Recent Advances in Robust Coarse Space Construction
An ASM type theory for P.L. Lions algorithm – Optimized Schwarz Methods

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$$-\Delta (u_1^{n+1}) = f \quad \text{in } \Omega_1,$$
$$u_1^{n+1} = 0 \quad \text{on } \partial \Omega_1 \cap \partial \Omega,$$
$$\left( \frac{\partial}{\partial n_1} + \alpha \right)(u_1^{n+1}) = \left( - \frac{\partial}{\partial n_2} + \alpha \right)(u_2^n) \quad \text{on } \partial \Omega_1 \cap \overline{\Omega}_2,$$

($n_1$ and $n_2$ are the outward normal on the boundary of the subdomains)

$$-\Delta (u_2^{n+1}) = f \quad \text{in } \Omega_2,$$
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$$\left( \frac{\partial}{\partial n_2} + \alpha \right)(u_2^{n+1}) = \left( - \frac{\partial}{\partial n_1} + \alpha \right)(u_1^n) \quad \text{on } \partial \Omega_2 \cap \overline{\Omega}_1.$$

with $\alpha > 0$. Overlap is not necessary for convergence. Parameter $\alpha$ can be optimized for.

1. (Recall) on Additive Schwarz Methods
2. Optimized Restricted Additive Schwarz Methods
3. SORAS-GenEO-2 coarse space
4. Numerical Results
5. Conclusion
The original Schwarz Method (H.A. Schwarz, 1870)

\[-\Delta(u) = f \quad \text{in } \Omega\]

\[u = 0 \quad \text{on } \partial \Omega.\]

Schwarz Method : \((u^n_1, u^n_2) \rightarrow (u^{n+1}_1, u^{n+1}_2)\) with

\[-\Delta(u^{n+1}_1) = f \quad \text{in } \Omega_1\]
\[u^{n+1}_1 = 0 \quad \text{on } \partial \Omega_1 \cap \partial \Omega\]
\[u^{n+1}_1 = u^n_2 \quad \text{on } \partial \Omega_1 \cap \Omega_2.\]

\[-\Delta(u^{n+1}_2) = f \quad \text{in } \Omega_2\]
\[u^{n+1}_2 = 0 \quad \text{on } \partial \Omega_2 \cap \partial \Omega\]
\[u^{n+1}_2 = u^{n+1}_1 \quad \text{on } \partial \Omega_2 \cap \Omega_1.\]

Parallel algorithm.
Consider the discretized Poisson problem: \( Au = f \in \mathbb{R}^n \).

Given a decomposition of \([1; n]\), \((\mathcal{N}_1, \mathcal{N}_2)\), define:
- the restriction operator \( R_i \) from \( \mathbb{R}^{[1:n]} \) into \( \mathbb{R}^{\mathcal{N}_i} \),
- \( R_i^T \) as the extension by 0 from \( \mathbb{R}^{\mathcal{N}_i} \) into \( \mathbb{R}^{[1:n]} \).

\( u^m \rightarrow u^{m+1} \) by solving concurrently:

\[
\begin{align*}
  u_1^{m+1} &= u_1^m + A_1^{-1} R_1 (f - Au^m) \\
  u_2^{m+1} &= u_2^m + A_2^{-1} R_2 (f - Au^m)
\end{align*}
\]

where \( u_i^m = R_i u^m \) and \( A_i := R_i A R_i^T \).
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\[ u^m \rightarrow u^{m+1} \text{ by solving concurrently:} \]

\[ u_1^{m+1} = u_1^m + A_1^{-1} R_1(f - Au^m) \quad u_2^{m+1} = u_2^m + A_2^{-1} R_2(f - Au^m) \]

where \( u_i^m = R_i u^m \) and \( A_i := R_i A R_i^T \).
An introduction to Additive Schwarz

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\]

where \( u_i^m = R_i u^m \) and \( A_i := R_i A R_i^T \).
We have effectively divided, but we have yet to conquer.

_Duplicated_ unknowns coupled via a _partition of unity:_

\[ I = \sum_{i=1}^{N} R_i^T D_i R_i. \]

Then, \( u^{m+1} = \sum_{i=1}^{N} R_i^T D_i u_i^{m+1}. \)

\[ M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i. \]

RAS algorithm (Cai & Sarkis, 1999)
An introduction to Additive Schwarz II

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*Duplicate*ed unknowns coupled via a *partition of unity*:

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Then, \( u^{m+1} = \sum_{i=1}^{N} R_i^T D_i u_i^{m+1} \). \( M^{-1}_{RAS} = \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i \).

RAS algorithm (Cai & Sarkis, 1999)
Schwarz algorithm iterates on a pair of local functions \((u_1^m, u_2^m)\).  
RAS algorithm iterates on the global function \(u^m\).

**Schwarz and RAS**

Discretization of the classical Schwarz algorithm and the iterative RAS algorithm:

\[
U^{n+1} = U^n + M^{-1}_{RAS} r^n, \quad r^n := F - A U^n.
\]

are equivalent

\[
U^n = R_1^T D_1 U_1^n + R_2^T D_2 U_2^n.
\]

(Efstathiou and Gander, 2002).

Operator \(M^{-1}_{RAS}\) is used as a preconditioner in Krylov methods for non symmetric problems.
Schwarz algorithm iterates on a pair of local functions \((u^1_m, u^2_m)\).

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**Schwarz and RAS**

Discretization of the classical Schwarz algorithm and the iterative RAS algorithm:

\[
U^{n+1} = U^n + M_{\text{RAS}}^{-1} r^n, \quad r^n := F - A U^n.
\]

are equivalent

\[
U^n = R_1^T D_1 U^n_1 + R_2^T D_2 U^n_2.
\]

(Efstathiou and Gander, 2002).

Operator \(M_{\text{RAS}}^{-1}\) is used as a preconditioner in Krylov methods for non symmetric problems.
ASM: a symmetrized version of RAS

\[
M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i.
\]

A symmetrized version: Additive Schwarz Method (ASM),

\[
M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i
\]  \hspace{1cm} (1)

is used as a preconditioner for the conjugate gradient (CG) method.

Although RAS is more efficient, ASM is amenable to to condition number estimates.

**Chronological curiosity:** First paper on Additive Schwarz dates back to 1989 whereas RAS paper was published in 1998.
Adding a coarse space

One level methods are not scalable.
We add a coarse space correction (aka second level)
Let $V_H$ be the coarse space and $Z$ be a basis, $V_H = \text{span } Z$,
writing $R_0 = Z^T$ we define the two level preconditioner as:

$$M_{ASM,2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i.$$ 

The Nicolaides approach (1987) is to use the kernel of the operator as a coarse space, this is the constant vectors, in local form this writes:

$$Z := (R_i^T D_i R_i 1)_{1 \leq i \leq N}$$

where $D_i$ are chosen so that we have a partition of unity:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id.$$ 

Key notion: Stable splitting (J. Xu, 1989)
Theoretical convergence result

Theorem (Widlund, Dryija)

Let $M_{ASM,2}^{-1}$ be the two-level additive Schwarz method:

$$\kappa(M_{ASM,2}^{-1} A) \leq C \left(1 + \frac{H}{\delta}\right)$$

where $\delta$ is the size of the overlap between the subdomains and $H$ the subdomain size.

This does indeed work very well

<table>
<thead>
<tr>
<th>Number of subdomains</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASM</td>
<td>18</td>
<td>35</td>
<td>66</td>
<td>128</td>
</tr>
<tr>
<td>ASM + Nicolaides</td>
<td>20</td>
<td>27</td>
<td>28</td>
<td>27</td>
</tr>
</tbody>
</table>

Fails for highly heterogeneous problems
You need a larger and adaptive coarse space
Strategy

Define an appropriate coarse space $V_{H2} = \text{span}(Z_2)$ and use the framework previously introduced, writing $R_0 = Z_2^T$ the two level preconditioner is:

$$P_{ASM2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i.$$ 

The coarse space must be

- Local (calculated on each subdomain) $\rightarrow$ parallel
- Adaptive (calculated automatically)
- Easy and cheap to compute
- Robust (must lead to an algorithm whose convergence is proven not to depend on the partition nor the jumps in coefficients)
Adaptive Coarse space for highly heterogeneous Darcy and (compressible) elasticity problems:

**Geneo .EVP** per subdomain:

Find $V_{j,k} \in \mathbb{R}^{N_j}$ and $\lambda_{j,k} \geq 0$:

\[
D_j R_j A R_j^T D_j V_{j,k} = \lambda_{j,k} A_j^{Neu} V_{j,k}
\]

---

**In the two-level ASM**, let $\tau$ be a user chosen parameter:

Choose eigenvectors $\lambda_{j,k} \geq \tau$ per subdomain:

\[
Z := (R_j^T D_j V_{j,k})_{j=1,\ldots,N}^{\lambda_{j,k} \geq \tau}
\]

This automatically includes Nicolaides CS made of Zero Energy Modes.
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\]

This automatically includes Nicolaides CS made of Zero Energy Modes.
Two technical assumptions.

**Theorem (Spillane, Dolean, Hauret, N., Pechstein, Scheichl (Num. Math. 2013))**

*If for all $j$: $0 < \lambda_j, m_{j+1} < \infty$*

$$\kappa(M_{ASM,2}^{-1}A) \leq (1 + k_0) \left[ 2 + k_0 (2k_0 + 1) \left( 1 + \tau \right) \right]$$

Possible criterion for picking $\tau$: (used in our Numerics)

$$\tau := \min_{j=1,\ldots,N} \frac{H_j}{\delta_j}$$

$H_j \ldots$ subdomain diameter, $\delta_j \ldots$ overlap
Channels and inclusions: $1 \leq \alpha \leq 1.5 \times 10^6$, the solution and partitionings (Metis or not)
Outline

1. (Recall) on Additive Schwarz Methods
2. Optimized Restricted Additive Schwarz Methods
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P.L. Lions’ Algorithm (1988)

\[-\Delta (u_1^{n+1}) = f \quad \text{in } \Omega_1,\]

\[u_1^{n+1} = 0 \quad \text{on } \partial \Omega_1 \cap \partial \Omega,\]

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\((n_1 \text{ and } n_2 \text{ are the outward normal on the boundary of the subdomains})\)

\[-\Delta (u_2^{n+1}) = f \quad \text{in } \Omega_2,\]

\[u_2^{n+1} = 0 \quad \text{on } \partial \Omega_2 \cap \partial \Omega\]

\[
\left( \frac{\partial}{\partial n_2} + \alpha \right)(u_2^{n+1}) = \left( - \frac{\partial}{\partial n_1} + \alpha \right)(u_1^n) \quad \text{on } \partial \Omega_2 \cap \overline{\Omega_1}.
\]

with \(\alpha > 0\). Overlap is not necessary for convergence.
Parameter \(\alpha\) can be optimized for.
Extended to the Helmholtz equation (B. Desprès, 1991)
GOAL of this work

(Recap) \( A_i := R_i A R_i^T \), \( 1 \leq i \leq N \)

1. Schwarz algorithm at the continuous level (partial differential equation)
2. Algebraic reformulation \( \Rightarrow M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i \)
3. Symmetric variant \( \Rightarrow M_{AS}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i \)
4. Adaptive Coarse space with prescribed targeted convergence rate
\( \Rightarrow \) Find \( V_{j,k} \in \mathbb{R}^{N_j} \) and \( \lambda_{j,k} \geq 0 \):

\[
D_j A_j D_j V_{j,k} = \lambda_{j,k} A_j^{Neu} V_{j,k}
\]

GOAL: Develop a theory and computational framework for P.L. Lions algorithm similar to what was done for ASM for a S.P.D. matrix \( A \).
1. P.L. Lions algorithm at the continuous level (partial differential equation)

2. Algebraic formulation for overlapping subdomains ⇒ Let $B_i$ be the matrix of the Robin subproblem in each subdomain $1 \leq i \leq N$, define $M^{-1}_{\text{ORAS}} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$, Optimized multiplicative, additive, and restricted additive Schwarz preconditioning, St Cyr et al, 2007

3. Symmetric variant ⇒
   - $M^{-1}_{\text{OAS}} := \sum_{i=1}^{N} R_i^T B_i^{-1} R_i$ (Natural but K.O.)
   - $M^{-1}_{\text{SORAS}} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i$ (O.K.)

4. Adaptive Coarse space with prescribed targeted convergence rate ⇒ ???
ORAS: Optimized RAS

1. P.L. Lions algorithm at the continuous level (partial differential equation)

2. Algebraic formulation for overlapping subdomains \( \Rightarrow \) Let \( B_i \) be the matrix of the Robin subproblem in each subdomain \( 1 \leq i \leq N \), define \( M_{\text{ORAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i \), Optimized multiplicative, additive, and restricted additive Schwarz preconditioning, St Cyr et al, 2007

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4. Adaptive Coarse space with prescribed targeted convergence rate
\( \Rightarrow ??? \)
Provided subdomains overlap, discretization of the classical P.L. Lions algorithm and the iterative ORAS algorithm:

\[ U^{n+1} = U^n + M_{\text{ORAS}}^{-1} r^n, \quad r^n := F - A U^n. \]

are equivalent

\[ U^n = R_1^T D_1 U^n_1 + R_2^T D_2 U^n_2, \]

(St Cyr, Gander and Thomas, 2007).

- **Huge** simplification in the implementation: no boundary right hand side discretization
- Operator \( M_{\text{ORAS}}^{-1} \) is used as a preconditioner in Krylov methods for non symmetric problems.
- **First step in a global theory**
1. (Recall) on Additive Schwarz Methods
2. Optimized Restricted Additive Schwarz Methods
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Lemma (Fictitious Space Lemma, Nepomnyaschikh 1991)

Let $H$ and $H_D$ be two Hilbert spaces. Let $a$ be a symmetric positive bilinear form on $H$ and $b$ on $H_D$. Suppose that there exists a linear operator $\mathcal{R} : H_D \to H$, such that

- $\mathcal{R}$ is surjective.
- there exists a positive constant $c_R$ such that
  \[ a(\mathcal{R}u_D, \mathcal{R}u_D) \leq c_R \cdot b(u_D, u_D) \quad \forall u_D \in H_D. \quad (2) \]
- \textbf{Stable decomposition:} there exists a positive constant $c_T$ such that for all $u \in H$ there exists $u_D \in H_D$ with $\mathcal{R}u_D = u$ and
  \[ c_T \cdot b(u_D, u_D) \leq a(\mathcal{R}u_D, \mathcal{R}u_D) = a(u, u). \quad (3) \]
We introduce the adjoint operator $\mathcal{R}^* : H \to H_D$ by $(\mathcal{R} u_D, u) = (u_D, \mathcal{R}^* u)_D$ for all $u_D \in H_D$ and $u \in H$. Then we have the following spectral estimate

$$c_T \cdot a(u, u) \leq a\left(\mathcal{R} B^{-1} \mathcal{R}^* A u, u\right) \leq c_R \cdot a(u, u), \quad \forall u \in H \quad (4)$$

which proves that the eigenvalues of operator $\mathcal{R} B^{-1} \mathcal{R}^* A$ are bounded from below by $c_T$ and from above by $c_R$. 
FSL lemma is the Lax-Milgram lemma of domain decomposition methods.

In combination with GenEO techniques it yields adaptive coarse spaces with a targeted condition number.

- Additive Schwarz method
- Hybrid Schwarz method
- Balancing Neumann Neumann and FETI
- Optimized Schwarz method

For a comprehensive presentation:

FSL and one level SORAS

- $H := \mathbb{R}^{\#N}$ and the $a$-bilinear form:
  \[ a(U, V) := V^T A U. \]  
  \hspace{1cm} (5)

  where $A$ is the matrix of the problem we want to solve.

- $H_D$ is a product space and $b$ a bilinear form defined by
  \[ H_D := \prod_{i=1}^{N} \mathbb{R}^{\#N_i} \text{ and } b(U, V) := \sum_{i=1}^{N} V_i^T B_i U_i. \]  
  \hspace{1cm} (6)

- The linear operator $\mathcal{R}_{SORAS}$ is defined as
  \[ \mathcal{R}_{SORAS} : H_D \longrightarrow H, \mathcal{R}_{SORAS}(U) := \sum_{i=1}^{N} R_i^T D_i U_i. \]  
  \hspace{1cm} (7)

We have: $M_{SORAS}^{-1} = \mathcal{R}_{SORAS} B^{-1} \mathcal{R}_{SORAS}^*$. 
Let $k_0$ be the maximum number of neighbors of a subdomain and $\gamma_1$ be defined as:

$$
\gamma_1 := \max_{1 \leq i \leq N} \max_{u_i \in \mathbb{R}^{\#N_i} \setminus \{0\}} \frac{(D_i u_i)^T A_i (D_i u_i)}{u_i^T B_i u_i}
$$

We can take $c_R := k_0 \gamma_1$.

Let $k_1$ be the maximum multiplicity of the intersection between subdomains and $\tau_1$ be defined as:

$$
\tau_1 := \min_{1 \leq i \leq N} \min_{u_i \in \mathbb{R}^{\#N_i} \setminus \{0\}} \frac{u_i^T A_i^{Neu} u_i}{u_i^T B_i u_i}
$$

We can take $c_T := \frac{\tau_1}{k_1}$.

We have:

$$
\frac{\tau_1}{k_1} \leq \lambda(M_{SORAS}^{-1} A) \leq k_0 \gamma_1.
$$
Control of the upper bound

Definition (Generalized Eigenvalue Problem for the upper bound)

Find \((U_{ik}, \mu_{ik}) \in \mathbb{R}^{\#N_i \setminus \{0\}} \times \mathbb{R}\) such that

\[
D_i A_i D_i U_{ik} = \mu_{ik} B_i U_{ik}.
\]

(8)

Let \(\gamma > 0\) be a user-defined threshold, we define \(Z_{\text{geneo}}^\gamma \subset \mathbb{R}^{\#N}\) as the vector space spanned by the family of vectors \((R_i^T D_i U_{ik})_{\mu_{ik} > \gamma}, 1 \leq i \leq N\) corresponding to eigenvalues larger than \(\gamma\).
Control of the lower bound

Definition (Generalized Eigenvalue Problem for the lower bound)

For each subdomain $1 \leq j \leq N$, we introduce the generalized eigenvalue problem

$$\text{Find } (V_{jk}, \lambda_{jk}) \in \mathbb{R}^{\#N_j} \setminus \{0\} \times \mathbb{R} \text{ such that } A_{j}^{\text{Neu}} V_{jk} = \lambda_{jk} B_{j} V_{jk}. \quad (9)$$

Let $\tau > 0$ be a user-defined threshold, we define $Z_{\text{geneo}}^{\tau} \subset \mathbb{R}^{\#N}$ as the vector space spanned by the family of vectors $(R_{j}^{T} D_{j} V_{jk})_{\lambda_{jk} < \tau, 1 \leq j \leq N}$ corresponding to eigenvalues smaller than $\tau$. 
Definition (Two level SORAS-GENEO-2 preconditioner)

Let $P_0$ denote the $a$-orthogonal projection on the SORAS-GENEO-2 coarse space

$$Z_{GenEO-2} := Z_{geneo}^T \bigoplus Z_{geneo}^\gamma,$$

the two-level SORAS-GENEO-2 preconditioner is defined:

$$M^{-1}_{SORAS,2} := P_0 A^{-1} + (I_d - P_0) M^{-1}_{SORAS} (I_d - P_0^T)$$

where $P_0 A^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0$, see J. Mandel, 1992.
Theorem (Haferssas, Jolivet and N., 2015)

Let $\gamma$ and $\tau$ be user-defined targets. Then, the eigenvalues of the two-level SORAS-GenEO-2 preconditioned system satisfy the following estimate

$$
\frac{1}{1 + \frac{k_1}{\tau}} \leq \lambda(M_{\text{SORAS,2}}^{-1} A) \leq \max(1, k_0 \gamma)
$$

What if one level method is $M_{\text{OAS}}^{-1}$:

Find $(V_{jk}, \lambda_{jk}) \in \mathbb{R}^{\#N_j \setminus \{0\}} \times \mathbb{R}$ such that

$$
A_j^{\text{Neu}} V_{jk} = \lambda_{jk} D_j B_j D_j V_{jk}.
$$
Outline

1. (Recall) on Additive Schwarz Methods
2. Optimized Restricted Additive Schwarz Methods
3. SORAS-GenEO-2 coarse space
4. Numerical Results
   - Comparisons
   - Scalability tests
5. Conclusion
Nearly incompressible elasticity

Material properties: Young modulus $E$ and Poisson ratio $\nu$ or alternatively by its Lamé coefficients $\lambda$ and $\mu$:

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}.$$ 

For $\nu$ close to $1/2$, the variational problem consists in finding $(u_h, p_h) \in \mathcal{V}_h := \mathbb{P}_2^d \cap H_0^1(\Omega) \times \mathbb{P}_1$ such that for all $(v_h, q_h) \in \mathcal{V}_h$

$$\begin{cases} 
\int_\Omega 2\mu\varepsilon(u_h)^T : \varepsilon(v_h) \, dx - \int_\Omega p_h \text{div} (v_h) \, dx = \int_\Omega f v_h \, dx \\
\int_\Omega \text{div} (u_h) q_h \, dx - \int_\Omega \frac{1}{\lambda} p_h q_h = 0
\end{cases}$$

$$\Rightarrow A\mathbf{U} = \begin{bmatrix} H & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} = \mathbf{F}.$$ 

$A$ is symmetric but no longer positive.
"Robin" interface condition for nearly incompressible elasticity

(Lube, 1998)

\[ \sigma(u) \cdot n + L(\alpha) u = 0 \text{ on } \partial\Omega_i \setminus \partial\Omega \]

Where \( L \) is constructed from the Lamé coefficient of the material and it is defined as follows

\[ L(\alpha, \lambda, \mu) := \frac{2\alpha\mu(2\mu + \lambda)}{\lambda + 3\mu}. \]

Parameter \( \alpha \) in the range \((1., 10.)\).
Comparisons (with FreeFem++)

Figure: 2D Elasticity: Sandwich of steel $(E_1, \nu_1) = (210 \cdot 10^9, 0.3)$ and rubber $(E_2, \nu_2) = (0.1 \cdot 10^9, 0.4999)$.

Metis partitioning

<table>
<thead>
<tr>
<th>Nb DOFs</th>
<th>Nb subdom</th>
<th>AS iteration</th>
<th>SORAS iteration</th>
<th>AS+CS(ZEM) iteration</th>
<th>SORAS +CS(ZEM) iteration</th>
<th>AS-GenEO iteration</th>
<th>SORAS -GenEO-2 iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>35841</td>
<td>8</td>
<td>150</td>
<td>184</td>
<td>117</td>
<td>79</td>
<td>110</td>
<td>13</td>
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<td>70590</td>
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<td>++1000</td>
<td>369</td>
<td>++1000</td>
<td>++1000</td>
<td>36</td>
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</tbody>
</table>

Table: 2D Elasticity. GMRES iteration counts
Numerical results via a Domain Specific Language

FreeFem++ (http://www.freefem.org/ff++), F. Hecht interfaced with

- Metis Karypis and Kumar 1998
- SCOTCH Chevalier and Pellegrini 2008
- UMFPACK Davis 2004
- ARPACK Lehoucq et al. 1998
- MPI Snir et al.

Intel MKL
- PARDISO Schenk et al. 2004
- MUMPS Amestoy et al. 1998
- PETSc solvers Balay et al.
- Slepc via PETSc

Runs on PC (Linux, OSX, Windows, Smartphones) and HPC (Babel@CNRS, HPC1@LJLL, Titane@CEA via GENCI PRACE)

Why use a DS(E)L instead of C/C++/Fortran/..?
- performances close to low-level language implementation,
- hard to beat something as simple as:

  \[
  \text{varf } a(u, v) = \int_{\text{mesh}} \left( \left[ dx(u), dy(u), dz(u) \right]' \cdot \left[ dx(v), dy(v), dz(v) \right] \right) \\
  + \int_{\text{mesh}} (f \cdot v) + \text{on(boundary mesh)}(u = 0)
  \]
Rubber Steel sandwich with automatic mesh partition

200 millions unknowns in 3D wall-clock time: 200. sec.
IBM/Blue Gene Q machine with 1.6 GHz Power A2 processors.
Hours provided by an IDRIS-GENCI project.
Stokes problem with automatic mesh partition. Driven cavity problem

<table>
<thead>
<tr>
<th>N</th>
<th>Factorization</th>
<th>Deflation</th>
<th>Solution</th>
<th># of it.</th>
<th>Total</th>
<th># of d.o.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>79.2 s</td>
<td>229.0 s</td>
<td>76.3 s</td>
<td>45</td>
<td>387.5 s</td>
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</tr>
<tr>
<td>2048</td>
<td>29.5 s</td>
<td>76.5 s</td>
<td>34.8 s</td>
<td>42</td>
<td>143.9 s</td>
<td>50.63 · 10^6</td>
</tr>
<tr>
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<td>11.1 s</td>
<td>45.8 s</td>
<td>19.8 s</td>
<td>42</td>
<td>80.9 s</td>
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</tr>
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<td>26.1 s</td>
<td>14.9 s</td>
<td>41</td>
<td>56.8 s</td>
<td></td>
</tr>
</tbody>
</table>

Peak performance: 50 millions d.o.f.’s in 3D in 57 sec.
IBM/Blue Gene Q machine with 1.6 GHz Power A2 processors.
Hours provided by an IDRIS-GENCI project.

HPDDM [https://github.com/hpddm/hpddm](https://github.com/hpddm/hpddm) is a framework in C++/MPI for high-performance domain decomposition methods with a Plain Old Data (POD) interface
1. (Recall) on Additive Schwarz Methods
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Summary

- **SORAS preconditioner**

\[ M^{-1}_{\text{SORAS}} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i \]

is amenable to a fruitful theory for OSM

- Using two generalized eigenvalue problems, we are able to achieve a targeted convergence rate for OSM

- Freely available via HPDDM library or FreeFem++

Future work

- Another look at parameter \( \alpha \) optimization

- Nonlinear time dependent problem (Coarse space reuse)

- Multigrid like three (or more) level methods
Preprints available on HAL and Software on freefem.org and github:


THANK YOU FOR YOUR ATTENTION!
Preprints available on HAL and Software on freefem.org and github:


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