the base model and in the predicate. The proof entails a well-known technique from geometric stability theory, namely the group configuration theorem [9]. We follow the approach of [2], where T. Blossier and A. Martin-Pizarro characterized definable and interpretable groups in pairs of algebraically closed fields. Furthermore, as the main example we prove that the pair (K, G) satisfies these tameness assumptions where K is an algebraically closed field and G is a multiplicative subgroup of K^\times with the Mann property. The results are based on the author's PhD thesis [8].

Now we fix our setting and notations. Let \mathcal{M} be an \mathcal{L} -structure whose theory T is ω -stable. Let

$$\widetilde{\mathcal{M}} = (\mathcal{M}, \mathcal{P})$$

be an expansion of \mathcal{M} by a predicate \mathcal{P} , where \mathcal{P} is an \mathcal{L}' -structure in a sublanguage \mathcal{L}' of \mathcal{L} . Thus $\widetilde{\mathcal{M}}$ is an $\mathcal{L}(U)$ -structure where $\mathcal{L}(U) = \mathcal{L} \cup \{U\}$ and U is a unary relation whose interpretation in \mathcal{M} is \mathcal{P} . Let \widetilde{T} be the complete theory of $\widetilde{\mathcal{M}}$. We always assume that the

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theory \widetilde{T} is stable. We call \mathcal{P} small (see [4]) if there is (N, B) elementarily equivalent to $(\mathcal{M}, \mathcal{P})$ such that for every finite subset b of N every \mathcal{L} -type over Bb is realized in N . To characterize definable and interpretable groups in $\widetilde{\mathcal{M}}$, we will need some of the following assumptions.

Assumptions:

- (1) (EI and Smallness) T has elimination of imaginaries and \mathcal{P} is small.
- (2) (Forking) (i) Let \downarrow^T , $\downarrow^{\widetilde{T}}$ and \downarrow^{pr} be the independence relations in \mathcal{M} , $\widetilde{\mathcal{M}}$ and \mathcal{P} respectively. Let A and B be two algebraically closed sets in $\widetilde{\mathcal{M}}$. Then $A \downarrow_{A \cap B}^{\widetilde{T}} B$ if and only if $A \downarrow_{A \cap B, \mathcal{P}}^T B$ and $\mathcal{P}_A \downarrow_{\mathcal{P}_{A \cap B}}^{pr} \mathcal{P}_B$, where \mathcal{P}_D is defined to be $D \cap \mathcal{P}$ for a subset D of \mathcal{M} .
(ii) Let A be an algebraically closed set in $\widetilde{\mathcal{M}}$. Then $A \downarrow_{\mathcal{P}_A}^{\widetilde{T}} \mathcal{P}$.
- (3) (Algebraic Closure) Let A and B be two algebraically closed sets in $\widetilde{\mathcal{M}}$. Then the independence $A \downarrow_{A \cap B}^{\widetilde{T}} B$ implies that $\text{acl}_{\widetilde{T}}(A, B) = \text{acl}_T(A, B)$ and $\mathcal{P}_{\text{acl}_{\widetilde{T}}(A, B)} = \text{acl}_{\mathcal{P}}(\mathcal{P}_A, \mathcal{P}_B)$ where acl_T , $\text{acl}_{\widetilde{T}}$ and $\text{acl}_{\mathcal{P}}$ are algebraic closures in \mathcal{M} , $\widetilde{\mathcal{M}}$ and \mathcal{P} respectively.
- (4) (Tame Definable Groups) Let H be a connected \widetilde{T} -type-definable subgroup of a definable group in \mathcal{M} , all definable over an algebraically closed set A in $\widetilde{\mathcal{M}}$. Let a be the generic of H over A which lies in some translate of H that is also definable over A . If $\mathcal{P}_{\text{acl}_{\widetilde{T}}(a, A)} = \mathcal{P}_A$, then H is a definable group in \mathcal{M} .
- (5) (Tame Imaginaries) The structure \mathcal{P} is ω -stable and for every $e \in \widetilde{\mathcal{M}}^{\text{eq}}$ imaginary element, there is a finite real tuple $d \in \mathcal{M}$ such that e is algebraic over d , the type of d over e in the expansion, denoted by $\text{tp}^{\widetilde{T}}(d/e)$, is almost \mathcal{P} -internal and $d \downarrow_e^{\widetilde{T}} \mathcal{P}$.

Whenever we assume any of the above, we always work within a sufficiently saturated model of \widetilde{T} , which is also denoted by $\widetilde{\mathcal{M}}$. An important fact is that the assumptions above (1) – (5) are satisfied by pairs of algebraically closed fields, see [1, Remark 7.2 and Proposition 7.3], [2, Lemma 1.2 and Lemma 2.1] and [15, Lemma 2.4 and Lemma 2.5].

Next, we define the *Mann property*. Let K be an algebraically closed field, the field \mathbb{F} its prime field and G be a multiplicative subgroup of K^\times . A fundamental notion about multiplicative groups is the *Mann property*. To define this property, consider an equation

$$(1) \quad a_1 x_1 + \cdots + a_n x_n = 1$$

with $n \geq 2$ and $a_i \in \mathbb{F}$. A solution (g_1, \dots, g_n) of this equation is called *non-degenerate* if for every non-empty subset I of $\{1, 2, \dots, n\}$, the sum $\sum_{i \in I} a_i g_i$ is not zero. We say that G has the *Mann property* if every such equation (1) has only finitely many non-degenerate solutions in G . In [14], H. Mann showed that the set of complex roots of unity μ has the Mann property and his proof is effective. The rank of an abelian group G is the dimension of the \mathbb{Q} -vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$, where G is viewed as a \mathbb{Z} -module. Later on, H. Mann's result was generalized and it was proved that any multiplicative group of finite rank (note that μ has rank 0) in any field of

characteristic zero has the Mann property [6]. To illustrate, every finitely generated multiplicative subgroup of \mathbb{C}^\times has the Mann property, such as $2^{\mathbb{Z}}3^{\mathbb{Z}}$. The result above is not true in the positive characteristic. For instance, the multiplicative group of the algebraic closure of a finite field has rank 0; and yet it does not have the Mann property since the equation $x + y = 1$ has infinitely many non-degenerate solutions. More generally, the multiplicative group of an infinite field does not have the Mann property. However, any cyclic group has the Mann property in all characteristics. The group $\exp(\overline{\mathbb{Q}})$ has the Mann property by Lindemann's theorem and it is of infinite rank. The unit circle in \mathbb{C} also does not have the Mann property. It was B. Zilber who first considered the model theory of the pair (\mathbb{C}, μ) in 1990. In his unpublished note, B. Zilber [22] showed that the pair (\mathbb{C}, μ) is ω -stable, using H. Mann's result [14] as his main tool.

From now on, we fix K and G with the Mann property as above. As before, by the pair (K, G) , we mean the structure $(K, G, +, -, \cdot, 0, 1)$. So our language is $L(U) = \{+, -, \cdot, 0, 1, U\}$ where U is a unary relation whose interpretation in K is G . The model theory of the pair (K, G) was first studied in the paper of L. van den Dries and A. Günaydın [5], and it was proved that G is small in K ; in other words, by changing the model if necessary, we may assume that K is $|G|^+$ -saturated as a field. Moreover, we suppose that the pair (K, G) is κ -saturated and strongly homogeneous for some uncountable cardinal $\kappa \geq \omega_1$. In this paper, we will be working within this sufficiently saturated model. Among other things, in [5] an axiomatization of the theory of (K, G) was obtained by adding the constants to denote the collection of non-degenerate solutions of the equation (1). L. van den Dries and A. Günaydın [5] generalized B. Zilber's result to (K, G) , where K is an algebraically closed field and G has the Mann property. That is to say, the theory of (K, G) , denoted by T_P , is stable and if G is superstable (ω -stable) in the pure group language, then so is the pair (K, G) . In [5], it was also proved that the Mann property is global, which means that we can choose a_i to be in K in (1) and this still yields finitely many non-degenerate solutions in G . Furthermore, L. van den Dries and A. Günaydın [5] showed that every subset of G^n definable in (K, G) is also definable in the abelian group G . In other words, the induced structure on G is just the pure abelian group structure.

In sections 3 and 4, we characterize algebraic closure and forking in the pair (K, G) in terms of algebraic closure and independence in the pure field K . Characterization of the model-theoretic algebraic closure and the independence in (K, G) enable us to show that $\widetilde{\mathcal{M}} = (K, G)$ satisfies assumptions (1) – (3) (see Theorem (4.10)). In section 5, we show that the pair (K, G) also satisfies assumption (4) and we turn our attention to definable groups in $\widetilde{\mathcal{M}}$ up to isogeny (see Definition (5.4) for the definition of isogeny). Thus we have the following theorem (see Theorem (5.8)) which yields the characterization of definable groups in (K, G) and also in pairs of algebraically closed fields [2]:

Theorem A. *Suppose that we have assumptions (1) – (4). Any type-definable group in $\widetilde{\mathcal{M}}$ is isogenous to a subgroup of a group definable in \mathcal{M} . Moreover, any type-definable group H in $\widetilde{\mathcal{M}}$ is, up to isogeny, an extension of a type-interpretable group H_1 in \mathcal{P} by a definable group V in \mathcal{M} :*

$$0 \longrightarrow V(\mathcal{M}) \longrightarrow H \longrightarrow H_1(\mathcal{P}) \longrightarrow 0.$$

Next, in section 6 we study imaginaries in the pair (K, G) via canonical bases as studied by A. Pillay [15] and we give a description in terms of real elements. In the case where G is ω -stable with the algebraic closure of the empty set infinite in G , we can show that the pair $\widetilde{\mathcal{M}} = (K, G)$ satisfies assumptions (1) – (5), see Theorem (6.16). Then we prove the following theorem with a method based on [2] (see Theorem (6.18)):

Theorem B. *Suppose that we have assumptions (1) – (5). Every type-interpretable group H in $\widetilde{\mathcal{M}}$ is, up to isogeny, an extension of an interpretable group H_1 in \mathcal{P} by a \widetilde{T} -type-interpretable group N , which is a quotient of a definable group W in \mathcal{M} by a subgroup N_1 , which is an interpretable group H_2 in \mathcal{P} :*

$$0 \longrightarrow N \longrightarrow H \longrightarrow H_1(\mathcal{P}) \longrightarrow 0$$

with

$$0 \longrightarrow H_2(\mathcal{P}) \longrightarrow W(\mathcal{M}) \longrightarrow N \longrightarrow 0.$$

2. PRELIMINARIES

We assume the reader's familiarity with geometric stability theory, though we give the necessary definitions. Details can be found in [16].

Definition 2.1. A formula $\psi(x)$ is called *almost strongly minimal* if there is a strongly minimal formula $\varphi(x)$ defined over a set of parameters B such that, for every $\mathbb{M} \models T$ containing B , we have

$$\psi(\mathbb{M}) \subseteq \text{acl}(\varphi(\mathbb{M}), B).$$

Now we recall stationarity.

Definition 2.2. Let T be a stable theory and p be a type over A . The type p is said to be *stationary* if there is a unique non-forking extension to every B containing A .

The next definition plays an important role in the classification of geometries.

Definition 2.3. The theory T is *one-based* if for every real set A and every real tuple c the canonical base $\text{Cb}(c/A)$ is algebraic over c .

It is known that pure abelian groups are one-based. However *ACF* (the theory of algebraically closed fields) is not one-based; see [20, Chapter 10, Proposition 10.3.6].

Another important notion concerning stable groups is the notion of generics which is defined next. For the details, we refer the reader to [17] and [16, Chapter 1, Section 6].

Definition 2.4. Let T be a stable theory and M be a model of T . Let H be a type-definable group in M . An element g in H is called *generic* over a given set of parameters A if whenever h is in H with $g \downarrow_A h$, then

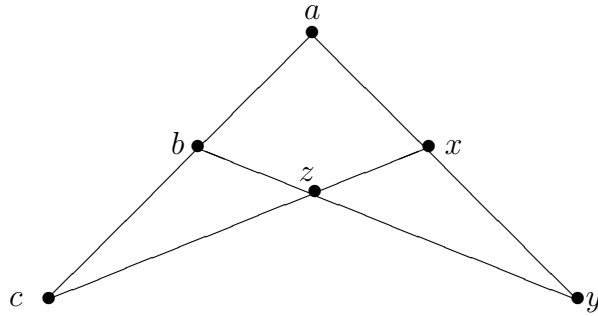
$$h \cdot g \downarrow_A h.$$

The group H is *connected* over A if it has a unique generic over A .

Every non-forking extension of a generic is also generic. Every element of H can be written as a product of two generics. The product of two independent generics over A is also a generic over A and it is also independent from each factor over A .

Next, we give the group configuration theorem from geometric stability theory which is useful to construct a group from some independent elements. Let T be a stable theory, \mathcal{U} be its sufficiently saturated model and $A \subset \mathcal{U}$ with $|A| < |\mathcal{U}|$. By a **-tuple*, we mean a possibly infinite tuple $(a_i)_{i \in I}$ in \mathcal{U}^{eq} where the index set I has cardinality less than the saturation of \mathcal{U} . The type $\text{tp}((a_i)_{i \in I}/A)$ is called an *infinitary type* and it is the union of types of the form $\text{tp}(b/A)$ where b is a finite subset of $\{a_i : i \in I\}$. By a **-definable set*, we mean a collection of **-tuples* (each tuple being indexed by some fixed I), which is the set of realizations of a partial infinitary type. By a **-definable group*, we mean a group H such that both H and the graph of multiplication are **-definable*. A **-definable group* H is *connected*, if it has a unique generic in terms of the independence which is given in (2.4).

Definition 2.5. Let T be a stable theory and \mathcal{U} its sufficiently saturated model. By a *group configuration* over A we mean 6-tuple of points (possibly infinite) (in \mathcal{U}^{eq}) (a, b, c, x, y, z) such that



- any triple of non-collinear points are A -independent,
- $\text{acl}^{\text{eq}}(A, a, b) = \text{acl}^{\text{eq}}(A, a, c) = \text{acl}^{\text{eq}}(A, b, c)$,
- x and y are interalgebraic over Aa , the elements y and z are interalgebraic over Ab , the elements z and x are interalgebraic over Ac ,
- $a \in \text{acl}^{\text{eq}}(x, y, A)$, $b \in \text{acl}^{\text{eq}}(y, z, A)$ and $c \in \text{acl}^{\text{eq}}(x, z, A)$.

Now we give the group configuration theorem, which states that every group configuration arises from a definable group with its independent generic elements. Details can be found in [9, 12].

Theorem 2.6. [9] (*The Group Configuration Theorem*) Let T be a stable theory and \mathcal{U} its sufficiently saturated model. Suppose $M \subset \mathcal{U}$ to be a $|T|^+$ -saturated model of T , and suppose (a, b, c, x, y, z) is a group configuration over M . Then there is a **-definable group* H in \mathcal{U}^{eq} over M , and there are elements a', b', c', x', y', z' of H which form a group configuration, each generic over M , such that the element a is interalgebraic with a' over M and the same holds for the other elements. Moreover, we have $a'x' = y'$, $b'y' = z'$, $c'x' = z'$ and $b'a' = c'$.

Next we give a lemma concerning just pure abelian groups. Let A be an abelian group written additively and $S \subseteq A$ be a set of parameters. An *equation* over S is a formula $\gamma(\bar{x})$ of

the form

$$(2) \quad m_1x_1 + \cdots + m_nx_n = a,$$

where $m_i \in \mathbb{Z}$ and a is in the subgroup generated by S . Note that if a is zero in (2), then this formula defines a subgroup of A^n where the addition is done by coordinatewise. A *positive primitive formula* over S is of the form

$$\exists \bar{y}(\gamma_1 \wedge \dots \wedge \gamma_k)$$

where $\gamma_i(\bar{x}\bar{y})$ are equations over S . Similarly if $\gamma(\bar{x}\bar{y})$ is an equation with $a = 0$, then the formula $\exists \bar{y} \gamma(\bar{x}\bar{y})$ also defines a subgroup of A^m for some $m \geq 1$ as it is the projection of a subgroup of some cartesian power of A . The next lemma is based on the fact that every formula in a pure abelian group is equivalent to Boolean combinations of positive primitive formulas. For details, see [19, Chapter 2, 2.1] and [21, Theorem 4.2.8].

Lemma 2.7. *Let A be an abelian group written additively and B, C be algebraically closed subgroups of A . Then $B + C$ is also algebraically closed in the sense of the pure abelian group A .*

Proof. Let $\varphi(x, b, c)$ be an algebraic formula over $B + C$ where b is a tuple in B with $|b| = s$ and c is a tuple in C with $|c| = t$. Since every formula in an abelian group is equivalent to Boolean combinations of positive primitive formulas and since the formula $\varphi(x, b, c)$ is algebraic, we may assume that $\varphi(x, y, z) = \alpha + D$ where $\alpha \in \text{acl}_A(\emptyset)$ and D is a subgroup of $A \times A^s \times A^t$. As both B and C are algebraically closed, the tuple α is contained in $B \cap C$. Note that $B_1 = \{g : (g, b, 0) \in \alpha + D\}$ is finite and so it lies in B . Similarly $C_1 = \{g : (g, 0, c) \in D\}$ is finite and hence it is a subset of C . As $B_1 + C_1 = \{g : (g, b, c) \in \alpha + D\} \subseteq B + C$, we are done. \square

3. CHARACTERIZATION OF ALGEBRAIC CLOSURE IN (K, G)

In this section, we characterize algebraically closed structures in the pair (K, G) which will be used frequently for all other proofs. Now we fix some notations and give more definitions. For a substructure A in the sense of the pair (K, G) , we denote $G_A = A \cap G$ and call it the G -part of A . By $\text{acl}(A)$ we mean the algebraic closure A^{ac} of A in the field sense, and $\text{acl}_P(A)$ signifies the algebraic closure of A in the pair (K, G) . By \perp we mean the independence in the pure field K , and \perp^P denotes the independence in the pair. If A is a subset of G , the algebraic closure of A in G will be denoted by $\text{acl}_G(A)$. By \perp^G we represent the independence in the pure group G . Let a be a tuple in K and B be a set of parameters. Unless otherwise stated, the type $\text{tp}(a/B)$ denotes the type of a over B in the pure field sense and T indicates the theory of K , in other words ACF_p , where p is the characteristic of K . By $\text{tp}^P(a/B)$, we mean the type of a over B in (K, G) and $\text{tp}_G(\gamma/C)$ indicates a type in the pure group G where γ and C lie in G . For a subset A of K , the notation $A(G)$ denotes the field generated by A and G . In other words it is the smallest subfield of K containing A and G .

Next, we define linear disjointness. Let E, F and L be three fields where $L \subseteq E \cap F$. We say that E is *linearly disjoint* from F over L , denoted by $E \underset{L}{\downarrow}^{ld} F$, if each tuple (a_1, \dots, a_m) in E^m which is linearly independent over L is also linearly independent over F . Now we list some basic properties of linear disjointness which we will use frequently in the rest of the paper. For the details, we refer the reader to [11, Chapter 6, Section 2] and [7, Chapter 2].

- (Symmetry) We have $E \underset{L}{\downarrow}^{ld} F$ if and only if $F \underset{L}{\downarrow}^{ld} E$.
- (Transitivity) For a subfield S of F containing L , we have $E \underset{L}{\downarrow}^{ld} F$ if and only if $ES \underset{S}{\downarrow}^{ld} F$ and $E \underset{L}{\downarrow}^{ld} S$.
- If $E \underset{L}{\downarrow}^{ld} F$ then we have the algebraic independence $E \underset{L}{\downarrow} F$.
- If L is an algebraically closed field, then the independence $E \underset{L}{\downarrow} F$ also implies $E \underset{L}{\downarrow}^{ld} F$.

The next remark will be subsequently used throughout the paper.

Remark 3.1. Let \mathbb{F} be the prime subfield of K . Let Nd denote the collection of all non-degenerate solutions in G of the equation (1) in section 1 for every $n \geq 2$ and a_1, \dots, a_n in the prime subfield \mathbb{F} . Then for every natural number n and elements of the prime field a_1, \dots, a_n , the set

$$\{(g_1, \dots, g_n) : a_1 g_1 + \dots + a_n g_n = 0\} \subseteq G^n$$

is definable over Nd in the pure abelian group structure G . Now let C be a subset of G containing the set Nd . This in turn gives that every group automorphism of G over C extends to a ring automorphism of the ring $\mathbb{F}[G]$, which further extends to a field automorphism of the field $\mathbb{F}(G)$. In particular, since every algebraically closed structure A in the sense of the pair (K, G) contains Nd , every group automorphism of G over G_A extends to a field automorphism of the field $\mathbb{F}(G)$.

For the following lemma see [3] and this lemma will be crucial in order to characterize algebraic closure in the pair (K, G) .

Lemma 3.2. [3, Lemma 2.1] *Let $T_1 \subset T_2$ be stable theories. Suppose that T_1 eliminates imaginaries. Let M be a model of T_2 and a, b be tuples in M . If C is an algebraically closed set in the sense of T_2 , then $a \underset{C}{\downarrow}^{T_2} b$ implies $a \underset{C}{\downarrow}^{T_1} b$.*

In particular, if C is algebraically closed in (K, G) and $a, b \in K$, then the independence $a \underset{C}{\downarrow}^P b$ implies the algebraic independence $a \underset{C}{\downarrow} b$.

In order to characterize algebraic closure in the pair (K, G) , we need the stability of the pair (K, G) which we know by [5, Corollary 6.2], and we apply Lemma (3.2). After characterizing algebraic closure, we will characterize forking in the next section. Thus, our method will be different from [1], though we use a similar technique for the characterization of forking. We begin with a definition.

Definition 3.3. We say that a substructure A is G -independent if $A \underset{\mathbb{F}(G_A)}{\overset{ld}{\perp}} \mathbb{F}(G)$.

Lemma 3.4. *Let A be algebraically closed in the sense of the pair (K, G) . Then A is G -independent.*

Proof. Let a_1, \dots, a_n be in A and $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ be in $\mathbb{F}(G)$ such that

$$a_1 \Sigma_1 + a_2 \Sigma_2 + \dots + a_n \Sigma_n = 0.$$

By multiplying with the denominators, we may assume that $\Sigma_i = k_{i1}g_1 + \dots + k_{im}g_m$ for some natural number $m \geq 1$ where k_{ij} in \mathbb{F} and g_j in G for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus we obtain that $c_1g_1 + \dots + c_mg_m = 0$ where $c_i = a_1k_{i1} + a_2k_{i2} + \dots + a_nk_{in}$ which is in A . Without loss of the generality, we may assume that no proper subsum of $c_1g_1 + \dots + c_mg_m$ is 0. Then we have

$$\frac{-c_2}{c_1} \frac{g_2}{g_1} + \dots + \frac{-c_m}{c_1} \frac{g_m}{g_1} = 1.$$

Note that the tuple $(\frac{g_2}{g_1}, \dots, \frac{g_m}{g_1})$ is a non-degenerate solution of the equation

$$\frac{-c_2}{c_1} x_2 + \dots + \frac{-c_m}{c_1} x_m = 1.$$

As G has the Mann property over K , we know that this equation has only finitely many solutions in G . Since A is algebraically closed, we deduce that the tuple $(\frac{g_2}{g_1}, \dots, \frac{g_m}{g_1})$ is in A and hence in G_A . Hence we conclude that

$$a_1 \frac{\Sigma_1}{g_1} + \dots + a_n \frac{\Sigma_n}{g_1} = 0$$

and $\frac{\Sigma_i}{g_1}$ is in $\mathbb{F}(G_A)$ for $1 \leq i \leq n$. □

Next, we give the characterization of algebraic closure in the pair (K, G) . We apply a similar method as in the paper [5] where they apply a back-and-forth argument. We also benefit from the stability of the pair and Lemma (3.2). In [5, Corollary 3.7], they prove that (A, G_A) is an elementary substructure of the pair (K, G) if and only if A is an algebraically closed field, the group G_A is an elementary substructure of G and A is G -independent. The next proposition is inspired by their result.

Proposition 3.5. *(Algebraic Closure in (K, G)) Let $A \subset K$. Then A is algebraically closed in the sense of the pair (K, G) if and only if A is an algebraically closed field, the group G_A is algebraically closed in G containing the set Nd and A is G -independent.*

Proof. Clearly if A is algebraically closed in the sense of the pair (K, G) , then A is an algebraically closed field and G_A is an algebraically closed subgroup of G containing Nd . Moreover by Lemma (3.4), we know that A is G -independent. Conversely, suppose that A is an algebraically closed field, the group G_A is an algebraically closed subgroup of G containing Nd and A is G -independent. Let α be in K but not in A .

Case 1: Let $\alpha \in G \setminus G_A$. Then since G_A is algebraically closed, we know that α has infinitely many conjugates in G . Choose a conjugate $\beta \in G$ of α . Then, there is an automorphism $f \in \text{Aut}(G/G_A)$ sending α to β . Since G has the Mann property, by remark (3.1) f extends

to a ring automorphism of $\mathbb{F}[G]$, which also extends to a field automorphism of $\mathbb{F}(G)$. Since A is G -independent, by linear disjointness the former automorphism extends to a field automorphism of $A(G)$ over A and this further extends to an automorphism of K over A which is actually an automorphism of the pair (K, G) over A . This yields that α is not in $\text{acl}_P(A)$. In particular, we have $G_{\text{acl}_P(A)} = G_A$.

Case 2: Let $\alpha \in \text{acl}(A, G) \setminus A$. Then there exist $g_1, \dots, g_n \in G$ such that α is in $A(g_1, \dots, g_n)^{\text{ac}} \setminus A$. So there is a rational polynomial $r(x_0, x_1, \dots, x_n)$ with coefficients from A with $r(\alpha, g_1, \dots, g_n) = 0$. We may assume that g_1, \dots, g_n are algebraically independent over A . Therefore by the first case, we know that g_i is not in $\text{acl}_P(A)$ for $1 \leq i \leq n$. Thus the type $p = \text{tp}^P(g_1, \dots, g_n / \text{acl}_P(A))$ is not algebraic. Now take $(h_1, \dots, h_n) \models p$ such that $h_1, \dots, h_n \downarrow_{\text{acl}_P(A)}^P g_1, \dots, g_n$. By Lemma (3.2) we obtain that $h_1, \dots, h_n \downarrow_{\text{acl}_P(A)} g_1, \dots, g_n$. Moreover, since $\text{acl}_P(A)$ is G -independent by Lemma (3.4) and $G_{\text{acl}_P(A)} = G_A$, by transitivity we get that

$$h_1, \dots, h_n \downarrow_A g_1, \dots, g_n.$$

Since there is a pair automorphism over A sending (g_1, \dots, g_n) to (h_1, \dots, h_n) , this gives a conjugate β of α owing to the polynomial equation $r = 0$. Observe that β is different from α thanks to the independence $h_1, \dots, h_n \downarrow_A g_1, \dots, g_n$. Choosing other independent conjugates, as a result we conclude that α has infinitely many conjugates over A and hence α is not in $\text{acl}_P(A)$.

Case 3: The element α is not in $\text{acl}(A, G)$. Note that every field automorphism fixing G is an automorphism of the pair (K, G) . This shows that $\text{acl}(A, G) = \text{acl}_P(A, G)$, and hence α is not in $\text{acl}_P(A)$. Thus we are done. \square

Proposition (3.5) yields the following corollary.

Corollary 3.6. *For any subset D in K ,*

$$\text{acl}_P(D) = \text{acl}(D, G_{\text{acl}_P(D)}).$$

In particular, if H is an algebraically closed substructure of G containing Nd in the sense of the pure group, then $\text{acl}_P(H) = \text{acl}(H)$.

Proof. As $\text{acl}(D, G_{\text{acl}_P(D)}) \subseteq \text{acl}_P(D)$ and $\text{acl}_P(D)$ is G -independent by Lemma (3.4), we conclude by applying Proposition (3.5). The proof of Proposition (3.5) case 1 shows that $G_{\text{acl}_P(H)} = H$. Since $\text{acl}_P(H)$ is G -independent by Lemma (3.4), applying Proposition (3.5) again we deduce that $\text{acl}_P(H) = \text{acl}(H)$. \square

4. FORKING IN (K, G)

In this section, we characterize forking in the pair (K, G) and prove that (K, G) satisfies assumptions (1) – (3). Since we have the following equivalence

$$A \downarrow_C^P B \iff \text{acl}_P(A, C) \downarrow_{\text{acl}_P(C)}^P \text{acl}_P(B, C),$$

we characterize forking for algebraically closed sets in terms of the algebraic independence in K . The next lemma affirms when two algebraically closed structures have the same type over a common algebraically closed substructure.

Lemma 4.1. *Let B_1, B_2 and $C \subseteq B_1 \cap B_2$ be three algebraically closed sets in the sense of the pair (K, G) . Then $\text{tp}^P(B_1/C) = \text{tp}^P(B_2/C)$ if and only if there is a field automorphism over C sending B_1 to B_2 with G_{B_1} to G_{B_2} , and $\text{tp}_G(G_{B_1}/G_C) = \text{tp}_G(G_{B_2}/G_C)$.*

Proof. \Leftarrow : Suppose that there is a field automorphism f sending B_1 to B_2 with G_{B_1} to G_{B_2} over C , and a group automorphism h of G sending G_{B_1} to G_{B_2} over G_C . Since G has the Mann property, by Remark (3.1) the automorphism h extends to a field automorphism of $\mathbb{F}(G)$. As C is G -independent by Lemma (3.4), the automorphism h further extends to an automorphism of $C(G)$ over C . Since both B_1 and B_2 are G -independent, we obtain $B_i \downarrow_{C(G_{B_i})}^{ld} C(G)$ for each $i = 1, 2$. The automorphisms f and h now extend to an isomorphism between $B_1(G)$ and $B_2(G)$ over C which further extends to K . Thus we conclude that $\text{tp}^P(B_1/C) = \text{tp}^P(B_2/C)$. The other direction is clear, as a pair automorphism is a field automorphism fixing the group G setwise. \square

Now we prove a lemma which will be crucial for the characterization of forking and the characterization of definable groups, and it will lead to assumption (3) for $\widetilde{\mathcal{M}} = (K, G)$.

Lemma 4.2. *Let $C = A \cap B$ and all of them be algebraically closed in the sense of the pair (K, G) . If $A \downarrow_{C, G} B, G$, then $\text{acl}_P(A, B) = \text{acl}(A, B)$ and $G_{\text{acl}_P(A, B)} = \text{acl}_G(G_A, G_B) = G_A G_B$.*

Proof. Since A is algebraically closed and it contains C , by Lemma (3.4) we have $A \downarrow_{C(G_A)}^{ld} C(G)$. As linear disjointness implies algebraic independence, we obtain that $A \downarrow_{C, G_A} C, G$. Transitivity of the algebraic independence yields that $A \downarrow_{C, G_A} B, G$ and so $A \downarrow_{B, G_A} G$. As B is algebraically closed, similarly we have that $B \downarrow_{G_B} G$ and therefore $B \downarrow_{G_A G_B} G$. By transitivity of the algebraic independence again, we obtain that

$$A, B \downarrow_{G_A G_B} G$$

which in turn gives

$$\text{acl}(A, B) \downarrow_{\text{acl}(G_A G_B)} \text{acl}(G).$$

As the base set $\text{acl}(G_A G_B)$ is algebraically closed, we also have the linear disjointness

$$(3) \quad \text{acl}(A, B) \downarrow_{\text{acl}(G_A G_B)}^{ld} \text{acl}(G).$$

Note that by Lemma (2.7), the group $G_A G_B$ is algebraically closed in G . By Corollary (3.6), we see that $\text{acl}_P(G_A G_B) = \text{acl}(G_A G_B)$ and the G -part of $\text{acl}(G_A G_B)$ is $G_A G_B$. By Lemma (3.4), the set $\text{acl}(G_A G_B)$ is also G -independent. Now by (3), transitivity and in terms of linear

disjointness, we conclude that

$$\text{acl}(A, B) \underset{\mathbb{F}(G_A G_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

Hence by Proposition (3.5), we deduce that $\text{acl}_P(A, B) = \text{acl}(A, B)$ and $G_{\text{acl}_P(A, B)} = G_A G_B = \text{acl}_G(G_A, G_B)$. \square

The next lemma is somewhat surprising, however it holds in the pair (K, G) contrary to pairs of algebraically closed fields.

Lemma 4.3. *Let $C = A \cap B$ and all of them be algebraically closed in the sense of the pair (K, G) . If $A \underset{C, G}{\downarrow} B, G$ then $A \underset{C}{\downarrow} B$.*

Proof. Since A is algebraically closed in the sense of the pair, by Lemma (3.4) and transitivity we can replace $A \underset{C, G}{\downarrow} B, G$ by $A \underset{C, G_A}{\downarrow} B, G$. As linear disjointness implies algebraic independence,

by transitivity it is enough to show that $B \underset{C}{\downarrow} C(G_A)$. Let $b_1, \dots, b_n \in B$ and $g_1, \dots, g_n \in G_A$ be such that

$$b_1 g_1 + \dots + b_n g_n = 0.$$

We may suppose that no proper subsum of this equation is 0. Thus similar to the proof of Lemma (3.4), we obtain that $\frac{g_i}{g_1} \in B$ for $i = 1, 2, \dots, n$. As they are also in G_A , we deduce that $\frac{g_i}{g_1} \in G_C$. Hence we are done. \square

Remark 4.4. Lemma (4.3) does not hold in pairs of algebraically closed fields. This is one of the points where the pair (K, G) and pairs of algebraically closed fields differ from each other. This will be the main reason why the independence in (K, G) is more plain than the independence in pairs of algebraically closed fields. More precisely, let K and E be two algebraically closed fields such that E is small in K , that is to say E is not K . Let $C = A \cap B$ and all of them be algebraically closed in the pair (K, E) . Note also that all the fields E_A, E_B and E_C are also algebraically closed in the pair (K, E) . Then we do not necessarily have

$$B \underset{C}{\downarrow} C(E_A),$$

since C and E_C are both algebraically closed, by transitivity the previous independence is equivalent to

$$B \underset{E_C}{\downarrow} E_A.$$

As B is algebraically closed, by transitivity the latter independence is equivalent to the independence

$$E_B \underset{E_C}{\downarrow} E_A.$$

Since as a field E is not one-based, the independence $E_B \underset{E_C}{\downarrow} E_A$ does not always hold. A counter example can be found in [20, Chapter 10, Proposition 10.3.6]. This is one of the differences between (K, G) and (K, E) , and therefore we have just one independence in the field sense in order to characterize forking in the next Theorem (4.5). This results from the fact that E^\times

does not have the Mann property as it is infinite and the equation $x + y = 1$ has infinitely many non-degenerate solutions in E^\times . Recall that in (K, E) ,

$$A \underset{C}{\overset{(K,E)}{\downarrow}} B$$

if and only if $A \underset{C,E}{\downarrow} B, E$ and $E_A \underset{E_C}{\downarrow} E_B$. For details, we refer the reader to [1, 18].

Now we are ready to characterize forking in the pair (K, G) using the characterization of algebraic closure (3.5), Lemma (4.2) and Lemma (4.3). The proof of the following result is motivated from [1, 7.3].

Theorem 4.5. (*Forking*) *Let $C = A \cap B$ and all of them be algebraically closed in the sense of the pair (K, G) . Then the following are equivalent:*

- (i) $A \underset{C}{\overset{P}{\downarrow}} B,$
- (ii) $A \underset{C,G}{\downarrow} B, G,$
- (iii) $A \underset{C(G_A)}{\overset{ld}{\downarrow}} B(G).$

Proof. We first show the equivalence of (i) and (ii) and then that of (ii) and (iii). Now suppose that $A \underset{C}{\overset{P}{\downarrow}} B$. Moreover, suppose for a contradiction that $A \not\underset{C,G}{\downarrow} B, G$. Let $q = \text{tp}(B/C \cup G_B)$ and $\lambda \geq \omega_1$. By saturation, there exists a sequence $(B_i)_{i \leq \lambda}$ with $B = B_0$ such that $B_i \models q$ and $(B_i)_{i \leq \lambda}$ is independent over $C \cup G$ in the field sense, and in particular we have $B_i \underset{C, G_B}{\downarrow} C, G$ and $G_B \subseteq G_{B_i}$ for all i . On the other hand, by the independence $B_i \underset{C, G_B}{\downarrow} C, G$ we have that $G_{B_i} \subseteq \text{acl}(C, G_B) \subseteq B$. Thus we obtain the equality $G_B = G_{B_i}$ for all i . Since C is a subset of B , we have that $\text{acl}(C, G_B) \subseteq B$ and G -part of $\text{acl}(C, G_B)$ is G_B . As B is G -independent by Lemma (3.4), we also see that

$$(4) \quad \text{acl}(C, G_B) \underset{\mathbb{F}(G_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

As $B_i \underset{C, G_B}{\downarrow} G$, by (4), transitivity and in terms of linear disjointness, we obtain that

$$B_i \underset{\mathbb{F}(G_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

Therefore by Proposition (3.5), this proves that B_i is algebraically closed in the sense of the pair for all i . Then by Lemma (4.1), we conclude that $\text{tp}^P(B_i/C) = \text{tp}^P(B/C)$. By the Erdős-Rado theorem, we may assume that $(B_i)_{i \leq \lambda}$ is C -indiscernible in the sense of T_P . Let p_i be the pullback of the type $\text{tp}^P(A/B)$ under $B_i \rightarrow B$. Since $A \underset{C}{\overset{P}{\downarrow}} B$, we know that $\bigcup_{i \leq \lambda} p_i(x, B_i)$ is consistent. So there exists A_1 such that $\text{tp}^P(A_1 B_i) = \text{tp}^P(AB)$ for all i . Now, the sequence $(B_i)_{i \leq \lambda}$ is independent over $C \cup G$ and $A_1 \not\underset{C,G}{\downarrow} B_i$ for each B_i . This contradicts the stability of the field K .

Conversely, assume that $A \underset{C,G}{\perp} B, G$. By Lemma (4.3), we know that $A \underset{C}{\perp} B$. Let $(B_i)_i$ be a Morley sequence over C in the sense of the pair (K, G) where $B_0 = B$. Note that the sequence $(B_i, G_{B_i})_i$ is also a Morley sequence over C , but for simplicity we write $(B_i)_i$ instead. The independence $A \underset{C}{\perp} B$ yields that $G_A \underset{C}{\perp} B$. By stationarity over algebraically closed sets (they are models of ACF) and since K is ω -stable, we may assume that $(B_i)_i$ is a Morley sequence over $C \cup G_A$ in the field sense. Since $A \underset{C}{\perp} B$, we also have $A \underset{C, G_A}{\perp} B, G_A$. Let $p(x) = \text{tp}(A/B \cup G_A)$ and $p_i(x)$ be the copy over $B_i \cup G_A$. Then by $A \underset{C, G_A}{\perp} B, G$ and by saturation, there exists an infinite tuple $d \models \bigcup_i p_i(x)$ such that

$$(5) \quad d \underset{C, G_A}{\perp} B_i, G$$

for all i . Observe that $G_d = G_A$ as d and A has the same type over BG_A and by (5). Moreover, there is a field automorphism f_i of K over CG_A sending ABG_B to $dB_iG_{B_i}$. As A is G -independent by Lemma (3.4), in particular we have that

$$(6) \quad \text{acl}(C, G_A) \underset{\mathbb{F}(G_A)}{\overset{ld}{\perp}} \mathbb{F}(G).$$

As we also have that $d \underset{C, G_A}{\perp} G$, by (6), transitivity and in terms of linear disjointness, we

derive that $d \underset{\mathbb{F}(G_A)}{\overset{ld}{\perp}} \mathbb{F}(G)$. By Proposition (3.5) again, we deduce that d is algebraically closed in the sense of the pair (K, G) . By Lemma (4.2), we know that $\text{acl}_P(A, B) = \text{acl}(A, B)$ and $G_{\text{acl}_P(A, B)} = G_A G_B$, and also that

$$(7) \quad \text{acl}(A, B) \underset{\mathbb{F}(G_A G_B)}{\overset{ld}{\perp}} \mathbb{F}(G).$$

By the choice of d and Lemma (4.2) once again, we also have that

$$(8) \quad \text{acl}(d, B_i) \underset{\mathbb{F}(G_A G_{B_i})}{\overset{ld}{\perp}} \mathbb{F}(G).$$

Now let g_i be a field automorphism of $\mathbb{F}(G)$ sending $G_A G_B$ to $G_d G_{B_i}$ which can be deduced from f_i and Remark (3.1). Now by the two previous linear disjointness (7) and (8), we obtain a field automorphism fixing G . Since this is an automorphism of the pair, we get that $\text{tp}^P(dB_iG_A) = \text{tp}^P(ABG_A)$ for all i . Hence we obtain $A \underset{C}{\overset{P}{\perp}} B$.

Now we prove that (ii) and (iii) are equivalent. Clearly (iii) implies (ii) as linear disjointness implies algebraic independence. So suppose (ii). By Lemma (4.2), we know that $\text{acl}_P(A, B) = \text{acl}(A, B)$ and $G_{\text{acl}_P(A, B)} = G_A G_B$, and also $\text{acl}(A, B) \underset{\mathbb{F}(G_A G_B)}{\overset{ld}{\perp}} \mathbb{F}(G)$. Therefore we obtain that

$AB \underset{B(G_A)}{\downarrow}^{ld} B(G)$. Moreover since C is algebraically closed and $A \underset{C}{\downarrow} B$ by (4.3), we see that $A \underset{C}{\downarrow}^{ld} B$ and hence $A \underset{C(G_A)}{\downarrow}^{ld} B(G_A)$. By transitivity we conclude that $A \underset{C(G_A)}{\downarrow}^{ld} B(G)$. \square

Next we give some corollaries of Theorem (4.5).

Corollary 4.6. *Let a and b be finite tuples from K and C be a subset of K . If $a \underset{C}{\downarrow}^P b$ then $a \underset{C,G}{\downarrow} b$. Moreover, if we have*

$$\text{acl}_P(a, C) \cap \text{acl}_P(b, C) = \text{acl}_P(C),$$

then $a \underset{C,G}{\downarrow} b$ also implies $a \underset{C}{\downarrow}^P b$.

Proof. Observe that for any subset D of K we have that

$$\text{acl}(D, G) = \text{acl}(\text{acl}_P(D), G) = \text{acl}_P(D, G).$$

Now we conclude by Theorem (4.5). \square

Remark 4.7. Note that if $a \underset{C}{\downarrow}^P b$, then we do not necessarily have $a \underset{C}{\downarrow} b$ unless $C = \text{acl}_P(C)$.

Corollary 4.8. *For every $a \in K$, we have the independence $a \underset{G_{\text{acl}_P(a)}}{\downarrow}^P G$.*

Proof. Since $\text{acl}_P(a)$ is G -independent by Lemma (3.4), we have the independence

$$\text{acl}_P(a) \underset{\text{acl}(G_{\text{acl}_P(a)})}{\downarrow} \text{acl}(G)$$

and $\text{acl}_P(a) \cap \text{acl}(G) = \text{acl}(G_{\text{acl}_P(a)})$. By (3.6), we see that $\text{acl}(G_{\text{acl}_P(a)}) = \text{acl}_P(G_{\text{acl}_P(a)})$ and $\text{acl}(G) = \text{acl}_P(G)$. Therefore we have

$$\text{acl}_P(a) \underset{\text{acl}_P(G_{\text{acl}_P(a)})}{\downarrow} \text{acl}_P(G).$$

With the help of Theorem (4.5) we finish the corollary. \square

Next, we prove that the independence in the pair implies the independence in G .

Lemma 4.9. *Let $C = A \cap B$ and all of them be algebraically closed in the sense of the pair (K, G) and $A \underset{C}{\downarrow}^P B$. Then we have the independence $G_A \underset{G_C}{\downarrow}^G G_B$ in G .*

Proof. As $A \underset{C}{\downarrow}^P B$, we have $G_A \underset{C}{\downarrow}^P G_B$. Corollary (4.8) and transitivity of the independence yield that $G_A \underset{G_C}{\downarrow}^P G_B$. Hence we conclude that $G_A \underset{G_C}{\downarrow}^G G_B$. \square

Combining Lemma (4.2), Corollary (4.8), Lemma (4.9) and Theorem (4.5), we have proved the following theorem:

Theorem 4.10. *The pair (K, G) satisfies assumptions (1) – (3).*

5. DEFINABLE GROUPS IN $\widetilde{\mathcal{M}}$

In this section, we give the characterization of type-definable groups in the pair $\widetilde{\mathcal{M}}$ under assumptions (1) – (4). We first prove that (K, G) satisfies assumption (4). As a result, it turns out that any definable group in (K, G) , up to isogeny, is an extension of a type-interpretable abelian group in G by an algebraic group over K . In plain words, definable and interpretable groups in K and G give rise to all definable groups in the pair (K, G) . In the proof, we apply the group configuration theorem (2.6). For more on the applications of the group configuration theorem, we refer the reader to [12].

5.1. Generics and Isogeny. The generics of a group are useful to construct a group in stable theories. They will play a significant role in the characterization of definable groups in the pair.

Lemma 5.1. [17, 5.4] *Suppose that H is a stable group. Every formula $\varphi(x, y)$ can be associated with a natural number $n = n(\varphi)$ such that, if A is a generic subset of H defined by a formula $\varphi(x, a)$, then H is covered by n translates of A .*

The next lemma states that when a type-definable group in the pair is actually an algebraic group and thus we show that the pair (K, G) satisfies assumption (4). The motivation for the proof comes from [2, Lemma 2.1].

Lemma 5.2. *Let H be a connected T_P -type-definable subgroup of an algebraic group V , all definable over an algebraically closed set A in the sense of the pair (K, G) . Let a be the generic over A which lies in some translate of H which is also definable over A . If $G_{\text{acl}_P(a, A)} = G_A$, then H is an algebraic group. In particular, H is definable in K .*

Proof. First we may assume that $a \in H$ as follows. Suppose that $a \in bH$. Let a' be such that $\text{tp}^P(a'/A) = \text{tp}^P(a/A)$ and $a' \downarrow_A^P a$. Then we have $a^{-1}a' \downarrow_A^P a$ and $a^{-1}a' \in H$ is the generic over A . As we have $a', A \downarrow_A^P a, A$, therefore by Theorem (4.5) and Lemma (4.2), we see that

$$G_{\text{acl}_P(a^{-1}a', A)} \subseteq G_{\text{acl}_P(a', a, A)} \subseteq \text{acl}_G(G_{\text{acl}_P(a', A)}, G_{\text{acl}_P(a, A)}).$$

Since $G_{\text{acl}_P(a', A)} = G_{\text{acl}_P(a, A)} = G_A$, we obtain that $G_{\text{acl}_P(a^{-1}a', A)} = G_A$. So we may assume that $a \in H$. Put $p = \text{tp}^P(a/A)$ and p_0 its T -reduct. Let H_0 be the smallest algebraic group containing H which exists by the assumption and the ω -stability of K . Note that $H = \text{stab}_P(p) \subseteq \text{stab}_T(p_0)$. So $H_0 \subseteq \text{stab}_T(p_0)$. On the other hand since $p_0(x)$ implies that $x \in H_0$, we get that $\text{stab}_T(p_0) \subseteq H_0$. Thus we have the equality and moreover H_0 is T -connected.

To prove the lemma, it is enough to show that p is the unique generic of H_0 since this implies that H_0 is T_P -connected and $H_0 = \text{stab}_P(p) = H$. Let h be a generic of H_0 over A in the sense of the pair (K, G) and put $q = \text{tp}^P(h/A)$. Observe that $p_0 \subseteq q$ since H_0 is T -connected.

Claim: We have the following independence $h \downarrow_A^P G$.

Proof of the claim: First, note that $a \in H_0$ and since the algebraic closure is G -independent and also by the assumption $G_{\text{acl}_P(a, A)} = G_A$, we obtain that $a, A \downarrow_{G_A}^P G$ and so $a \downarrow_A^P G$. As a result, the element a is a generic over $A \cup G$. Now if $h \not\downarrow_A^P G$, then there exists a T -formula

$\varphi(x, g) \in \text{tp}(h/A \cup G)$ with parameters from A which is not generic in H_0 . Put $n = n(\varphi)$ as in Lemma (5.1) and let

$$\theta(y) = \exists h_1 \dots \exists h_n \in H_0 (\forall x \in H_0 \bigvee_{i \leq n} h_i \varphi(x, y))$$

and $\phi(x, y) = \neg\theta(y) \wedge \varphi(x, y)$. Observe that for any tuple b , the formula $\phi(x, b)$ is not generic in H_0 . However, the formula $\psi(x) = \exists y (U(y) \wedge \phi(x, y))$ with parameters from A is realized by h , and so it is generic in H_0 . Thus a finite number of translates of $\psi(x)$ cover H_0 , say $H_0 = \bigcup_{i \leq k} \alpha_i \psi(x)$. Take a' such that $\text{tp}^P(a'/A) = \text{tp}^P(a/A)$ and $a' \underset{A}{\downarrow}^P \alpha_1, \dots, \alpha_k$. Thus for certain $\alpha \in H_0$, we may suppose that $a \in \alpha \psi(x)$ and $a \underset{A}{\downarrow}^P \alpha$. So $a \in \alpha \phi(x, g')$ for some $g' \in G$. By the characterization of the independence (4.5), we have that $a \underset{A, G}{\downarrow} \alpha$ and by transitivity we get the independence $a \underset{A}{\downarrow} \alpha, G$. This is a contradiction since the formula $\alpha \phi(x, g')$ is not generic in H_0 . Hence we have the claim.

Now as A is G -independent by Lemma (3.4), by transitivity of the independence, we get that $h, A \underset{G_A}{\downarrow} G$. In terms of linear disjointness, this gives that $\text{acl}(h, A) \underset{\mathbb{F}(G_A)^{ac}}{\downarrow}^{ld} \mathbb{F}(G)^{ac}$. By Corollary (3.6), we see that $\text{acl}_P(G_A) = \mathbb{F}(G_A)^{ac}$. Moreover, since the algebraic closure is G -independent, by transitivity we get that

$$\text{acl}(h, A) \underset{\mathbb{F}(G_A)}{\downarrow}^{ld} \mathbb{F}(G).$$

Therefore by Proposition (3.5), we deduce that $\text{acl}_P(h, A) = \text{acl}(h, A)$ and $G_{\text{acl}_P(h, A)} = G_A$. Since there is a field automorphism sending a to h over A , by linear disjointness this in turn gives rise to a field automorphism over $A \cup G$. We conclude that $q = p$, which is determined by p_0 . Hence $H = H_0$ as required. \square

Remark 5.3. The group G does not satisfy the conditions of the previous lemma even though it is connected, as for any a in $G \setminus A$ we do not have the equality $G_{\text{acl}_P(a, A)} = G_A$. This is expected as G is not an algebraic group.

Definition 5.4. (Isogeny) Let D and H be two type-definable groups in a stable theory. We say that D and H are *isogenous* (or there is an *isogeny* between them) if there is a type-definable subgroup S of $D \times H$ such that

- The projection of S to D , denoted by D_S , has a bounded index (the index is less than the saturation cardinal κ) in D ,
- The projection of S to H , denoted by H_S , has a bounded index in H ,
- The kernel $\ker(S) = \{g \in D : (g, 1) \in S\}$ and the co-kernel $\text{coker}(S) = \{h \in H : (1, h) \in S\}$ are finite.

Note that if D and H are isogenous, then there is an isomorphism between $D_S/\ker(S)$ and $H_S/\text{coker}(S)$.

Remark 5.5. The isogeny relation is an equivalence relation. Every group is isogenous to its connected component and every isogeny of the connected component gives rise to an isogeny of the group.

The following lemma and remark are in [2] and they allow us to construct an isogeny between two type-definable groups in a stable theory.

Lemma 5.6. [2, 2.4 and 2.5] *Let G_1 and G_2 be two groups type-definable (type-interpretable) in a stable theory. If there exist parameters $C = \text{acl}^{\text{eq}}(C)$ and elements a_1, b_1 of G_1 and a_2, b_2 of G_2 such that*

- (1) a_1 and a_2 , b_1 and b_2 , a_1b_1 and a_2b_2 are C -interalgebraic
- (2) a_1 , b_1 and a_1b_1 are pairwise independent over C ,

then the element a_1 (respectively a_2) is generic in a unique translate of a connected subgroup H_1 of G_1 (respectively H_2 of G_2), all definable over C and there is an isogeny between H_1 and H_2 given by the stabilizer of the type $\text{tp}(a_1, a_2/C)$. In the condition (1), if we just have that a_2 is algebraic over C , a_1 (respectively for b_2 and a_2b_2), then there is a type-interpretable projection from H_1 to a quotient of H_2 by a finite subgroup.

Remark 5.7. The previous lemma can be generalized to the case when both G_1 and G_2 are $*$ -interpretable. In the previous Lemma (5.6), suppose that G_1 is a connected type-definable group and G_2 is a connected $*$ -interpretable group. If the kernel of the projection N is definable, by stability there is a projection with the same kernel from G_1 to a connected type-interpretable subgroup D of G_2 whose generic is C -interalgebraic with the generic of G_2 . This can be seen by compactness as G_1/N is a type-interpretable group whose generic is a finite tuple and it is C -interalgebraic with the generic of G_2 which is an infinite tuple.

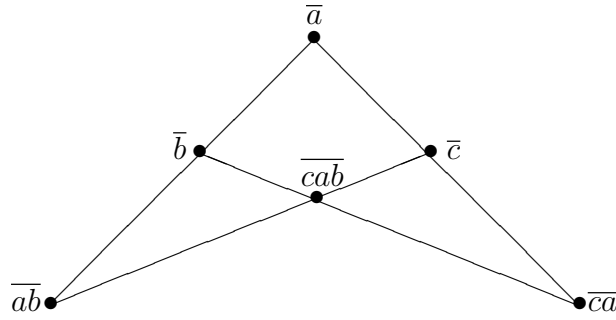
5.2. Characterization of Definable Groups. Now we are ready for the characterization of type-definable groups in the pair $\widetilde{\mathcal{M}}$ under assumptions (1) – (4). In particular, we characterize definable groups in (K, G) by Theorem (4.10) and Lemma (5.2).

The following theorem and its proof are based on [2, Proposition 2.6].

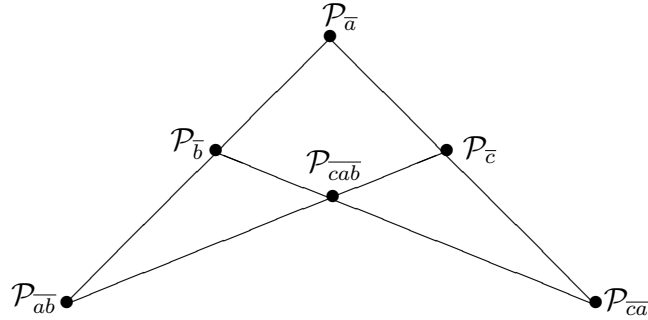
Theorem 5.8. *(Definable Groups in $\widetilde{\mathcal{M}}$) Suppose that we have assumptions (1) – (4). Any type-definable group in $\widetilde{\mathcal{M}}$ is isogenous to a subgroup of a group definable in \mathcal{M} . Moreover any type-definable group H in $\widetilde{\mathcal{M}}$ is, up to isogeny, an extension of a type-interpretable group H_1 in \mathcal{P} by a definable group V in \mathcal{M} :*

$$0 \longrightarrow V(\mathcal{M}) \longrightarrow H \longrightarrow H_1(\mathcal{P}) \longrightarrow 0.$$

Proof. Recall that T is the theory of \mathcal{M} and \widetilde{T} is the theory of $\widetilde{\mathcal{M}}$. By Remark (5.5), it is enough to assume that H is a connected type-definable group in $\widetilde{\mathcal{M}}$ over some parameters. We will work over a model containing these parameters which we will omit. Given two independent generics a and b of H , we write \bar{a} , \bar{b} and \overline{ab} instead of their algebraic closures in the sense of the pair $\widetilde{\mathcal{M}}$ respectively. By assumption (3), the set \overline{ab} is T -algebraic over $\bar{a} \cup \bar{b}$ since \bar{a} , \bar{b} are two independent algebraically closed subsets. With the help of the third generic c which is independent from a, b , we obtain the following diagram:



We know that by assumption (1), the theory T has elimination of imaginaries. Then by Lemma (3.2), we have a T -group configuration for the diagram above. Therefore by the group configuration theorem (2.6) and Lemma (5.6), there exists a $*$ -interpretable group V in T , whose generic is \tilde{T} -interalgebraic with the generic of H . Thus by elimination of imaginaries in \mathcal{M} , by the ω -stability of T and the use of Lemma (5.6) again, we may assume that there exists a definable group in which H embeds up to isogeny. In other words, up to isogeny, we may suppose that H is a subgroup of a definable group in \mathcal{M} . By assumption (3), the set $\mathcal{P}_{\bar{ab}}$ is \mathcal{P} -algebraic over $\mathcal{P}_{\bar{a}} \cup \mathcal{P}_{\bar{b}}$. Moreover by assumption (2), we have the following diagram in \mathcal{P} :



So by the group configuration theorem (2.6) and Lemma (5.6), we obtain a connected $*$ -interpretable group H_1 in \mathcal{P} and a projection π from H to H_1 . Also, the generic h of H_1 is \mathcal{P} -interalgebraic with $\mathcal{P}_{\bar{a}}$. Furthermore, we may assume that the generic h of H_1 is \tilde{T} -interalgebraic with $\mathcal{P}_{\bar{a}}$. Finally, we show that the connected component N of the kernel $\ker(\pi)$ is a definable group in \mathcal{M} by assumption (4). Let n be a generic of N over a in the sense of the pair. So we have $n \downarrow_{\tilde{T}} a$ and $na \downarrow_{\tilde{T}} a$. Observe that $na \in Na$ is a generic also. Since the tuple $(n, 1)$ is in the stabilizer of $\text{tp}^{\tilde{T}}(a, h)$, we have that the tuples (na, h) and (a, h) have the same \tilde{T} -type. In particular, we see that $\mathcal{P}_{\bar{na}} = \mathcal{P}_{\bar{a}}$. Moreover by assumption (3), the set $\mathcal{P}_{\text{acl}_{\tilde{T}}(na, a)}$ is in the \mathcal{P} -algebraic closure of $\mathcal{P}_{\bar{na}}$ and $\mathcal{P}_{\bar{a}}$. Therefore we obtain that $\mathcal{P}_{\text{acl}_{\tilde{T}}(na, a)} = \mathcal{P}_{\bar{a}}$. Hence, the type $\text{tp}^{\tilde{T}}(na/\bar{a})$ satisfies assumption (4) and we conclude that N is a definable group in \mathcal{M} . Now by Remark (5.7), the group H_1 can be taken to be type-interpretable. \square

Remark 5.9. In stable theories, every type-definable group is an intersection of definable groups in this theory.

Recall the result proved by E. Hrushovski and A. Pillay [10]:

Theorem 5.10. [10] *Let A be an interpretable group in a one-based stable theory. Then the connected component of A is abelian.*

As a result of Theorem (5.8), by Remark (5.9) and Theorem (5.10) we have the characterization of definable groups in the pair (K, G) .

Theorem 5.11. (*Definable Groups in (K, G)*) Let K be an algebraically closed field and G be a multiplicative subgroup of K^\times with the Mann property. Any type-definable group in (K, G) is isogenous to a subgroup of an algebraic group. Moreover any type-definable group H is, up to isogeny, an extension of a type-interpretable abelian group H_1 in G by an algebraic group V :

$$0 \longrightarrow V(K) \longrightarrow H \longrightarrow H_1(G) \longrightarrow 0.$$

Remark 5.12. Note that if G is ω -stable in the pure group language, then every type-interpretable group in G is interpretable in G . For example if G is an elementary extension of μ (the set of complex roots of unity), then every definable group in the pair is an extension of an interpretable group in G by an algebraic group.

Example 5.13. (Some definable groups) The additive group of K , the multiplicative group K^\times , any algebraic group over K , the group G and its powers, the cartesian product $K \times G$ and

$$SL(2, G, K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \in G \right\}$$

are all definable in the pair. Observe that they all satisfy the conclusion of Theorem (5.11).

6. IMAGINARIES IN (K, G) AND INTERPRETABLE GROUPS IN $\widetilde{\mathcal{M}}$

In this section, we characterize interpretable groups in $\widetilde{\mathcal{M}}$ under assumptions (1) – (5). We first prove that the pair (K, G) satisfies all assumptions (1) – (5). To achieve this, we need (K, G) to be ω -stable since we will use the existence of the Morley rank in Lemma (6.14). By [5, Corollary 6.2], it is sufficient to suppose that G is ω -stable as a pure abelian group. Moreover by [13], we see that G is divisible by finite. This means $G = DF$ where D is divisible and F is a finite group and $D \cap F = 1$. Furthermore, the group D is the connected component of G , it is definable and strongly minimal. In other words, G is almost strongly minimal (see Definition (2.1)). Moreover, any infinite algebraically closed subset A of G contains F , and it is of the form $D_A F$ where $D_A = A \cap D$ is also divisible. Hence, any infinite algebraically closed subset of G is an elementary substructure.

From now on, we assume that G is ω -stable in the pure group language. As an example, we can take (K, G) to be an elementary extension of the pair (\mathbb{C}, μ) . As noted in [5, 6.4], the pair (K, G) is ω -stable and $\text{MR}(K, G) = \omega$. Thus the pair (K, G) cannot be uncountably categorical since uncountably categorical theories have finite Morley rank. Moreover if K and L are countable algebraically closed fields where L is a proper extension of K , then we have $(K, G) \preceq (L, G)$ by [5, Corollary 3.7]. Thus the pair (K, G) is not categorical for any infinite cardinal.

We will provide a description for imaginaries in (K, G) , which allows us to characterize interpretable groups in (K, G) . Our description of imaginaries will be via canonical bases as in [15]. The pair (K, G) does not eliminate imaginaries which will be explained in Remark (7.9), as one can not eliminate the quotient K^\times/G .

Now we recall the notion of internality from geometric stability theory.

Definition 6.1. Let T be a stable theory. Let a be a tuple and A be set of parameters (possibly containing imaginary elements), and D be a definable subset. We say that the type $\text{tp}(a/A)$ is *almost D -internal* if there exists a set of parameters B such that $a \downarrow_A B$ and $a \in \text{acl}(A, B, D)$.

Note that every extension of an almost D -internal type is almost D -internal and every non-forking restriction is also almost D -internal.

Remark 6.2. For a stable theory T , if $p = \text{tp}(a/A)$ is a type and D is a definable set, then it is known that p is almost D -internal if and only if there is some B such that if a' realizes p then $a' \in \text{acl}(B, D)$. Details can be found in [17, Lemma 2.17].

Lemma 6.3. *Suppose that we have assumptions (1) and (2). Let a be a real tuple from \mathcal{M} . If $\text{tp}^{\tilde{T}}(a/A)$ is almost \mathcal{P} -internal over a real set of parameters A , then $a \in \text{acl}_T(A, \mathcal{P})$.*

Proof. Take B containing A such that $a \downarrow_A^{\tilde{T}} B$ and $a \in \text{acl}_{\tilde{T}}(B, \mathcal{P}) = \text{acl}_T(B, \mathcal{P})$. Assumption (2) yields that $a \downarrow_{A, \mathcal{P}}^T B, \mathcal{P}$ and therefore we obtain that $a \in \text{acl}_T(A, \mathcal{P})$. \square

Remark 6.4. Note that Lemma (6.3) may fail if the parameter set A contains imaginary elements.

The following lemma is based on [2, Remark 3.3] and it will be vital for the characterization of interpretable groups in $\widetilde{\mathcal{M}}$.

Lemma 6.5. *Suppose that we have assumptions (1) – (4). Let H be a type-definable group in $\widetilde{\mathcal{M}}$. If a generic of H is almost \mathcal{P} -internal then H is isogenous to a type-interpretable group in \mathcal{P} .*

Proof. If necessary we may assume that H is \emptyset -definable by adding the parameters where H is defined. Let a be the generic of H . By almost internality and as $a \downarrow_{\mathcal{P}_{\text{acl}_{\tilde{T}}(a)}}^{\tilde{T}} \mathcal{P}$ by assumption (2)(ii), we see that a and $\mathcal{P}_{\text{acl}_{\tilde{T}}(a)}$ are interalgebraic. We conclude the lemma as we did the proof of Theorem (5.8). \square

As a special case and from another point of view, we have the following remark:

Remark 6.6. Let H be a definable group in (K, G) . If a generic of H is almost G -internal then H is isogenous to an interpretable group in G .

Proof. By almost internality, we deduce that H is of finite Morley rank. Since infinite algebraic groups have infinite Morley rank in the pair, by Theorem (5.11) and ω -stability we conclude the lemma. \square

6.1. Canonical Base Analysis. In this subsection, we prove several lemmas for the properties of canonical bases in (K, G) . These lemmas will be analogous to the lemmas in [15] for the pair (K, G) .

Lemma 6.7. *Let B be an elementary substructure of (K, G) . Suppose that*

$$d = \text{Cb}(\text{tp}(a/\text{acl}(B, G))).$$

Then we have $a \underset{d}{\downarrow}^P B, G$.

Proof. By elimination of imaginaries in K , we may assume that d is in K . First observe that

$$\text{acl}_P(B, G) = \text{acl}(B, G)$$

and $a \underset{d}{\downarrow} B, G$. Note also that $\text{acl}(d, G_{\text{acl}_P(d)})$ is algebraically closed in the sense of the pair (K, G) by Corollary (3.6). For the same reason, the set $\text{acl}(G_{\text{acl}_P(d)})$ is also algebraically closed in (K, G) . In particular, they are both G -independent by Lemma (3.4) and their G -parts are $G_{\text{acl}_P(d)}$. The independence $a \underset{d}{\downarrow} B, G$ yields that

$$\text{acl}(a, d, G_{\text{acl}_P(d)}) \underset{d, G_{\text{acl}_P(d)}}{\downarrow} G,$$

and since $\text{acl}(d, G_{\text{acl}_P(d)}) \underset{G_{\text{acl}_P(d)}}{\downarrow} G$ by Lemma (3.4), by transitivity and in terms of linear disjointness we obtain that

$$\text{acl}(a, d, G_{\text{acl}_P(d)}) \underset{\text{acl}(G_{\text{acl}_P(d)})}{\downarrow}^{ld} \text{acl}(G).$$

As $\text{acl}(G_{\text{acl}_P(d)}) = \text{acl}_P(G_{\text{acl}_P(d)})$ is G -independent, by transitivity one sees that

$$\text{acl}(a, d, G_{\text{acl}_P(d)}) \underset{\mathbb{F}(G_{\text{acl}_P(d)})}{\downarrow}^{ld} \mathbb{F}(G).$$

Thus by Proposition (3.5), we deduce that

$$\text{acl}_P(a, d, G_{\text{acl}_P(d)}) = \text{acl}(a, d, G_{\text{acl}_P(d)}) \text{ and } G_{\text{acl}_P(a, d)} = G_{\text{acl}_P(d)}.$$

Now since $a \underset{d}{\downarrow} B, G$, we have that

$$\text{acl}(a, d, G_{\text{acl}_P(d)}) \underset{\text{acl}(d, G_{\text{acl}_P(d)})}{\downarrow} \text{acl}(B, G).$$

By Corollary (3.6) again, we see that $\text{acl}(d, G_{\text{acl}_P(d)}) = \text{acl}_P(d)$ and obtain that

$$\text{acl}_P(a, d) \underset{\text{acl}_P(d)}{\downarrow} \text{acl}_P(B, G).$$

We finish the lemma by applying the characterization of the independence Theorem (4.5). \square

The idea of the next corollary comes from [1, Proposition 7.5].

Corollary 6.8. *Let B be an elementary substructure of (K, G) and let a be a finite tuple from K . Set $d = \text{Cb}(\text{tp}(a/\text{acl}(B, G)))$. Then $\text{Cb}(\text{tp}^P(a/B))$ is interalgebraic in (K, G) with $\text{Cb}(\text{tp}^P(d/B))$.*

Proof. Set $p_B = \text{tp}^P(a/B)$ and $q_B = \text{tp}^P(d/B)$. Let $e_1 = \text{Cb}(p)$ and $e_2 = \text{Cb}(q)$. Note that both e_1 and e_2 are in B^{eq} . By Lemma (6.7), we know that $a \underset{d}{\downarrow}^P B, G$. So $a \underset{d}{\downarrow}^P B, d$ and $a \underset{e_2, d}{\downarrow}^P B, d$. As

$d \underset{e_2}{\downarrow}^P B$, by transitivity we conclude that $a \underset{e_2}{\downarrow}^P B$. This yields that e_1 is algebraic over e_2 . Now we show the converse. Take B_1 such that $\text{tp}^P(B_1/e_1) = \text{tp}^P(B/e_1)$ and $B_1 \underset{e_1}{\downarrow}^P B$. Let p_{B_1} and q_{B_1} be the corresponding types. Choose an element $a_1 \models p_B \cup p_{B_1}$ such that $a_1 \underset{e_1}{\downarrow}^P B, B_1$. Put $d_1 = \text{Cb}(a_1/\text{acl}(B, B_1, G))$, the element $d_2 = \text{Cb}(a_1/\text{acl}(B, G))$ and $d_3 = \text{Cb}(a_1/\text{acl}(B_1, G))$. The independence $a_1 \underset{e_1}{\downarrow}^P B, B_1$ yields that $a_1 \underset{B}{\downarrow}^P B_1$ and by the characterization of the independence Theorem (4.5), we obtain that

$$(9) \quad a_1 \underset{B, G}{\downarrow} B_1.$$

Similarly we have

$$(10) \quad a_1 \underset{B_1, G}{\downarrow} B.$$

If the tuple a_1 is algebraically independent over $\text{acl}(B, B_1, G)$, then from (9) and (10) we see that $d_1 = d_2 = d_3 = \text{acl}(\emptyset)$. Now suppose that a_1 satisfies a polynomial in $\text{acl}(B, B_1, G)$. By (9) and (10), we see that the tuple a_1 satisfies a polynomial in $\text{acl}(B, G)$ and in $\text{acl}(B_1, G)$. This gives again that $d_1 = d_2 = d_3$. As a result, in all cases we conclude that $d_1 = d_2 = d_3$. By the choice of the element a_1 , we see that $d_1 \models q_B \cup q_{B_1}$. Furthermore, the element d_1 is in $\text{acl}_P(a_1, B) \cap \text{acl}_P(a_1, B_1)$ as it is the canonical base of the types $\text{tp}(a_1/\text{acl}(B, G))$ and $\text{tp}(a_1/\text{acl}(B_1, G))$. Now, from $a_1 \underset{B_1}{\downarrow}^P B$ and $d_1 \in \text{acl}_P(a_1, B_1)$, we obtain that $d_1 \underset{B_1}{\downarrow}^P B$. By the independence $B_1 \underset{e_1}{\downarrow}^P B$ and transitivity, we get that $d_1 \underset{e_1}{\downarrow}^P B$. Hence e_2 is algebraic over e_1 . \square

The next lemma states that, up to interalgebraicity, an imaginary element is a canonical base of a type over itself and this type is almost G -internal.

Lemma 6.9. *Let $e \in (K, G)^{\text{eq}}$ be an imaginary element. Then there is $e' \in (K, G)^{\text{eq}}$ interalgebraic with e , such that for some finite tuple d' from K we have $e' = \text{Cb}(\text{tp}^P(d'/e'))$ and $\text{tp}^P(d'/e')$ is almost G -internal.*

Proof. Let a be a tuple in K such that $e = f(a)$ for some \emptyset -definable function in $(K, G)^{\text{eq}}$. Set $e_1 = \text{Cb}(\text{tp}^P(a/\text{acl}_P^{\text{eq}}(e)))$. Observe that e_1 is algebraic over e . As $e = f(a)$ and $a \underset{e_1}{\downarrow}^P e$, we obtain that $e \underset{e_1}{\downarrow}^P e$ and hence e and e_1 are interalgebraic. Now let (B, G_B) be an elementary substructure of (K, G) such that $e_1 \in (B, G_B)^{\text{eq}}$ and $a \underset{e_1}{\downarrow}^P B$. Let $d' = \text{Cb}(\text{tp}(a/\text{acl}(B, G)))$. We may assume d' to be a finite tuple in K by the ω -stability and elimination of imaginaries in K . Let $e' = \text{Cb}(\text{tp}^P(d'/B))$. Then by Corollary (6.8) and $a \underset{e_1}{\downarrow}^P B$, we see that e_1 and e' are interalgebraic. As a result, we deduce that e and e' are interalgebraic. Note that the type

$\text{tp}^P(d'/B)$ is almost G -internal. Thus as we have $d' \underset{e'}{\downarrow}^P B$, the type $\text{tp}^P(d'/e')$ is also almost G -internal and its canonical base is e' . Thus we have the lemma. \square

In the following parts of this section, we assume that G is ω -stable with $\text{acl}_G(\emptyset)$ infinite in the pure group language. The next lemma is related to [15, Lemma 1.8].

Lemma 6.10. *(Coheir) Suppose that G is ω -stable with $\text{acl}_G(\emptyset)$ infinite in the pure group language. Let $e \in (K, G)^{\text{eq}}$ and $B = \text{acl}_P(e) \cap G$. Let c be a tuple from G . Then $\text{tp}^P(c/B, e)$ is finitely satisfiable in B .*

Proof. First, note that $B = G_{\text{acl}_P(e)}$ is an elementary substructure of G . By Proposition (3.5) and Corollary (3.6), we see that $\text{acl}(B) = \text{acl}(G_{\text{acl}_P(e)})$ is algebraically closed in the pair, it is G -independent and

$$(11) \quad \text{acl}(B) \cap G = G_{\text{acl}_P(e)} = B.$$

Now by [5, Corollary 3.7], by stability and as G has the Mann property, we deduce that $\text{acl}(B)$ is an elementary substructure of K in the sense of the pair (K, G) . By Corollary (4.8), we get that

$$(12) \quad \text{acl}_P(e) \underset{B}{\downarrow}^P G.$$

Now by the equation (12) and [20, Corollary 8.3.7], we have that the type $\text{tp}^P(c/\text{acl}(B), e)$ is finitely satisfiable in $\text{acl}(B)$. On the other hand, if c is in G and $\text{acl}(B)$, then it is in B by (11). Hence we conclude that any $c \in \text{acl}(B)$ which satisfies a finite part of $\text{tp}^P(c/B, e)$ must be in B and we are done. \square

Now, we define the U-rank, orthogonality and regular types.

Definition 6.11. Let T be a stable theory. For a type p , we define its U-rank $U(p)$ as follows:

- 1- The U-rank of p is always ≥ 0 ,
- 2- The U-rank $U(p) \geq \alpha + 1$ if and only if there is a forking extension q of p such that $U(q) \geq \alpha$,
- 3- For limit ordinal δ , the U-rank $U(p) \geq \delta$ if and only if $U(p) \geq \alpha$ for all $\alpha < \delta$.

We say that $U(q) = \alpha$ when the $U(q) \geq \alpha$ but not $U(q) \geq \alpha + 1$. If $U(q) \geq \alpha$ for all ordinals α , we say the U-rank is unbounded.

If $p = \text{tp}(a/A)$, then $U(p)$ is denoted by $U(a/A)$. If T is an ω -stable theory then we always have that $U(p) \leq \text{MR}(p)$. Now we define the symmetric sum \oplus for ordinals. If α is an ordinal, we can write α as a finite sum $\sum_{i=1}^n \omega^{\alpha_i} m_i$, where $\alpha_1 > \dots > \alpha_n$ and $m_i \in \mathbb{N}$. If $\alpha = \sum_{i=1}^n \omega^{\alpha_i} m_i$

and $\beta = \sum_{i=1}^n \omega^{\alpha_i} n_i$, then $\alpha \oplus \beta$ is defined to be

$$\sum_{i=1}^n \omega^{\alpha_i} (m_i + n_i).$$

Note that $\alpha + \beta \leq \alpha \oplus \beta$. For example, the sum $1 + \omega = \omega$ while $1 \oplus \omega = \omega + 1 > \omega$. One of the nice properties of the U-rank is Lascar's inequalities [16, Chapter 1, 3.26]:

$$U(a/bC) + U(b/C) \leq U(ab/C) \leq U(a/bC) \oplus U(b/C).$$

Next we define orthogonality and regular types.

Definition 6.12. 1- If $p, q \in S(A)$, then p and q are said to be *almost orthogonal* if whenever a and b realize p and q respectively, then a and b are independent over A .

2- Two stationary types p and q are said to be *orthogonal* if all their non-forking extensions to common domains are almost orthogonal.

3- A stationary type p is called *regular* if it is non-algebraic and orthogonal to all its forking extensions.

The following theorem connects Morley rank and U-rank, and also ensures the definability of Morley rank.

Theorem 6.13. [21, Chapter 4, 4.7.10] *Let T be an ω -stable theory. Suppose that there exist strongly minimal formulas ϕ_1, \dots, ϕ_k such that any type is non-orthogonal to a certain ϕ_i . Then T has finite Morley rank which coincides with U-rank and Morley rank is definable, that is to say for every formula $\theta(x, y)$ and every natural number $n < \omega$, the set $\{a : \text{MR}(\theta(x, a)) = n\}$ is definable.*

The following lemma and its proof are based on [15, 2.4 and 2.5].

Lemma 6.14. *Suppose that G is ω -stable with $\text{acl}_G(\emptyset)$ infinite in the pure group language. Let $e \in (K, G)^{\text{eq}}$ be an imaginary element. There is a tuple d from K , an $L(U)$ -definable function $f(x)$ over \emptyset , an $L(U)$ -formula $\psi(y)$ over e and an $L(U)$ -definable function $h(y, z)$ over e such that*

- (i) $f(d) = e$,
- (ii) $\psi(y) \in \text{tp}(d/e)$,
- (iii) $(\forall y, y')(\psi(y) \wedge \psi(y') \implies \exists z(U(z) \wedge h(y, z) = y'))$
- (iv) *Furthermore, the element d is independent from G over e in (K, G) .*

Proof. For (i), (ii) and (iii) we refer the reader to [15, 2.4]. Now we prove (iv). Choose d such that $\text{MR}(\text{tp}^P(d/e))$ is minimized and also (i)-(iii) are satisfied with respect to a fixed choice of (f, ψ, h) . Moreover by Lemma (6.9), we can assume that $q = \text{tp}^P(d/e)$ is almost G -internal. Thus Remark (6.2) yields that there is some set u such that if $d' \models q$ then $d' \in \text{acl}(u, G)$. We will show that d is independent from G over e . Suppose not and choose $b \in G$ such that d forks with b over e . Note that by almost internality and as G has Morley rank 1, we deduce that $\text{MR}(\text{tp}^P(d/e))$ is finite. Let $m = \text{MR}(\text{tp}^P(d/e, b)) < \text{MR}(\text{tp}^P(d/e))$. Note that Morley rank is definable in some formula in the type q by Theorem (6.13), as G is almost strongly minimal and the type q is almost G -internal. Let $\chi(y, z)$ be a formula over e such that $\chi(d, b)$ holds, and for any c we have $\text{MR}(\chi(y, c)) = m$ if it is consistent. Let $\Delta(z)$ be the formula

$$\exists y(f(y) = e \wedge \psi(y)) \wedge (\forall y, y')(\psi(y) \wedge \psi(y') \implies (h(y, z) = y' \wedge U(z))).$$

Therefore $\Delta(b)$ holds. Let $B = \text{acl}_P(e) \cap G$. By Lemma (6.10), there is $b_1 \in B$ such that $\Delta(b_1)$ holds. Then we find d_1 satisfying (i), (ii) and (iii) of the lemma with $\chi(d_1, b_1)$ holds. As b_1 is algebraic over e , we have that $\text{MR}(\text{tp}^P(d_1/e)) \leq m$, contradicting the choice of d . \square

Therefore combining Lemmas (6.9) and (6.14), we have the following theorem describing imaginary elements in the pair (K, G) :

Theorem 6.15. *Let K be an algebraically closed field and G be an ω -stable multiplicative subgroup of K^\times with $\text{acl}_G(\emptyset)$ infinite and with the Mann property. Let $e \in (K, G)^{\text{eq}}$ be an imaginary element. There is a finite real tuple d such that e is algebraic over d , the type $\text{tp}^P(d/e)$ is almost G -internal and d is independent from G over e in the sense of the pair (K, G) .*

By the previous Theorem (6.15), Theorem (4.10) and Lemma (5.2), we finally deduce that:

Theorem 6.16. *Let K be an algebraically closed field and G be an ω -stable multiplicative subgroup of K^\times with $\text{acl}_G(\emptyset)$ infinite and with the Mann property. Then the pair $\widetilde{\mathcal{M}} = (K, G)$ satisfies assumptions (1) – (5).*

6.2. Characterization of Interpretable Groups. The following lemma is from [2]:

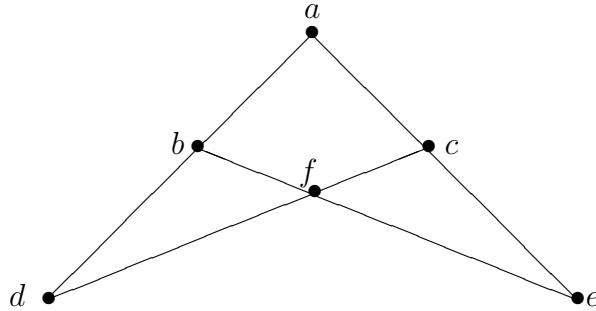
Lemma 6.17. [2, Lemma 3.1] *Let H be a connected type-interpretable group in a stable theory. Let α, β and γ be three independent generics of H and a_0 be a real element such that α is algebraic over a_0 . Then there exist real tuples a, b, c, d, e and f such that*

$$(a, \alpha) \equiv (b, \beta) \equiv (c, \gamma) \equiv (d, \alpha\beta) \equiv (e, \gamma\alpha) \equiv (f, \gamma\alpha\beta) \equiv (a_0, \alpha)$$

and

$$a \perp_{\alpha} b, c, d, e, f$$

and the same for the other tuples. Moreover in the following diagram:



all non-linear triples are independent and each point is independent from each line which does not contain it.

Recall that T is the theory of \mathcal{M} and \widetilde{T} is the theory of $\widetilde{\mathcal{M}}$. Now we will characterize type-interpretable groups in the pair $\widetilde{\mathcal{M}}$. The following theorem and its proof are based on [2, Theorem 3.5].

Theorem 6.18. *(Type-Interpretable groups in $\widetilde{\mathcal{M}}$) Suppose that we have assumptions (1) – (5). Every type-interpretable group H in $\widetilde{\mathcal{M}}$ is, up to isogeny, an extension of an interpretable group H_1 in \mathcal{P} by a \widetilde{T} -type-interpretable group N , which is a quotient of a definable group W in \mathcal{M} by a subgroup N_1 which is an interpretable group H_2 in \mathcal{P} :*

$$0 \longrightarrow N \longrightarrow H \longrightarrow H_1(\mathcal{P}) \longrightarrow 0$$

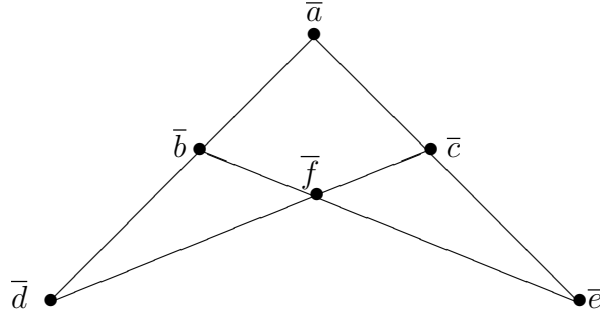
with

$$0 \longrightarrow H_2(\mathcal{P}) \longrightarrow W(\mathcal{M}) \longrightarrow N \longrightarrow 0.$$

Proof. Let H be a type-interpretable group in $\widetilde{\mathcal{M}}$. By Remark (5.5), we may assume that H is connected. Again we work over a small model that we omit. Let α, β and γ be three independent generics of H in the sense of the pair $\widetilde{\mathcal{M}}$. By assumption (5), the generic α is algebraic over a real tuple a_0 which is independent from \mathcal{P} over α in the sense of the pair $\widetilde{\mathcal{M}}$, and the type $\text{tp}^{\widetilde{T}}(a_0/\alpha)$ is almost \mathcal{P} -internal. Then by Lemma (6.17), there are real tuples a, b, c, d, e and f such that

$$(a, \alpha) \equiv^{\widetilde{T}} (b, \beta) \equiv^{\widetilde{T}} (c, \gamma) \equiv^{\widetilde{T}} (d, \alpha\beta) \equiv^{\widetilde{T}} (e, \gamma\alpha) \equiv^{\widetilde{T}} (f, \gamma\alpha\beta) \equiv^{\widetilde{T}} (a_0, \alpha)$$

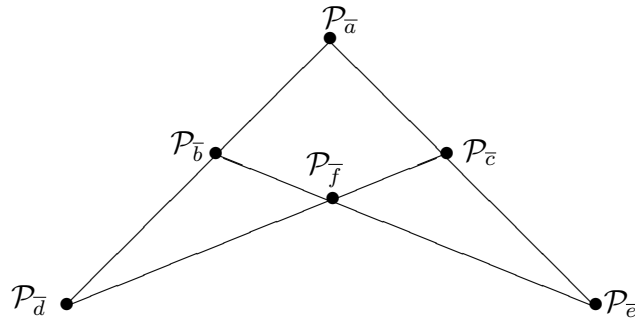
and if we put $\bar{a} = \text{acl}_{\widetilde{T}}(a)$ and the same for the others, we have the following diagram:



such that all non-linear triples are \widetilde{T} -independent and each point is \widetilde{T} -independent from the lines which do not contain it. Since $a \downarrow_{\alpha}^{\widetilde{T}} \mathcal{P}$, we see that $\mathcal{P}_{\bar{a}} \subseteq \text{acl}_{\widetilde{T}}(\alpha) \subseteq \bar{a}$. Therefore, we obtain that $\mathcal{P}_{\bar{a}} = \text{acl}_{\widetilde{T}}(\alpha) \cap \mathcal{P}$. Moreover by assumption (3), we have that $\mathcal{P}_{\text{acl}_{\widetilde{T}}(\bar{a}, \bar{b})} = \text{acl}_{\mathcal{P}}(\mathcal{P}_{\bar{a}}, \mathcal{P}_{\bar{b}})$. Since

$$\mathcal{P}_{\bar{d}} = \text{acl}_{\widetilde{T}}(\alpha\beta) \cap \mathcal{P} \subseteq \mathcal{P}_{\text{acl}_{\widetilde{T}}(\bar{a}, \bar{b})},$$

we get that $\mathcal{P}_{\bar{d}} \subset \text{acl}_{\mathcal{P}}(\mathcal{P}_{\bar{a}}, \mathcal{P}_{\bar{b}})$. This is true for all other tuples and by assumption (2), the set $\mathcal{P}_{\bar{a}}$ is \mathcal{P} -independent from $\text{acl}_{\mathcal{P}}(\mathcal{P}_{\bar{b}}, \mathcal{P}_{\bar{e}})$. The same holds for the others. So we have the following diagram:



and by the group configuration theorem (2.6), we have a connected $*$ -interpretable group $H_1(\mathcal{P})$ in \mathcal{P} whose generic h is \mathcal{P} -interalgebraic with $\mathcal{P}_{\bar{a}}$. By Lemma (5.6) and ω -stability of \mathcal{P} , we have a type-definable surjection

$$\pi : H \rightarrow H_1(\mathcal{P})$$

and up to isogeny we may assume that H_1 is interpretable and its generic h is \tilde{T} -interalgebraic with $\mathcal{P}_{\bar{a}}$.

Next we show that the points $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ give a T -group configuration with the help of the parameter set \mathcal{P} . We know that any three non-collinear points among them are independent in the sense of the pair $\tilde{\mathcal{M}}$, and hence they are T -independent over \mathcal{P} by assumption (2). As β is algebraic over b and $\alpha\beta$ is algebraic over d , we have that α is algebraic over b, d . Moreover by assumption (3), we know that $\text{acl}_T(\bar{b}, \bar{d}) = \text{acl}_{\tilde{T}}(\bar{b}, \bar{d})$. Since the type $\text{tp}^{\tilde{T}}(a/\alpha)$ is almost \mathcal{P} -internal and α is algebraic over b, d , we observe that the type $\text{tp}^{\tilde{T}}(a/\text{acl}_T(\bar{b}, \bar{d}))$ is also almost \mathcal{P} -internal. Thus by Lemma (6.3) we obtain that $a \in \text{acl}_T(\mathcal{P}, \bar{b}, \bar{d})$. The same holds for the other tuples.

Therefore we obtain a connected $*$ -interpretable group V in \mathcal{M} and two independent generics a_1, b_1 of V such that a_1 is T -interalgebraic with \bar{a} over \mathcal{P} , the element b_1 is T -interalgebraic with \bar{b} and a_1b_1 is T -interalgebraic with \bar{d} . Since the tuples α, β and γ are algebraic over the finite tuples a_1, b_1 and a_1b_1 respectively and as V is a $*$ -interpretable group, by ω -stability and elimination of imaginaries in T , there exist a connected definable group W in \mathcal{M} and two independent generics a_2, b_2 such that α is algebraic over a_2 and the same for the others. Note that a_2 is T -algebraic over \mathcal{P}, \bar{a} and the same holds for the others. Moreover, since $\bar{a}, \bar{b}, \bar{d}$ are pairwise \tilde{T} -independent over \mathcal{P} , so are a_2, b_2 and a_2b_2 . As α is algebraic over a and $a \underset{\mathcal{P}_{\bar{a}}}{\downarrow}^{\tilde{T}} \mathcal{P}$ by assumption (2)(ii), we see that $\alpha \underset{\mathcal{P}_{\bar{a}}}{\downarrow}^{\tilde{T}} \mathcal{P}$. Now let N be the connected component of $\ker(\pi)$. Then α is generic in $N\alpha$ over $\text{acl}_{\tilde{T}}(h) = \text{acl}_{\tilde{T}}(\mathcal{P}_{\bar{a}})$, therefore α is also generic over the set \mathcal{P} .

Now we apply Lemma (5.6) to the tuples (a_2, α) and (b_2, β) . This yields a type-definable surjection ϕ from W to N , up to isogeny. Lastly, we prove that the connected component N_1 of $\ker(\phi)$ is isogenous to an interpretable group in \mathcal{P} . Let n_1 be a generic of N_1 over \mathcal{P}, a_2 . Then the point $(n_1, 1_N)$ is in the stabilizer of the type $\text{tp}^{\tilde{T}}(a_2, \alpha/\text{acl}^{\text{eq}}(\mathcal{P}))$ and so $\text{tp}^{\tilde{T}}(n_1a_2/\alpha) = \text{tp}^{\tilde{T}}(a_2/\alpha)$. Since $\text{tp}^{\tilde{T}}(a_2/\alpha)$ is almost \mathcal{P} -internal and as a_2 is algebraic over \mathcal{P}, a then the type $\text{tp}^{\tilde{T}}(n_1a_2/\alpha)$ is also almost \mathcal{P} -internal. As α is algebraic over \mathcal{P}, a_2 , the type $\text{tp}^{\tilde{T}}(n_1/\mathcal{P}, a_2)$ is almost \mathcal{P} -internal. Owing to the independence $n_1 \underset{\mathcal{P}}{\downarrow}^{\tilde{T}} a_2$, we conclude that $\text{tp}^{\tilde{T}}(n_1/\mathcal{P})$ is also almost \mathcal{P} -internal. Then by Lemma (6.5) and ω -stability, we have that N_1 is isogenous to an interpretable group in \mathcal{P} . \square

As a result of the previous theorem, Theorem (6.16), Theorem (5.10) and ω -stability, we have the characterization of interpretable groups in (K, G) .

Theorem 6.19. (*Interpretable groups in (K, G))* Let K be an algebraically closed field and G be an ω -stable multiplicative subgroup of K^\times with $\text{acl}_G(\emptyset)$ infinite and with the Mann property. Every interpretable group H in (K, G) is, up to isogeny, an extension of an interpretable abelian group H_1 in G by a $T_{\mathcal{P}}$ -interpretable group N , which is a quotient of an algebraic group W by a subgroup N_1 which is an interpretable abelian group H_2 in G :

$$0 \longrightarrow N \longrightarrow H \longrightarrow H_1(G) \longrightarrow 0$$

with

$$0 \longrightarrow H_2(G) \longrightarrow W(K) \longrightarrow N \longrightarrow 0.$$

7. MORLEY RANK AND U-RANK

In this section, we assume that G is ω -stable as a group. We compute Morley Rank of the types in the pair (K, G) in terms of Morley Rank in K and G . We show that Morley rank and U-rank coincide. This will be a generalization of the result $\text{MR}(K, G) = \omega$ from [5, Theorem 6.4]. We also prove that types over algebraically closed sets are stationary.

Note that the type $p \in S(A)$ is regular if and only if the independence over A is a pregeometry on the set of realizations of p ; see [16, Chapter 1, Lemma 4.5.1] and [16, Chapter 7, Remark 1.1].

Proposition 7.1. *Let a be in K . Let $p = \text{tp}^P(a/A)$ be a non-algebraic type such that $a \notin A(G)^{ac}$ and $A = \text{acl}_P(A)$ be a parameter set. Then p is stationary and it is a regular type with the pregeometry $\text{cl}(C) = \text{acl}(C, G)$ on the set of realizations of p . Furthermore, we have*

$$\text{U}(p) = \text{MR}(p) = \omega.$$

Proof. Let A' be a parameter set containing A . Let a_1 and a_2 be such that $\text{tp}^P(a_1/A) = \text{tp}^P(a_2/A) = \text{tp}^P(a/A)$ and $a_i \downarrow_A^P A'$ for $i = 1, 2$. By Theorem (4.5), we see that a_1 and a_2 are not in $A'(G)^{ac}$. Since there is only one transcendental type over $A'(G)^{ac}$ in the pure field K and since every field automorphism fixing G is an automorphism of the pair (K, G) , we conclude that the type p is stationary. Let R be the set of realizations of the type p in K . It is sufficient to show that (R, cl) is a pregeometry where

$$\text{cl}(B) = \{b \in R : b \not\downarrow_A^P B\}$$

for $B \subseteq R$. First observe that if $b \in R$ then we also have that $b \notin A(G)^{ac}$. Moreover as $b \downarrow_A^P G$ and A is G -independent, by transitivity we obtain that $b, A \downarrow_{G_A}^P G$. By Corollary (3.6) and in terms of linear disjointness we see that

$$\text{acl}(b, A) \downarrow_{\mathbb{F}(G_A)}^{ld} \mathbb{F}(G).$$

Therefore by Proposition (3.5), we deduce that $\text{acl}_P(b, A) = \text{acl}(b, A)$ and $G_{\text{acl}_P(b, A)} = G_A$. Now by applying Corollary (4.6), we conclude that $b \not\downarrow_A^P B$ if and only if $b \not\downarrow_{A, G}^P B$ if and only if $b \in B(G)^{ac} = \text{acl}(B, G)$. So we have a pregeometry since (K, acl) is a pregeometry. Hence the type p is regular.

Let C be a set containing A . Since $\text{acl}_P(a, A) = \text{acl}(a, A)$ and $G_{\text{acl}_P(a, A)} = G_A$ as shown before, by Theorem (4.5) we see that $a \underset{A}{\not\perp}^P C$ if and only if $a \underset{A, G}{\not\perp} C$ iff $a \in \text{acl}(C, G)$. Now take d_1, \dots, d_n which are algebraically independent over $A(G)$. Let g_1, \dots, g_n be independent generics of G over $D = Ad_1 \dots d_n$. Put $a' = d_1 g_1 + \dots + d_n g_n$. Note that no proper subsum of $d_1 g_1 + \dots + d_n g_n$ is 0. Moreover, the elements a and a' have the same type over A and a' forks with D over A in (K, G) by Theorem (4.5). Since G has the Mann property over K , we see that $g_1, \dots, g_n \in \text{acl}_P(a, D)$. This proves that $U(a/D) \geq U(g_1, \dots, g_n/D) = n$. Thus we obtain that $U(p) \geq \omega$. Finally, since the theory (K, G) has Morley rank ω , we conclude that $U(p) \leq \text{MR}(p) \leq \omega$ and hence

$$U(p) = \text{MR}(p) = \omega.$$

□

Remark 7.2. In [5, Lemma 6.5, Lemma 6.6 and the proof of Theorem 6.4 on page 27], it was shown that if $a \in A(G)^{ac}$ then $\text{MR}(a/A)$ is finite, and if b is not in $A(G)^{ac}$ then $\text{MR}(b/A) \leq \omega$ since there is only one such type.

Remark 7.3. Note that if $a \in A(G)^{ac} \setminus A$, then the type $q = \text{tp}^P(a/A)$ need not be regular. To see this, let g_1 and g_2 be two independent generics of G over A and put $a = g_1 + g_2$. Since G has the Mann property, we see that $g_1, g_2 \in \text{acl}_P(a)$. However, each g_i forks with a over A since they are not in A . Hence it cannot be regular.

Next we compute the rank of an element from K .

Lemma 7.4. *If $a \in A(G)^{ac}$ where $A = \text{acl}_P(A)$, then*

$$U(a/A) = \text{MR}(a/A) = \text{MR}_G(G_{\text{acl}_P(a, A)}/G_A).$$

Proof. As G is almost strongly minimal and a is in $A(G)^{ac}$, by Theorem (6.13) we see that $U(a/A) = \text{MR}(a/A)$. Now we prove the other equality. Since $a \in A(G)^{ac}$, there are g_1, \dots, g_n from G such that $a \in A(g_1, \dots, g_n)^{ac}$. We may assume that g_1, \dots, g_n are algebraically independent over A and n is minimal. So without loss of generality, we may suppose that $a = d_1 h_1 + \dots + d_m h_m$ where h_i is in G and d_i is in the field generated by a and A , the element a is algebraic over d_1, \dots, d_m, A and no proper subsum of $d_1 h_1 + \dots + d_m h_m$ is 0. Since G has the Mann property, this yields that $h_1, \dots, h_n \in \text{acl}_P(a, A)$. Thus $\text{MR}(a/A) = \text{MR}(h_1, \dots, h_n/A) = \text{MR}(G_{\text{acl}_P(a, A)}/A)$. Now by Corollary (4.8) and as the induced structure on G is the pure abelian group structure, we conclude that

$$\text{MR}(G_{\text{acl}_P(a, A)}/A) = \text{MR}_G(G_{\text{acl}_P(a, A)}/G_A).$$

□

Combining Proposition (7.1) and Lemma (7.4) we have the following formula for the ranks in the pair:

Corollary 7.5. *Let a be finite tuple from K and $A = \text{acl}_P(A)$ be a set of parameters. Then*

$$U(a/A) = \text{MR}(a/A) = \omega \cdot \text{tr.deg}(a/A(G)) + \text{MR}_G(G_{\text{acl}_P(a, A)}/G_A).$$

Proof. We have already proved it if a is an element in K by (7.1) and (7.4). Now let a be a finite tuple from K . In this case, we split the tuple into two parts a_1 and b_1 where a_1 is algebraically independent over $A(G)$ and $b_1 \in \text{acl}(a_1, A, G)$. Then we can replace b_1 by a tuple g in G as we did in (7.4). Thus we obtain that

$$U(a/A) = U(a_1b_1/A) = U(a_1g/A)$$

and similarly for the Morley rank. Note also that a_1 is independent from g over A in the sense of the pair (K, G) . Since also a_1 is independent over A and the U-rank is additive, we obtain the formula for the finite tuples for the U-rank. For the Morley rank, we proceed by induction and use a similar argument as in Remark (7.2) to obtain that $\text{MR}(a_1g/A) \leq \omega \cdot \text{tr.deg}(a_1/A(G)) + \text{MR}(g/A)$. Since we know the formula for the U-rank and U-rank is always smaller than the Morley rank, this yields the formula for finite tuples. \square

The following result characterizes definable subfields of K .

Proposition 7.6. *If F is an infinite definable subfield of K in the sense of the pair (K, G) , then $F = K$.*

Proof. Note that F is an algebraically closed field since it is ω -stable. So its additive group and multiplicative group are connected. First, we show that the Morley rank of F is infinite. If the Morley rank of F is finite, then by Proposition (7.1), the generic of the additive group of F and the multiplicative group of F are G -internal. By Remark (6.6), they are isogeneous to a group interpretable in G . However, since G is an abelian group, it cannot interpret an infinite field. This indicates that $\text{MR}(F)$ is infinite. Thus we obtain that $\text{MR}(F) = \omega$ as $\text{MR}(K) = \omega$. So the extension K/F cannot be infinite. Since F is algebraically closed, we conclude that $F = K$. \square

7.1. Stationarity. In this last subsection, we prove that types over algebraically closed sets are stationary if G has WEI. Recall that a theory T has *weak elimination of imaginaries*, WEI for short, if for every imaginary element e there is a real tuple c such that $e \in \text{dcl}^{\text{eq}}(c)$ and $c \in \text{acl}(e)$. A theory T has *geometric elimination of imaginaries*, GEI for short, if every imaginary element is interalgebraic with a real element.

We need two basic lemmas.

Lemma 7.7. (*Shelah lemma*) *Let T be a stable theory. If the type $\text{tp}(A/C)$ is stationary and $B \downarrow_C A$, given two C -elementary maps $f : A \rightarrow A_1$ and $g : B \rightarrow B_1$ such that $A_1 \downarrow_C B_1$, then $f \cup g$ is also C -elementary.*

Proof. Since g is C -elementary, we have that $g(A) \downarrow_C B_1$ and $g(A) \equiv_C A \equiv_C A_1$. By stationarity there is an automorphism h fixing B_1 and sending $g(A)$ to A_1 . Now one can see that $h \circ g$ restricted to $A \cup B$ is $f \cup g$. \square

The next lemma states that under WEI, types over algebraically closed sets are stationary in stable theories.

Lemma 7.8. *Suppose that T is stable and $M \models T$ has weak elimination of imaginaries (WEI). Then every type over an algebraically closed set is stationary.*

Proof. Let A be an algebraically closed set in M . Since T has WEI, we have that $\text{acl}^{\text{eq}}(A) = \text{dcl}^{\text{eq}}(A)$. Since in a stable theory types over $\text{acl}^{\text{eq}}(A)$ are stationary, we are done. \square

The next remark does not require that the group G has the Mann property but smallness.

Remark 7.9. The pair (K, G) does not have GEI.

Proof. To see that (K, G) does not have GEI, we consider the quotient K^\times/G . Now let a be an element of K which is not in $\text{acl}(G)$. Put $e = aG$ which is an imaginary element. Suppose there is a real element c such that $e \in \text{acl}_P^{\text{eq}}(c)$ and $c \in \text{acl}_P(e)$. If we have $c \in \text{acl}(G)$, then as a is not in $\text{acl}(G)$, via some automorphism we can send a to any element b which is also not in $\text{acl}(G)$. Since there are infinitely many elements which are not in $\text{acl}(G)$ as G is small in K , this contradicts the assumption $e \in \text{acl}_P^{\text{eq}}(c)$. This yields that c is not in $\text{acl}(G)$. Since $e \in \text{dcl}_P(a)$, we see that $c \in \text{acl}_P(e) \subseteq \text{acl}_P(a) \subseteq \text{acl}(a, G)$. Thus there is a polynomial f such that $f(a, c, g) = 0$ where g is a tuple from G . So there are polynomials f_0, \dots, f_k such that a satisfies the polynomial $f_k(c, g)X^k + \dots + f_1(c, g)X + f_0(c, g)$ and $f_k(c, g)$ is not 0. Since $c \in \text{acl}_P(e)$, there are finitely many conjugates of c under e , say $c = c_1, \dots, c_m$. Note also that any automorphism fixing G and sending a to ah fixes e , where $h \in G$. Therefore, every such automorphism gives finitely many polynomials with coefficients from g and c_1, \dots, c_m , but infinitely many elements satisfying these polynomials, a contradiction. \square

With the help of Lemmas (7.7), (7.8) and assuming that G has WEI, we are able to prove that the types in (K, G) over algebraically closed sets are stationary although (K, G) does not have WEI.

Corollary 7.10. *Suppose that G has WEI. Let A be algebraically closed in the sense of the pair (K, G) and $a \in K$. Then the type $\text{tp}^P(a/A)$ is stationary.*

Proof. Let B be a set containing A . We may suppose that B is algebraically closed in the sense of the pair. Let a_1 and a_2 be such that $\text{tp}^P(a_1/A) = \text{tp}^P(a_2/A)$, and we have the following two independence $a_1 \underset{A}{\downarrow}^P B$ and $a_2 \underset{A}{\downarrow}^P B$. Note that $\text{acl}_P(a_1, A)$ and $\text{acl}_P(a_2, A)$ have the same type over A in the sense of the pair (K, G) . By the characterization of the independence Theorem (4.5), we see that $a_i, A \underset{A(G)}{\downarrow}^{ld} B(G)$ for $i = 1, 2$. By Lemma (4.9), we also have $G_{\text{acl}_P(a_i, A)} \underset{G_A}{\downarrow}^G G_B$ for $i = 1, 2$. Since $\text{tp}_G(G_{\text{acl}_P(a_1, A)}/G_A) = \text{tp}_G(G_{\text{acl}_P(a_2, A)}/G_A)$ and G has WEI, we obtain that

$$\text{tp}_G(G_{\text{acl}_P(a_1, A)}/G_B) = \text{tp}_G(G_{\text{acl}_P(a_2, A)}/G_B)$$

with the help of Lemma (7.8). Let f be an automorphism of G over G_B sending $G_{\text{acl}_P(a_1, A)}$ to $G_{\text{acl}_P(a_2, A)}$. As G has the Mann property, by Remark (3.1) the map f extends to a field automorphism of $\mathbb{F}(G)$ over $\mathbb{F}(G_B)$. Moreover since B is G -independent by Lemma (3.4), the map f further extends to an automorphism of $B(G)$ over B . Now since $a_i, A \underset{A(G)}{\downarrow}^{ld} B(G)$ for $i = 1, 2$ and $\text{tp}^P(a_1/A) = \text{tp}^P(a_2/A)$, we have an isomorphism from $B(a_1, G)$ to $B(a_2, G)$

sending a_1 to a_2 over B . Since this also extends to an automorphism of K , we conclude that $\text{tp}^P(a_1/B) = \text{tp}^P(a_2/B)$. \square

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