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► **To cite this version:**

Romain Abraham, Aymen Bouaziz, Jean-François Delmas. Local limits of galton-watson trees conditioned on the number of protected nodes. 2015. hal-01195701v1

HAL Id: hal-01195701

<https://hal.science/hal-01195701v1>

Preprint submitted on 8 Sep 2015 (v1), last revised 25 Apr 2016 (v2)

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LOCAL LIMITS OF GALTON-WATSON TREES CONDITIONED ON THE NUMBER OF PROTECTED NODES

ROMAIN ABRAHAM, AYMEN BOUAZIZ, AND JEAN-FRANÇOIS DELMAS

ABSTRACT. We consider a marking procedure of the vertices of a tree where each vertex is marked independently from the others with a probability that depends only on its out-degree. We prove that a critical Galton-Watson tree conditioned on having a large number of marked vertices converges in distribution to the associated size-biased tree. We then apply this result to give the limit in distribution of a critical Galton-Watson tree conditioned on having a large number of protected nodes.

1. INTRODUCTION

In [6], Kesten proved that a critical or sub-critical Galton-Watson (GW) tree conditioned on reaching at least height h converges in distribution (for the local topology on trees) as $h \rightarrow +\infty$ toward the so-called sized-biased tree (that we call here Kesten's tree and whose distribution is described in Section 3.2). Since then, other conditionings have been considered, see [1, 2, 4] and the references therein for recent developments on the subject.

A protected node is a node that is not a leaf and none of its offsprings is a leaf. Precise asymptotics for the number of protected nodes in a conditioned GW tree have already been obtained in [3, 5] for instance. Let $A(\mathbf{t})$ be the number of protected nodes in the tree \mathbf{t} . Remark that this functional A is clearly monotone in sense of [4] (using for instance (10)); therefore, using Theorem 2.1 of [4], we immediately get that a critical GW tree τ conditioned on $\{A(\tau) > n\}$ converges in distribution toward Kesten's tree as n goes to infinity. Conditioning on $\{A(\tau) = n\}$ needs extra work and is the main objective of this paper. Using the general result of [1], if we have the following limit

$$(1) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(A(\tau) = n + 1)}{\mathbb{P}(A(\tau) = n)} = 1,$$

then the critical GW tree τ conditioned on $\{A(\tau) = n\}$ converges in distribution also toward Kesten's tree, see Theorem 5.1.

In fact, the limit (1) can be seen as a special case of a more general problem: conditionally given the tree, we mark the nodes of the tree independently of the rest of the tree with a probability that depends only on the number of offsprings of the nodes. Then we prove that a critical GW tree conditioned on the total number of marked nodes being large converges in distribution toward Kesten's tree, see Theorem 3.3.

Date: September 8, 2015.

2010 Mathematics Subject Classification. 60J80, 60B10, 05C05.

Key words and phrases. Galton-Watson tree; random trees; local limits; protected nodes.

The paper is then organized as follows: we first recall briefly the framework of discrete trees, then we consider in Section 3 the problem of a marked GW tree and the proofs of the results are given in Section 4. In particular, we prove the limit (1) when A is the number of marked nodes in Lemma 4.1 and we deduce the convergence of a critical GW tree conditioned on the number of marked nodes toward Kesten's tree in Theorem 3.3. We finally explain in Section 5 how the problem of protected nodes can be viewed as a problem on marked nodes and deduce the convergence in distribution of a critical GW tree conditioned on the number of protected nodes toward Kesten's tree in Theorem 5.1.

2. TECHNICAL BACKGROUND ON GW TREES

2.1. The set of discrete trees. We recall Neveu's formalism [7] for ordered rooted trees. Let $\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$ be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. For $u \in \mathcal{U}$, its length or generation $|u| \in \mathbb{N}$ is defined by $u \in (\mathbb{N}^*)^{|u|}$. If u and v are two sequences of \mathcal{U} , we denote by uv the concatenation of the two sequences, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$. The set of ancestors of u is the set

$$\text{An}(u) = \{v \in \mathcal{U}, \exists w \in \mathcal{U} \text{ such that } u = vw\}.$$

Notice that u belongs to $\text{An}(u)$. For two distinct elements u and v of \mathcal{U} , we denote by $u < v$ the lexicographic order on \mathcal{U} i.e. $u < v$ if $u \in \text{An}(v)$ and $u \neq v$ or if $u = wiu'$ and $v = wjv'$ for some $i, j \in \mathbb{N}^*$ with $i < j$. We write $u \leq v$ if $u = v$ or $u < v$.

A tree \mathbf{t} is a subset of \mathcal{U} that satisfies:

- $\emptyset \in \mathbf{t}$.
- If $u \in \mathbf{t}$, then $\text{An}(u) \subset \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists $k_u(\mathbf{t}) \in \mathbb{N}$ such that, for every $i \in \mathbb{N}^*$, $ui \in \mathbf{t}$ iff $1 \leq i \leq k_u(\mathbf{t})$.

The vertex \emptyset is called the root of \mathbf{t} . The integer $k_u(\mathbf{t})$ represents the number of offsprings of the vertex $u \in \mathbf{t}$. The set of children of a vertex $u \in \mathbf{t}$ is given by:

$$(2) \quad C_u(\mathbf{t}) = \{ui, 1 \leq i \leq k_u(\mathbf{t})\}.$$

By convention, we set $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$.

A vertex $u \in \mathbf{t}$ is called a leaf if $k_u(\mathbf{t}) = 0$. We denote by $\mathcal{L}_0(\mathbf{t})$ the set of leaves of \mathbf{t} . A vertex $u \in \mathbf{t}$ is called a *protected node* if $C_u(\mathbf{t}) \neq \emptyset$ and $C_u(\mathbf{t}) \cap \mathcal{L}_0(\mathbf{t}) = \emptyset$, that is u is not a leaf and none of his children is a leaf. For $u \in \mathbf{t}$, we define the fringe subtree $S_u(\mathbf{t})$, or the subtree of \mathbf{t} "above" u , as $S_u(\mathbf{t}) = \{v \in \mathcal{U}, uv \in \mathbf{t}\}$. Notice $S_u(\mathbf{t})$ is a tree. We denote by \mathbb{T} the set of trees and by $\mathbb{T}_0 = \{\mathbf{t} \in \mathbb{T}, \text{Card}(\mathbf{t}) < +\infty\}$ the subset of finite trees.

We say that a sequence of trees $(\mathbf{t}_n, n \in \mathbb{N})$ converges locally to a tree \mathbf{t} if and only if $\lim_{n \rightarrow \infty} k_u(\mathbf{t}_n) = k_u(\mathbf{t})$ for all $u \in \mathcal{U}$. Let $(T_n, n \in \mathbb{N})$ and T be \mathbb{T} -valued random variables. We denote by $\text{dist}(T)$ the distribution of the random variable T and write

$$\lim_{n \rightarrow +\infty} \text{dist}(T_n) = \text{dist}(T)$$

for the convergence in distribution of the sequence $(T_n, n \in \mathbb{N})$ to T with respect to the local topology.

If $\mathbf{t}, \mathbf{t}' \in \mathbb{T}$ and $x \in \mathcal{L}_0(\mathbf{t})$ we denote by

$$(3) \quad \mathbf{t} \otimes_x \mathbf{t}' = \{u \in \mathbf{t}\} \cup \{xv, v \in \mathbf{t}'\}$$

the tree obtained by grafting the tree \mathbf{t}' on the leaf x of the tree \mathbf{t} .

2.2. Galton Watson trees. Let $p = (p(n), n \in \mathbb{N})$ be a probability distribution on \mathbb{N} . We assume that

$$(4) \quad p(0) > 0, p(0) + p(1) < 1, \text{ and } \mu := \sum_{n=0}^{+\infty} np(n) < +\infty.$$

A \mathbb{T} -valued random variable τ is a GW tree with offspring distribution p if the distribution of $k_\emptyset(\tau)$ is p and it enjoys the branching property: for $n \in \mathbb{N}^*$, conditionally on $\{k_\emptyset(\tau) = n\}$, the subtrees $(S_1(\tau), \dots, S_n(\tau))$ are independent and distributed as the original tree τ .

The GW tree and the offspring distribution are called critical (resp. sub-critical, super-critical) if $\mu = 1$ (resp. $\mu < 1$, $\mu > 1$).

3. CONDITIONING ON THE NUMBER OF MARKED VERTICES

3.1. Definition of the marking procedure. We begin with a fixed tree \mathbf{t} . We add marks on the vertices of \mathbf{t} in an independent way such that the probability of adding a mark on a node u depends only on the number of children of u . More precisely, we consider a mark function $q : \mathbb{N} \rightarrow [0, 1]$ and a family of independent Bernoulli random variables $(Z_u(\mathbf{t}), u \in \mathbf{t})$ such that for all $u \in \mathbf{t}$:

$$\mathbb{P}(Z_u(\mathbf{t}) = 1) = 1 - \mathbb{P}(Z_u(\mathbf{t}) = 0) = q(k_u(\mathbf{t})).$$

The vertex u is said to have a mark if $Z_u(\mathbf{t}) = 1$. We denote by $\mathcal{M}(\mathbf{t}) = \{u \in \mathbf{t}; Z_u(\mathbf{t}) = 1\}$ the set of marked vertices and by $M(\mathbf{t})$ its cardinal. We call $(\mathbf{t}, \mathcal{M}(\mathbf{t}))$ a marked tree.

A marked GW trees with offspring distribution p and mark function q is a couple $(\tau, \mathcal{M}(\tau))$, with τ a GW trees with offspring distribution p and conditionally on $\{\tau = \mathbf{t}\}$ the set of marked vertices $\mathcal{M}(\tau)$ is distributed as $\mathcal{M}(\mathbf{t})$.

Remark 3.1. Notice that for $\mathcal{A} \subseteq \mathbb{N}$, if we set $q(k) = \mathbf{1}_{\{k \in \mathcal{A}\}}$, then the set $\mathcal{M}(\mathbf{t})$ is just the set of vertices with out-degree (i.e. number of offsprings) in \mathcal{A} considered in [1, 8]. Hence, the above construction can be seen as an extension of this case.

3.2. Kesten's tree. Let p be an offspring distribution satisfying Assumption (4) with $\mu \leq 1$ (i.e. the associated GW process is critical or sub-critical). We denote by $p^* = (p^*(n) = np(n)/\mu, n \in \mathbb{N})$ the corresponding size-biased distribution.

We define an infinite random tree τ^* (the size-biased tree that we call Kesten's tree in this paper) whose distribution is described as follows:

There exists a unique infinite sequence $(v_k, k \in \mathbb{N}^*)$ of positive integers such that, for every $h \in \mathbb{N}$, $v_1 \cdots v_h \in \tau^*$, with the convention that $v_1 \cdots v_h = \emptyset$ if $h = 0$. The joint distribution of $(v_k, k \in \mathbb{N}^*)$ and τ^* is determined recursively as follows. For each

$h \in \mathbb{N}$, conditionally given (v_1, \dots, v_h) and $\{u \in \tau^*, |u| \leq h\}$ the tree τ^* up to level h , we have:

- The number of children $(k_u(\tau^*), u \in \tau^*, |u| = h)$ are independent and distributed according to p if $v \neq v_1 \cdots v_h$ and according to p^* if $u = v_1 \cdots v_h$.
- Given $\{u \in \tau^*, |u| \leq h + 1\}$ and (v_1, \dots, v_h) , the integer v_{h+1} is uniformly distributed on the set of integers $\{1, \dots, k_{v_1 \cdots v_h}(\tau^*)\}$.

Remark 3.2. Notice that by construction, a.s. τ^* has a unique infinite spine. And following Kesten [6], the random tree τ^* can be viewed as the tree τ conditioned on non extinction.

3.3. Main theorem.

Theorem 3.3. *Let p be a critical offspring distribution that satisfies Assumption (4). Let $(\tau, \mathcal{M}(\tau))$ be a marked GW tree with offspring distribution p and mark function q such that $p(k)q(k) > 0$ for some $k \in \mathbb{N}$. For every $n \in \mathbb{N}^*$, let τ_n be a tree whose distribution is the conditional distribution of τ given $\{M(\tau) = n\}$. Let τ^* be a Kesten's tree associated with p . Then we have:*

$$\lim_{n \rightarrow +\infty} \text{dist}(\tau_n) = \text{dist}(\tau^*),$$

where the limit has to be understood along a subsequence for which $\mathbb{P}(M(\tau) = n) > 0$.

Remark 3.4. If for every $k \in \mathbb{N}$, $0 < q(k) < 1$, then $\mathbb{P}(M(\tau) = n) > 0$ for every $n \in \mathbb{N}$, hence the above conditioning is always valid.

4. PROOF OF THEOREM 3.3

Set $\gamma = \mathbb{P}(M(\tau) > 0)$. Since there exists $k \in \mathbb{N}$ with $p(k)q(k) > 0$, we have $\gamma > 0$. To simplify the proof, we suppose that $\mathbb{P}(M(\tau) = n) > 0$ for every n large enough but it is possible to consider periodicity as in [1] and then restrict to the subsequence such that $\mathbb{P}(M(\tau) = n) > 0$. A sufficient condition (but not necessary) to have $\mathbb{P}(M(\tau) = n) > 0$ for every n large enough is to assume that $\gamma < 1$ (see Lemma 4.2 and Section 4.4). Therefore, we shall assume $\gamma < 1$ and leave the case $\gamma = 1$ to the interested reader who then will take care of the periodicity problem. (Taking $q = \mathbf{1}_{\mathcal{A}}$, see Remark 3.1 for $0 \subset \mathcal{A} \subset \mathbb{N}$ implies $\gamma = 1$; see then [1] for how to handle the possible periodicity problem.)

The main ingredient for the proof of Theorem 3.3 is the following lemma.

Lemma 4.1. *We have*

$$(5) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(M(\tau) = n + 1)}{\mathbb{P}(M(\tau) = n)} = 1.$$

We then notice that:

- (i) $M(\mathbf{t}) \leq \text{Card}(\mathbf{t})$.
- (ii) For every $\mathbf{t} \in \mathbb{T}_0$, $x \in \mathcal{L}_0(\mathbf{t})$ and $\mathbf{t}' \in \mathbb{T}$, we have that $M(\mathbf{t} \otimes_x \mathbf{t}')$ is distributed as $\hat{M}(\mathbf{t}') + M(\mathbf{t}) - \mathbf{1}_{\{Z_x(\mathbf{t})=1\}}$, where $\hat{M}(\mathbf{t}')$ is distributed as $M(\mathbf{t}')$ and is independent of $\mathcal{M}(\mathbf{t})$.

Property (ii) is the analogue in the random case of the additive property in [1]. A straightforward adaptation of the proof of Theorem 3.1 of [1] using Lemma 4.1 and properties (i) and (ii) of M gives Theorem 3.3.

The end of this section is devoted to the proof of Lemma 4.1, see Section 4.4, which follows the ideas of the proof of Theorem 5.1 of [1].

4.1. Transformation of a subset of a tree onto a tree. We recall Rizzolo's map [8] which from $\mathbf{t} \in \mathbb{T}_0$ and a non-empty subset A of \mathbf{t} builds a tree \mathbf{t}_A such that $\text{Card}(A) = \text{Card}(\mathbf{t}_A)$. We will give a recursive construction of this map $\phi: (\mathbf{t}, A) \mapsto \mathbf{t}_A = \phi(\mathbf{t}, A)$. We will check in the next section that this map is such that if τ is a GW tree then τ_A will also be a GW tree for well chosen subset A of τ .

For a vertex $u \in \mathbf{t}$, recall $C_u(\mathbf{t})$ is the set of children of u in \mathbf{t} . We define for $u \in \mathbf{t}$:

$$R_u(\mathbf{t}) = \bigcup_{w \in \text{An}(u)} \{v \in C_w(\mathbf{t}); u < v\}$$

the vertices of \mathbf{t} which are larger than u for the lexicographic order and are children of u or of one of its ancestors. For a vertex $u \in \mathbf{t}$, we shall consider A_u the set of elements of A in the fringe subtree above u :

$$(6) \quad A_u = \{v \in S_u(\mathbf{t}), uv \in A\}.$$

Let $\mathbf{t} \in \mathbb{T}_0$ and $A \subset \mathbf{t}$ such that $A \neq \emptyset$. We shall define $\mathbf{t}_A = \phi(\mathbf{t}, A)$ recursively. Let u_0 be the smallest (for the lexicographic order) element of A . Consider the fringe subtrees of \mathbf{t} rooted at the vertices in $R_{u_0}(\mathbf{t})$ and among them those which contains at least one vertex of A that is $(S_u(\mathbf{t}); u \in R_{u_0}^A(\mathbf{t}))$, with

$$R_{u_0}^A(\mathbf{t}) = \{u \in R_{u_0}(\mathbf{t}); A_u \neq \emptyset\} = \{u \in R_{u_0}(\mathbf{t}); \exists v \in A \text{ and } u \leq v\}.$$

Define the number of children of the root of tree \mathbf{t}_A as the number of those fringe subtrees:

$$k_\emptyset(\mathbf{t}_A) = \text{Card}(R_{u_0}^A(\mathbf{t})).$$

If $k_\emptyset(\mathbf{t}_A) = 0$ set $\mathbf{t}_A = \emptyset$ otherwise let $u_1 < \dots < u_{k_\emptyset(\mathbf{t}_A)}$ be the ordered elements of $R_{u_0}^A(\mathbf{t})$ with respect to the lexicographic order on \mathcal{U} . And we define $\mathbf{t}_A = \phi(\mathbf{t}, A)$ recursively by:

$$(7) \quad S_i(\mathbf{t}_A) = \phi(S_{u_i}(\mathbf{t}), A_{u_i}) \quad \text{for } 1 \leq i \leq k_\emptyset(\mathbf{t}_A).$$

Since $\text{Card}(A_{u_i}) < \text{Card}(A)$, we deduce $\mathbf{t}_A = \phi(\mathbf{t}, A)$ is well defined and it is a tree by construction. Furthermore, we clearly have that A and \mathbf{t}_A have the same cardinal:

$$(8) \quad \text{Card}(\mathbf{t}_A) = \text{Card}(A).$$

4.2. Law of the tree $\tau_{\mathcal{M}(\tau)}$. Let $(\tau, \mathcal{M}(\tau))$ be a marked GW tree with critical offspring distribution satisfying (4) and mark function q . Recall $\gamma = \mathbb{P}(M(\tau) > 0) = \mathbb{P}(\mathcal{M}(\tau) \neq \emptyset)$ and $\gamma \in (0, 1)$.

Let $((X_i, Z_i), i \in \mathbb{N}^*)$ be i.i.d. random variables such that X_i is distributed according to p and Z_i is conditionally on X_i Bernoulli with parameter $q(X_i)$. We define:

- $T = \inf\{k \in \mathbb{N}^*, \sum_{i=1}^k (X_i - 1) = -1\}$.

- $N = \inf\{k \in \mathbb{N}^*, Z_k = 1\}$.
- \tilde{X} a random variable distributed as $1 + \sum_{i=1}^N (X_i - 1)$ conditionally on $\{N \leq T\}$.
- Y a random variable which is conditionally on \tilde{X} binomial with parameter (\tilde{X}, γ) .

Notice N has finite expectation as $\mathbb{P}(Z_1 = 1) > 0$ and thus \tilde{X} has finite expectation. We say a probability distribution on \mathbb{N} is aperiodic if the greatest common divisor of its support restricted to \mathbb{N}^* is 1. The following result is immediate as the distribution p of X_1 satisfies (4).

Lemma 4.2. *The distribution of Y satisfies (4) and if $\gamma < 1$ then it is aperiodic.*

Recall that for a tree $\mathbf{t} \in \mathbb{T}_0$, we have:

$$(9) \quad \sum_{u \in \mathbf{t}} (k_u(\mathbf{t}) - 1) = -1$$

and $\sum_{u \in \tilde{\mathbf{t}}, u < v} (k_u(\mathbf{t}) - 1) > -1$ for any $v \in \mathbf{t}$. We deduce that T is distributed according to $\text{Card}(\tau)$ and thus (T, N) is distributed as $(\text{Card}(\tau), M(\tau))$. In particular, we have:

$$\gamma = \mathbb{P}(N \leq T).$$

We denote by $(\tau^0, \mathcal{M}(\tau^0))$ a random marked tree distributed as $(\tau, \mathcal{M}(\tau))$ conditioned on $\{\mathcal{M}(\tau) \neq \emptyset\}$. By construction, $\text{Card}(\tau^0)$ is distributed as \tilde{X} .

Lemma 4.3. *Under the hypothesis of this section, we have that $\tau_{\mathcal{M}(\tau^0)}^0 = \phi(\tau^0, \mathcal{M}(\tau^0))$ is a critical GW tree with the law of Y as offspring distribution.*

This result generalizes Theorem 6 and Lemma 6 in [8]. Its proof, which is given in Section 4.3, follows the same ideas.

4.3. Proof of Lemma 4.3. In order to simplify notation, we write $\tilde{\tau}$ for $\tau_{\mathcal{M}(\tau^0)}^0 = \phi(\tau^0, \mathcal{M}(\tau^0))$ and for $u \in \tau^0$, we set R_u for $R_u(\tau^0)$.

Lemma 4.4. *The random tree $\tilde{\tau}$ is a GW tree with offspring distribution the law of Y .*

Proof. Let u_0 be the smallest (for the lexicographic order) element of $\mathcal{M}(\tau^0)$. The branching property of GW trees implies that the fringe subtrees of τ^0 rooted at the vertices in R_{u_0} , $(S_u(\tau^0), u \in R_{u_0})$ are independent and distributed as τ . Recall notation (6) so that the marked vertices of the fringe subtree rooted at u is $\mathcal{M}_u(\tau^0) = \{v \in S_u(\tau^0), uv \in \mathcal{M}(\tau^0)\}$. Then, the construction of the marks $\mathcal{M}(\tau)$ implies that the corresponding marked fringe subtrees $((S_u(\tau^0), \mathcal{M}_u(\tau^0)), u \in R_{u_0})$ are independent and distributed as $(\tau, \mathcal{M}(\tau))$. Notice that for $u \in R_{u_0}$, the fringe subtree $S_u(\mathbf{t})$ contains at least one mark iff u belongs to $R_{u_0}^{\mathcal{M}(\tau^0)} = \{u \in R_{u_0}; \exists v \in \mathcal{M}(\tau^0) \text{ and } u \leq v\}$. Then by considering only the fringe subtrees containing at least one mark, we get that, conditionally on $R_{u_0}^{\mathcal{M}(\tau^0)}$, the subtrees $((S_u(\tau^0), \mathcal{M}_u(\tau^0)), u \in R_{u_0}^{\mathcal{M}(\tau^0)})$ are independent and distributed as $(\tau^0, \mathcal{M}(\tau^0))$. We deduce from the recursive construction of the map ϕ , see (7), that $\tilde{\tau}$ is a GW tree. Notice the offspring distribution of $\tilde{\tau}$ is given by the distribution of the cardinal of $R_{u_0}^{\mathcal{M}(\tau^0)}$.

We now compute the corresponding offspring distribution. We first give an elementary formula for the cardinal of $R_u(\mathbf{t})$. Let $\mathbf{t} \in \mathbb{T}_0$ and $u \in \mathbf{t}$. Consider the tree $\mathbf{t}' = R_u(\mathbf{t}) \cup \{v \in \mathbf{t}, v \leq u\}$. Using (9) for \mathbf{t}' , we get:

$$-1 = \sum_{v \in \mathbf{t}} (k_v(\mathbf{t}') - 1) = \sum_{v \in \mathbf{t}, v \leq u} (k_v(\mathbf{t}') - 1) + \sum_{v \in R_u(\mathbf{t})} (-1).$$

This gives $\text{Card}(R_u(\mathbf{t})) = 1 + \sum_{v \in \mathbf{t}, v \leq u} (k_v(\mathbf{t}') - 1)$. We deduce from the definition of \tilde{X} that $\text{Card}(R_{u_0})$ is distributed as \tilde{X} . We deduce from the first part of the proof that conditionally on $\text{Card}(R_{u_0})$, the distribution of $\text{Card}(R_{u_0}^{\mathcal{M}(\tau^0)})$ is binomial with parameter $(\text{Card}(R_{u_0}(\tau^0)), \gamma)$. This gives that the offspring distribution of $\tilde{\tau}$ is given by the law of Y . \square

Lemma 4.5. *The GW tree $\tilde{\tau}$ is critical.*

Proof. Since the offspring distribution is the law of Y we need to check that $\mathbb{E}[Y] = 1$ that is $\gamma \mathbb{E}[\tilde{X}] = 1$ since Y is conditionally on \tilde{X} binomial with parameter (\tilde{X}, γ) .

Recall N has finite expectation as $\mathbb{P}(Z_1 = 1) > 0$ and is a stopping time with respect to the filtration generated by $((X_i, Z_i), i \in \mathbb{N}^*)$. Using Wald's equality and $\mathbb{E}[X_i] = 1$, we get $\mathbb{E} \left[\sum_{i=1}^N (X_i - 1) \right] = 0$ and thus using the definition of \tilde{X} :

$$\gamma \mathbb{E}[\tilde{X}] = \gamma + \mathbb{E} \left[\sum_{i=1}^N (X_i - 1) \mathbf{1}_{\{N \leq T\}} \right] = \gamma - \mathbb{E} \left[\sum_{i=1}^N (X_i - 1) \mathbf{1}_{\{N > T\}} \right].$$

We have:

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^N (X_i - 1) \mathbf{1}_{\{N > T\}} \right] &= \mathbb{E} \left[\sum_{i=1}^T (X_i - 1) \mathbf{1}_{\{N > T\}} \right] + \mathbb{P}(N > T) \mathbb{E} \left[\sum_{i=1}^N (X_i - 1) \right] \\ &= -\mathbb{P}(N > T) \\ &= \gamma - 1, \end{aligned}$$

where we used the strong Markov property of $((X_i, Z_i), i \in \mathbb{N}^*)$ at the stopping time T for the first equation, the definition of T and Wald's equality for the second. We deduce that $\mathbb{E}[Y] = \gamma \mathbb{E}[\tilde{X}] = 1$, which ends the proof. \square

4.4. Proof of (5). According to Lemma 4.3 and (8), we have that $M(\tau)$ is distributed as the total size of a critical GW whose offspring distribution satisfies (4) and is aperiodic as $\gamma < 1$ thanks to Lemma 4.2. Moreover, the proof of Proposition 4.3 of [1] entails that if τ' is a critical GW tree with aperiodic offspring distribution, then:

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(\text{Card}(\tau') = n + 1)}{\mathbb{P}(\text{Card}(\tau') = n)} = 1.$$

Thus we deduce that (5) holds if $\gamma < 1$.

5. PROTECTED NODES

Recall that a node of a tree \mathbf{t} is protected if it is not a leaf and none of its offsprings is a leaf. We denote by $A(\mathbf{t})$ the number of protected nodes of the tree \mathbf{t} .

Theorem 5.1. *Let τ be a critical GW tree with offspring distribution p satisfying (4) and let τ^* be the associated Kesten's tree. Let τ_n be a random tree distributed as τ conditionally given $\{A(\tau) = n\}$. Then:*

$$\lim_{n \rightarrow +\infty} \text{dist}(\tau_n) = \text{dist}(\tau^*).$$

Proof. Notice that $\mathbb{P}(A(\tau) = n) > 0$ for all $n \in \mathbb{N}$. Notice that the functional A satisfies the additive property of [1], namely for every $\mathbf{t} \in \mathbb{T}$, every $x \in \mathcal{L}_0(\mathbf{t})$ and every $\mathbf{t}' \in \mathbb{T}$ that is not reduced to the root, we have

$$(10) \quad A(\mathbf{t} \otimes_x \mathbf{t}') = A(\mathbf{t}) + A(\mathbf{t}') + D(\mathbf{t}, x)$$

where $D(\mathbf{t}, x) = 1$ if x is the only child of its first ancestor which is a leaf (therefore this ancestor becomes a protected node in $\mathbf{t} \otimes_x \mathbf{t}'$) and $D(\mathbf{t}, x) = 0$ otherwise. According to Theorem 3.1 of [1], to end the proof it is enough to check that

$$(11) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(A(\tau) = n + 1)}{\mathbb{P}(A(\tau) = n)} = 1.$$

For a tree $\mathbf{t} \neq \{\emptyset\}$, let $\mathbf{t}_{\mathbb{N}^*} = \phi(\mathbf{t}, \mathbf{t} \setminus \mathcal{L}_0(\mathbf{t}))$ be the tree obtained from \mathbf{t} by removing the leaves. Using Theorem 6 and Corollary 2 of [8] with $A = \mathbb{N}^*$ (or Theorem 3.3 with $q(k) = \mathbf{1}_{\{k > 0\}}$), we have that $\tau_{\mathbb{N}^*}$ is a critical GW tree with offspring distribution:

$$p_{\mathbb{N}^*}(k) = \sum_{n=\max(k,1)}^{+\infty} p(n) \binom{n}{k} (p(0))^{n-k} (1-p(0))^{k-1}, \quad k \in \mathbb{N}.$$

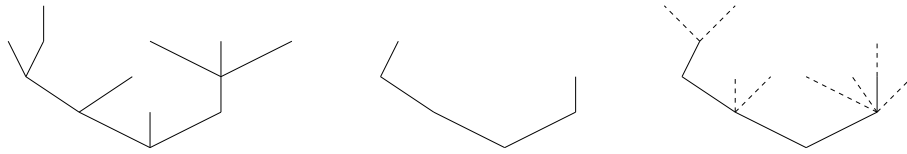
Conditionally given $\{\tau_{\mathbb{N}^*} = \mathbf{t}\}$, we consider independent random variables $(W(u), u \in \mathbf{t})$ taking values in \mathbb{N}^* whose distributions are given for all $u \in \mathbf{t}$ by $\mathbb{P}(W(u) = 0) = 0$ for $k_u(\mathbf{t}) = 0$ and otherwise for $k_u(\mathbf{t}) + n > 0$:

$$\mathbb{P}(W(u) = n) = \frac{p(k_u(\mathbf{t}) + n)}{p_{\mathbb{N}^*}(k_u(\mathbf{t}))} \binom{k_u(\mathbf{t}) + n}{n} p(0)^n (1-p(0))^{k_u(\mathbf{t})-1}.$$

In particular for $k_u(\mathbf{t}) > 0$, we have:

$$(12) \quad \mathbb{P}(W(u) = 0) = \frac{p(k_u(\mathbf{t}))}{p_{\mathbb{N}^*}(k_u(\mathbf{t}))} (1-p(0))^{k_u(\mathbf{t})-1}.$$

Then, we graft leaves on every node u of $\tau_{\mathbb{N}^*}$ $W(u)$ so that we get a new tree $\hat{\tau}$, see Figure 1. The distributions of the W 's and the way we graft these leaves are such that the new tree $\hat{\tau}$ is distributed as the original tree τ . Notice that the protected nodes of $\hat{\tau}$ are exactly the nodes of $\tau_{\mathbb{N}^*}$ on which we did not add leaves i.e. for which $W(u) = 0$. If we set $\mathcal{M}(\tau_{\mathbb{N}^*}) = \{u \in \tau_{\mathbb{N}^*}, W(u) = 0\}$, we have $M(\tau_{\mathbb{N}^*}) = A(\hat{\tau})$.

FIGURE 1. The trees τ , $\tau_{\mathbb{N}^*}$ and $\hat{\tau}$

Using (12), we get the corresponding mark function q is given by:

$$q(k) = \frac{p(k)(1-p(0))^{k-1}}{p_{\mathbb{N}^*}(k)} \mathbf{1}_{\{k \geq 1\}}.$$

As $\hat{\tau}$ is distributed as τ , we have:

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(A(\tau) = n+1)}{\mathbb{P}(A(\tau) = n)} = \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(A(\hat{\tau}) = n+1)}{\mathbb{P}(A(\hat{\tau}) = n)} = \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(M(\tau_{\mathbb{N}^*}) = n+1)}{\mathbb{P}(M(\tau_{\mathbb{N}^*}) = n)}.$$

As $\tau_{\mathbb{N}^*}$ is a critical GW tree, we deduce from Lemma 4.1 that

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(M(\tau_{\mathbb{N}^*}) = n+1)}{\mathbb{P}(M(\tau_{\mathbb{N}^*}) = n)} = 1$$

which gives (11) and hence ends the proof. \square

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