



HAL
open science

Estimating the Division Kernel of a Size-Structured Population

van Ha Hoang

► **To cite this version:**

van Ha Hoang. Estimating the Division Kernel of a Size-Structured Population. 2015. hal-01196427v1

HAL Id: hal-01196427

<https://hal.science/hal-01196427v1>

Preprint submitted on 9 Sep 2015 (v1), last revised 18 May 2016 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Estimating the Division Kernel of a Size-Structured Population

Van Ha Hoang

September 9, 2015

Abstract

We consider a size-structured population describing the cell divisions. The cell population is described by an empirical measure and we observe the divisions in the continuous time interval $[0, T]$. We address here the problem of estimating the division kernel h (or fragmentation kernel) in case of complete data. An adaptive estimator of h is constructed based on a kernel function K with a fully data-driven bandwidth selection method. We obtain an oracle inequality and an exponential convergence rate, for which optimality is considered.

Keywords: random size-structured population, division kernel, nonparametric estimation, Goldenshluger-Lepski's method, adaptive estimator, penalization, optimal rate.

1 Introduction

In this paper, we are interested in a stochastic individual-based model of size-structured population in continuous time where individuals are cells undergoing binary divisions. Individuals are characterized by their sizes, *i.e.* variables that grow deterministically with time (such as volume, length, level of certain proteins, DNA content, *etc.* ...). Such models have been studied in Athreya and Ney [1], Harris [9], Jagers [16] and make the object of an abundant literature (*e.g.* Bansaye *et al.* [3], Cloez [5], Tran [20]). Here, we have in mind that each cell contains some toxicities which play the role of the size, in the spirit of the study of Stewart *et al.* [18].

Let us give a brief description of our model. A cell containing a toxicity $x \in \mathbb{R}_+$ divides at a rate $R > 0$. The toxicity grows inside the cell with rate $\alpha > 0$. In general, the division rate and the growth rate can be non constant but we consider here the constant case. When a cell divides, a random fraction $\Gamma \in [0, 1]$ of the toxicity goes in the first daughter cell and $1 - \Gamma$ in the second one. If $\Gamma = \frac{1}{2}$, the daughters are the same with toxicity

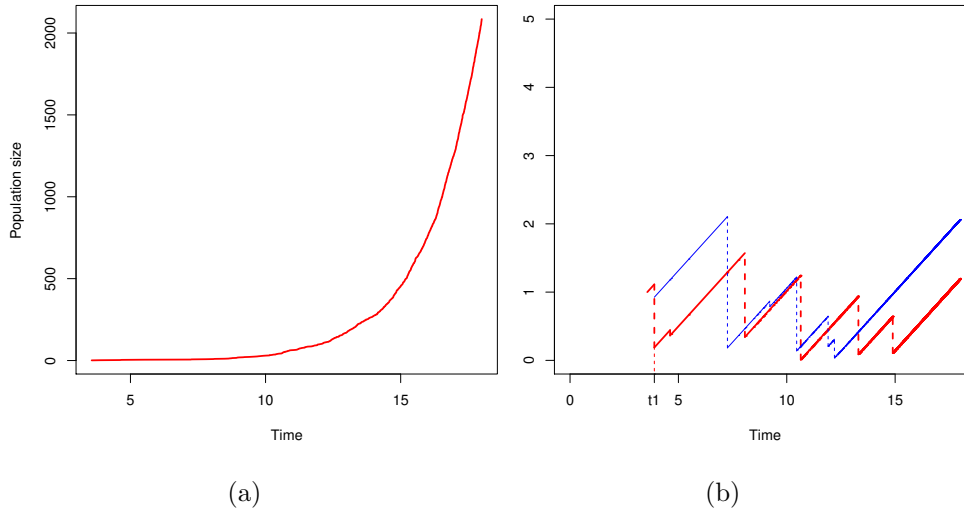


Figure 1: (a) Population size. (b) Trajectories of two offsprings, separating after the first division at time t_1 .

$\frac{x}{2}$. We assume that Γ has a symmetric distribution on $[0, 1]$ with a density h with respect to Lebesgue measure such that $\mathbb{P}(\Gamma = 0) = \mathbb{P}(\Gamma = 1) = 0$. The more h puts weight in the neighbourhood of 0 and 1, the more asymmetric the divisions are. Figure 1 shows the trajectories of two daughter cells after a division and we also observe that the population size increases with an exponential rate.

Individual-based models provide a natural framework for statistical estimation. Estimation of the division rate is, for instance, the subject of Doumic et al. [6, 7], Hoffmann and Olivier [10]. Here, the density h is the kernel division that we want to estimate. The interest of estimating h is because the kurtosis of h provides indication on the age of the lineage and to detect aging phenomena such as the one put into light by Stewart et al. [18]. If the division kernel is not peaked at $1/2$, the division is asymmetric and we can consider that one daughter is older. Using the number of divisions since the older daughter was formed, Stewart et al. [18] assigned to the cells an age in divisions. Hence, if individual cells are followed over time and if the density h of Γ and the growth rate α are known, we can study the aging effect. More discussions about the influence of the kernel division to aging effect will be presented in Section 3 with simulation results.

Assuming that we observe the divisions of cells in continuous time on the interval $[0, T]$, with $T > 0$, we propose an adaptive kernel estimator \hat{h} of

h for which we obtain an oracle inequality in Theorem 2. The construction of \hat{h} is detailed in the sequel. From oracle inequality we can infer adaptive exponential rates of convergence with respect to T depending on β the smoothness of the density. Most of the time, nonparametric rates are of the form $n^{-\frac{2\beta}{2\beta+1}}$ (see for instance Tsybakov [21]) and exponential rates are not often encountered in the literature. The exponential rates are due to binary splitting, the number of cells *i.e* the sample size increases exponentially in $\exp(RT)$ (see section 2.3). By comparison, in [10] Hoffmann and Olivier obtain a similar rate of convergence $\exp\left(-\lambda_B \frac{\varsigma}{2\varsigma+1} T\right)$ of the kernel estimator of their division rate $B(x)$, where λ_B is the Malthus parameter and $\varsigma > 0$ is the smoothness of $B(x)$. However, their estimator \hat{B}_T of B is not adaptive since the choice of their optimal bandwidth still depends on ς . Our estimator is adaptive with an "optimal" bandwidth $\hat{\ell}$ chosen from a data-driven method.

This paper is organized as follows. In Section 2, we introduce a stochastic differential equation driven by a Poisson point measure to describe the population of cells. Then, we construct the estimator of h and obtain upper and lower bound for the MISE and thus the rate of convergence under L_2 -risk. Our main results are stated in Theorems 3 and 4. Numerical results and discussions about aging effect are presented in Section 3. The main proofs are shown in Section 4.

2 Microscopic model and Kernel estimator of h

2.1 The model

We recall the Ulam-Harris-Neveu notation used to describe the genealogical tree. The first cell is labeled by \emptyset and when the cell i divides, the two descendants are labeled by $i0$ and $i1$. The set of labels is

$$J = \{\emptyset\} \cup \bigcup_{m=1}^{\infty} \{0, 1\}^m. \quad (1)$$

We denote V_t be the set of cells alive at time t , and $V_t \subset J$.

Let $\mathcal{M}_F(\mathbb{R}_+)$ be the space of finite measures on \mathbb{R}_+ embedded with the topology of weak convergence and X_t^i be the quantity of toxicity in the cell i at time t , we describe the population of cells at time t by a random point measure in $\mathcal{M}_F(\mathbb{R}_+)$:

$$Z_t(dx) = \sum_{i=1}^{N_t} \delta_{X_t^i}(dx), \quad \text{where} \quad N_t = \langle Z_t, 1 \rangle = \int_{\mathbb{R}_+} Z_t(dx) \quad (2)$$

is the number of individuals living at time t . For a measure $\mu \in \mathcal{M}_F(\mathbb{R}_+)$ and a positive function f , we use the notation $\langle \mu, f \rangle = \int_{\mathbb{R}_+} f d\mu$.

Along branches of the genealogical tree, the toxicity $(X_t, t \geq 0)$ satisfies

$$dX_t = \alpha dt, \quad (3)$$

with $X_0 = x_0$. When the cells divide, the toxicity is shared between the daughter cells. This is described by the following stochastic differential equation (SDE).

Let $Z_0 \in \mathcal{M}_F(\mathbb{R}_+)$ be an initial condition such that

$$\mathbb{E}(\langle Z_0, 1 \rangle) < +\infty, \quad (4)$$

and let $Q(ds, di, d\gamma)$ be a Poisson point measure on $\mathbb{R}_+ \times \mathcal{E} := \mathbb{R}_+ \times J \times [0, 1]$ with intensity $q(ds, di, d\gamma) = R ds n(di) H(d\gamma)$. $n(di)$ is the counting measure on J and ds is Lebesgue measure on \mathbb{R}_+ . We denote $\{\mathcal{F}_t\}_{t \geq 0}$ the canonical filtration associated with the Poisson point measure and the initial condition. The stochastic process $(Z_t)_{t \geq 0}$ can be described by a SDE as follows.

Definition 1. *For every test function $f_t(x) = f(x, t) \in \mathcal{C}_b^{1,1}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ (bounded of class \mathcal{C}^1 in t and x with bounded derivatives), the population of cells is described by:*

$$\begin{aligned} \langle Z_t, f_t \rangle &= \langle Z_0, f_0 \rangle + \int_0^t \int_{\mathbb{R}_+} (\partial_s f_s(x) + \alpha \partial_x f_s(x)) Z_s(dx) ds \\ &+ \int_0^t \int_{\mathcal{E}} \mathbb{1}_{\{i \leq N_{s-}\}} \left[f_s(\gamma X_{s-}^i) + f_s((1-\gamma)X_{s-}^i) - f_s(X_{s-}^i) \right] Q(ds, di, d\gamma). \end{aligned} \quad (5)$$

The second term in the right hand side of (5) corresponds to the growth of toxicities in the cells and the third term gives a description of cell divisions where the sharing of toxicity into two daughter cells depends on the random fraction Γ .

We now state some properties of N_t that are useful in the sequel.

Proposition 1. *Let $T > 0$, and assume the initial condition N_0 , the number of mother cells at time $t = 0$, is deterministic, for the sake of simplicity. We have*

i) Let T_j be the j^{th} -jump time. Then:

$$\lim_{j \rightarrow +\infty} T_j = +\infty \text{ and } \lim_{T \rightarrow +\infty} N_T = +\infty \quad (a.s). \quad (6)$$

ii)

$$\mathbb{E}[N_T] \leq N_0 e^{RT}. \quad (7)$$

iii)

$$\frac{1}{N_0}e^{-RT} \leq \mathbb{E} \left[\frac{1}{N_T} \right] \leq \frac{1}{N_0}e^{-\frac{N_0}{N_0+1}RT}, \quad (8)$$

Proof.

i) Let us prove that $\lim_{T \rightarrow +\infty} N_T = \lim_{j \rightarrow +\infty} N_{T_j} = +\infty$. Since our model has only births and no death, $(N_t)_{t \in [0, T]}$ is a non-decreasing process: $N_{T_j} = N_0 + j$. We deduce from the estimate $\sup_{t \in [0, T]} \mathbb{E}[N_t] < +\infty$ for all $T > 0$ that $T_j \xrightarrow{j \rightarrow +\infty} +\infty$ a.s. Hence, $\lim_{T \rightarrow +\infty} N_T = +\infty$ a.s.

ii) From (5), taking $f_t(x) = 1$, we have

$$N_T = N_0 + \int_0^T \int_{\mathcal{E}} \mathbb{1}_{\{i \leq N_{s-}\}} Q(ds, di, d\gamma). \quad (9)$$

Thus,

$$\mathbb{E}[N_T] = N_0 + \mathbb{E} \left[\int_0^T RN_s ds \right] = N_0 + R \int_0^T \mathbb{E}[N_s] ds.$$

Applying Gronwall's inequality, we obtain the result.

iii) Applying Itô formula for jump processes (see [11], Theorem 5.1 on p.67) to (9), we obtain

$$\begin{aligned} \frac{1}{N_T} &= \frac{1}{N_0} + \int_0^T \int_{\mathcal{E}} \left(\frac{1}{N_{s-} + 1} - \frac{1}{N_{s-}} \right) \mathbb{1}_{\{i \leq N_{s-}\}} Q(ds, di, d\gamma) \\ &= \frac{1}{N_0} - \int_0^T \int_{\mathcal{E}} \frac{1}{N_{s-} (N_{s-} + 1)} \mathbb{1}_{\{i \leq N_{s-}\}} Q(ds, di, d\gamma). \end{aligned}$$

Hence,

$$\mathbb{E} \left[\frac{1}{N_T} \right] = \frac{1}{N_0} - \mathbb{E} \left[\int_0^T \frac{1}{N_s (N_s + 1)} RN_s ds \right] = \frac{1}{N_0} - R \int_0^T \mathbb{E} \left[\frac{1}{N_s + 1} \right] ds. \quad (10)$$

Since $N_s \geq N_0$, we have

$$\frac{1}{N_s} \frac{N_0}{N_0 + 1} \leq \frac{1}{N_s + 1} \leq \frac{1}{N_s}.$$

Therefore, (10) implies that

$$\frac{1}{N_0} - R \int_0^T \mathbb{E} \left[\frac{1}{N_s} \right] ds \leq \mathbb{E} \left[\frac{1}{N_T} \right] \leq \frac{1}{N_0} - \frac{N_0 R}{N_0 + 1} \int_0^T \mathbb{E} \left[\frac{1}{N_s} \right] ds. \quad (11)$$

By comparison $\mathbb{E} [1/N_T]$ with the solutions of the following ODE's

$$\frac{du_1(T)}{dT} = -Ru_1(T) \text{ and } \frac{du_2(T)}{dT} = -\frac{N_0 R}{N_0 + 1} u_2(T),$$

where $u_1(0) = u_2(0) = 1/N_0$, we finally obtain

$$\frac{1}{N_0} e^{-RT} \leq \mathbb{E} \left[\frac{1}{N_T} \right] \leq \frac{1}{N_0} e^{-\frac{N_0}{N_0+1} RT}.$$

□

2.2 Influence of age

In this section, we study the aging effect via the mean age which is defined as follows.

Definition 2. *The mean age of the cell population up to time $t \in \mathbb{R}_+$ is defined by:*

$$\bar{X}_t = \frac{1}{N_t} \sum_{i=1}^{N_t} X_t^i = \frac{\langle Z_t, f \rangle}{N_t}, \quad (12)$$

where $f(x) = x$.

Following the work of Bansaye et al. [2], we note that the long time behavior of the mean age is related to the law of an auxiliary process Y started at $Y_0 = \frac{X_0}{N_0}$ with infinitesimal generator characterized by $\forall f \in \mathcal{C}_b^{1,1}(\mathbb{R}_+, \mathbb{R})$,

$$Af(x) = \alpha f'(x) + 2R \int_0^1 (f(\gamma x) - f(x)) h(\gamma) d\gamma. \quad (13)$$

The empirical distribution of $\frac{1}{N_t} \sum_{i=1}^{N_t} \delta_{X_t^i}$ gives the law of the path of a particle chosen at random at time t . Heuristically, the distribution of Y restricted to $[0, t]$ approximates this distribution. Hence, this explains the coefficient 2 which is a size-biased phenomenon, *i.e.* when one chooses a cell in the population at time t , a cell belonging to a branch with more descendants is more likely to be chosen.

We will show that the auxiliary process Y satisfies ergodic properties (Section 4.1) which entails the following theorem.

Theorem 1. *For $t \in \mathbb{R}_+$,*

$$\lim_{t \rightarrow +\infty} \bar{X}_t = \lim_{t \rightarrow +\infty} \mathbb{E}(Y_t) = \frac{\alpha}{R}. \quad (14)$$

Theorem 1 is a consequence of the ergodic properties of Y , of Theorem 4.2 in Bansaye et al. [2] and of the Lemma 1. It shows that the average of the mean age tends to the constant α/R when the time t is large. Simulations in section 3 illustrate the results.

Lemma 1. *Let Y be the auxiliary process with infinitesimal generator (13), for $t \in \mathbb{R}_+$,*

$$Y_t = \left(Y_0 - \frac{\alpha}{R} \right) e^{-Rt} + \frac{\alpha}{R} + \int_0^t e^{-R(t-s)} dM_s. \quad (15)$$

where M_t is a square-integrable martingale.

Proof. By symmetry of h with respect to $1/2$, we have:

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \left(\alpha + 2R \int_0^1 (\gamma Y_s - Y_s) h(\gamma) d\gamma \right) ds + M_t \\ &= Y_0 + \int_0^t \left(\alpha - 2RY_s \int_0^1 \gamma h(\gamma) d\gamma \right) ds + M_t \\ &= Y_0 + \int_0^t (\alpha - RY_s) ds + M_t. \end{aligned}$$

where M_t is a square-integrable martingale.

Let $\tilde{Y}_t = Y_t e^{Rt}$, $\tilde{Y}_0 = Y_0$. By Itô formula, we get

$$\tilde{Y}_t = \tilde{Y}_0 + \frac{\alpha}{R} (e^{Rt} - 1) + \int_0^t e^{Rs} dM_s,$$

Replacing \tilde{Y}_t by $Y_t e^{Rt}$, we obtain

$$Y_t = \left(Y_0 - \frac{\alpha}{R} \right) e^{-Rt} + \frac{\alpha}{R} + \int_0^t e^{-R(t-s)} dM_s.$$

□

Corollary 1. *From (15), we note that*

$$\mathbb{E}[Y_t] = \left(Y_0 - \frac{\alpha}{R} \right) e^{-Rt} + \frac{\alpha}{R}, \quad (16)$$

and

$$\lim_{t \rightarrow \infty} \mathbb{E}[Y_t] = \frac{\alpha}{R}. \quad (17)$$

Remark 1. *When the population is large, we are interested in studying the asymptotic behavior of the random point measure. As in Doumic et al. [7], we can show that our stochastic model is approximated by a growth-fragmentation partial differential equation. This problem is a work in progress.*

2.3 Estimation of the kernel division

Data and construction of the estimator

Suppose that we observe the evolution of the cell population in a given time interval $[0, T]$. At the i^{th} division time t_i , let us denote j_i the individual who splits into two daughters $X_{t_i}^{j_i 1}$ and $X_{t_i}^{j_i 2}$ and define

$$\Gamma_i^1 = \frac{X_{t_i}^{j_i 1}}{X_{t_i}^{j_i}} \quad \text{and} \quad \Gamma_i^2 = \frac{X_{t_i}^{j_i 2}}{X_{t_i}^{j_i}},$$

the random fractions that go into the daughter cells, with the convention $\frac{0}{0} = 0$.

Γ_i^1 and Γ_i^2 are exchangeable with $\Gamma_i^1 + \Gamma_i^2 = 1$, Γ_i^1 and Γ_i^2 are thus not independent but the couples $(\Gamma_i^1, \Gamma_i^2)_{i \in \mathbb{N}^*}$ are independent and identically distributed with distribution (Γ^1, Γ^2) where $\Gamma^1 \sim H(d\gamma)$ and $\Gamma^2 = 1 - \Gamma^1$.

Since h is a density function, it is natural to use a kernel method. We define an estimator \hat{h}_ℓ of h based on the data $(\Gamma_i^1, \Gamma_i^2)_{i \in \mathbb{N}^*}$ as follows.

Definition 3. Let N_T be the random number of divisions in the time interval $[0, T]$. For all $\gamma \in (0, 1)$, define

$$\hat{h}_\ell(\gamma) = \frac{1}{N_T} \sum_{i=1}^{N_T} K_\ell(\gamma - \Gamma_i^1), \quad (18)$$

where $K_\ell = \frac{1}{\ell} K(\cdot/\ell)$ is a bounded kernel function, $\ell > 0$ is the bandwidth to be chosen.

In (18), \hat{h}_ℓ depends also on T . However, we omit T for the sake of notation. The estimator \hat{h}_ℓ will satisfy the following properties.

Proposition 2.

i) The expectation and variance of $\hat{h}_\ell(\gamma)$, conditional variance of $\hat{h}_\ell(\gamma)$ given N_T are:

$$\mathbb{E}[\hat{h}_\ell(\gamma)] = K_\ell \star h(\gamma) \quad \text{and} \quad \mathbb{E}[\hat{h}_\ell(\gamma) | N_T] = K_\ell \star h(\gamma), \quad (19)$$

$$\text{Var}[\hat{h}_\ell(\gamma)] = \mathbb{E}\left[\frac{1}{N_T}\right] \text{Var}\left[K_\ell(\gamma - \Gamma_1^1)\right], \quad (20)$$

$$\text{Var}[\hat{h}_\ell(\gamma) | N_T] = \frac{1}{N_T} \text{Var}\left[K_\ell(\gamma - \Gamma_1^1)\right]. \quad (21)$$

ii) For all $\gamma \in (0, 1)$,

$$\lim_{T \rightarrow +\infty} \hat{h}_\ell(\gamma) = K_\ell \star h(\gamma) \quad (a.s.). \quad (22)$$

Proof. To prove (19), let us remark that the number of random divisions N_T is independent of $(\Gamma_i^1)_{i \in \mathbb{N}^*}$, because the division rate R is constant and the construction of our stochastic process. Therefore, we have

$$\begin{aligned}\mathbb{E}[\hat{h}_\ell | N_T] &= \mathbb{E}\left[\frac{1}{N_T} \sum_{i=1}^{N_T} K_\ell(\gamma - \Gamma_i^1) | N_T\right] = \frac{N_T \mathbb{E}[K_\ell(\gamma - \Gamma_1^1)]}{N_T} \\ &= \mathbb{E}[K_\ell(\gamma - \Gamma_1^1)] = K_\ell \star h(\gamma),\end{aligned}$$

and $\mathbb{E}[\hat{h}_\ell] = \mathbb{E}\left[\mathbb{E}[\hat{h}_\ell | N_T]\right] = K_\ell \star h(\gamma)$. By similar calculations as (19), we obtain (20) and (21).

To prove *ii*), by the Strong Law of Large Numbers, we have

$$\frac{1}{n} \sum_{i=1}^n K_\ell(\gamma - \Gamma_i^1) \xrightarrow{\text{a.s.}} \mathbb{E}[K_\ell(\gamma - \Gamma_1^1)] \quad \text{as } n \rightarrow +\infty.$$

Since $\lim_{T \rightarrow +\infty} N_T = +\infty$ (a.s), this yields

$$\frac{1}{N_T} \sum_{i=1}^{N_T} K_\ell(\gamma - \Gamma_i^1) \xrightarrow{\text{a.s.}} \mathbb{E}[K_\ell(\gamma - \Gamma_1^1)] = K_\ell \star h(\gamma).$$

□

Adaptive estimation of h by GL method

Let \hat{h}_ℓ be the kernel estimator of h as in Definition 3. Our problem is bandwidth selection. The objective is to find a bandwidth that minimizes the L_2 -risk. Since N_T is random, we first study the MISE conditionally to N_T .

Proposition 3. *The L_2 -risk of \hat{h}_ℓ given N_T satisfies:*

$$\mathbb{E}\left[\|\hat{h}_\ell - h\|_2 | N_T\right] \leq \|h - K_\ell \star h\|_2 + \frac{\|K\|_2}{\sqrt{N_T \ell}}. \quad (23)$$

Proof. We have

$$\mathbb{E}\left[\|\hat{h}_\ell - h\|_2 | N_T\right] \leq \|h - K_\ell \star h\|_2 + \mathbb{E}\left[\|\hat{h}_\ell - \mathbb{E}[\hat{h}_\ell]\|_2 | N_T\right].$$

For the variance term, using that $\mathbb{E}[\hat{h}_\ell(\gamma)] = \mathbb{E}[\hat{h}_\ell(\gamma) | N_T]$

$$\begin{aligned}\mathbb{E}\left[\|\hat{h}_\ell - \mathbb{E}[\hat{h}_\ell]\|_2^2 | N_T\right] &= \mathbb{E}\left[\int_0^1 |\hat{h}_\ell(\gamma) - \mathbb{E}[\hat{h}_\ell(\gamma)]|^2 d\gamma | N_T\right] \\ &= \int_0^1 \text{Var}\left[\hat{h}_\ell(\gamma) | N_T\right] d\gamma \\ &= \frac{1}{N_T} \int_0^1 \text{Var}\left[K_\ell(\gamma - \Gamma_1^1)\right] d\gamma \\ &\leq \frac{1}{N_T} \int_0^1 \mathbb{E}\left[K_\ell^2(\gamma - \Gamma_1^1)\right] d\gamma = \frac{\|K\|_2^2}{N_T \ell}.\end{aligned} \quad (24)$$

Hence, applying Cauchy-Schwarz's inequality, we obtain (23). □

In the right hand side of (23), the first term decreases when $\ell \rightarrow 0$ and the second one increases when $\ell \rightarrow 0$. The best choice of ℓ should minimize the bias-variance decomposition. Thus, from a finite family of bandwidths H , we propose a bandwidth $\bar{\ell}$ where

$$\bar{\ell} := \operatorname{argmin}_{\ell \in H} \left\{ \|h - K_\ell \star h\|_2 + \frac{\|K\|_2}{\sqrt{N_T \ell}} \right\}. \quad (25)$$

The bandwidth $\bar{\ell}$ is called oracle bandwidth since it depends on h and then it cannot be used in practice. Goldenshluger and Lepski [8] developed a fully data-driven bandwidth selection method (GL method). In a similar fashion, Doumic et al. [7] and Reynaud-Bouret et al. [17] have used this method. To apply this method, we set for any $\ell, \ell' \in H$:

$$\hat{h}_{\ell, \ell'} := \frac{1}{N_T} \sum_{i=1}^{N_T} (K_\ell \star K_{\ell'}) (\gamma - \Gamma_i^1) = (K_\ell \star \hat{h}_{\ell'}) (\gamma).$$

Finally, the adaptive bandwidth and the estimator of h are selected as follows:

Definition 4. Given $\epsilon > 0$ and setting $\chi := (1 + \epsilon)(1 + \|K\|_1)$, we define

$$\hat{\ell} := \operatorname{argmin}_{\ell \in H} \left\{ A(\ell) + \frac{\chi \|K\|_2}{\sqrt{N_T \ell}} \right\}, \quad (26)$$

where, for any $\ell \in H$,

$$A(\ell) := \sup_{\ell' \in H} \left\{ \|\hat{h}_{\ell, \ell'} - \hat{h}_{\ell'}\|_2 - \frac{\chi \|K\|_2}{\sqrt{N_T \ell'}} \right\}_+, \quad (27)$$

Then, the estimator \hat{h} is given by putting

$$\hat{h} := \hat{h}_{\hat{\ell}}. \quad (28)$$

Since $A(\ell)$ depends only on $\hat{h}_{\ell, \ell'}$ and $\hat{h}_{\ell'}$, the estimator \hat{h} can be computed in practice.

We shall now state the oracle inequality.

Theorem 2. Let $T > 0$ and assume that observations are taken on $[0, T]$. Let N_0 be the number of mother cells at the beginning of divisions and N_T is the random number of divisions in $[0, T]$. Consider H a denumerable subset of $\{D^{-1} : D = 1, \dots, D_{\max}\}$ in which we choose the bandwidths and $D_{\max} = \lfloor \delta N_T \rfloor$ for some $\delta > 0$. Assume $h \in L^\infty([0, 1])$ and let \hat{h} be a kernel estimator defined with the kernel $K_{\hat{\ell}}$ where $\hat{\ell}$ is chosen by the GL method. Then, given $\epsilon > 0$,

$$\mathbb{E} \left[\|\hat{h} - h\|_2^2 \right] \leq C_1 \inf_{\ell \in H} \left\{ \|K_\ell \star h - h\|_2^2 + \frac{\|K\|_2^2}{\ell} e^{-\frac{N_0}{N_0+1} RT} \right\} + C_2 e^{-\frac{N_0}{N_0+1} RT}. \quad (29)$$

where C_1 is a constant depending on $\|K\|_1$ and ϵ and C_2 is a constant depending on δ , ϵ , $\|K\|_1$, $\|K\|_2$, $\|h\|_\infty$.

The term $\|K_\ell \star h - h\|_2^2$ is an approximation term, $\frac{\|K\|_2^2}{\ell} e^{-\frac{N_0}{N_0+1}RT}$ is a variance term and the last term $e^{-\frac{N_0}{N_0+1}RT}$ is asymptotically negligible. Hence the right hand side of the oracle inequality corresponds to a bias variance trade-off.

We now establish upper and lower bounds for the MISE. The lower bound is obtained by perturbation methods (Theorem 4) and is valid for any estimator \hat{h}_T of h , thus indicating the optimal convergence rate. The upper bound is obtained (Theorem 3) thanks to the key oracle inequality of Theorem 2.

For the rate of convergence, it is necessary to assume that the density h and the kernel function K satisfy some regularity conditions introduced in the following definitions.

Definition 5. Let $\beta > 0$ and $L > 0$. The Hölder class of smoothness β and radius L is defined by

$$\mathcal{H}(\beta, L) = \left\{ f : f \text{ has } k = \lfloor \beta \rfloor \text{ derivative and } \forall x, y \in \mathbb{R} \right. \\ \left. |f^{(k)}(y) - f^{(k)}(x)| \leq L|x - y|^{\beta-k} \right\}.$$

Definition 6. Let $\beta^* > 0$. A function $K : \mathbb{R} \rightarrow \mathbb{R}$ is a kernel of order β^* if

- $\int K(x)dx = 1$,
- $\int |x|^{\beta^*} |K(x)|dx < \infty$,
- For $k = \lfloor \beta^* \rfloor$, $\forall 1 \leq j \leq k$, $\int x^j K(x)dx = 0$.

Then, the following theorem gives us the rate of convergence of the adaptive estimator \hat{h} .

Theorem 3. Assume that h belongs to $\mathcal{H}(\beta, L)$ and K is a kernel of order β^* with $\beta \in (0, \beta^*)$. Let $\hat{\ell}$ be the adaptive bandwidth defined in (26). Then, for any $T > 0$, the kernel estimator \hat{h} satisfies

$$\sup_{h \in \mathcal{H}(\beta, L)} \mathbb{E} \|\hat{h} - h\|_2^2 \leq C_3 \exp \left(-\frac{2\beta}{2\beta + 1} \frac{N_0}{N_0 + 1} RT \right), \quad (30)$$

where C_3 is a constant depending on δ , ϵ , $\|K\|_1$, $\|K\|_2$, $\|h\|_\infty$, β and L .

Remark 2. Theorem 3 illustrates adaptive properties of our procedure: it achieves the rate $\exp \left(-\frac{2\beta}{2\beta+1} \frac{N_0}{N_0+1} RT \right)$ over the Hölder class $\mathcal{H}(\beta, L)$ as soon as β is smaller than β^* . So, it automatically adapts to the unknown smoothness of the signal to estimate.

Theorem 4. For any $T > 0$, $\beta > 0$ and $L > 0$. Assume that $h \in \mathcal{H}(\beta, L)$, then there exists a constant $C_4 > 0$ such that for any estimator \hat{h}_T of h

$$\sup_{h \in \mathcal{H}(\beta, L)} \mathbb{E} \|\hat{h}_T - h\|_2^2 \geq C_4 \exp\left(-\frac{2\beta}{2\beta+1}RT\right). \quad (31)$$

Remark 3. Contrarily to the classical cases of nonparametric estimation (e.g. Tsybakov [21], ...), the number of observations N_T is a random variable that converges to $+\infty$ when $T \rightarrow +\infty$ which is one of the main difficulty here. From Theorem 3, the upper bound is in $\exp\left(-\frac{2\beta}{2\beta+1}\frac{N_0}{N_0+1}RT\right)$ that differs from the rate in the lower bound, $\exp\left(-\frac{2\beta}{2\beta+1}RT\right)$ that is the best rate that we expect. The term $\exp\left(-\frac{N_0}{N_0+1}RT\right)$ in (29) appears as an upper bound of $\mathbb{E}\left[\frac{N_0}{N_T}\right]$ when we used (8). The effect of the random finite population size on fixed compact time intervals $[0, T]$ is uniformly upper bounded in time thanks to the correction $\frac{N_0}{N_0+1}$. If we consider the MISE on a time interval $[\tau, \tau+T]$, the correction term will become $\frac{N_\tau}{N_\tau+1}$ that tends to 1 when $\tau \rightarrow +\infty$ since the population size is increasing to $+\infty$ with time. Hence the rate $\exp\left(-\frac{2\beta}{2\beta+1}\right)$ is the “optimal asymptotic rate” that is asymptotically achieved on the time window $[\tau, \tau+T]$ when $\tau \rightarrow +\infty$.

3 Numerical simulations

3.1 Numerical computation of \hat{h}

We use **R** software to implement simulations with two original distributions of kernel division h and compare with their estimators. On the interval $[0, 1]$, the first distribution to test is Beta(2, 2). Beta(a, b) distribution on $[0, 1]$ are characterized by their densities

$$h_{\text{Beta}(a,b)}(x) = \frac{x^a(1-x)^b}{\mathcal{B}(a,b)}.$$

where $\mathcal{B}(a, b)$ is the renormalization constant.

Since h is symmetric, we only consider the distributions with $a = b$. Generally, asymmetric divisions correspond to $a < 1$ and symmetric divisions with kernels concentrated around $\frac{1}{2}$ correspond to $a > 1$. The smaller the parameter a , the more asymmetric the divisions. For the second density, we choose a Beta mixture distribution as

$$\frac{1}{2} \text{Beta}(2, 6) + \frac{1}{2} \text{Beta}(6, 2).$$

This choice give us a bimodal density corresponding to very asymmetric divisions.

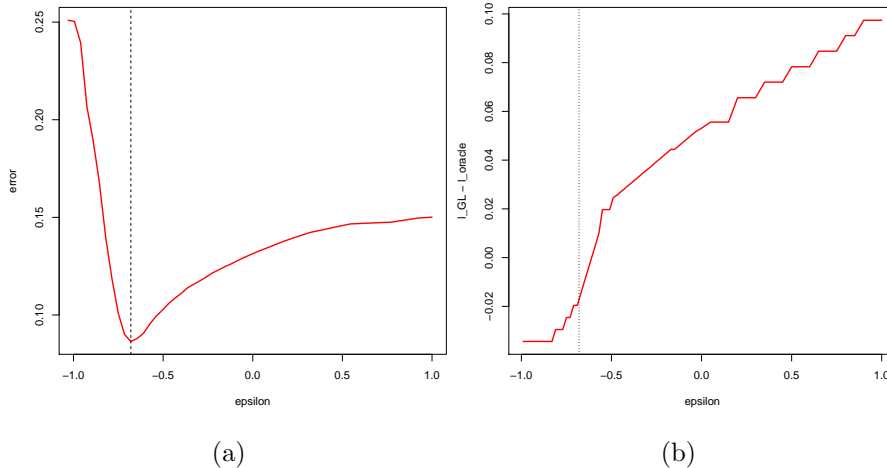
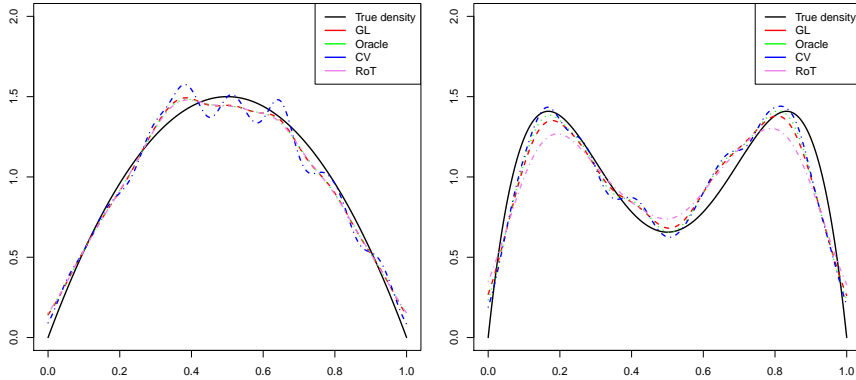


Figure 2: (a): MISE's for as a function of ϵ . (b): $\hat{\ell} - \ell_{oracle}$ as a function of ϵ . The dotted lines indicate the optimal value of ϵ which is used in all simulations.

We estimate \hat{h} by using (18) and we take the classical Gaussian kernel $K(x) = (2\pi)^{-1/2} \exp(-x^2/2)$. For the choice of bandwidth, we apply the GL method with the family $H = \{1, 2^{-1}, \dots, [\delta N_T]^{-1}\}$ for some $\delta > 0$ small enough when N_T is large to reduce the time of numerical simulation. We have $\|K\|_1 = 1$, $\|K\|_2 = 2^{-1/2}\pi^{-1/4}$ and $K_\ell \star K_{\ell'} = K_{\sqrt{\ell^2 + \ell'^2}}$, hence it is not difficult to calculate in practice $\hat{h}_{\ell, \ell'}$ as well as $\hat{h}_{\ell'}$. Finally, the value of ϵ in $\chi = (1 + \epsilon)(1 + \|K\|_1)$ is chosen to find an optimal value of the MISE. To do this, we implement a preliminary simulation to calibrate ϵ in which we choose $\epsilon > -1$ to assure that $1 + \epsilon > 0$. We compute the MISE and $\hat{\ell} - \ell_{oracle}$ as functions of ϵ where $\ell_{oracle} = \operatorname{argmin}_{\ell \in H} \mathbb{E}[\|\hat{h}_\ell - h\|_2^2]$ and h is the density of Beta(2, 2). In Figure 2a, simulation results show that the risk has minimum value at $\epsilon = -0.68$. This value is not justified from a theoretical point of view. The theoretical choice $\epsilon > 0$ (see Theorem 2) does not give bad results but this choice is too conservative for non-asymptotic practical purposes as often met in the literature (see Bertin et al. [4] for more details about the GL methodology). Moreover, following the discussion in Lacour and Massart [13] we investigate (see Figure 2b) the difference $\hat{\ell} - \ell_{oracle}$ and observe some explosion close to $\epsilon = -0.68$. Consequently, we choose $\epsilon = -0.68$ for all following simulations.

Figure 3 illustrates a reconstructions for the density of Beta(2, 2) and beta mixture $\frac{1}{2} \text{Beta}(2, 6) + \frac{1}{2} \text{Beta}(6, 2)$ when $T = 13$. We choose here the division rate and the growth rate $R = 0.5$ and $\alpha = 0.35$ respectively. We

also compare the estimated densities by the GL bandwidth with those by the cross-validation (CV) bandwidth, the rule-of-thumb (RoT) bandwidth and oracle bandwidth which is found by assuming that we know the true density.



(a) *Reconstruction of Beta(2, 2)* (b) *Reconstruction of beta mixture*

Figure 3: *Reconstruction of kernel divisions with $T = 13$.*

To estimate the MISE, we implement Monte-Carlo simulations with respect to $T = 13, 17$ and 20 . The number of repetitions for each simulation is $M = 100$. Then, we compute the mean of relative error $\bar{e} = (1/M) \sum_{i=1}^M e_i$ and the standard deviation $\sigma_e = \sqrt{(1/M) \sum_{i=1}^M (e_i - \bar{e})^2}$ where

$$e_i = \frac{\|\hat{h} - h\|_2}{\|h\|_2}, \quad i = 1, \dots, M. \quad (32)$$

The MISE's are computed for estimated densities using the GL bandwidth, the oracle bandwidth, the CV bandwidth and the RoT bandwidth. For a further comparison, in the reconstruction of Beta(2, 2), we compute the relative error in a parametric setting by comparing the true density h with the density of Beta(\hat{a}, \hat{a}) where \hat{a} is the Maximum Likelihood (ML) estimation of a . The simulation results are displayed in Table 1 and Table 2. For the density of Beta mixture, we only compute the error with $T = 13$ and $T = 17$. The boxplot in Figure 4 illustrates the MISE's in Table 1 when $T = 17$.

From Tables 1 and 2, we can note that the accuracy of the estimation of Beta(2, 2) and Beta mixture by the GL bandwidth increases for larger T . In Figure 5, we illustrate on a log-log scale the mean relative error and the rate of convergence versus time T . This shows that the error is close to the exponential rate predicted by the theory. Moreover, we can observe that

the error of Beta mixture is larger than those of Beta(2, 2) with the same T due to the complexity of its density. In both cases, the error estimated by using oracle bandwidth is always smaller than the others. The GL error is slightly smaller than the CV error. The RoT can show very good behavior but lacks of stability. Overall, we conclude that the GL method has a good behavior when compared to the cross validation method and rule-of-thumb. As usual, we also see that the ML error is quite smaller than those of nonparametric approach but the magnitude of the mean \bar{e} remains similar.

| | | GL | Oracle | CV | RoT | ML method |
|----------|----------------|---------------|---------------|---------------|---------------|-----------|
| $T = 13$ | \bar{e} | 0.1001 | 0.0840 | 0.1009 | 0.0900 | 0.0610 |
| | σ_e | 0.0585 | 0.0481 | 0.0599 | 0.0577 | 0.0724 |
| | $\tilde{\ell}$ | 0.0920 | 0.0845 | 0.0824 | 0.0727 | |
| $T = 17$ | \bar{e} | 0.0458 | 0.0397 | 0.0459 | 0.0405 | 0.0166 |
| | σ_e | 0.0260 | 0.0230 | 0.0297 | 0.0237 | 0.0171 |
| | $\tilde{\ell}$ | 0.0485 | 0.0497 | 0.0478 | 0.0470 | |
| $T = 20$ | \bar{e} | 0.0261 | 0.0241 | 0.0262 | 0.0245 | 0.0088 |
| | σ_e | 0.0140 | 0.0114 | 0.0132 | 0.00121 | 0.0091 |
| | $\tilde{\ell}$ | 0.0377 | 0.0359 | 0.0345 | 0.0354 | |

Table 1: Mean of relative error and its standard deviation for the reconstruction of Beta(2, 2). $\tilde{\ell}$ is the average of bandwidths for $M = 100$ samples.

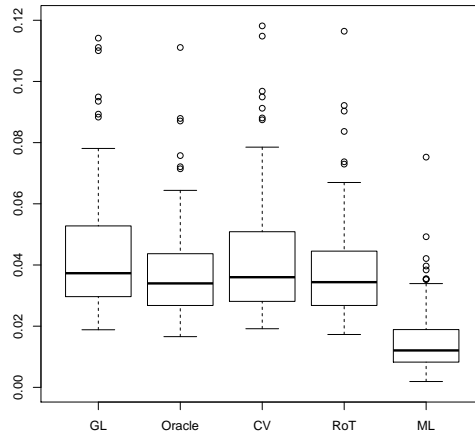


Figure 4: Errors of estimated densities of Beta(2, 2) when $T = 17$.

| | | GL | Oracle | CV | RoT |
|----------|--------------|---------------|---------------|---------------|---------------|
| $T = 13$ | \bar{e} | 0.1361 | 0.1245 | 0.1379 | 0.1686 |
| | σ_e | 0.0672 | 0.0562 | 0.0815 | 0.0537 |
| | $\hat{\ell}$ | 0.0618 | 0.0527 | 0.0522 | 0.0948 |
| $T = 17$ | \bar{e} | 0.0539 | 0.0534 | 0.0550 | 0.0919 |
| | σ_e | 0.0180 | 0.0168 | 0.0168 | 0.00223 |
| | $\hat{\ell}$ | 0.0309 | 0.0272 | 0.0264 | 0.0590 |

Table 2: Mean of relative error and its standard deviation for the reconstruction of beta mixture $\frac{1}{2}\text{Beta}(2, 6) + \frac{1}{2}\text{Beta}(6, 2)$.

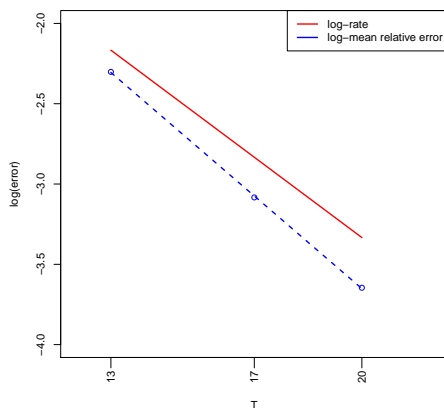


Figure 5: The log-mean relative error for the reconstruction of Beta(2, 2) compared to the log-rate (solid line) computed with $\beta = 1$.

Since h is symmetric on $[0, 1]$ with respect to $\frac{1}{2}$, the estimator \hat{h} can be improved and we can introduce

$$\tilde{h}(x) = \frac{1}{2} \left(\hat{h}(x) + \hat{h}(1 - x) \right),$$

which is symmetric by construction and satisfies also (30). We compute the mean of relative error for the estimator \tilde{h} with the estimation of Beta(2, 2) and Beta mixture. The results are displayed in Table 3. Compared with the error in Table 1 and 2, one can see as expected that the errors for the reconstruction of \tilde{h} is smaller. However, these errors are the same order, indicating that the estimator \hat{h} had already good symmetric properties.

3.2 Influence of the distribution on the mean age

For $t \geq 0$, recall the mean age defined in (12). To study the influence of the distribution on the mean age, we simulate $n = 50$ trees with respect to $t = 6, 6 + \Delta t, \dots, 24$ with $\Delta t = 0.36$. For each sample $(\bar{x}_t^{(1)}, \dots, \bar{x}_t^{(n)})$, we

| | | GL | Oracle | CV | RoT |
|--------------|----------|--------|--------|--------|--------|
| Beta(2, 2) | $T = 13$ | 0.0785 | 0.0634 | 0.0762 | 0.0644 |
| | $T = 17$ | 0.0356 | 0.0309 | 0.0356 | 0.0309 |
| Beta mixture | $T = 13$ | 0.1117 | 0.0953 | 0.1030 | 0.1584 |
| | $T = 17$ | 0.0450 | 0.0414 | 0.0417 | 0.0893 |

Table 3: Mean of relative error for the reconstruction of \tilde{h} .

compute the average mean, the 1st (Q_{25}) quartile and 3rd (Q_{75}) quartile. Figure 6a and 6b show the simulation results corresponding to the density of Beta(2, 2) with $\alpha = 0.45$ and $R = 0.4$. One can see that the average of mean age and the mean age converge to $\frac{\alpha}{R} = 0.125$ for larger t . This agrees with the theoretical result proved in Section 2.2.

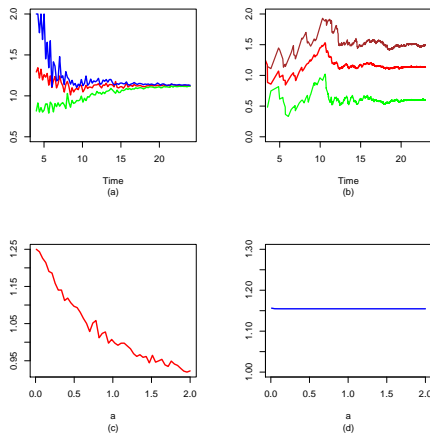


Figure 6: (a) Average mean, 1st and 3rd quartiles for the sample of means for 50 trees. (b) Average mean, 1st and 3rd quartiles for one tree. (c) Average of $Q_{75} - Q_{25}$ with $a \in [0, 2]$ at $t = 12$. (d) Mean age with $a \in [0, 2]$ at $t = 12$.

Moreover, Q_{25} and Q_{75} vary when the parameter a changes. In Figure 6c, we draw a fitted curve of the average of $(Q_{75} - Q_{25})$ when a varies from 0 to 2. As we mentioned in the introduction, if divisions are more asymmetric corresponding to small values of a , the toxicities concentrate on few cells, *i.e.* we have more older cells after the divisions. This explains the decreasing trend in the average of $(Q_{75} - Q_{25})$. Finally, Figure 6d displays the average of mean ages with respect to a . One can note that it does not change when we replace the kernel distribution, *e.g.* Beta(0.6, 0.6) instead of Beta(2, 2).

4 Proofs

4.1 Proof of Theorem 1

We will show that the process Y satisfies ergodicity and integrability assumptions in Bansaye et al. [2] (see (H1) - (H4), Section 4). More precisely:

1. $\mathbb{E}[Y_t] < +\infty$ for all $t \geq 0$.
2. There exists $\varpi < R$ and $c > 0$ such that $\mathbb{E}[Y_t^2] < ce^{\varpi t}$ for all $t \geq 0$.

From (16) we note that $\mathbb{E}[Y_t] < +\infty$ for all $t \geq 0$. To prove the second point, from (13) we have

$$\begin{aligned} \mathbb{E}[Y_t^2] &= \mathbb{E} \left[Y_0^2 + \int_0^t \left(2\alpha Y_s + 2R \int_0^1 (\gamma^2 Y_s^2 - Y_s^2) h(\gamma) d\gamma \right) ds \right] \\ &= Y_0^2 + 2\alpha \int_0^t \mathbb{E}[Y_s] ds - 2\theta R \int_0^t \mathbb{E}[Y_s^2] ds, \end{aligned} \quad (33)$$

with $\theta = \int_0^1 (1 - \gamma^2) h(\gamma) d\gamma$ and $0 < \theta < 1$.

Substituting $\mathbb{E}[Y_t] = (Y_0 - \alpha/R)e^{-Rt} + \alpha/R$ into (33), we see that $\mathbb{E}(Y_t^2)$ solves the following equation:

$$\frac{d\mathbb{E}[Y_t^2]}{dt} = -2\theta R \mathbb{E}[Y_t^2] + \left(2\alpha Y_0 - \frac{2\alpha^2}{R} \right) e^{-Rt} + \frac{2\alpha^2}{R}. \quad (34)$$

The solution of the equation (34) is:

$$\mathbb{E}[Y_t^2] = e^{-2\theta Rt} \left[Y_0^2 + \int_0^t e^{2\theta Rs} \left((2\alpha Y_0 - \frac{2\alpha^2}{R}) e^{-Rs} + \frac{2\alpha^2}{R} \right) ds \right]. \quad (35)$$

Hence, if $\theta = \frac{1}{2}$, we have

$$\begin{aligned} \mathbb{E}[Y_t^2] &= Y_0^2 e^{-Rt} + (2\alpha Y_0 - \frac{2\alpha^2}{R}) t e^{-Rt} + \frac{2\alpha^2}{\theta R^2} (1 - e^{-Rt}) \\ &\leq Y_0^2 e^{-Rt} + (2\alpha Y_0 - \frac{2\alpha^2}{R}) e^{-(R-\theta)t} + \frac{2\alpha^2}{R^2} \\ &\leq \left(Y_0^2 + 2\alpha Y_0 + \frac{2\alpha^2}{R} + \frac{2\alpha^2}{R^2} \right) e^{(0 \vee (\theta - R))t} = c_1 e^{\varpi t}, \end{aligned}$$

with $\varpi = 0 \vee (\theta - R)$.

If $\theta \neq \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}[Y_t^2] &= e^{-2\theta Rt} \left[Y_0^2 + (2\alpha Y_0 - \frac{2\alpha^2}{R}) \int_0^t e^{(2\theta-1)Rs} ds + \frac{2\alpha^2}{R} \int_0^t e^{2\theta Rs} ds \right] \\ &= Y_0^2 e^{-2\theta Rt} + (2\alpha Y_0 - \frac{2\alpha^2}{R}) \frac{1}{(2\theta-1)R} (e^{-Rt} - e^{-2\theta Rt}) \\ &\quad + \frac{\alpha^2}{\theta R^2} (1 - e^{-2\theta Rt}) \\ &\leq \left(Y_0^2 + (2\alpha Y_0 + \frac{2\alpha^2}{R}) \frac{1}{|2\theta-1|R} + \frac{\alpha^2}{\theta R^2} \right) = c_2,\end{aligned}$$

Thus, if we set $c = \max(c_1, c_2)$ then $\mathbb{E}[Y_t^2] < ce^{\varpi t}$ for all $t \geq 0$.

For $f(x) = x$, we check that the infinitesimal generator of the auxiliary process Y satisfies

$$Af(x) \leq \mathcal{C}f(x) + \mathcal{D},$$

for some $\mathcal{C}, \mathcal{D} < +\infty$. Hence, by Theorem 5.3 of Meyn and Tweedie [15], there exists $\pi \in \mathcal{M}_F(\mathbb{R}_+)$ such that $\lim_{t \rightarrow +\infty} E[Y_t] = \langle \pi, f \rangle = \frac{\alpha}{R}$. Finally, applying Theorem 4.2 of [2], we obtain the result

$$\lim_{t \rightarrow +\infty} \frac{\langle Z_t, f \rangle}{N_t} = \langle \pi, f \rangle = \frac{\alpha}{R}.$$

4.2 Proof of Theorem 2

This proof is inspired by the proof of Doumic et al. [7]. However, our problem here is that the number of observations N_T is random. To overcome this difficulty, we work conditionally to N_T to get concentration inequalities.

Hereafter, we refer $\int f$ to $\int_{\mathbb{R}} f$ and since the support of h in $(0, 1)$, we can express $\int h(\gamma)d\gamma$ instead of $\int_0^1 h(\gamma)d\gamma$. Recall that

$$A(\ell) := \sup_{\ell' \in H} \left\{ \|\hat{h}_{\ell, \ell'} - \hat{h}_{\ell'}\|_2 - \frac{\chi \|K\|_2}{\sqrt{N_T \ell'}} \right\}_+.$$

Then, for any $\ell \in H$, we have

$$\|\hat{h} - h\|_2 \leq A_1 + A_2 + A_3,$$

where

$$\begin{aligned}A_1 &:= \|\hat{h} - \hat{h}_{\hat{\ell}, \ell}\|_2 \leq A(\ell) + \frac{\chi \|K\|_2}{\sqrt{N_T \hat{\ell}}}, \\ A_2 &:= \|\hat{h}_{\hat{\ell}, \ell} - \hat{h}_{\ell}\|_2 \leq A(\hat{\ell}) + \frac{\chi \|K\|_2}{\sqrt{N_T \hat{\ell}}}, \\ A_3 &:= \|\hat{h}_{\ell} - h\|_2.\end{aligned}$$

By definition of $\hat{\ell}$, we have

$$A_1 + A_2 \leq 2A(\ell) + 2\frac{\chi\|K\|_2}{\sqrt{N_T\ell}}, \quad (36)$$

and

$$\begin{aligned} A(\ell) &\leq \sup_{\ell' \in H} \left\{ \|(\hat{h}_{\ell, \ell'} - \mathbb{E}[\hat{h}_{\ell, \ell'}]) - (\hat{h}_{\ell'} - \mathbb{E}[\hat{h}_{\ell'}])\|_2 \right. \\ &\quad \left. + \|\mathbb{E}[\hat{h}_{\ell, \ell'}] - \mathbb{E}[\hat{h}_{\ell'}]\|_2 - \frac{\chi\|K\|_2}{\sqrt{N_T\ell'}} \right\}_+ \\ &\leq \xi_T(\ell) + \sup_{\ell' \in H} \left\{ \|\mathbb{E}[\hat{h}_{\ell, \ell'}] - \mathbb{E}[\hat{h}_{\ell'}]\|_2 \right\}, \end{aligned} \quad (37)$$

where

$$\xi_T(\ell) = \sup_{\ell' \in H} \left\{ \|(\hat{h}_{\ell, \ell'} - \mathbb{E}[\hat{h}_{\ell, \ell'}]) - (\hat{h}_{\ell'} - \mathbb{E}[\hat{h}_{\ell'}])\|_2 - \frac{\chi\|K\|_2}{\sqrt{N_T\ell'}} \right\}_+. \quad (38)$$

For the term $\sup_{\ell' \in H} \left\{ \|\mathbb{E}[\hat{h}_{\ell, \ell'}] - \mathbb{E}[\hat{h}_{\ell'}]\|_2 \right\}$, we have

$$\begin{aligned} \mathbb{E}[\hat{h}_{\ell, \ell'}] - \mathbb{E}[\hat{h}_{\ell'}] &= \int (K_\ell \star K_{\ell'})(\gamma - u)h(u)du - \int K_{\ell'}(\gamma - v)h(v)dv \\ &= \int \int K_\ell(\gamma - u - t)K_{\ell'}(t)h(u)dtdu - \int K_{\ell'}(\gamma - v)h(v)dv \\ &= \int \int K_\ell(v - u)K_{\ell'}(\gamma - v)h(u)dudv - \int K_{\ell'}(\gamma - v)h(v)dv \\ &= \int K_{\ell'}(\gamma - v) \left(\int K_\ell(v - u)h(u)du - h(v) \right) dv \\ &= \int K_{\ell'}(\gamma - v) (K_\ell \star h(v) - h(v)) dv. \end{aligned}$$

Hence, we derive

$$\|\mathbb{E}[\hat{h}_{\ell, \ell'}] - \mathbb{E}[\hat{h}_{\ell'}]\|_2 = \|K_{\ell'} \star (K_\ell \star h - h)\|_2 \leq \|K\|_1 \|K_\ell \star h - h\|_2, \quad (39)$$

where the right hand side does not depend on ℓ' allowing us to take $\sup_{\ell' \in H}$ in the left hand side.

Thus, (36), (38) and (39) give

$$A_1 + A_2 \leq 2\xi_T(\ell) + 2\|K\|_1 \|K_\ell \star h - h\|_2 + 2\frac{\chi\|K\|_2}{\sqrt{N_T\ell}}.$$

Then,

$$\mathbb{E} \left[(A_1 + A_2)^2 \right] \leq 12\mathbb{E}[\xi_T^2(\ell)] + 12\|K\|_1^2 \|K_\ell \star h - h\|_2^2 + 12\frac{\chi^2\|K\|_2^2}{\ell} \mathbb{E} \left[\frac{1}{N_T} \right]. \quad (40)$$

For the term A_3 ,

$$\begin{aligned}\mathbb{E} \left[A_3^2 \right] &= \|\mathbb{E}[\hat{h}_\ell] - h\|_2^2 + \mathbb{E} \left[\|\hat{h}_\ell - \mathbb{E}[\hat{h}_\ell]\|_2^2 \right] \\ &\leq \|K_\ell \star h - h\|_2^2 + \frac{\|K\|_2^2}{\ell} \mathbb{E} \left[\frac{1}{N_T} \right].\end{aligned}$$

Finally, replacing χ by $(1 + \epsilon)(1 + \|K\|_1)$, we have for any $\ell \in H$

$$\begin{aligned}\mathbb{E} \left[\|\hat{h} - h\|_2^2 \right] &\leq 2\mathbb{E} \left[(A_1 + A_2)^2 \right] + 2\mathbb{E} \left[A_3^2 \right] \\ &\leq 24\mathbb{E} \left[\xi_T^2(\ell) \right] + 2 \left(1 + 12\|K\|_1^2 \right) \|K_\ell \star h - h\|_2^2 \\ &\quad + 2 \left(1 + 12(1 + \epsilon)^2(1 + \|K\|_1)^2 \right) \frac{\|K\|_2^2}{\ell} \mathbb{E} \left[\frac{1}{N_T} \right] \\ &\leq 24\mathbb{E} \left[\xi_T^2(\ell) \right] + C_1 \left(\|K_\ell \star h - h\|_2^2 + \frac{\|K\|_2^2}{\ell} \mathbb{E} \left[\frac{1}{N_T} \right] \right),\end{aligned}\quad (41)$$

with C_1 a constant depending on ϵ and $\|K\|_1$.

It remains to examine with the term $\mathbb{E} \left[\xi_T^2(\ell) \right]$ where $\xi_T(\ell)$ is defined in (38),

$$\begin{aligned}\xi_T(\ell) &\leq \sup_{\ell' \in H} \left\{ \|\hat{h}_{\ell, \ell'} - \mathbb{E}[\hat{h}_{\ell, \ell'}]\|_2 + \|\hat{h}_{\ell'} - \mathbb{E}[\hat{h}_{\ell'}]\|_2 - \frac{\chi \|K\|_2}{\sqrt{N_T \ell'}} \right\}_+ \\ &\leq \sup_{\ell' \in H} \left\{ \|\hat{h}_{\ell'} - \mathbb{E}[\hat{h}_{\ell'}]\|_2 \|K\|_1 + \|\hat{h}_{\ell'} - \mathbb{E}[\hat{h}_{\ell'}]\|_2 - \frac{\chi \|K\|_2}{\sqrt{N_T \ell'}} \right\}_+ \\ &\leq \sup_{\ell' \in H} \left\{ (1 + \|K\|_1) \|\hat{h}_{\ell'} - \mathbb{E}[\hat{h}_{\ell'}]\|_2 - \frac{(1 + \epsilon)(1 + \|K\|_1) \|K\|_2}{\sqrt{N_T \ell'}} \right\}_+ \\ &\leq (1 + \|K\|_1) S_T,\end{aligned}$$

where

$$S_T := \sup_{\ell \in H} \left\{ \|\hat{h}_\ell - \mathbb{E}[\hat{h}_\ell]\|_2 - \frac{(1 + \epsilon) \|K\|_2}{\sqrt{N_T \ell}} \right\}_+.$$

Hence,

$$\mathbb{E} \left[\xi_T^2(\ell) \right] \leq (1 + \|K\|_1)^2 \mathbb{E} \left[\mathbb{E} \left[S_T^2 | N_T \right] \right].\quad (42)$$

If we show that

$$\mathbb{E} \left[S_T^2 | N_T = n \right] \leq C_* \frac{1}{n},\quad (43)$$

then

$$\mathbb{E} \left[\xi_T^2(\ell) \right] \leq C_* (1 + \|K\|_1)^2 \mathbb{E} \left[\frac{1}{N_T} \right]\quad (44)$$

where C_* is a constant.

Let us establish (43). When $N_T = n$, $\forall n \in \mathbb{N}^*$,

$$\mathbb{E}[S_T^2 | N_T = n] = \mathbb{E}[S_n^2]$$

where

$$S_n := \sup_{\ell \in H} \left\{ \|Z_\ell\|_2 - \frac{(1+\epsilon)\|K\|_2}{\sqrt{n\ell}} \right\}_+,$$

with

$$Z_\ell = \hat{h}_\ell - \mathbb{E}[\hat{h}_\ell] = \frac{1}{n} \sum_{i=1}^n K_\ell(\gamma - \Gamma_i^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_i^1)].$$

Then,

$$\begin{aligned} \mathbb{E}[S_n^2] &= \mathbb{E} \left[\sup_{\ell \in H} \left\{ \|Z_\ell\|_2 - \frac{(1+\epsilon)\|K\|_2}{\sqrt{n\ell}} \right\}_+^2 \right] \\ &\leq \int_0^{+\infty} \mathbb{P} \left[\sup_{\ell \in H} \left\{ \|Z_\ell\|_2 - \frac{(1+\epsilon)\|K\|_2}{\sqrt{n\ell}} \right\}_+ \geq x \right] dx \\ &\leq \sum_{\ell \in H} \int_0^{+\infty} \mathbb{P} \left[\left\{ \|Z_\ell\|_2 - \frac{(1+\epsilon)\|K\|_2}{\sqrt{n\ell}} \right\}_+ \geq x \right] dx. \end{aligned}$$

We bound this with Talagrand's inequality.

Let \mathcal{A} be a countable dense subset of the unit ball of $\mathbb{L}_2([0, 1])$. We express the norm $\|Z_\ell\|_2$ as

$$\begin{aligned} \|Z_\ell\|_2 &= \sup_{a \in \mathcal{A}} \int a(\gamma) Z_\ell(\gamma) d\gamma \\ &= \sup_{a \in \mathcal{A}} \sum_{i=1}^n \int a(\gamma) \frac{1}{n} \left(K_\ell(\gamma - \Gamma_i^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_i^1)] \right) d\gamma. \end{aligned}$$

Let

$$V_{i,\Gamma} = \int a(\gamma) \frac{1}{n} \left(K_\ell(\gamma - \Gamma_i^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_i^1)] \right) d\gamma.$$

Then $V_{i,\Gamma}$, $i = 1, \dots, n$ is a sequence of i.i.d random variables with zero mean. Thus, we can apply Talagrand's inequality (see [14, p. 170]) to $\|Z_\ell\|_2 = \sup_{a \in \mathcal{A}} \sum_{i=1}^n V_{i,\Gamma}$. For all $\eta, x > 0$, one has

$$\mathbb{P} \left(\|Z_\ell\|_2 \geq (1+\eta)\mathbb{E}[\|Z_\ell\|_2] + \sqrt{2\nu x} + c(\eta)bx \right) \leq e^{-x},$$

where $c(\eta) = 1/3 + \eta^{-1}$,

$$\nu = \frac{1}{n} \sup_{a \in \mathcal{A}} \mathbb{E} \left[\left(\int a(\gamma) \left(K_\ell(\gamma - \Gamma_1^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_1^1)] \right) d\gamma \right)^2 \right],$$

and,

$$b = \frac{1}{n} \sup_{y \in (0,1), a \in \mathcal{A}} \int a(\gamma) \left(K_\ell(\gamma - y) - \mathbb{E}[K_\ell(\gamma - \Gamma_1^1)] \right) d\gamma.$$

Next, we calculate the terms $\mathbb{E}[\|Z_\ell\|_2]$, ν and b . Applying Cauchy - Schwarz's inequality, we get

$$\begin{aligned} \mathbb{E}[\|Z_\ell\|_2] &\leq \left(\mathbb{E}[\|Z_\ell\|_2^2] \right)^{1/2} \\ &\leq \left(\mathbb{E} \left[\int \left(\frac{1}{n} \sum_{i=1}^n K_\ell(\gamma - \Gamma_i^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_i^1)] \right)^2 d\gamma \right] \right)^{1/2} \\ &\leq \frac{1}{n} \left(\int \mathbb{E} \left[\left(\sum_{i=1}^n K_\ell(\gamma - \Gamma_i^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_i^1)] \right)^2 \right] d\gamma \right)^{1/2} \\ &\leq \frac{1}{n} \left(\int \sum_{i=1}^n \mathbb{E} \left[\left(K_\ell(\gamma - \Gamma_i^1) - \mathbb{E}[K_\ell(\gamma - \Gamma_i^1)] \right)^2 \right] d\gamma \right)^{1/2} \\ &\leq \frac{1}{n} \left(n \int \mathbb{E} \left[K_\ell(\gamma - \Gamma_1^1)^2 \right] d\gamma \right)^{1/2} \leq \frac{\|K\|_2}{\sqrt{n\ell}}. \end{aligned}$$

For the term ν , we have

$$\begin{aligned} \nu &\leq \frac{1}{n} \sup_{a \in \mathcal{A}} \mathbb{E} \left[\left(\int a(\gamma) K_\ell(\gamma - \Gamma_1^1) d\gamma \right)^2 \right] \\ &\leq \frac{1}{n} \sup_{a \in \mathcal{A}} \mathbb{E} \left[\int |K_\ell(\gamma - \Gamma_1^1)| d\gamma \times \int a^2(\gamma) |K_\ell(\gamma - \Gamma_1^1)| d\gamma \right] \\ &\leq \frac{\|K\|_1}{n} \sup_{a \in \mathcal{A}} \mathbb{E} \left[\int a^2(\gamma) |K_\ell(\gamma - \Gamma_1^1)| d\gamma \right] \\ &\leq \frac{\|K\|_1}{n} \sup_{a \in \mathcal{A}} \int a^2(\gamma) \mathbb{E} \left[|K_\ell(\gamma - \Gamma_1^1)| \right] d\gamma \\ &\leq \frac{\|K\|_1}{n} \sup_{a \in \mathcal{A}} \int \int a^2(\gamma) |K_\ell(\gamma - u)| h(u) du d\gamma \\ &\leq \frac{\|h\|_\infty \|K\|_1^2}{n}. \end{aligned}$$

For the term b , we have

$$\begin{aligned} b &= \frac{1}{n} \sup_{y \in (0,1)} \|K_\ell(\cdot - y) - \mathbb{E}[K_\ell(\cdot - \Gamma_1^1)]\|_2 \\ &\leq \frac{1}{n} \left(\sup_{y \in (0,1)} \|K_\ell(\cdot - y)\|_2 + \left(\mathbb{E} \left[\int K_\ell^2(\gamma - \Gamma_1^1) d\gamma \right] \right)^{1/2} \right) \leq \frac{2\|K\|_2}{n\sqrt{\ell}}. \end{aligned}$$

So, for all $\eta, x > 0$, we have

$$\mathbb{P} \left(\|Z_\ell\|_2 \geq (1 + \eta) \frac{\|K\|_2}{\sqrt{n\ell}} + \|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{2x}{n}} + 2c(\eta) \frac{\|K\|_2 x}{n\sqrt{\ell}} \right) \leq e^{-x}.$$

Let H_ℓ be some strictly positive weights, we apply the previous inequality to $x = H_\ell + u$ for $u > 0$. We have

$$\begin{aligned} \mathbb{P} \left(\|Z_\ell\|_2 \geq (1 + \eta) \frac{\|K\|_2}{\sqrt{n\ell}} + \|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{H_\ell}{n}} + 2c(\eta) \frac{\|K\|_2 H_\ell}{n\sqrt{\ell}} \right. \\ \left. + \|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{u}{n}} + 2c(\eta) \frac{\|K\|_2 u}{n\sqrt{\ell}} \right) \leq e^{-H_\ell - u}. \end{aligned}$$

If we set

$$M_\ell = (1 + \eta) \frac{\|K\|_2}{\sqrt{n\ell}} + \|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{H_\ell}{n}} + 2c(\eta) \frac{\|K\|_2 H_\ell}{n\sqrt{\ell}},$$

then,

$$\mathbb{P} \left(\|Z_\ell\|_2 - M_\ell \geq \|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{u}{n}} + 2c(\eta) \frac{\|K\|_2 u}{n\sqrt{\ell}} \right) \leq e^{-H_\ell - u}.$$

Let

$$R = \mathbb{E} \left[\sup_{\ell \in H} (\|Z_\ell\|_2 - M_\ell)_+^2 \right] = \int_0^{+\infty} \mathbb{P} \left[\sup_{\ell \in H} (\|Z_\ell\|_2 - M_\ell)_+^2 \geq x \right] dx.$$

An upper bound of R is given by

$$R \leq \sum_{\ell \in H} \int_0^{+\infty} \mathbb{P} \left[(\|Z\|_2 - M_\ell)_+^2 \geq x \right] dx.$$

Let us take u such that

$$x = f(u)^2 = \left(\|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{u}{n}} + 2c(\eta) \frac{\|K\|_2 u}{n\sqrt{\ell}} \right)^2.$$

So,

$$dx = 2f(u) \left(\|h\|_\infty^{1/2} \|K\|_1 \frac{1}{2\sqrt{nu}} + 2c(\eta) \frac{\|K\|_2}{n\sqrt{\ell}} \right) du.$$

Hence,

$$\begin{aligned}
R &\leq \sum_{\ell \in H} \int_0^{+\infty} e^{-H_\ell - u} 2f(u) \left(\|h\|_\infty^{1/2} \|K\|_1 \frac{1}{2\sqrt{nu}} + 2c(\eta) \frac{\|K\|_2}{n\sqrt{\ell}} \right) du \\
&\leq \sum_{\ell \in H} \int_0^{+\infty} e^{-H_\ell - u} 2f(u) \left(\|h\|_\infty^{1/2} \|K\|_1 \sqrt{\frac{u}{n}} + 2c(\eta) \frac{\|K\|_2 u}{n\sqrt{\ell}} \right) u^{-1} du \\
&\leq 2 \sum_{\ell \in H} e^{-H_\ell} \int_0^{+\infty} f^2(u) e^{-u} u^{-1} du \\
&\leq C_\eta \sum_{\ell \in H} e^{-H_\ell} \left(\|h\|_\infty \|K\|_1^2 \int_0^{+\infty} e^{-u} du + \frac{\|K\|_2^2}{\ell^2} \int_0^{+\infty} u e^{-u} du \right) \times \frac{1}{n} \\
&\leq C_\eta \sum_{\ell \in H} e^{-H_\ell} \left(\|h\|_\infty \|K\|_1^2 + \frac{\|K\|_2^2}{\ell^2} \right) \times \frac{1}{n}. \tag{45}
\end{aligned}$$

We need to choose H_ℓ and η such that

$$\mathbb{E}[S_n^2] = \mathbb{E} \left[\sup_{\ell \in H} \left\{ \|Z_\ell\|_2 - \frac{(1+\epsilon)\|K\|_2}{\sqrt{n\ell}} \right\}_+^2 \right] \leq R \tag{46}$$

Let $\theta > 0$, we choose

$$H_\ell = \frac{\theta^2 \|K\|_2^2}{2 \|h\|_\infty \|K\|_1^2 \sqrt{\ell}},$$

the we have

$$M_\ell = (1+\eta) \frac{\|K\|_2}{\sqrt{n\ell}} + \frac{\theta \|K\|_2}{\sqrt{2n\sqrt{\ell}}} + \frac{c(\eta)\theta^2 \|K\|_2^3}{\|h\|_\infty \|K\|_1^2} \frac{1}{n\ell}.$$

Obviously, the series in (45) is finite and for any $\ell \in H$, since $\ell \leq 1$, we have

$$\begin{aligned}
M_\ell &\leq (1+\eta+\theta) \frac{\|K\|_2}{\sqrt{n\ell}} + \frac{c(\eta)\theta^2 \|K\|_2^3}{\|h\|_\infty \|K\|_1^2} \frac{1}{n\ell} \\
&\leq \left(1+\eta+\theta + \frac{c(\eta)\theta^2 \|K\|_2^2}{\|h\|_\infty \|K\|_1^2} \frac{1}{\sqrt{n\ell}} \right) \frac{\|K\|_2}{\sqrt{n\ell}}.
\end{aligned}$$

Since $H \subset \{D^{-1}, D = 1, \dots, D_{\max}\}$, if we choose $D_{\max} = \delta n$ for some $\delta > 0$, then $\ell_{\min} = D_{\max}^{-1}$ and we obtain

$$M_\ell \leq \left(1+\eta+\theta + \frac{c(\eta)\theta^2 \|K\|_2^2 \sqrt{\delta}}{\|h\|_\infty \|K\|_1^2} \right) \frac{\|K\|_2}{\sqrt{n\ell}}.$$

It remains to choose $\eta = \epsilon/2$ and θ small enough such that

$$\theta + \frac{c(\eta)\theta^2 \|K\|_2^2 \sqrt{\delta}}{\|h\|_\infty \|K\|_1^2} = \frac{\epsilon}{2},$$

then

$$M_\ell \leq (1 + \epsilon) \frac{\|K\|_2}{\sqrt{n\ell}},$$

and we get

$$\mathbb{E} \left[S_n^2 \right] \leq C_* \times \frac{1}{n},$$

where C_* is a constant depending on $\delta, \epsilon, \|h\|_\infty, \|K\|_1$ and $\|K\|_2$. Hence, we get (44).

Combining (41), (44) and using (8), we obtain: for any $\ell \in H$,

$$\mathbb{E} \left[\|\hat{h} - h\|_2^2 \right] \leq C_1 \left(\|K_\ell \star h - h\|_2^2 + \frac{\|K\|_2^2}{\ell} e^{-\frac{N_0}{N_0+1}RT} \right) + C_2 e^{-\frac{N_0}{N_0+1}RT}.$$

This ends the proof of Theorem 2.

4.3 Proof of Theorem 3

We begin with the bias term $\|K_\ell \star h - h\|_2$ in the right hand side of the oracle inequality (29). For any $\ell \in H$ and $\gamma \in (0, 1)$, let $k = \lfloor \beta \rfloor$ and $b(\gamma) = K_\ell \star h(\gamma) - h(\gamma)$, then we have

$$h(\gamma + u\ell) = h(\gamma) + h'(\gamma)u\ell + \dots + \frac{(u\ell)^k}{(k-1)!} \int_0^1 (1-\theta)^{k-1} h^{(k)}(\gamma + \theta u\ell) d\theta.$$

Since K is a kernel of order β^* and $\beta \in (0, \beta^*)$, we get

$$b(\gamma) = \int K(u) \frac{(u\ell)^k}{(k-1)!} \left[\int_0^1 (1-\theta)^{k-1} \left(h^{(k)}(\gamma + \theta u\ell) - h^{(k)}(\gamma) \right) d\theta \right] du.$$

Setting $E_{k,\ell}(u) = |K(u)| \frac{|u\ell|^k}{(k-1)!}$ for the sake of notation. Since $h \in \mathcal{H}(\beta, L)$ and applying twice the generalized Minskowki's inequality, we obtain

$$\begin{aligned} \|h - \mathbb{E}[\hat{h}]\|_2^2 &= \int b^2(\gamma) d\gamma \\ &\leq \int \left(\int E_{k,\ell}(u) \left[\int_0^1 (1-\theta)^{k-1} |h^{(k)}(\gamma + \theta u\ell) - h^{(k)}(\gamma)| d\theta \right] du \right)^2 d\gamma \\ &\leq \left(\int E_{k,\ell}(u) \left[\int \left(\int_0^1 (1-\theta)^{k-1} |h^{(k)}(\gamma + \theta u\ell) - h^{(k)}(\gamma)| d\theta \right)^2 d\gamma \right]^{1/2} du \right)^2 \\ &\leq \left(\int E_{k,\ell}(u) \left[\int_0^1 (1-\theta)^{k-1} \left(\int |h^{(k)}(\gamma + \theta u\ell) - h^{(k)}(\gamma)|^2 d\gamma \right)^{1/2} d\theta \right] du \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left(\int E_{k,\ell}(u) \left[\int_0^1 (1-\theta)^{k-1} L(\theta u \ell)^{\beta-k} d\theta \right] du \right)^2 \\
&\leq \left(\int |K(u)| \frac{|u\ell|^k}{(k-1)!} \left[\int_0^1 (1-\theta)^{k-1} L(u\ell)^{\beta-k} d\theta \right] du \right)^2 \\
&\leq C_{K,L,\beta} \ell^{2\beta},
\end{aligned}$$

where $C_{K,L,\beta} = \left(\frac{L}{k!} \int |u|^\beta |K(u)| du \right)^2$.

Finally, we have

$$\mathbb{E} \left[\|\hat{h} - h\|_2^2 \right] \leq C_1 \inf_{\ell \in H} \left\{ C_{K,L,\beta} \ell^{2\beta} + \frac{\|K\|_2^2}{\ell} e^{-\frac{N_0}{N_0+1} RT} \right\} + C_2 e^{-\frac{N_0}{N_0+1} RT}. \quad (47)$$

Taking the derivative of the expression inside the $\inf_{\ell \in H}$ of (47) with respect to ℓ , we obtain the minimizer

$$\ell^* = \left(\frac{\|K\|_2^2}{2\beta C_{K,L,\beta}} \right)^{\frac{1}{2\beta+1}} e^{-\frac{1}{2\beta+1} \frac{N_0}{N_0+1} RT}.$$

Since the optimal bandwidth $\hat{\ell}$ is proportional to ℓ^* up to a multiplicative constant. Therefore, by substituting ℓ by $\hat{\ell}$ in the right hand side of (47), we obtain

$$\mathbb{E} \left[\|\hat{h} - h\|_2^2 \right] \leq C_3 \exp \left(-\frac{2\beta}{2\beta+1} \frac{N_0}{N_0+1} RT \right),$$

with C_3 a constant depending on δ , ϵ , $\|K\|_1$, $\|K\|_2$, $\|h\|_\infty$, β and L . This ends the proof of Theorem 3.

4.4 Proof of Theorem 4

For $T > 0$, let us denote by \hat{h}_T the estimator of h . To prove the Theorem 4, we apply the general reduction scheme proposed by Tsybakov [21] (Section 2.2, p.79). We will show the existence of a family $\mathcal{H}_{M,T} = \{h_{j,T} : j = 0, 1, \dots, M\}$ such that:

- 1) $h_{j,T} \in \mathcal{H}(\beta, L)$, $j = 0, \dots, M$.
- 2) $\|h_{j,T} - h_{k,T}\|_2 \geq 2c e^{-\frac{\beta}{2\beta+1} RT}$, $0 \leq j < k \leq M$.
- 3) $\frac{1}{M} \sum_{j=1}^M K(P_j, P_0) \leq \alpha \log(M)$ for $0 < \alpha < 1/8$. P_j and P_0 are the distribution of observations when the division kernels are $h_{j,T}$ and

h_0 , respectively. $K(P, Q)$ denotes the Kullback-Leibler divergence between two measures P and Q :

$$K(P, Q) = \begin{cases} \int \log \frac{dP}{dQ} dP, & \text{if } P \ll Q \\ +\infty, & \text{otherwise.} \end{cases}$$

Under the preceding conditions 1, 2, 3, Tsybakov [21] (Theorem 2.5, p.99) show that

$$\inf_{\hat{h}_T} \max_{h \in \mathcal{H}_{M,T}} \mathbb{P} \left(\|\hat{h}_T - h\|_2^2 \geq c^2 e^{-\frac{2\beta}{2\beta+1} RT} \right) \geq C'. \quad (48)$$

where the infimum is taken over all estimators \hat{h}_T and positive constant C' is independent of T . This will be sufficient to obtain Theorem 4 by [21, Theorem 2.7]. The proof ends with proposing a family $\mathcal{H}_{M,T}$ and checking the assumptions 1, 2, 3.

Construction of the family $\mathcal{H}_{M,T}$:

The idea is construct a family of perturbations around h_0 which is a symmetric density w.r.t $\frac{1}{2}$ and belongs to $\mathcal{H}(\frac{L}{2}, \beta)$. For the simplification, we choose $h_0(\gamma) = \mathbb{1}_{[0,1]}(\gamma)$.

Let $c_0 > 0$ be a real number, and let $\gamma \in [0, 1]$, $f(\gamma) = LD^{-\beta}g(D\gamma)$ where g is a regular function having support $[0, 1]$ and $\int g(\gamma)d\gamma = 0$, $g \in \mathcal{H}(\frac{1}{2}, \beta)$, we define

$$D = \lceil c_0 e^{\frac{RT}{2\beta+1}} \rceil \quad \text{and} \quad f_k(\gamma) = f \left(\gamma - \frac{(k-1)}{D} \right),$$

By definition, the functions f_k 's have disjoint support and one can check that the functions $f_k \in \mathcal{H}(\frac{L}{2}, \beta)$.

Then, the function $h_{j,T}$ will be chosen in

$$\mathcal{D} = \left\{ h_\delta(\gamma) = h_0(\gamma) + c_1 \sum_{k=1}^D \delta_k f_k(\gamma) : \delta = (\delta_1, \dots, \delta_D) \in \{0, 1\}^D \right\},$$

where

$$c_1 = \min \left(\frac{1}{LD^{-\beta} \|g\|_\infty}, 1 \right). \quad (49)$$

We now check that h_δ is a density, since $\int h_\delta(\gamma)d\gamma = \int h_0(\gamma)d\gamma = 1$, it remains to verify that $h_\delta(\gamma) \geq 0 \forall \gamma$. We have

$$\begin{aligned} \inf_{[0,1]} h_\delta(\gamma) &\geq \inf_{[0,1]} h_0 - \|c_1 \sum_{k=1}^D \delta_k f_k\|_\infty \\ &\geq 1 - c_1 LD^{-\beta} \max_k \sup_\gamma |\delta_k| g(D\gamma - (k-1)) \\ &\geq 1 - c_1 LD^{-\beta} \|g\|_\infty \geq 0, \end{aligned}$$

by the choice of c_1 . Thus the family of densities \mathcal{D} is well-defined.

1) *The condition $h_{j,T} \in \mathcal{H}(\beta, L)$.*

Let us denote $q = \lfloor \beta \rfloor$, then for all $\gamma, \gamma' \in [0, 1]$ we have

$$\begin{aligned} \left| h_\delta^{(q)}(\gamma) - h_\delta^{(q)}(\gamma') \right| &= \left| h_0^{(q)}(\gamma) - h_0^{(q)}(\gamma') + c_1 \sum_{k=1}^D \delta_k (f_k^{(q)}(\gamma) - f_k^{(q)}(\gamma')) \right| \\ &\leq c_1 \sum_{k=1}^D |\delta_k| \left| f_k^{(q)}(\gamma) - f_k^{(q)}(\gamma') \right| \\ &\leq c_1 \max_k \left| f_k^{(q)}(\gamma) - f_k^{(q)}(\gamma') \right| \\ &\leq c_1 L D^{-\beta} \max_k D^q \left| g^{(q)}(D\gamma - (k-1)) - g^{(q)}(D\gamma' - (k-1)) \right| \\ &\leq c_1 L D^{\lfloor \beta \rfloor - \beta} D^{\beta - \lfloor \beta \rfloor} |\gamma - \gamma'|^{\beta - \lfloor \beta \rfloor} \leq L |\gamma - \gamma'|^{\beta - \lfloor \beta \rfloor}, \end{aligned}$$

which is always satisfied with $c_1 = \min\left(\frac{1}{LD^{-\beta}\|g\|_\infty}, 1\right)$, thus $h_\delta \in \mathcal{H}(L, \beta)$.

2) *The condition $\|h_{j,T} - h_{k,T}\|_2 \geq 2c e^{-\frac{\beta}{2\beta+1}RT}$.*

For all $\delta, \delta' \in \{0, 1\}^D$, we have

$$\begin{aligned} \|h_\delta - h_{\delta'}\|_2 &= \left[\int_0^1 (h_\delta(\gamma) - h_{\delta'}(\gamma))^2 d\gamma \right]^{1/2} = \left[\int_0^1 \left(c_1 \sum_{k=1}^D (\delta_k - \delta'_k) f_k(\gamma) \right)^2 d\gamma \right]^{1/2} \\ &= c_1 \left[\int_0^1 \sum_{k=1}^D (\delta_k - \delta'_k)^2 f_k^2(\gamma) d\gamma \right]^{1/2} = c_1 \left[\sum_{k=1}^D (\delta_k - \delta'_k)^2 \int_{\frac{k-1}{D}}^{\frac{k}{D}} f_k^2(\gamma) d\gamma \right]^{1/2} \\ &= c_1 \left[\sum_{k=1}^D (\delta_k - \delta'_k)^2 \int_{\frac{k-1}{D}}^{\frac{k}{D}} L^2 D^{-2\beta} g^2(D\gamma - (k-1)) d\gamma \right]^{1/2} \\ &= c_1 L D^{-\beta-1/2} \|g\|_2 \left[\sum_{k=1}^D (\delta_k - \delta'_k)^2 \right]^{1/2} = c_1 L D^{-\beta-1/2} \|g\|_2 \sqrt{d_H(\delta, \delta')}, \end{aligned}$$

where $d_H(\delta, \delta') = \sum_{k=1}^D \mathbf{1}\{\delta_k \neq \delta'_k\}$ is the Hamming distance between δ and δ' .

According to the Lemma of Varshamov-Gilbert (cf. Tsybakov [21], p.104), there exist a subset $\{\delta^{(0)}, \dots, \delta^{(M)}\}$ of $\{0, 1\}^D$ with cardinal (50) such that $\delta^{(0)} = (0, \dots, 0)$,

$$M \geq 2^{D/8}. \quad (50)$$

and

$$d_H(\delta^{(j)}, \delta^{(k)}) \geq \frac{D}{8}, \quad \forall 0 \leq j < k \leq M, \quad (51)$$

Then, by setting $h_{j,T}(x) = h_{\delta^{(j)}}(x)$, $j = 0, \dots, M$, we obtain

$$\begin{aligned} \|h_{j,T} - h_{k,T}\|_2 &= c_1 L D^{-\beta - \frac{1}{2}} \|g\|_2 \sqrt{d_H(\delta^{(j)}, \delta^{(k)})} \\ &\geq c_1 L D^{-\beta - 1/2} \|g\|_2 \sqrt{\frac{D}{8}} \\ &\geq \frac{c_1 L}{4} \|g\|_2 D^{-\beta}, \end{aligned}$$

whenever $D \geq 8$.

Suppose that, $N_T \geq N_{T^*}$ where $T^* = \log\left(\frac{7}{c_0}\right) \frac{2\beta+1}{R}$. Then, $D \geq 8$ and $D^\beta \leq (2c_0)^\beta \exp\left(\frac{\beta}{2\beta+1} RT\right)$. This implies:

$$\|h_{j,T} - h_{k,T}\|_2 \geq \frac{c_1 L}{4} \|g\|_2 (2c_0)^{-\beta} e^{-\frac{\beta}{2\beta+1} RT},$$

But,

$$\min\left(\frac{1}{L\|g\|_\infty}, 1\right) \leq c_1 \leq 1$$

Hence, we obtain

$$\|h_{j,T} - h_{k,T}\|_2 \geq 2c e^{-\frac{\beta}{2\beta+1} RT},$$

where

$$c = \frac{\min(1, L\|g\|_2)}{8} (2c_0)^{-\beta}.$$

3) The condition $\frac{1}{M} \sum_{j=1}^M K(P_j, P_0) \leq \alpha \log(M)$ for $0 < \alpha < 1/8$.

We need to show that for all $\delta \in \{0, 1\}^D$,

$$K(P_\delta, P_0) \leq \alpha \log(M),$$

where

$$K(P_\delta, P_0) = \mathbb{E} \left[\log \frac{dP_\delta}{dP_0} \Big|_{\mathcal{F}_T}(Z) \right],$$

and where $(Z_t)_{t \in [0, T]}$ is defined in (5) with the random measure Q having intensity $q(ds, di, d\gamma) = R h_\delta(\gamma) ds n(di) d\gamma$.

Here, the difficulty comes from the fact that N_T is variable because the observations result from a stochastic process Z_t . The law of these observations is not a probability distribution on a fixed \mathbb{R}^n where n would be the sample size, but rather a probability distribution on a path space. P_δ is the probability distribution when the Poisson point measure Q has intensity

$Rh_\delta(\gamma)ds n(di)d\gamma$. Thus a natural tool is to use Girsanov's theorem (see [12], Theorem 3.24, p. 159) saying that P_δ is absolutely continuous with respect to P_0 on \mathcal{F}_T with

$$\frac{dP_\delta}{dP_0}\Big|_{\mathcal{F}_T} = D_T^\delta,$$

where $(D_t^\delta)_{t \in [0, T]}$ is the unique solution of the following SDE (see Proposition 4.17 of [19] for a similar SDE):

$$D_T^\delta = 1 + \int_0^T \int_{\mathcal{E}} D_{s-}^\delta \mathbf{1}_{\{i \leq N_{s-}\}} \left(\frac{h_\delta(\gamma)}{h_0(\gamma)} - 1 \right) Q(ds, di, d\gamma). \quad (52)$$

Apply Itô formula for jump processes to (52), we get

$$\begin{aligned} \log D_T^\delta &= \int_0^T \int_{\mathcal{E}} \mathbf{1}_{\{i \leq N_{s-}\}} \left[\log \left(D_{s-}^\delta - \left(\frac{h_\delta(\gamma)}{h_0(\gamma)} - 1 \right) D_{s-}^\delta \right) - \log D_{s-}^\delta \right] Q(ds, di, d\gamma) \\ &= \int_0^T \int_{\mathcal{E}} \mathbf{1}_{\{i \leq N_{s-}\}} \log \frac{h_\delta(\gamma)}{h_0(\gamma)} Q(ds, di, d\gamma) = \sum_{i=1}^{N_T} \log \frac{h_\delta(\Gamma_i^1)}{h_0(\Gamma_i^1)}, \end{aligned}$$

by definition of $(\Gamma_1^1, \dots, \Gamma_{N_T}^1)$.

Then,

$$\begin{aligned} K(P_\delta, P_0) &= \mathbb{E}_\delta \left[\log D_T^\delta \right] = \mathbb{E}_\delta \left[\sum_{i=1}^{N_T} \log \frac{h_\delta(\Gamma_i^1)}{h_0(\Gamma_i^1)} \right] \\ &= \mathbb{E}[N_T] \mathbb{E}_\delta \left[\log \frac{h_\delta(\Gamma_1^1)}{h_0(\Gamma_1^1)} \right] = \mathbb{E}[N_T] \int_0^1 h_\delta(\gamma) \log \frac{h_\delta(\gamma)}{h_0(\gamma)} d\gamma. \end{aligned}$$

Here, $\mathbb{E}[N_T]$ does not depend on h_δ and we have $\mathbb{E}[N_T] \leq N_0 \exp(RT)$. Thus,

$$\begin{aligned} K(P_\delta, P_0) &\leq N_0 e^{RT} \int_0^1 h_\delta(\gamma) \log(h_\delta(\gamma)) d\gamma \\ &\leq N_0 e^{RT} \int_0^1 \left(1 + c_1 \sum_{k=1}^D \delta_k f_k(\gamma) \right) \log \left(1 + c_1 \sum_{k=1}^D \delta_k f_k(\gamma) \right) d\gamma \\ &\leq N_0 e^{RT} \sum_{k=1}^D \int_{\frac{k-1}{D}}^{\frac{k}{D}} (1 + c_1 \delta_k f_k(\gamma)) \log(1 + c_1 \delta_k f_k(\gamma)) d\gamma \\ &\leq N_0 e^{RT} \sum_{k=1}^D \delta_k \int_0^{1/D} (1 + c_1 f(\gamma)) \log(1 + c_1 f(\gamma)) d\gamma \\ &\leq N_0 e^{RT} D \int_0^{1/D} (1 + c_1 f(\gamma)) c_1 f(\gamma) d\gamma \end{aligned}$$

$$\begin{aligned}
&\leq N_0 e^{RT} \left[c_1 L D^{-\beta} \int_0^{1/D} g(D\gamma) D d\gamma + c_1^2 L^2 D^{-2\beta} \int_0^{1/D} g^2(D\gamma) D d\gamma \right] \\
&\leq N_0 e^{RT} c_1^2 L^2 D^{-2\beta} \int_0^1 g^2(\gamma) d\gamma \\
&\leq N_0 c_1^2 L^2 \|g\|_2^2 e^{RT} c_0^{-2\beta} e^{-\frac{2\beta}{2\beta+1} RT} \\
&\leq N_0 L^2 \|g\|_2^2 c_0^{-2\beta-1} D, \text{ since } c_1 \leq 1.
\end{aligned}$$

From (50), we have $M \geq 2^{D/8}$ then

$$D \leq \frac{8 \log(M)}{\log(2)}.$$

Hence, if we set

$$c_0 = \left(\frac{8N_0 L^2 \|g\|_2^2}{\alpha \log(2)} \right)^{1/(2\beta+1)},$$

we obtain $K(P_\delta, P_0) \leq \alpha \log(M)$. This ends the proof of Theorem 4.

Acknowledgements

The author is deeply grateful to V.C. Tran, V. Rivoirard and T.M. Pham Ngoc for the guidance and useful suggestions. The author would like to express his sincere thanks to P. Massart for constructive comments that improved the results. The author would like to thank Centre de Ressources Informatiques de Lille 1 (CRI) for the computational cluster to implement the numerical simulations. This work is supported by Program 911 of Vietnam Ministry of Education and Training and is partly supported by the French Agence Nationale de la Recherche (ANR 2011 BS01 010 01 projet Calibration).

References

- [1] K.B. Athreya and P.E. Ney. *Branching Processes*. Springer edition, 1970.
- [2] V. Bansaye, J.-F. Delmas, L. Marsalle, and V. C. Tran. Limit theorems for markov processes indexed by continuous time galton-watson trees. *The Annals of Applied Probability*, 21, 2011.
- [3] V. Bansaye and V. C. Tran. Branching feller diffusion for cell division with parasite infection. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 8, 2011.
- [4] K. Bertin, C. Lacour, and V. Rivoirard. Adaptive pointwise estimation of conditional density function. *to appear in the Ann. Inst. H. Poincar Probab. Statist.*, 2015.
- [5] B. Cloez. Limit theorems for some branching measure-valued processes. *hal-00598030*, 2011.

- [6] M. Doumic, M. Hoffmann, N. Krell, and L. Robert. Statistical estimation of a growth-fragmentation model observed on a genealogical tree. *arXiv:1210.3240*, 2012.
- [7] M. Doumic, M. Hoffmann, P. Reynaud-Bouret, and V. Rivoirard. Nonparametric estimation of the division rate of a size-structured population. *SIAM Journal on Numerical Analysis*, 50, 2012.
- [8] A. Goldenshluger and O. Lepski. Bandwidth selection in kernel density estimation: oracle inequalities and adaptive minimax optimality. *The Annals of Statistics*, 39(3):1608–1632, 2011.
- [9] T.E. Harris. *The Theory of Branching Processes*. Springer, Berlin, 1963.
- [10] M. Hoffmann and A. Olivier. Nonparametric estimation of the division rate of an age dependent branching process. *arXiv:1412.5936*, 2014.
- [11] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. volume 24 of North-Holland Mathematical Library.
- [12] J. Jacob and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin, 1987.
- [13] C. Lacour and P. Massart. Minimal penalty for goldenshluger-lepski method. *arXiv:1503.00946v1*, (4:3), 2015.
- [14] P. Massart. *Concentration Inequalities and Model Selection*. Lectures from the 33rd Summer School on Probability Theory held in Saint-Flour, 6-23, 2003. Springer, 2007.
- [15] S. P. Meyn and R. L. Tweedie. Stability of markovian processes iii: Foster - lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, 25, 1993.
- [16] J. Peter. A general stochastic model for population development. *Scandinavian Actuarial Journal*, 1969(1-2):84–103, 1969.
- [17] P. Reynaud-Bouret, V. Rivoirard V, F. Grammont, and C. Tuleau-Malot. Goodness-of-fit tests and nonparametric adaptive estimation for spike train analysis. *Journal of Mathematical Neuroscience*, (4:3), 2014.
- [18] E. J. Stewart, R. Madden, G. Paul, and F. Taddei. Aging and Death in an Organism That Reproduces by Morphologically Symmetric Division. *PLOS Biology*, 3, 2005.
- [19] V. C. Tran. *Modèles particuliers stochastiques pour des problèmes d'évolution adaptive et pour l'approximation de solutions statistiques*. PhD dissertation, Université Paris X - Nanterre, <http://tel.archives-ouvertes.fr/tel-00125100>, 2006.
- [20] V. C. Tran. Large population limit and time behaviour of a stochastic particle model describing an age-structured population. *ESAIM: Probability and Statistics*, 12, 2008.
- [21] A. B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer series in Statistics. Springer, 2004.