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## Auxiliary Variable-based Balancing (AVB) for source term treatment in open channel simulations

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bed slope source term (DBF) [37], the quasi-steady wave propagation method [29], asymptotic balancing [12] or the source term projection technique in discontinuous Galerkin techniques [27]. Examples of the latter are the well-balanced approach [1, 2, 8, 24, 31, 35], flux and source term splitting [11], characteristics-based approximate-state and augmented Riemann solvers [10, 16, 18, 20, 30], the homogeneous approach [28] and other static equilibrium-preserving techniques [6, 19, 40].

Various solutions have also been proposed to enforce the  $C$ -property in finite volume-based discretizations. One of the earliest solutions, proposed in [33] for the solution of the SWE and later extended in [40], consists in replacing the water depth with the free surface elevation. This option can be extended to the open channel equations in arbitrary-shaped channels, as shown in the present paper. It has the drawback that simple flow configurations such as uniform flow over a constant slope cannot be computed accurately (see section 3.2). Another option is to approximate the variations in the cross-sectional area with a consistent estimate taken from the balance between the specific force and the source term in the momentum equation [6]. The estimate is defined in such a way that it is zero under steady state conditions. Very similar formulae to that of [6] have been obtained using completely different approaches in [28, 30]. The approaches [6, 28, 30] have the common point that the gradient in one of the flow variables is replaced with the gradient in another variable, called auxiliary variable hereafter. This gradient is zero under static conditions. That different approaches yield the same formulae lead to wonder whether a general methodology can be derived to define auxiliary variables.

In the present paper, the principle of Auxiliary Variable-based Balancing (AVB) is presented for one-dimensional free surface flow calculations in non-prismatic, trapezoidal channels. This is motivated by the fact that in industrial open channel packages, the cross-sectional geometry is broken into a set of trapezoidal elements. The AVB approach is used to derive flux formulae that allow non-static, steady state flow conditions to be preserved, even at low orders of discretization, that is, when first-order schemes are used.

The principle of the AVB method is presented for the water hammer and one-dimensional SWE in [25]. However, the one-dimensional shallow water equations are a very simplified description of free surface flows in natural channels. Besides, only one possible approach for source term discretization (a variant of source term upwinding) is considered in [25]. The applicability of the approach to more complex cross-sections and other source term discretization approaches is not investigated in [25]. The objectives of the present paper are (i) to present the methodology of Auxiliary Variable-based Balancing (AVB), (ii) to apply the AVB approach to the open channel flow equations in a well-balanced, finite volume framework, (iii) to provide the flux and source term discretizations for a variety of Riemann solvers, and (iv) to analyse the accuracy of the numerical solutions obtained using a number of various AVB-based discretizations. As mentioned above, first-order space discretizations are retained for the sake of computational rapidity.

The structure of the paper is as follows. Section 2 presents the governing equations and their discretisation. The AVB methodology is detailed in section 3 and its application to classical Riemann solvers presented in section 4. Section 5 provides computational examples, including steady-state configurations and transient test-cases as well as a convergence analysis for the classical dam-break problem (for which an analytical solution is available).

## 2. Governing equations and solution method

### 2.1. Governing equations

The purpose is to solve  $2 \times 2$  hyperbolic systems of conservation laws in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \quad (1)$$

where  $\mathbf{U}$ ,  $\mathbf{F}$  and  $\mathbf{S}$  are defined as

$$\mathbf{U} = \begin{bmatrix} A \\ Q \end{bmatrix}, \mathbf{F} = \begin{bmatrix} Q \\ M \end{bmatrix} = \begin{bmatrix} Q \\ \frac{Q^2}{A} + \frac{P}{\rho} \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 0 \\ (S_0 - S_f)gA + \frac{R}{\rho} \end{bmatrix} \quad (2)$$

where  $A$  is the cross-sectional area,  $g$  is the gravitational acceleration,  $M$  is the specific force,  $P$  is the pressure force exerted on the wetted cross-sectional area,  $S_0$  and  $S_f$  are respectively

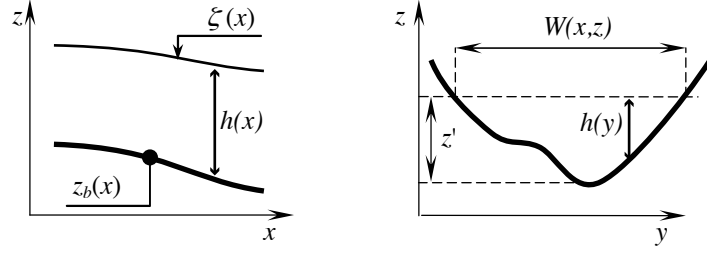


Figure 1: **Channel geometry.** Left) longitudinal view: bottom and water elevation. Right) transversal view: channel width and depth.

98 the bottom and energy slope,  $R$  is the  $x$ -component of the reaction of the walls onto the  
 99 water (if the channel is non-prismatic) and  $\rho$  is the water density.

100 The forces  $P$  and  $R$  are derived from the assumption of a hydrostatic pressure distribution  
 101 and obey the following definitions [14]:

$$\frac{P}{\rho} = \int_A (\zeta - z) g dA = \int_0^h (h - z') g W(z') dz' \quad (3)$$

$$\frac{R}{\rho} = \int_0^h (h - z') g \left( \frac{\partial W}{\partial x} \right)_{h-z'=\text{Const}} (z') dz' \quad (4)$$

102 where  $W(z)$  is the width of the channel at the elevation  $z$ ,  $h$  is the water depth (that is the  
 103 distance between the lowest point in the cross-section and the free surface),  $z' = z - z_b$  is  
 104 the elevation above the bottom lowest point and  $\zeta$  is the free surface elevation (Figure 1).

105 The energy slope is classically assumed to obey a turbulent-type friction law such as  
 106 Manning's law:

$$S_f = n_M^2 u^2 R_H^{-4/3} \quad (5)$$

107 where  $n_M$  is Manning's friction coefficient,  $u = Q/A$  is the flow velocity and  $R_H$  is the  
 108 hydraulic radius, defined as the ratio of the cross-sectional area  $A$  to the wetted perimeter  
 109  $\chi$ , yielding

$$S_f = n_M^2 Q^2 A^{-10/3} \chi^{4/3} \quad (6)$$

110 It is noted that the Jacobian matrix  $\mathbf{A}$  of  $\mathbf{F}$  with respect to  $\mathbf{U}$  is given by

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \quad (7)$$

111 where the speed  $c$  of the waves in still water is defined as

$$c^2 \equiv \frac{\partial(\frac{P}{\rho})}{\partial A} = \frac{gA}{b} \quad (8)$$

112 where  $b = W(\zeta)$  is the top width of the channel. The matrix  $\mathbf{A}$  can be diagonalized into a  
 113 matrix  $\mathbf{\Lambda}$  defined as:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{bmatrix} \quad (9a)$$

$$\lambda^{(1)} = u - c \quad (9b)$$

$$\lambda^{(2)} = u + c \quad (9c)$$

114 The problem is assumed to be properly posed hereafter, that is, the initial and boundary  
 115 conditions are specified such that Eq. (1) can be solved uniquely for  $\mathbf{U}$  at all points of a  
 116 computational domain  $[0, L]$  for all times  $t > 0$ .

117 *2.2. Finite volume discretization*

118 Eq. (1) is discretized using a finite volume formalism as

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x_i} \left( \mathbf{F}_{i-\frac{1}{2}}^{n+\frac{1}{2}} - \mathbf{F}_{i+\frac{1}{2}}^{n+\frac{1}{2}} \right) + \Delta t \mathbf{S}_i^{n+\frac{1}{2}} \quad (10)$$

119 where the subscript  $i$  denotes a cell average, subscripts  $i \pm \frac{1}{2}$  denote estimates at the interfaces  
 120 between the computational cells, the superscript  $n$  indicates that the variable is estimated  
 121 a time level  $n$ , and the superscript  $n + \frac{1}{2}$  denotes an average value between time levels  $n$   
 122 and  $n + 1$ , and where  $\Delta x_i$  is the width of the computational cell  $i$ . In explicit schemes, the  
 123 variables with superscripts  $n + \frac{1}{2}$  are computed using the known values at the time level  $n$ ;  
 124 in implicit schemes, the unknown values at the time level  $n + 1$  are used.

125 In what follows, non-prismatic, trapezoidal cross-sections are considered. The reason for  
 126 this is that in all commercial open channel packages, the channel geometry is discretized into  
 127 a series of trapezia. Consequently, the capability to deal with trapezoidal cross-sections is  
 128 seen as an indispensable prerequisite to a generalisation of the method to arbitrary-shaped  
 129 channels. Note that rectangular and triangular cross-sections are obtained as particular  
 130 cases of the proposed approach, as illustrated by a number of computational examples in  
 131 Section 5.

132 The geometric parameters of the cross-sections are defined at the interfaces between  
 133 the computational cells. They are interpolated linearly within the cells. Consequently, the  
 134 geometry is continuous at the cell interfaces. Assuming non-prismatic trapezoidal channel  
 135 geometry, the width  $W(x, z)$  at a given abscissa  $x$  and elevation  $z$  takes the form

$$W(x, z) = W_0(x) + (z - z_b(x)) W_1(x) \quad (11)$$

136 where  $W_0(x)$  is the bottom width of the channel at the abscissa  $x$ ,  $W_1(x)$  is the derivative  
 137 of  $W$  with respect to  $z$  and  $z_b$  is the bed elevation at the abscissa  $x$ . As mentioned above,  
 138  $W_0$ ,  $W_1$  and  $z_b$  are assumed to vary linearly with  $x$  within the cells. In the cell  $i$ , one has:

$$W_0(x) = W_{0,i-\frac{1}{2}} + (x - x_{i-\frac{1}{2}}) W_{0,i}^{(x)} \quad (12)$$

$$W_1(x) = W_{1,i-\frac{1}{2}} + (x - x_{i-\frac{1}{2}}) W_{1,i}^{(x)} \quad (13)$$

$$z_b(x) = z_{b,i-\frac{1}{2}} + (x - x_{i-\frac{1}{2}}) z_{b,i}^{(x)} \quad (14)$$

139 where the superscript  $(x)$  denotes the derivative with respect to  $x$ :

$$W_{0,i}^{(x)} = \frac{W_{0,i+\frac{1}{2}} - W_{0,i-\frac{1}{2}}}{\Delta x_i} \quad (15)$$

$$W_{1,i}^{(x)} = \frac{W_{1,i+\frac{1}{2}} - W_{1,i-\frac{1}{2}}}{\Delta x_i} \quad (16)$$

$$z_{b,i}^{(x)} = \frac{z_{b,i+\frac{1}{2}} - z_{b,i-\frac{1}{2}}}{\Delta x_i} = S_{0,i} \quad (17)$$

140 The cross-sectional area  $A$  is given as the integral of  $W$  between the bottom level and  
 141 the free surface elevation:

$$A(x) = \int_{z_b}^{\zeta} W(x, z) dz = \left[ W_0(x) + W_1(x) \frac{h(x)}{2} \right] h(x) \quad (18)$$

142 where  $h(x)$  is the water depth at the abscissa  $x$ . Assuming that the free surface is horizontal  
 143 in the cell  $i$  (which is true in the case of the first-order Godunov scheme), the average cell  
 144 value  $A$  is given by

$$A_i(\zeta) \equiv \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} A(x) dx = \alpha_i h_{i-\frac{1}{2}}^2 + \beta_i h_{i-\frac{1}{2}} + \gamma_i \quad (19a)$$

$$h_{i-\frac{1}{2}} = (\zeta - z_b)_{i-\frac{1}{2}} \quad (19b)$$

$$\alpha_i = \frac{1}{2} W_{1,i-\frac{1}{2}} + \frac{\Delta x_i}{4} W_{1,i}^{(x)} \quad (19c)$$

145

146

$$\beta_i = W_{0,i-\frac{1}{2}} + \frac{\Delta x_i}{2} \left( W_{0,i}^{(x)} + S_{0,i} W_{1,i-\frac{1}{2}} \right) + \frac{\Delta x_i^2}{3} S_{0,i} W_{1,i}^{(x)} \quad (19d)$$

147

$$\gamma_i = \frac{\Delta x_i}{2} S_{0,i} W_{0,i-\frac{1}{2}} + \frac{\Delta x_i^2}{6} \left( 2S_{0,i} W_{0,i}^{(x)} + S_{0,i}^2 W_{1,i-\frac{1}{2}} \right) + \frac{\Delta x_i^3}{8} S_{0,i}^2 W_{1,i}^{(x)} \quad (19e)$$

148 *2.3. Flux calculation*

149 Approximate Riemann solvers provide flux formulae that can be recast in the following  
150 form, derived from [17]:

$$\mathbf{F} = a\mathbf{F}_L + (1-a)\mathbf{F}_R + \mathbf{D}(\mathbf{U}_L - \mathbf{U}_R) \quad (20)$$

151 where  $L$  and  $R$  denote respectively the left and right states of the Riemann problem,  $a$  is a  
152 coefficient between 0 and 1 and  $\mathbf{D}$  is a diffusion matrix that contributes to stabilise the nu-  
153 merical solution. The left and right states are obtained from an appropriate reconstruction,  
154 the simplest possible option (the first-order Godunov scheme [23]) being to use the average  
155 cell values. For  $a = 1/2$ , Eq. (20) is the sum of a centred flux and a so-called artificial vis-  
156 cosity term. Both  $a$  and  $\mathbf{D}$  are functions of the wave speeds, in other words, the eigenvalues  
157 of the Jacobian matrix  $\mathbf{A}$  of  $\mathbf{F}$  with respect to  $\mathbf{U}$  (Eq. 9).

158 How the left and right states for the Riemann problem are to be computed from the  
159 average cell values is dealt with in Section 3.

160 *2.4. Source term discretization*

161 The momentum source term is discretized explicitly.

162 The friction source term is computed by applying explicit estimates to the terms in  
163 Eq. (6):

$$(S_f)_i^n = n_M^2 \left( u^2 R_H^{4/3} \right)_i^n = \left( \frac{W_0 + \frac{1}{2}W_1 h}{W_0 + h(W_1^2 + 4)^{1/2}} \right)_i^n \quad (21)$$

$$(S_f)_i^n = n_M^2 \left( Q^2 A^{2/3} \chi^{-4/3} \right)_i^n = n_m^2 \left( Q^2 \left( W_0 + \frac{1}{2}W_1 h \right)^{2/3} \left( W_0 + h\sqrt{W_1^2 + 4} \right)^{-4/3} \right)_i^n \quad (22)$$

164 The geometric source term  $gAS_0 + \frac{R}{\rho}$  is rather difficult to compute directly under the  
165 assumption of varying  $W_0$ ,  $W_1$  and  $z_b$ . However, it can be estimated cell-wise from simple  
166 balance considerations. Consider static equilibrium conditions, i.e.,  $Q=0$  in all cells, then  
167 Eqs. (1) and (10) yield

$$\left( gAS_0 + \frac{R}{\rho} \right)_i^n \Delta x_i = \left( \frac{P}{\rho} \right)_{i+\frac{1}{2}}^n - \left( \frac{P}{\rho} \right)_{i-\frac{1}{2}}^n \quad (23)$$

168 Given the definition (Eq 11) of the channel width, and the specific pressure force  $\frac{P}{\rho}$   
169 (Eq. 3), one has

$$\frac{P}{\rho} = g \int_0^h (h - z') (W_0(x) + z'W_1(x)) dz' = \frac{1}{2}gW_0(x)h^2 + \frac{1}{6}gW_1(x)h^3 \quad (24)$$

170 The value of the specific pressure force at the interface  $i - \frac{1}{2}$  is then easily computed as

$$\left( \frac{P}{\rho} \right)_{i-\frac{1}{2}}^n = \frac{g}{2}W_{0,i-\frac{1}{2}}h_{i-\frac{1}{2}}^2 + \frac{g}{6}W_{1,i-\frac{1}{2}}h_{i-\frac{1}{2}}^3 \quad (25)$$

171 with the definition (19b) for  $h_{i-\frac{1}{2}}$ . Since the purpose is to estimate the source term in the  
172 cell  $i$ , the free surface elevation to be used in Eq. (19b) is  $\zeta_i^n$ . The same formulation can  
173 be obtained at interface  $i + \frac{1}{2}$  yielding the final estimate for the source term to be used in  
174 Eq. (10):

$$\begin{aligned} \left( gAS_0 + \frac{R}{\rho} \right)_i^n \Delta x_i &= \frac{g}{2} \left( W_{0,i+\frac{1}{2}} + \frac{1}{3}W_{1,i+\frac{1}{2}}h_{i+\frac{1}{2},i} \right) h_{i+\frac{1}{2},i}^2 \\ &\quad - \frac{g}{2} \left( W_{0,i-\frac{1}{2}} + \frac{1}{3}W_{1,i-\frac{1}{2}}h_{i-\frac{1}{2},i} \right) h_{i-\frac{1}{2},i}^2 \end{aligned} \quad (26a)$$

175

$$h_{i-\frac{1}{2},i} = \zeta_i^n - z_{b,i-\frac{1}{2}} \quad (26b)$$

176

$$h_{i+\frac{1}{2},i} = \zeta_i^n - z_{b,i+\frac{1}{2}} \quad (26c)$$

177 *2.5. Balancing issues*

178 The discretization of the source term in the momentum equation usually poses no prob-  
 179 lem. This issue has been dealt with abundantly in the literature, within a very wide variety  
 180 of techniques [1, 2, 4, 6, 27, 28, 38, 40]. A remaining problem encountered in practical  
 181 applications is related to the continuity equation and the difference often observed between  
 182 the average cell values and the interface values for the volume discharge.

183 Consider a solution  $\mathbf{U}$  verifying steady state,  $\frac{\partial \mathbf{U}}{\partial t} = 0$ . The first component of Eq. (1)  
 184 imposes that  $Q$  be equal to a constant  $Q_0$  all throughout the computational domain. In  
 185 particular, the discharge  $Q$  computed at the cell interfaces should be identical to that in the  
 186 cell values. This, however, is not necessarily the case if Eq. (20) is used. Indeed, writing the  
 187 first component of Eq. (20) leads to the following formula for the interface flux

$$Q_0 = (a + D_{12}) Q_L + (1 - a - D_{12}) Q_R + D_{11}(A_L - A_R) \quad (27)$$

188 where  $D_{11}$  and  $D_{12}$  are the components on the first row of the artificial viscosity matrix  $\mathbf{D}$ .  
 189 Eq. (27) can be rewritten as

$$(a + D_{12}) Q_L + (1 - a - D_{12}) Q_R = Q_0 + D_{11}(A_R - A_L) \quad (28)$$

190 Assume that the discretized solution has reached steady state. If the geometry of the  
 191 channel is arbitrary (non-constant bottom slope and/or non-prismatic channel), in general  
 192  $A_L \neq A_R$ . It is then obvious from Eq. (28) that at least one of the discharges  $Q_L$  and  $Q_R$   
 193 is different from the uniform discharge  $Q_0$ . Consequently, a non-uniform discharge profile  
 194 is obtained. In particular, if the initial situation is static ( $Q_0 = 0$ ), non-zero discharges  
 195 are computed. Artificial oscillations appear and propagate throughout the computational  
 196 domain.

197 The ability of a numerical scheme to preserve static equilibrium conditions has been  
 198 introduced as the  $C$ -property in [4]. Specifying the  $C$ -property exactly or approximately  
 199 has proved to lead to efficient source term balancing techniques. The most widespread  
 200 approach consists in adapting the discretization of the source term to the formulation of  
 201 the flux so as to satisfy the  $C$ -property. In the Auxiliary Variable-based Balancing (AVB)  
 202 approach, the opposite approach is followed: the formulation of the flux gradients is adapted  
 203 to that of the source term.

204 **3. Auxiliary Variable-based Balancing method**205 *3.1. Principle*

206 The AVB method is based on the following requirements: (i) the artificial viscosity term  
 207 in Eq. (20) should be modified in such a way that diffusion becomes zero when steady state is  
 208 reached; (ii) the source term in the momentum equation should be discretized in such a way  
 209 that it does not influence the calculation of the flux in the intermediate region of constant  
 210 state. The second issue has been addressed in subsection 2.4 (source term discretization);  
 211 the first issue is dealt with in the following subsections.

212 AVB uses an auxiliary variable  $\mathbf{V}$  in the expression of the artificial viscosity term:

$$\mathbf{F} = a\mathbf{F}_L + (1 - a)\mathbf{F}_R + \mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) \quad (29)$$

213 where  $\mathbf{V}$  is a function of both the variable  $\mathbf{U}$  and the parameter  $\varphi$ ,  $\mathbf{V} = \mathbf{V}(\mathbf{U}, \varphi)$ . The dif-  
 214 fusion matrix  $\mathbf{D}_{\mathbf{V}}$  and the auxiliary variable  $\mathbf{V}$  are chosen such that the following conditions  
 215 are verified:

216 **(C1):** under steady state conditions,  $\mathbf{V}_L = \mathbf{V}_R$ .

217 **(C2):** for  $\varphi = \text{Const}$ ,  $\mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) = \mathbf{D}(\mathbf{U}_L - \mathbf{U}_R)$

218 Condition (C1) is the so-called enhanced consistency condition for steady state flow, which is  
 219 the desired property for scheme well-balancing. Condition (C2) means that the strengths of  
 220 the artificial viscosity terms in Eqs. (20) and (29) are identical, thus preserving the stability  
 221 properties of the numerical solution. The pending question is the determination of  $\mathbf{D}_V$ .

222 It is observed that the artificial viscosity terms in Eqs. (20) and (29) are approximations  
 223 of the following derivatives:

$$\mathbf{D}_V(\mathbf{V}_L - \mathbf{V}_R) = -\Delta x \mathbf{D}_V \frac{\partial \mathbf{V}}{\partial x} + \text{HOT}(\Delta x) \quad (30)$$

$$\mathbf{D}(\mathbf{U}_L - \mathbf{U}_R) = -\Delta x \mathbf{D} \frac{\partial \mathbf{U}}{\partial x} + \text{HOT}(\Delta x) \quad (31)$$

224 with

$$\text{HOT}(\Delta x) \xrightarrow{\Delta x \rightarrow 0} 0 \quad (32)$$

225 Noticing that  $\mathbf{V} = \mathbf{V}(\mathbf{U}, \varphi)$ , the derivative of  $\mathbf{V}$  with respect to  $x$  is expressed as

$$\frac{\partial \mathbf{V}}{\partial x} = \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{V}}{\partial \varphi} \frac{\partial \varphi}{\partial x} \quad (33)$$

226 Substituting Eq. (33) into Eq. (30), comparing with Eq. (31) and imposing condition (C2)  
 227 gives:

$$\mathbf{D}_V \frac{\partial \mathbf{V}}{\partial \mathbf{U}} = \mathbf{D} \quad (34)$$

228 Consequently,  $\mathbf{D}_V$  is given by

$$\mathbf{D}_V = \mathbf{D} \left( \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \right)^{-1} \quad (35)$$

### 229 3.2. Balancing option 1: free surface elevation

230 One of the earliest examples of the use of an auxiliary variable is found in [33] for the  
 231 solution of the shallow water equations where the free surface elevation  $\zeta = z_b + h$  is used  
 232 in place of the water depth  $h$ . The rationale is that under static conditions, the free surface  
 233 elevation is constant, consequently, both  $d\zeta$  and  $dQ$  are zero at equilibrium. Note that the  
 234 approach has been extended to the reconstruction technique in higher-order schemes in [40].  
 235 This leads to the following possible definition for the auxiliary variable

$$d\mathbf{V}_1 = \begin{bmatrix} d\zeta \\ dQ \end{bmatrix} \quad (36)$$

236 Since  $dA = bd\zeta$ , one has from the definition of  $\mathbf{U}$  in Eq. (2):

$$\frac{\partial \mathbf{V}_1}{\partial \mathbf{U}} = \begin{bmatrix} b^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \left( \frac{\partial \mathbf{V}_1}{\partial \mathbf{U}} \right)^{-1} = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \quad (37)$$

237 This leads to the following artificial viscosity term:

$$\mathbf{D}_V(\mathbf{V}_{1L} - \mathbf{V}_{1R}) = \mathbf{D} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_L - \zeta_R \\ Q_L - Q_R \end{bmatrix} = \mathbf{D} \begin{bmatrix} (\zeta_L - \zeta_R)b \\ Q_L - Q_R \end{bmatrix} \quad (38)$$

238 Note that in the case of the SWE,  $b = 1$  and Nujic's [33] approach is retrieved.

239 This option has the drawback that steady state, uniform flow cannot be maintained  
 240 exactly. Indeed, under uniform flow conditions,  $Q_L = Q_R$  but the free surface elevations  
 241 in two adjacent cells are not identical,  $\zeta_L \neq \zeta_R$ . Therefore, the artificial diffusion term in  
 242 the continuity equation is non-zero and the interface flux is not equal to  $Q_L = Q_R$ . This is  
 243 substantiated by the computational examples in Section 5.



244 *3.3. Balancing option 2: specific force*

245 This is the option explored in [6]. Similar formulae were obtained for the one-dimensional  
 246 shallow water equations in [28, 30, 18], albeit from different considerations. The latter  
 247 three approaches, however, focus on rectangular channels, while the proposed approach is  
 248 applicable to arbitrary-shaped channels.

249 The specific force is used in place of the cross-sectional area in the first component of  
 250 the auxiliary variable  $\mathbf{V}$ . The motivation is that under dynamic equilibrium (that is, under  
 251 steady state flow conditions), the variations in the specific force are balanced exactly by  
 252 the source terms in the momentum equation noted  $S_M$ . This leads to define the auxiliary  
 253 variable  $\mathbf{V}$  in differential form as

$$d\mathbf{V} = \begin{bmatrix} dM - S_M \\ dQ \end{bmatrix} \quad (39)$$

254 Since  $dM = (c^2 - u^2)dA + 2udQ$ , one has

$$\frac{\partial \mathbf{V}}{\partial \mathbf{U}} = \begin{bmatrix} c^2 - u^2 & 2u \\ 0 & 1 \end{bmatrix}, \left( \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \right)^{-1} = \begin{bmatrix} \frac{1}{c^2 - u^2} & -\frac{2u}{c^2 - u^2} \\ 0 & 1 \end{bmatrix} \quad (40)$$

255 This leads to the following artificial viscosity term:

$$\mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) = \mathbf{D} \left[ \frac{M_L - M_R - 2u(Q_L - Q_R) - S_M \Delta x}{c^2 - u^2} \right]_{Q_L - Q_R} \quad (41)$$

256 where the source term  $S_M$  is computed in average between the centres of the left- and  
 257 right-hand cells. It is simply estimated as the average of the cell values given by Eq. (26a).

258 Practical implementations [18, 30] indicate that in the neighbourhood of critical points,  
 259 Eq. (41) induces a downwinding of the discharge and a discontinuous switch between sub-  
 260 critical and supercritical flux formulae. Due to this, a different formula is proposed

$$d\mathbf{V} = \begin{bmatrix} dM - 2udQ - S_M \\ dQ \end{bmatrix} \quad (42)$$

261 This leads to

$$\frac{\partial \mathbf{V}}{\partial \mathbf{U}} = \begin{bmatrix} c^2 - u^2 & 0 \\ 0 & 1 \end{bmatrix}, \left( \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \right)^{-1} = \begin{bmatrix} \frac{1}{c^2 - u^2} & 0 \\ 0 & 1 \end{bmatrix} \quad (43)$$

262 and the following artificial viscosity term is obtained

$$\mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) = \mathbf{D} \left[ \frac{M_L - M_R - S_M \Delta x}{c^2 - u^2} \right]_{Q_L - Q_R} \quad (44)$$

263 This expression, however, remains invalid at critical points, for which  $c^2 = u^2$ . As  
 264 proposed in [6], in the case of 1D SWEs on rectangular channel, the final estimate for  $d\mathbf{V}$   
 265 is the minmod of the estimates given by the specific force option and the original approach:

$$d\mathbf{V}_{\mathbf{2}} = \text{minmod}(d\mathbf{V}, d\mathbf{U}) \quad (45)$$

266 where  $d\mathbf{V}$  is defined by Eq. (42) and the minmod operator by:

$$\text{minmod}(a, b) = \begin{cases} \min(|a|, |b|) & \text{if } ab \geq 0 \\ 0 & \text{if } ab < 0 \end{cases} \quad (46)$$

267 *3.4. Balancing option 3: hydraulic head*

268 In this option, presented in [25] (for 1D SWEs on rectangular channel), the hydraulic  
 269 head  $H = \zeta + \frac{u^2}{2g}$  is used as auxiliary variable:

$$d\mathbf{V} = \begin{bmatrix} dH - S_f \\ dQ \end{bmatrix} \quad (47)$$

270 Since  $dH = \left( \frac{1}{b} - \frac{u^2}{gA} \right) dA + \frac{u}{gA} dQ = \frac{1}{b} [(1 - F^2) dA + \frac{F}{c} dQ]$ , one has

$$\frac{\partial \mathbf{V}}{\partial \mathbf{U}} = \begin{bmatrix} \frac{1 - F^2}{b} & \frac{F}{bc} \\ 0 & 1 \end{bmatrix}, \left( \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \right)^{-1} = \begin{bmatrix} \frac{b}{1 - F^2} & -\frac{F}{1 - F^2} \\ 0 & 1 \end{bmatrix} \quad (48)$$

271 This leads to the following artificial viscosity term:

$$\mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) = \mathbf{D} \left[ \frac{(H_L - H_R - S_f \Delta x) b - \frac{F}{c} (Q_L - Q_R)}{1 - F^2} \frac{1}{Q_L - Q_R} \right] \quad (49)$$

272 For the same reasons as Option 2, the following variation is proposed for Option 3:

$$d\mathbf{V} = \begin{bmatrix} dH - S_f - \frac{F}{c} dQ \\ dQ \end{bmatrix} \quad (50)$$

273 The Jacobian matrix of  $\mathbf{V}$  with respect to  $\mathbf{U}$  is given by:

$$\frac{\partial \mathbf{V}}{\partial \mathbf{U}} = \begin{bmatrix} \frac{1-F^2}{b} & 0 \\ 0 & 1 \end{bmatrix}, \quad \left( \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \right)^{-1} = \begin{bmatrix} \frac{b}{1-F^2} & 0 \\ 0 & 1 \end{bmatrix} \quad (51)$$

274 This leads to the following artificial viscosity term:

$$\mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) = \mathbf{D} \left[ b \frac{H_L - H_R - S_f \Delta x}{1 - F^2} \frac{1}{Q_L - Q_R} \right] \quad (52)$$

275 As in option 2, this expression is not valid at critical points for which  $F = 1$ , the final  
276 estimate for  $d\mathbf{V}$  is thus

$$d\mathbf{V}_{\mathbf{3}} = \text{minmod}(d\mathbf{V}, d\mathbf{U}) \quad (53)$$

277 where  $d\mathbf{V}$  is defined by Eq. (50).

278 Note that if a singular head loss  $\Delta H_s$  is to be introduced, it can also be taken into  
279 account in the artificial viscosity term:

$$\mathbf{D}_{\mathbf{V}}(\mathbf{V}_L - \mathbf{V}_R) = \mathbf{D} \left[ b \frac{H_L - H_R - \Delta H_s - S_f \Delta x}{1 - F^2} \frac{1}{Q_L - Q_R} \right] \quad (54)$$

## 280 4. Application to classical approximate Riemann solvers

### 281 4.1. Application to the HLL solver

282 The HLL solver [26] can be written in the form (20) by defining  $a$  and  $\mathbf{D}$  as

$$a = \frac{\lambda^+}{\lambda^+ - \lambda^-} \quad (55a)$$

$$\mathbf{D} = -\frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-} \mathbf{I} \quad (55b)$$

283 where  $\mathbf{I}$  is the identity matrix and  $\lambda^-$ ,  $\lambda^+$  are respectively estimates of the fastest waves  
284  $\lambda^{(1)}$  and  $\lambda^{(2)}$  defined in Eqs. (9b, 9c) in the direction of negative and positive  $x$  [15, 17]:

$$\lambda^- = \min(u_L - c_L, u_R - c_R, 0) \quad (56a)$$

$$285 \lambda^+ = \max(u_L + c_L, u_R + c_R, 0) \quad (56b)$$

### 286 4.2. Application to Roe's solver

287 Roe's solver [34] can be written in the form (20) by setting

$$a = \frac{1}{2} \quad (57a)$$

$$\mathbf{D} = \frac{\tilde{\mathbf{A}}^\pm}{2} \quad (57b)$$

288 where  $\tilde{\mathbf{A}}^\pm$  is the matrix generated by the absolute values of the eigenvalues of  $\mathbf{A}$ :

$$\tilde{\mathbf{A}}^\pm = \tilde{\mathbf{K}} \left| \tilde{\Lambda} \right| \tilde{\mathbf{K}}^{-1} \quad (58)$$

289 with

$$\tilde{\mathbf{K}} = \begin{bmatrix} 1 & 1 \\ \tilde{\lambda}^{(1)} & \tilde{\lambda}^{(2)} \end{bmatrix}, \quad \tilde{\mathbf{K}}^{-1} = \frac{1}{\tilde{\lambda}^{(2)} - \tilde{\lambda}^{(1)}} \begin{bmatrix} \tilde{\lambda}^{(2)} & -1 \\ -\tilde{\lambda}^{(1)} & 1 \end{bmatrix}, \quad \left| \tilde{\Lambda} \right| = \begin{bmatrix} |\tilde{\lambda}^{(1)}| & 0 \\ 0 & |\tilde{\lambda}^{(2)}| \end{bmatrix} \quad (59)$$

290 leading to the following expression for  $\tilde{\mathbf{A}}^\pm$ :

$$\tilde{\mathbf{A}}^\pm = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (60a)$$

$$a_{11} = \frac{\tilde{\lambda}^{(2)} |\tilde{\lambda}^{(1)}| - \tilde{\lambda}^{(1)} |\tilde{\lambda}^{(2)}|}{\tilde{\lambda}^{(2)} - \tilde{\lambda}^{(1)}} \quad (60b)$$

$$a_{12} = \frac{|\tilde{\lambda}^{(2)}| - |\tilde{\lambda}^{(1)}|}{\tilde{\lambda}^{(2)} - \tilde{\lambda}^{(1)}} \quad (60c)$$

$$a_{21} = -\tilde{\lambda}^{(1)} \tilde{\lambda}^{(2)} a_{12} \quad (60d)$$

$$a_{22} = \frac{\tilde{\lambda}^{(2)} |\tilde{\lambda}^{(2)}| - \tilde{\lambda}^{(1)} |\tilde{\lambda}^{(1)}|}{\tilde{\lambda}^{(2)} - \tilde{\lambda}^{(1)}} \quad (60e)$$

291 In Roe's approach [34], the eigenvalues  $\tilde{\lambda}^{(1)} = (\tilde{u} - \tilde{c})$  and  $\tilde{\lambda}^{(2)} = (\tilde{u} + \tilde{c})$  in the diagonal  
292 matrix  $\tilde{\mathbf{A}}$  are obtained from Roe's averages [21, 22] :

$$\tilde{c} = \left[ \frac{g}{2} \left( \frac{A_L}{b_L} + \frac{A_R}{b_R} \right) \right]^{1/2} \quad (61a)$$

$$\tilde{u} = \frac{c_L u_L + c_R u_R}{c_L + c_R} \quad (61b)$$

#### 294 4.3. Application to the Q-scheme

295 The Q-scheme uses the same formula as Roe's formula, except that the matrix  $\tilde{\mathbf{A}}$  in  
296 Eq. (58) is estimated from the average of the left and right-hand cells

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}} \left( \frac{\mathbf{U}_L + \mathbf{U}_R}{2} \right) \quad (62)$$

297 yielding to the following approximation for the eigenvalues:

$$\tilde{c} = g \left( \frac{A_{LR}}{\tilde{b}} \right)^{1/2} \quad (63a)$$

$$\tilde{u} = \frac{Q_{LR}}{A_{LR}} \quad (63b)$$

299 where  $\tilde{b} = W_{0,i-\frac{1}{2}} + h_{LR} W_{1,i-\frac{1}{2}}$  and  $X_{LR} = (X_L + X_R)/2$  ( $X \in \{A, Q, h\}$ ).

#### 300 4.4. Summary of formulae - Algorithmic aspects

301 From an algorithmic point of view, the steps in the solution process are the following:

- 302 1. For each cell, compute the free surface elevations  $\zeta_L$  and  $\zeta_R$  from the left and right  
303 states  $\mathbf{U}_L$  and  $\mathbf{U}_R$ , using the correspondence between  $A$  and  $h$ , Eq (19a). Use the free  
304 surface elevations to compute the geometric source term from Eqs (26).
- 305 2. For each interface, compute the flux  $\mathbf{F}$  using Eq (29) with the fluxes on both sides of  
306 the interface ( $\mathbf{F}_L, \mathbf{F}_R$ ), the auxiliary variables  $\mathbf{V}_L, \mathbf{V}_R$  according to the AVB option  
307 chosen and with  $a$  depending on the solver.
- 308 3. Compute the friction source term as in Eq (21).
- 309 4. Apply the balance equation (10) to compute the hydrodynamic variable at the next  
310 time step.

Symbol	Meaning	Value
$g$	Gravitational acceleration	$9.81 \text{ m s}^{-2}$
$L$	Length of the domain	3,000 m
$S_0$	Bottom slope	$10^{-3}$
$W_0$	Channel width	1 m
$W_1$	Derivative of the width with respect to $z$	0
$h_0$	Initial water depth	1 m
$z_{\text{ds}}$	Prescribed surface elevation downstream	1 m
$Q_{\text{up}}$	Prescribed discharge upstream	$1 \text{ m}^3 \text{ s}^{-1}$
$n_M$	Manning's friction coefficient	$0.025 \text{ m}^{-1/3} \text{ s}$
$\Delta x$	Computational cell width	1 m
$\Delta t$	Simulation time	20,000 s

Table 1: Test 1 - steady state flow in a prismatic, rectangular channel. Parameters of the test case.

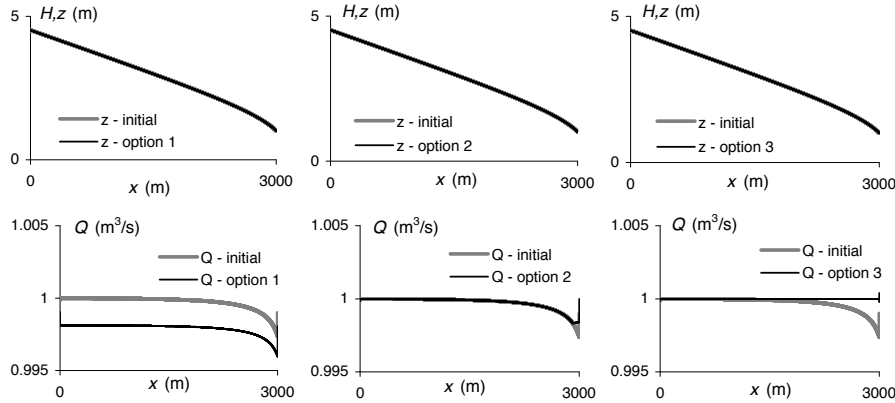


Figure 2: Test 1 - steady state flow in a prismatic, rectangular channel. Top: water elevation  $z$  and hydraulic head  $H$ , down: discharge  $Q$  obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options, using the HLL solver (the results obtained with the Roe's and the  $Q$ -scheme solvers are identical).

## 311 5. Computational examples

### 312 5.1. Steady state configurations

#### 313 5.1.1. Test 1: steady state flow in a prismatic, rectangular channel

314 In this test case, the various AVB options are applied to steady state flow in a prismatic,  
315 rectangular channel (i.e. with a constant value of  $W_0$  and with  $W_1 = 0$ ) including friction. A  
316 transient simulation is carried out from an initial state at rest until steady state is obtained.  
317 The parameters of the test case are given in Table 1.

318 Figure 2 shows the results obtained from the initial formulation *i.e.* with  $\mathbf{V} = \mathbf{U}$  and  
319 with the three different AVB options. Only the HLL solver is shown in this case because  
320 the results obtained with the two other solvers are identical. The profiles of the free surface  
321 elevation  $z$  and the hydraulic head  $H$  are identical regardless of the AVB option used (note  
322 that the water elevation and the hydraulic head are nearly identical because of a small  
323 velocity).

324 However, under steady-state conditions, the discharge  $Q$  is expected to be uniform over  
325 the entire domain and equal to  $Q_{\text{up}}$ . The only option that provides the correct value of  $Q$   
326 over the whole domain but the downstream boundary, is the third one, *i.e.* based on the  
327 hydraulic head.

328 Figure 3 shows results of the same test case but with the introduction of a singular  
329 head loss in the middle of the channel. The head loss is computed using a classical Borda  
330 relationship:

$$\Delta H_s = \alpha \frac{v^2}{2g}$$

331 where  $\alpha$  is arbitrarily chosen to  $\alpha = 5$  in this case, but can be estimated from any empirical  
332 law. Figure 3a presents results using HLL solver. Each option provides a good estimate of

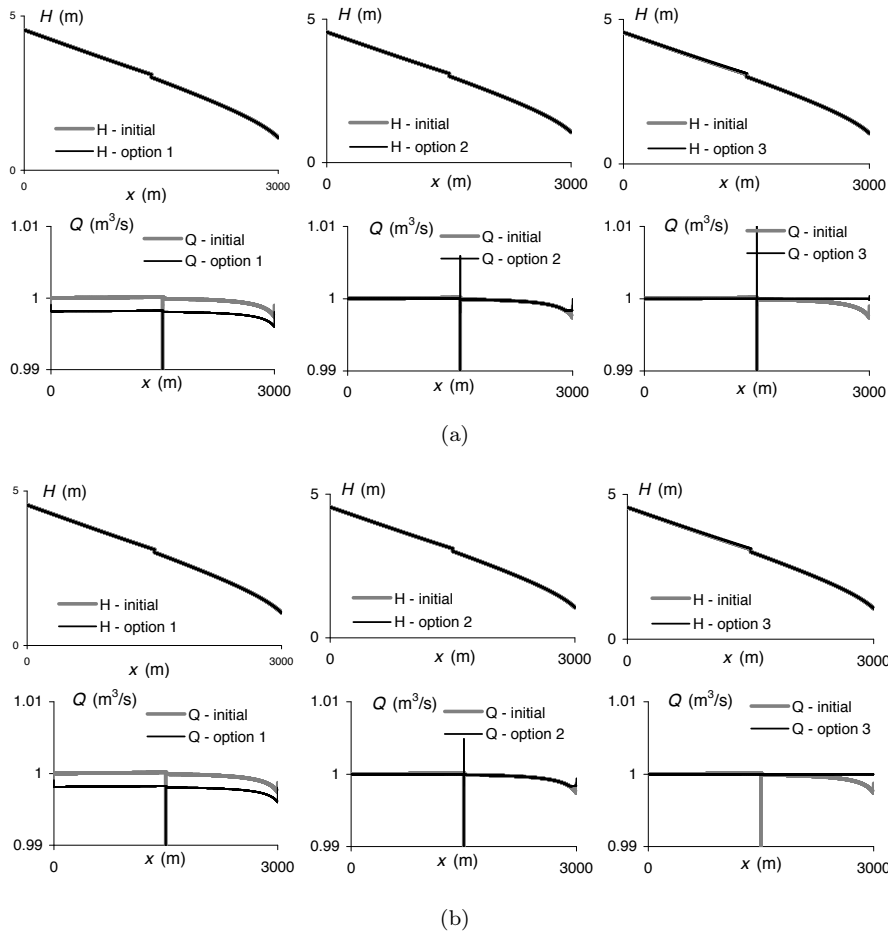


Figure 3: Test 1b - steady state flow in a prismatic, rectangular channel with an arbitrary singular head loss in the middle of the channel. Top: water elevation  $z$  and hydraulic head  $H$ , down: discharge  $Q$  obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options, using a) HLL solver, b) Roe's solver (identical to  $Q$ -scheme).

333 the hydraulic head and water elevation. In addition to the behaviour previously observed,  
 334 the singular head loss triggers a spike in the discharge profiles when the HLL solver is used.  
 335 Option 3, that explicitly takes into account the singular head loss in the flux computation,  
 336 is the only one that provides a constant value  $Q = Q_{\text{up}}$  with Roe's and  $Q$ -scheme solvers  
 337 (Figure 3b).

### 338 5.1.2. Test 2 : frictionless steady state flow in a non-prismatic, rectangular channel

339 The channel profile is shown in Figure 4; it contains two consecutive narrowings of the  
 340 cross-section: the first one due to the width narrowing (minimum width at 25% of the  
 341 channel length) and the second one to a bump in the bottom elevation (maximum elevation  
 342 at 75% of the channel length). This is a frictionless test case, the parameters of which are  
 343 given in Table 2. As for the first test case, a transient simulation is run for a sufficiently  
 344 long time, so that the transient regime vanishes and steady state is reached. The prescribed  
 345 discharge and downstream water level are chosen such that the flow regime is subcritical  
 346 upstream of both the narrowing and the bump, yielding two hydraulic jumps.

347 Figure 5 shows results obtained with the different AVB options and the three solvers  
 348 (note that the results obtained with Roe's solver and  $Q$ -scheme are identical). The profiles  
 349 obtained for the hydraulic head  $H$  and water elevation  $z$  with the different solvers and  
 350 options bear similarities except for the points upstream the channel narrowing. In contrast,  
 351 substantial differences can be observed for the discharge  $Q$ . As for the first test case, the  
 352 steady state configuration theoretically implies a constant value for the discharge. It can  
 353 be seen that the same profile is obtained using the initial formulation (i.e.  $\mathbf{V} = \mathbf{U}$ ) for the  
 354 three solvers, and that this profile is the most different from the constant value of  $Q = Q_{\text{up}}$ .  
 355 Option 2 also give a strongly variable discharge in space when used with HLLC, but not  
 356 with Roe's solver or  $Q$ -scheme. Option 3 gives better results: it is very close to  $Q = Q_{\text{up}}$

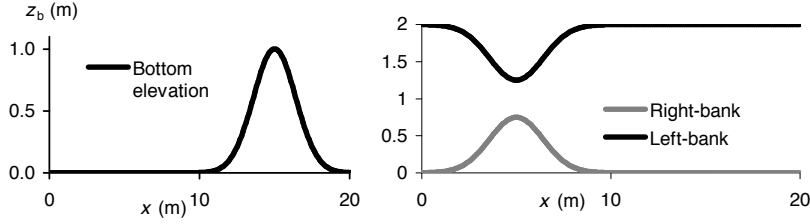


Figure 4: Test 2 - Frictionless steady state flow in a non-prismatic, rectangular channel. Channel profile: left, bottom elevation; right, left- and right-bank profile.

Symbol	Meaning	Value
$g$	Gravitational acceleration	$9.81 \text{ m s}^{-2}$
$L$	Length of the domain	20 m
$W_0, z_b$	Channel width and bottom elevation	Figure 4
$W_1$	Derivative of the width with respect to $z$	0
$z_0$	Initial free surface elevation	1.1 m
$z_{\text{ds}}$	Prescribed surface elevation downstream	1.1 m
$Q_{\text{up}}$	Prescribed discharge upstream	$2 \text{ m}^3 \text{ s}^{-1}$
$n_M$	Manning's friction coefficient	$0 \text{ m}^{-1/3} \text{ s}$
$\Delta x$	Computational cell width	0.1 m
$\Delta t$	Simulation time	400 s

Table 2: Test 2 - Frictionless steady state flow in a non-prismatic, rectangular channel. Parameters of the test case.

357 over the whole domain except in the immediate vicinity of the hydraulic jump ( $x \approx 17\text{m}$ ).

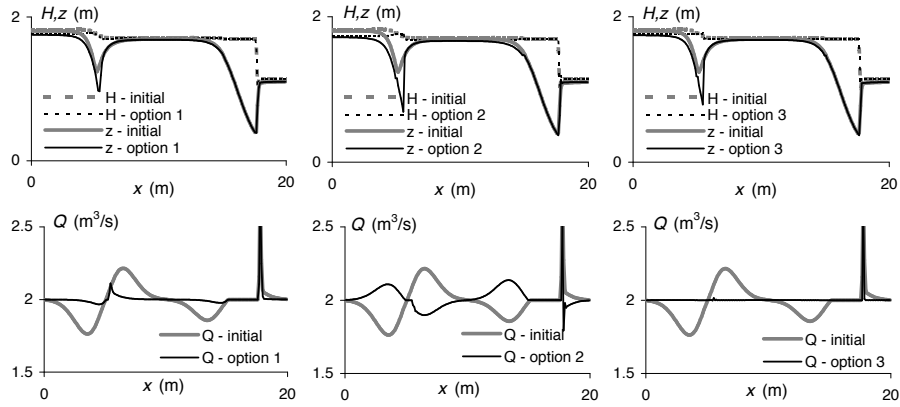
### 358 5.1.3. Test 3: frictionless steady-state flow in a non-prismatic trapezoidal channel

359 The channel profile is shown in Figure 6. It presents two simultaneous reductions of the  
360 cross-section (bump and width narrowing), located at the same abscissa. The channel is not  
361 prismatic with a variable bank slope yielding a transition from a trapezoidal shape at the  
362 boundaries to a rectangular shape at half length. The parameters used for this steady-state,  
363 frictionless test case are given in Table 3.

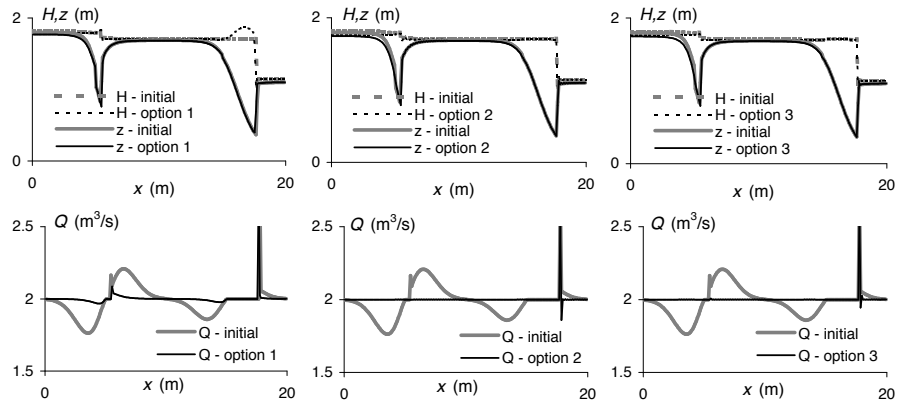
364 The simulated free surface elevation  $z$ , hydraulic head  $H$  and discharge  $Q$ , obtained  
365 with the three AVB options and the three solvers are given in Figure 7. In this case again,  
366 all three AVB options provide improved solutions compared to that given by the initial  
367 formulation, for which the transition from subcritical to supercritical conditions (and vice-  
368 versa) is observed to induce strong variations in the estimation of the discharge. This  
369 statement however is to be moderated concerning Option 2 combined with HLLC solver  
370 that also yields such variations. In a largely lesser extent, option 1 also exhibits some small  
371 variations in the discharge. Moreover, it can be seen that the abscissa of the hydraulic jump  
372 is not exactly located using Option 1 with Roe's solver or  $Q$ -scheme, with an increase in  
373 hydraulic head upstream the jump.

Symbol	Meaning	Value
$g$	Gravitational acceleration	$9.81 \text{ m s}^{-2}$
$L$	Length of the domain	20 m
$W_0, W_1$	Channel width and its derivative with respect to $z$	Figure 6
$z_b$	Bottom elevation	Figure 6
$z_0$	Initial free surface elevation	1.2 m
$z_{\text{ds}}$	Prescribed surface elevation downstream	1.2 m
$Q_{\text{up}}$	Prescribed discharge upstream	$4 \text{ m}^3 \text{ s}^{-1}$
$n_M$	Manning's friction coefficient	$0 \text{ m}^{-1/3} \text{ s}$
$\Delta x$	Computational cell width	0.1 m
$\Delta t$	Simulation time	200 s

Table 3: Test 3 - Frictionless steady state flow in a non-prismatic trapezoidal channel. Parameters of the test case.



(a) HLL



(b) Roe (identical to  $Q$ -scheme)

Figure 5: Test 2 - Frictionless steady state flow in a non-prismatic, rectangular channel. Up: water elevation  $z$  and hydraulic head  $H$ , down: discharge  $Q$  obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options, using a) HLL solver, b) Roe's solver. The  $Q$ -scheme gives similar results to the Roe's solver.

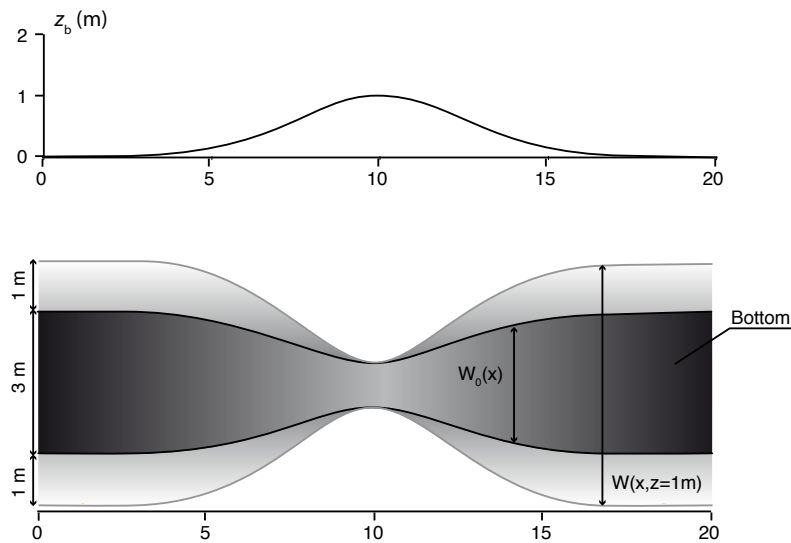


Figure 6: Test 3 - Frictionless steady state flow in a non-prismatic trapezoidal channel. Channel profile: top, bottom elevation; down,  $W_0(x)$  and  $W(x, z = 1)$ .

374 Option 3 gives a uniform value for the discharge everywhere, except across the hydraulic  
 375 jump, regardless the solver used. Once again, Option 3 is thus deemed more suitable to deal  
 376 with transcritical flows.

#### 377 5.1.4. Test 4: steady state flow in a Venturi flume

378 This test case involves the simultaneous presence of all source terms: friction, bottom  
 379 slope and width variation. It is a real world test case for which experiment validation has  
 380 been carried out in a channel of 67 cm wide. The Venturi flume used is 2.5 m long with  
 381 narrow section of 10 cm wide (Figure 8). The flume is made of aluminium plates, with a  
 382 Manning friction coefficient  $n_M = 10^{-2} \text{ m}^{-1/3} \text{ s}$  calibrated from experiments in a straight  
 383 channel made of the same material.

384 In the experiment, steady state was obtained under a discharge of 40 litres per second.  
 385 The elevation of the free surface along the walls and axis of the channel was measured every  
 386 5 cm. Figure 9 shows numerical results obtained with Roe's solver and the different AVB  
 387 options. The three AVB options give similar results. The unit-discharge is better estimated  
 388 upstream than with the initial formulation. Option 2 and 3 give erroneous results with the  
 389 HLL solver and the  $Q$ -scheme. Figure 10 shows the longitudinal profiles of the measured  
 390 and simulated free surface. As can be seen from the figure, the simulation agrees well with  
 391 the measurement upstream and downstream of the narrowing. In contrast, the free surface  
 392 elevation is overestimated by the numerical model in the narrow section of the Venturi flume.  
 393 Besides, the curvature of the simulated free surface profile is wrong. These results invalidate  
 394 the shallow water assumption of a hydrostatic pressure distribution but this is beyond the  
 395 scope of the present paper.

#### 396 5.2. Transient test cases

397 There is no guarantee that an accurate well-balanced approach for steady state flows,  
 398 gives correct results on transient configurations. The following transient test cases are thus  
 399 performed.

##### 400 5.2.1. Test 5: frictionless dam-break problem in a rectangular channel with flat bottom

401 The dam-break problem is an initial-value problem in which the water is initially at rest  
 402 and the water levels are different on both sides of the dam. The solution of the dam-break  
 403 problem in rectangular channels is similar to that of the one-dimensional shallow water  
 404 equations. The properties of the analytical solution are presented in [36]. The dam-break  
 405 problem is a Riemann problem defined as:

$$h(x, 0) = \begin{cases} h_L & \text{for } x \leq x_0 \\ h_R & \text{for } x > x_0 \end{cases} \quad (64a)$$

$$q(x, 0) = 0 \forall x \quad (64b)$$

406 The solution is made of a rarefaction wave and a moving shock separated by a region of  
 407 constant state. For the dam-break problem without source terms (friction or bottom slope),  
 408 the profile obeys the following equations in the rarefaction wave

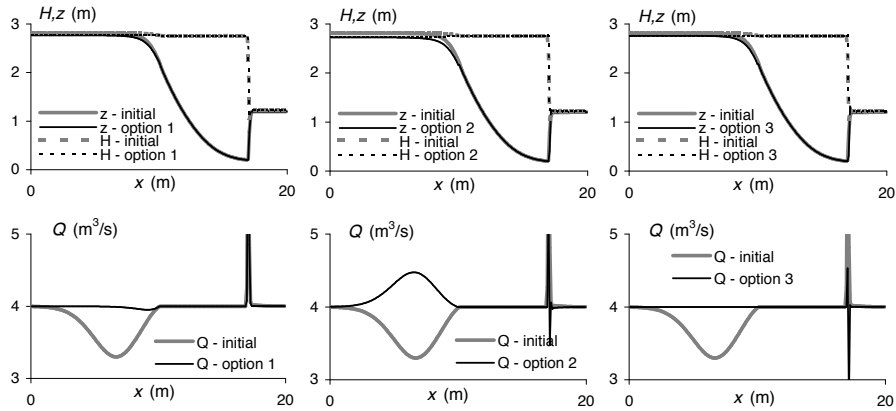
$$u(x, t) = \frac{2}{3} \left( c_L + \frac{x}{t} \right) \quad (65a)$$

$$c(x, t) = \frac{1}{3} \left( 2c_L - \frac{x}{t} \right) \quad (65b)$$

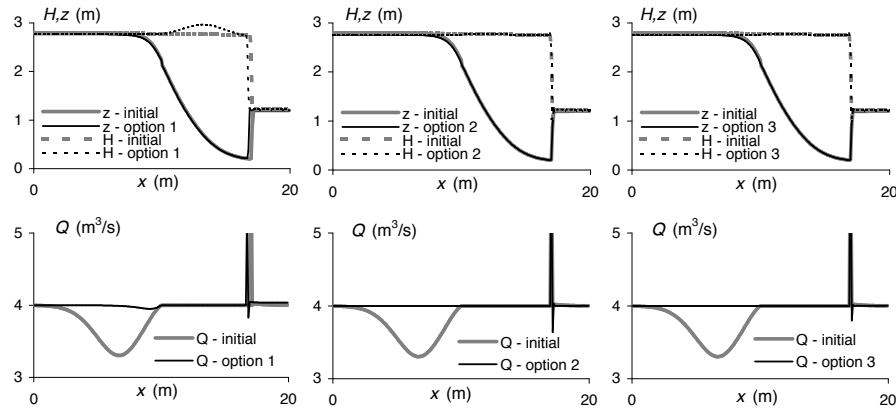
410 from which the expression of the flow solution  $\mathbf{U}$  is straightforward using  $A = c^2/g$  and  
 411  $Q = uA$ . In the other parts of the domain, the profile is piecewise constant (see [36] for  
 412 more details).

413 The parameters used in this test case are given in Table 4. Profiles of free surface  
 414 elevation, hydraulic head and discharge, obtained with the initial formulation and the three  
 415 AVB options are given in Figure 11 for the three solvers. Contrarily to previous test cases,  
 416 the discharge  $Q$  is correctly estimated by each option included initial formulation except  
 417 for the combinations HLL/Option 2 (Figure 11b) and  $Q$ -scheme/Option 3 (Figure 11d).  
 418 For these latter, the free surface elevation and hydraulic head are discontinuous across the  
 419 critical point (Note that this problem was pointed out in [30, 18] where a specific treatment  
 420 of the critical point was proposed).

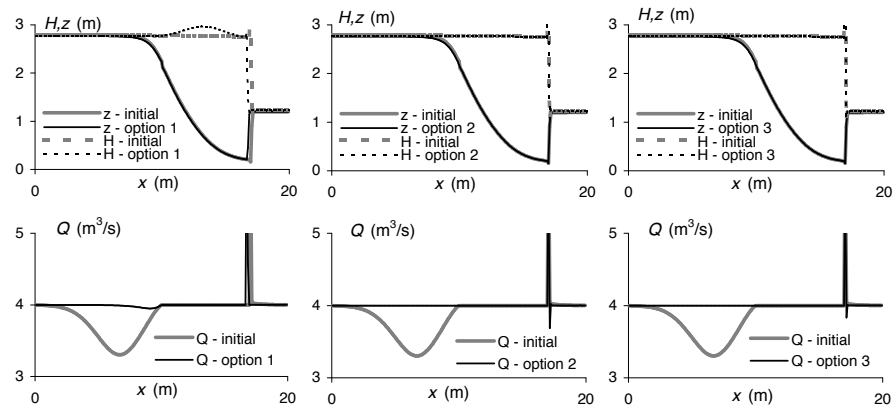




(a) HLL



(b) Roe



(c) Q-scheme

Figure 7: Test 3 - Frictionless steady state flow in a non-prismatic trapezoidal channel. Top: water elevation  $z$  and hydraulic head  $H$ , down: discharge  $Q$  obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options, using a) HLL solver, b) Roe's solver, c) Q-scheme.

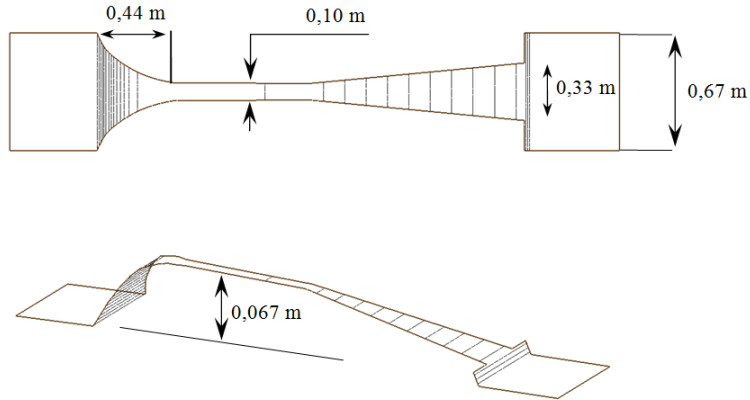


Figure 8: Test 4 - Dimensions of the Venturi flume used in the experiment. Top: plan view. Bottom: bird eye's view with a vertical scale magnified by a factor 5.

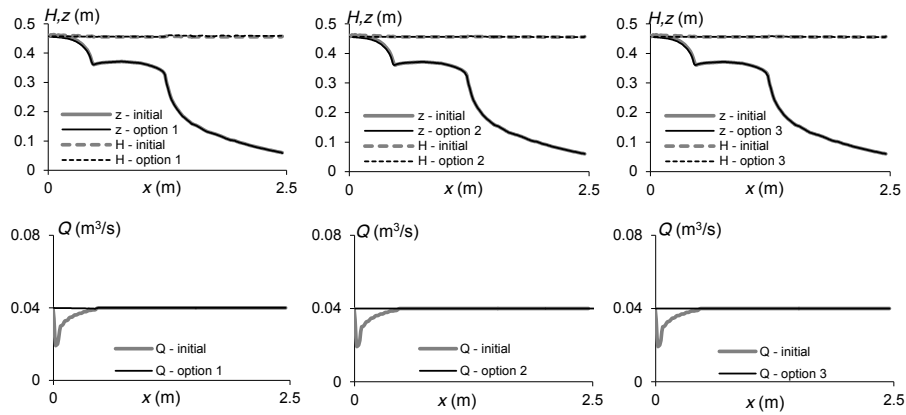


Figure 9: Test 4 - Steady state flow in a Venturi flume. Top: water elevation  $z$  and hydraulic head  $H$ , down: discharge  $Q$  obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options, using Roe's solver.

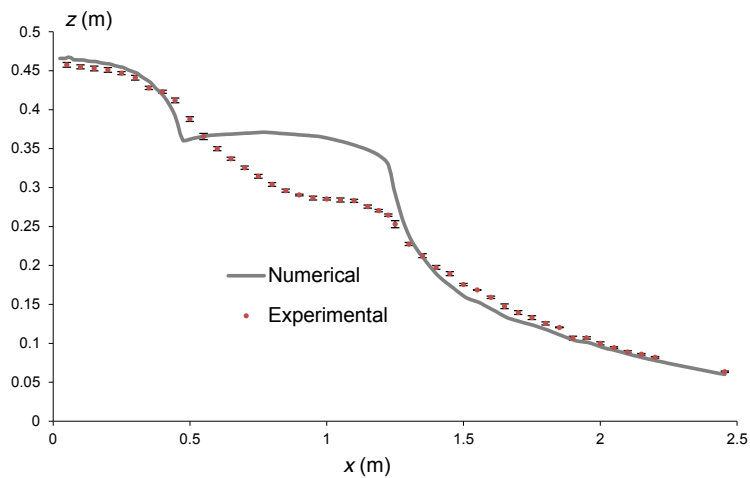
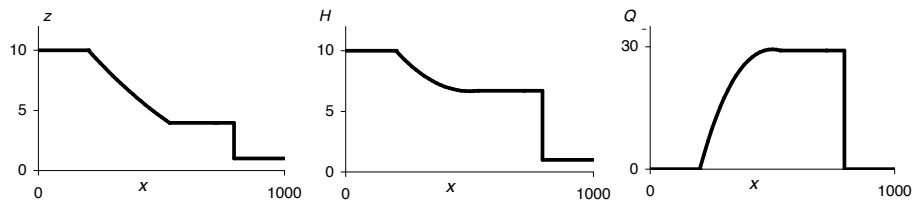
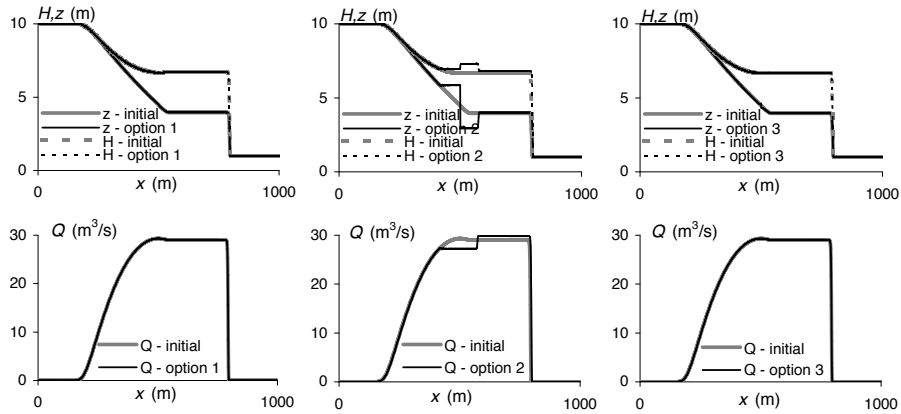


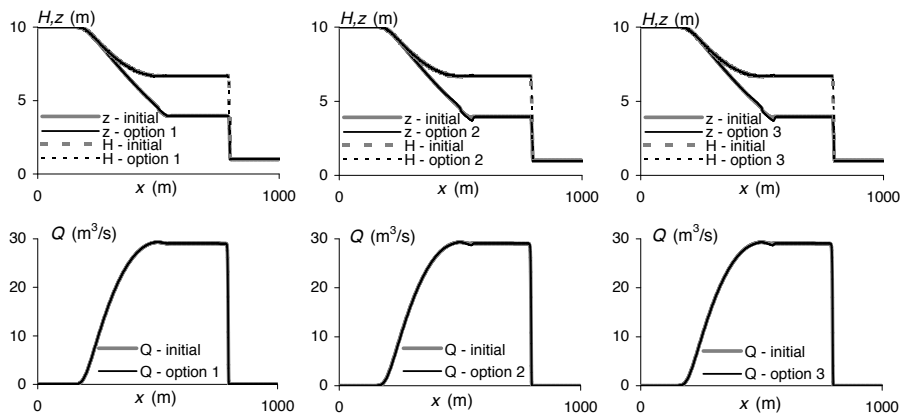
Figure 10: Test 4 - Steady state flow in a Venturi flume. Comparison between numerical results and experimental data.



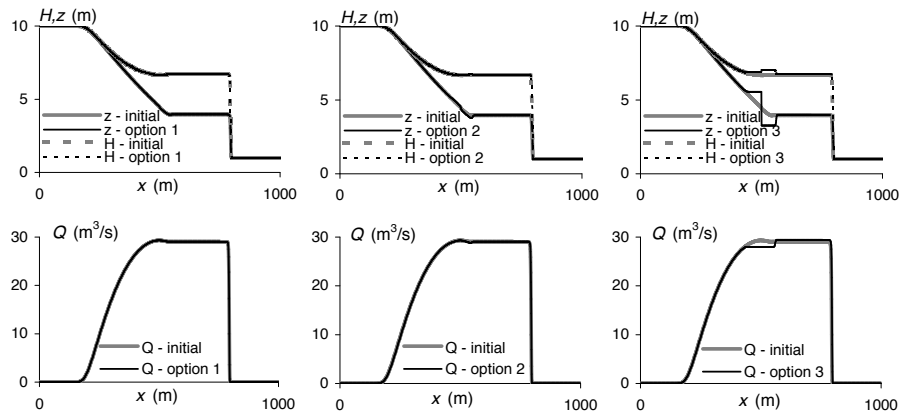
(a) Analytical solution



(b) HLL



(c) Roe



(d)  $Q$ -scheme

Figure 11: Test 5 - Dam-break problem in a rectangular channel. a) analytical solution; b) HLL solver; c) Roe's solver; d)  $Q$ -scheme. Top: water surface elevation  $z$  and hydraulic head  $H$ , bottom: discharge  $Q$ , obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options.

Symbol	Meaning	Value
$g$	Gravitational acceleration	$9.81 \text{ m s}^{-2}$
$L$	Length of the domain	1,000 m
$W_0$	Channel width	1 m
$W_1$	Derivative of the channel width with respect to $z$	0
$z_b$	Bottom elevation	0 m
$h_L$	Initial free surface elevation on the left-hand side of the dam	10 m
$h_R$	Initial free surface elevation on the right-hand side of the dam	1 m
$n_M$	Manning's friction coefficient	$0 \text{ m}^{-1/3} \text{ s}$
$\Delta x$	Computational cell width	1 m
$\Delta t$	Simulation time	30 s

Table 4: Test 5 - dam-break problem in a rectangular channel. Parameters of the test case.

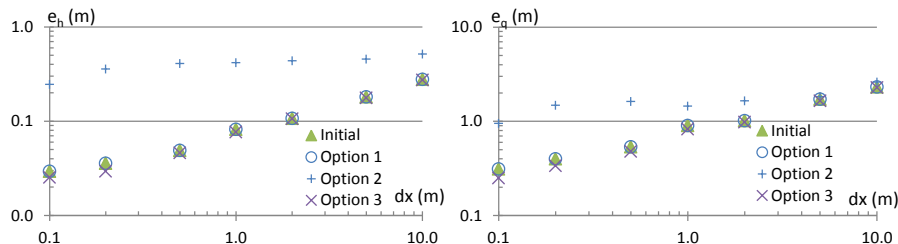


Figure 12: Test 5 - Dam-break problem in a rectangular channel. Convergence analysis using HLL solver and the three AVB options.  $L_2$ -norm between the computed output water depth (left) or unit discharge (right) and analytical solution.

421 Since the analytical solution is available for the dam-break problem, a convergence  
422 analysis is performed on this test case using the three AVB options and HLL solver. Figure 12  
423 shows that options 1 and 3 have almost the same convergence as initial formulation (slightly  
424 faster for Option 3), and confirms the non-convergence of Option 2 used with HLL solver.

#### 425 5.2.2. Test 6: frictionless dam-break problem in a triangular channel with flat bottom

426 This test case is identical to the previous one (dam-break problem in a rectangular  
427 channel, without bottom slope or friction) except that the cross-section of the channel has  
428 a triangular shape. The parameters of the test case are the same as given in Table 4 for  
429 Test 5 except that  $W_0 = 0$  and  $W_1 = 2$  in the whole domain.

430 The analytical solution is given by [25]:

$$431 \quad u_* + 4c_* = u_L + 4c_L \quad (66a)$$

$$432 \quad Q_* - Q_R = (A_* - A_R) c_s \quad (66b)$$

$$433 \quad \left( \frac{Q^2}{A} + \frac{gA^2}{2} \right)_* - \left( \frac{Q^2}{A} + \frac{gA^2}{2} \right)_R = (Q_* - Q_R) c_s \quad (66c)$$

434 where the subscript  $*$  denotes the intermediate region of constant state.

435 Equation (66a) expresses the invariance of the Riemann invariant ( $u + 4c$ ) across the  
436 rarefaction wave. Equations (66b) and (66c) are the jump relationships across the shock  
437 moving at the speed  $c_s$ . The unknown shock speed can be eliminated from the system by  
438 combining the second and third equations. The system can then be solved iteratively to  
439 find the values of  $A$  and  $Q$  in the intermediate region of constant state using  $A = c^2/g$   
 $Q = uA$ . Across the rarefaction wave,  $u$  and  $c$  verify:

$$440 \quad u + 4c = u_L + 4c_L \quad (67a)$$

$$441 \quad u - c = \frac{x}{t} \quad (67b)$$

yielding the following profile for  $u$  and  $c$  in the rarefaction wave:

$$442 \quad u(x, t) = \frac{4}{5} \left( c_L + \frac{x}{t} \right) \quad (68a)$$

Symbol	Meaning	Value
$L$	Length of the domain	1,000 m
$W_0$	Channel width	1 m
$W_1$	Derivative of the channel width with respect to $z$	0
$z_{bL}$	Bottom elevation on the left-hand side of the dam	0 m
$z_{bR}$	Bottom elevation on the right-hand side of the dam	5 m
$h_L$	Initial free surface elevation on the left-hand side of the dam	15 m
$h_R$	Initial free surface elevation on the right-hand side of the dam	1 m
$n_M$	Manning's friction coefficient	$0 \text{ m}^{-1/3} \text{ s}$
$\Delta x$	Computational cell width	1 m
$\Delta t$	Simulation time	30 s

Table 5: Test 5 - dam-break problem in a rectangular channel. Parameters of the test case.

442

$$c(x, t) = \frac{1}{5} \left( 4c_L - \frac{x}{t} \right) \quad (68b)$$

443

from which  $A$  and  $Q$  profiles can be determined.

444

Results of water elevation  $z$ , hydraulic head  $H$  and discharge  $Q$ , obtained with the three AVB options and the three solvers are given in Figure 13. In this case again, Roe's solver gives satisfactory results with the 3 options as well as the initial formulation. However, very strong discontinuities at the critical point can be seen with Option 2 and 3 combined with HLL solver and  $Q$ -scheme, yielding to an underestimation of the maximum discharge. Moreover, the shock is incorrectly located with Option 2/HLL.

445

### 450 5.2.3. Test 7: frictionless dam-break problem on a bottom step

451

The parameters of this test case are given in Table 5. A bottom step of 5 m is located at the same abscissa as the initial water depth discontinuity. The analytical solution (that can be found for example in [1, 5]) as well as results obtained with the three AVB options and Roe's solver are given in Figure 14. HLL solver and  $Q$ -scheme provide erroneous solutions with Option 2 and 3.

452

## 456 6. Discussion - Conclusions

457

In practical engineering applications, geometrical source terms arising e.g. from bottom slope or the non prismatic character of the channel are to be accounted for in the governing equations. These source terms can in general not be discretized independently of the conservation part. Riemann-solver based techniques compute the fluxes from the average cell values on the left and right hand of the interface. The flux can be seen as a combination of the average cell fluxes, augmented with a diffusion term involving the gradient in the conserved variable. Artificial oscillations may appear in the computed profiles if the gradients (and hence the diffusive part of the flux) is not estimated properly. The Auxiliary Variable-Based balancing, consists of using an "auxiliary" variable instead of the conserved one in the flux function, defined so as to allow the steady-state condition (of which static equilibrium is only a particular case) to be preserved. It is applied to the one-dimensional open channel equations in the present paper.

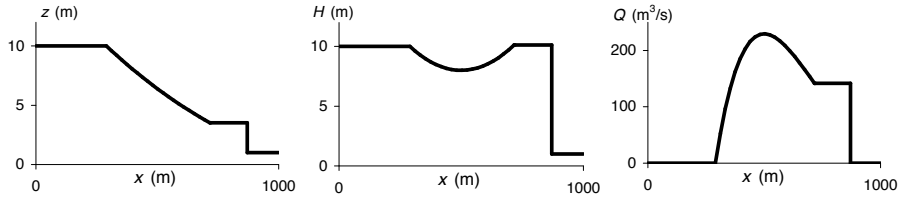
458

Three different options of AVB have been tested in this paper in addition to the classical flux formulation that uses the gradient of the conserved variables: 1) free surface elevation; 2) specific force and 3) hydraulic head. The application of the method to three classical approximate Riemann solvers (HLL, Roe and  $Q$ -scheme) is also presented.

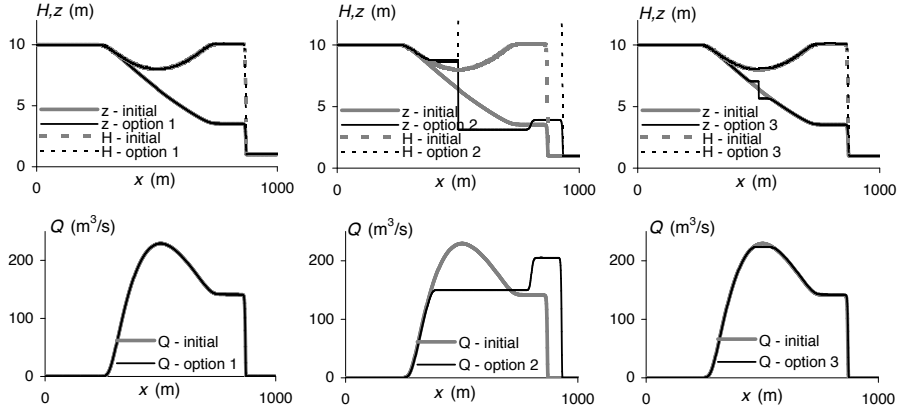
459

Various steady-state test cases including singular head losses, friction, bottom and width variation, non-prismatic configurations, have been implemented to assess the ability of the AVB approach to deal with transcritical flows, the critical points being well-known to introduce instabilities. In the steady-state test cases, the three options generally gives a better estimate of the uniform discharge than the initial formulation. Option 3, based on the hydraulic head, is the one that gives a uniform discharge equal to the prescribed one with the best accuracy, for each steady-state test case and in the whole domain, except across hydraulic jumps where a small spike remains. It is important to check the validity of these approaches for unsteady states configurations. Indeed, some examples in the literature that give correct results in steady state configurations (such as [28]) have revealed incorrect on

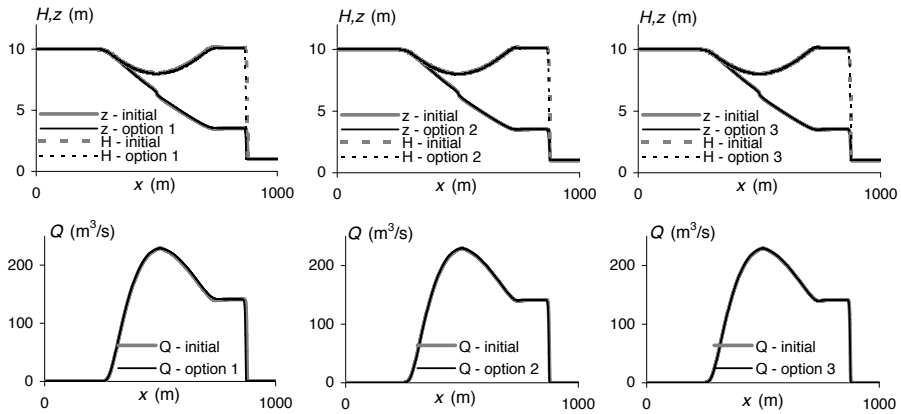
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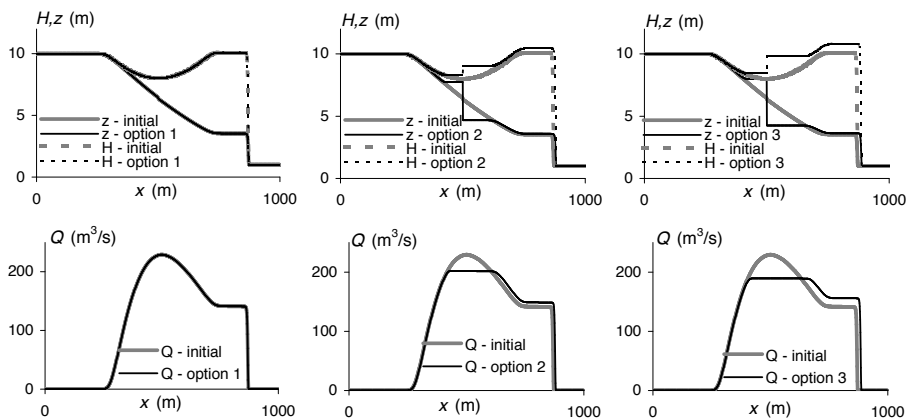
(a) Analytical solution



(b) HLL

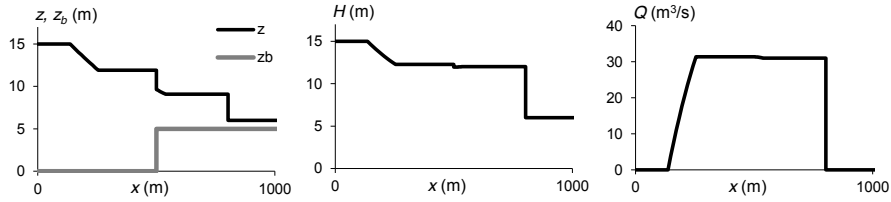


(c) Roe

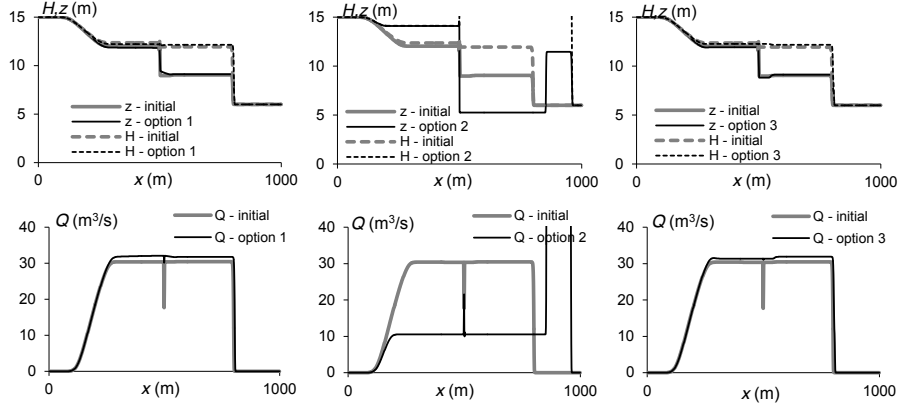


(d)  $Q$ -scheme

Figure 13: Test 6 - Dam-break problem in a triangular channel. a) Analytical solution, b) HLL solver, c) Roe's solver and d)  $Q$ -scheme. For each sub-figure, top: hydraulic head  $H$  and water elevation  $z$ , bottom: discharge  $Q$ , obtained with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options.



(a) Analytical solution



(b) Roe

Figure 14: Test 07 - Dam-break problem on a bottom step. a) Analytical solution; b) top: water surface elevation  $z$  and hydraulic head  $H$ , bottom: discharge  $Q$ , obtained using Roe's solver with  $\mathbf{V} = \mathbf{U}$  (Initial) and with the three different AVB options.

483 transient test cases, such as shown in the three unsteady configurations presented (dam-  
 484 break in rectangular or triangular channels and over a bottom step) where Option 2 gives  
 485 bad results when used with HLL or  $Q$ -scheme.

486 Finally, when used with Roe's solver, Option 3 is the only one that produces correct  
 487 results for all the test cases. An interesting feature of this option is that it allows head loss  
 488 functions (stemming from e.g. bridges or other singularities) to be accounted for directly  
 489 within the discretized equations. In contrast with what is classically done in commercially  
 490 available river packages, the AVB method eliminates the need for internal boundaries across  
 491 hydraulic singularities.

492  
 493 In the present paper, the geometric source term is accounted for by integrating the  
 494 bottom slope over the surface of the (non-horizontal) computational cell. This procedure is  
 495 rather easy to carry out when the cell is full. But in the case of wetting/drying, the water  
 496 does not occupy the full length of the cell. Computing the integral of the term  $ghS_0$  over  
 497 only part of the cell becomes a very complex and time-consuming task. The approach to  
 498 source term estimation proposed in the present paper is thus not the best possible option  
 499 to the discretization of real-world geometries and practical river problems. An alternative  
 500 option is currently under study. It consists in considering each cell as prismatic and lumping  
 501 the geometric source term at the cell interfaces. This approach, however, requires that a  
 502 proper splitting of the lumped source terms between the adjacent cells to the interfaces be  
 503 devised. This point is currently under study.

504 Finally, this paper deals with finite volume Godunov-type discretizations, but if higher-  
 505 order schemes are to be designed, the AVB method may be applied by reconstructing the  
 506 auxiliary variable.

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