Entropy rigidity of Hilbert and Riemannian metrics

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Abstract. In this paper we provide two new characterizations of real hyperbolic n-space using the Poincaré exponent of a discrete group and the volume growth entropy. The first characterization is in the space of Hilbert metrics and generalizes a result of Crampon. The second is in the space of Riemannian metrics with Ricci curvature bounded below and generalizes a result of Ledrappier and Wang.

1. Introduction

For any discrete group Γ acting by isometries on a proper metric space (X, d), we define the *Poincaré*, or *critical*, *exponent* of Γ as

$$\delta_\Gamma(X, d) := \limsup_{r\to+\infty} \frac{1}{r} \log \#\{\gamma \in \Gamma \mid d(o, \gamma \cdot o) \leqslant r\}$$

where $o \in X$ is some fixed point.

If X has a measure μ one can also define the *volume growth entropy* as

$$h_{vol}(X, d, \mu) := \limsup_{r\to+\infty} \frac{1}{r} \log \mu\left(B_r(o)\right).$$

If the measure μ is $\mathrm{Isom}(X, d)$-invariant, finite on bounded sets, and positive on open sets then

$$\delta_\Gamma(X, d) \leqslant h_{vol}(X, d, \mu)$$

and, in some cases, for instance when the action of Γ on (X, d) is cocompact, the Poincaré exponent and the volume growth entropy coincide.

These two invariants have a long and interesting history, as they are intimately related to the geometric and dynamical property of the space (X, d) (for instance [Man79, FM82]). Moreover, they are often linked to rigidity phenomenons (for instance [BCG95, BCG96]).

In this paper we present two new characterizations of real hyperbolic n-space using the Poincaré exponent of a discrete group and the volume growth entropy. The first characterization (Theorem B) is in the space of Hilbert metrics and generalizes a result of Crampon [Cra09]. The second characterization (Theorem D) is in the space of Riemannian metrics with Ricci curvature bounded below and generalizes a result of Ledrappier and Wang [LW10].

1.1. Hilbert metrics.

Given a proper convex subset $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$, we let H_Ω be the associated Hilbert metric. The Hilbert metric is a complete length metric on Ω which is invariant under the group of projective automorphisms of Ω

$$\mathrm{Aut}(\Omega) := \{\varphi \in \mathrm{PGL}_{n+1}(\mathbb{R}) : \varphi\Omega = \Omega\}.$$

Moreover, if Ω is projectively equivalent to the ball $\mathcal{B}$, then (Ω, H_Ω) is the Klein–Beltrami model of the real hyperbolic n-space.

Tholozan recently proved the volume entropy conjecture for Hilbert metrics:

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Date: September 22, 2015.
Key words and phrases.

Theorem 1.1. [Tho15] *If $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set then*

$$h_{vol}(\Omega, H_\Omega, \mu_B) \leqslant n-1.$$

where μ_B is the Busemann–Hausdorff volume associated with (Ω, H_Ω) (or any bi-Lipschitz equivalent measure).

In particular the volume growth entropy is maximized when Ω is projectively equivalent to the unit ball. There are many other examples which maximize volume growth entropy, for instance, Berck, Bernig and Vernicos [BBV10] proved that, if $\partial\Omega$ is $C^{1,1}$ then

$$h_{vol}(\Omega, H_\Omega, \mu_B) = n-1.$$

However, if Ω has "enough" symmetry then it is reasonable to expect that $h_{vol}(\Omega, H_\Omega, \mu_B) = n-1$ if and only if Ω is projectively equivalent to the unit ball. For instance, Crampon proved the following:

Theorem 1.2. [Cra09] *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper strictly convex open set and there exists a discrete group $\Gamma \leqslant \mathrm{Aut}(\Omega)$ that acts properly, freely, and cocompactly. Then $h_{vol}(\Omega, H_\Omega, \mu_B) \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

Remark 1.3. If $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set and there exists a discrete group $\Gamma \leqslant \mathrm{Aut}(\Omega)$ that acts properly, freely, and cocompactly then Benoist [Ben04] proved that Ω is strictly convex if and only if $\partial\Omega$ is C^1.

Our first new characterization of real hyperbolic space removes the strictly convex hypothesis from Crampon's theorem:

Theorem A. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set and there exists a discrete group $\Gamma \leqslant \mathrm{Aut}(\Omega)$ which acts properly, freely, and cocompactly. Then $h_{vol}(\Omega, H_\Omega, \mu_B) \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

Remark 1.4. For the Hilbert metric, strict convexity of Ω is somewhat analogous to negative curvature. In particular, for a strictly convex set the Hilbert metric is uniquely geodesic, that is every pair of points are joined by a unique geodesic. Moreover, Benoist [Ben04] proved that when Ω is strictly convex and has a cocompact quotient then the induced geodesic flow is Anosov and is $C^{1+\alpha}$. In his proof of Theorem 1.2, Crampon first shows that the topological entropy of this flow coincides with the volume growth entropy and then he uses techniques from hyperbolic dynamics to prove rigidity. For a general convex open set, the Hilbert metric may not be uniquely geodesic, but one can consider a natural "geodesic line" flow obtained by flowing along the geodesics that are lines segments in $\mathbb{P}(\mathbb{R}^d)$. However this flow is only C^0 and will have "parallel" flow lines. Thus Crampon's approach via smooth hyperbolic dynamics will not extend, at least directly, to the general case.

Our second new characterization of real hyperbolic space replaces compactness with finite volume, but with the cost of replacing h_{vol} with δ_Γ.

Theorem B. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set and there exists a discrete group Γ which preserves Ω and acts properly, freely, and with cofinite volume. Then $\delta_\Gamma \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

When $\Gamma\backslash\Omega$ is non compact but has finite volume, it is unclear whether or not $h_{vol}(\Omega, H_\Omega, \mu_B)$ and $\delta_\Gamma(\Omega, H_\Omega)$ coincide (for Riemannian negatively curved metrics, there exists groups acting with cofinite volume for which the volume entropy and the critical exponent are distinct [DPPS09]). However, when Ω has C^1 boundary and is strictly convex then Crampon and Marquis [CM14b, Théorème 9.2] proved that these two asymptotic invariants coincide. We will prove that in the finite quotient case having C^1 boundary and being strictly convex are equivalent and thus establish:

Corollary C. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set which is either strictly convex or has C^1 boundary and such that there exists a discrete group Γ which preserves Ω and acts properly, freely, and with cofinite volume. Then $h_{vol}(\Omega, H_\Omega, \mu_B) \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

Remark 1.5. This result was announced for surfaces by Crampon in [Cra11], but his proof was not complete in the finite volume case since some of the dynamical results used are only fully proved in the compact case.

1.2. **The Blaschke metric.** Associated to every Hilbert metric space (Ω, H_Ω) is a Riemannian distance B_Ω on Ω called the *Blaschke*, or *affine*, distance (see, for instance, [Lof01, BH13]). This Riemannian distance is $\mathrm{Aut}(\Omega)$-invariant and by a result of Calabi [Cal72] has Ricci curvature bounded below by $-(n-1)$. In particular, if $d\,\mathrm{Vol}$ is the associated Riemannian volume form then the Bishop-Gromov volume comparison theorem implies that

$$h_{vol}(\Omega, B_\Omega, d\,\mathrm{Vol}) \leqslant n-1.$$

Benoist and Hulin [BH13] showed that the Hilbert distance and the Blaschke distance are bi-Lipschitz equivalent. Tholozan recently proved the following new relation:

Theorem 1.6. [Tho15] *If $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex subset, then*

$$B_\Omega < H_\Omega + 1.$$

In particular,

$$h_{vol}(\Omega, H_\Omega, \mu) \leqslant h_{vol}(\Omega, B_\Omega, d\,\mathrm{Vol})$$

and if $\Gamma \leqslant \mathrm{Aut}(\Omega)$ is a discrete group then

$$\delta_\Gamma(\Omega, H_\Omega) \leqslant \delta_\Gamma(\Omega, B_\Omega).$$

Tholozan's result allows us to transfer from the Hilbert setting to the Riemannian setting where many more analytic tools are available.

1.3. **Riemannian metrics.** In the Riemannian setting we will prove the following characterization of real hyperbolic space:

Theorem D. *Let (X, g) be a complete, simply connected Riemannian n-manifold and Γ a group acting by isometries on X. Suppose that*

1. $\mathrm{Ric} \geqslant -(n-1)$*;*
2. *X has bounded curvature;*
3. *Γ acts properly and freely on X and $\Gamma\backslash X$ has finite volume;*
4. *the Poincaré exponent satisfies $\delta_\Gamma(X, g) = n-1$*

Then X is isometric to the real hyperbolic space $\mathbb{H}^n$.

Ledrappier and Wang [LW10] proved the above theorem when the quotient $\Gamma\backslash X$ is assumed to be compact, in which case you can replace the Poincaré exponent with the volume entropy. Although our proof will follow the general outline of their argument, only assuming finite volume introduces a number of technical complications. The bounded curvature assumption and finite volume assumptions are important assumptions for our argument, but it may be possible to remove them.

Acknowledgments. The first author would like to thank François Ledrappier and Nicolas Tholozan for helpful discussions. The second author was supported by the ANR Facettes and the ANR Finsler. The third author was partially supported by NSF grant 1400919.

2. Entropy rigidity for Riemannian metrics

This section is entirely devoted to the proof of Theorem D. It will follow from Proposition 2.7 and Proposition 2.13 below.

2.1. **The Busemann boundary.** In this subsection we describe the Busemann compactification of a non-compact complete Riemannian manifold X.

Fix a point $o \in X$. As in [Led10, LW10], we will normalize our Busemann functions so that $\xi(o) = 0$. Now, for each $y \in X$, define the Busemann function based at y to be

$$b_y(x) := d(x, y) - d(y, o).$$

As each b_y is 1-Lipschitz, the embedding $y \to b_y \in C(X)$ is relatively compact when $C(X)$ is equipped with the topology of uniform convergence on compact subsets. The *Busemann compactification* $\hat{X}$ of X is then defined to be the closure of X in $C(X)$. The *Busemann boundary* of X is the set $\partial\widehat{X} = \hat{X} \setminus X$. We begin by recalling some features of this compactification.

Theorem 2.1. *Let X be a non-compact complete simply connected Riemannian manifold. Then*

(1) *X is open in $\hat{X}$, hence the Busemann boundary $\partial\widehat{X}$ is compact.*
(2) *The action of $Isom(X)$ on X extends to an action on $\hat{X}$ by homeomorphisms and for $\gamma \in Isom(X)$ and $\xi \in \partial\hat{X}$ the action is given by*

$$(\gamma \cdot \xi)(x) = \xi(\gamma^{-1}x) - \xi(\gamma^{-1}o).$$

The first result can be found in [LW10, Proposition 1]. The second assertion is straightforward to prove.

2.2. **Patterson-Sullivan measures.**

Definition 2.2. Let (X, g) be a non-compact complete simply connected Riemannian manifold and $\Gamma \leqslant \mathrm{Isom}(X, g)$ a discrete subgroup with $\delta_\Gamma < \infty$. A family of measures $\{\nu_x : x \in X\}$ on $\partial\hat{X}$ is a (normalized) *Patterson-Sullivan measure* if

(1) $\nu_o(\partial\hat{X}) = 1$,
(2) for any $x, y \in X$ the measures ν_x, ν_y are in the same measure class and satisfy

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\delta_\Gamma(\xi(x)-\xi(y))},$$

(3) for any $g \in \Gamma$, $\nu_{gx} = g_*\nu_x$.

Following the standard construction of Pattersion-Sullivan measures via the Poincaré series (see for instance Section 2 of [LW10]) we obtain:

Proposition 2.3. *Let (X, g) be a non-compact complete simply connected Riemannian manifold and $\Gamma \leqslant \mathrm{Isom}(X, g)$ a discrete subgroup with $\delta_\Gamma < \infty$. Then there exists a Patterson-Sullivan measure $\{\nu_x : x \in X\}$ on $\partial\hat{X}$.*

2.3. **An integral formula.** Now suppose (X, g) is a non-compact complete simply connected Riemannian manifold and $\Gamma \leqslant \mathrm{Isom}(X, g)$ is a discrete subgroup with $\delta_\Gamma < \infty$. Moreover, assume that Γ acts properly and freely on X and the quotient manifold $M = \Gamma\backslash X$ has finite volume (with respect to the Riemannian volume form).

Following [Led10, LW10], we introduce the *laminated space*

$$X_M = (X \times \partial\hat{X})/\Gamma$$

where Γ acts diagonally on the product. The space X_M is laminated by the images of $X \times \{\xi\}$ under the projection. The leaves of this lamination inherit a smooth structure from X and using this structure we can define a gradient $\nabla^{\mathcal{W}}$, a divergence $\mathrm{div}^{\mathcal{W}}$, and a Laplacian $\Delta^{\mathcal{W}}$ in the leaf direction. A Patterson-Sullivan measure $\{\nu_x : x \in X\}$ yields a measure on the laminated space X_M as follows: by definition $d\nu_x(\xi) = e^{-\delta_\Gamma\xi(x)}d\nu_o(\xi)$ for all $x \in X$. In particular if dx is the Riemannian volume form on X, then the measure

$$d\widetilde{m}(x, \xi) = e^{-\delta_\Gamma\xi(x)}dxd\nu_o(\xi)$$

is Γ-invariant and descends to a measure ν on X_M.

Now the function $x \to \nu_x(\partial\widehat{X})$ is Γ-invariant so with a slight abuse of notation the measure ν has total mass

$$\nu(X_M) = \int_M \nu_x(\partial\widehat{X})dx.$$

Since $x \to \nu_x(\partial\widehat{X})$ is continuous, if $\Gamma\backslash M$ is compact then the measure ν is finite. For general finite volume quotients it is not clear when ν will be a finite measure, but we can prove the following:

Proposition 2.4. *With the notation above, if (X, g) has $\mathrm{Ric} \geqslant -(n-1)$ and $\delta_\Gamma = n-1$ then $\nu(X_M) < \infty$.*

Proof. Since $\mathrm{Ric} \geqslant -(n-1)$ the Laplacian comparison theorem implies for any $\xi \in \partial\widehat{X}$ we have

$$\Delta e^{-(n-1)\xi} \geqslant 0$$

in the sense of distribution (see for instance [LW10, Proposition 4]). So in particular, since $\delta_\Gamma = n-1$, the function

$$f(x) := \nu_x(\partial\widehat{X}) = \int_{\partial\widehat{X}} e^{-(n-1)\xi(x)}d\nu_o(x)$$

is such that $\Delta f \geqslant 0$ in the sense of distributions. However, thanks to the invariance of the Patterson-Sullivan measure, f is Γ-invariant and hence descends to a superharmonic function on $M = \Gamma \backslash X$. Since M has finite volume and f is a positive, superharmonic function, f must be constant [Ada92, Proposition 0.2]. Then,

$$\nu(X_M) = \int_M \nu_x(\partial\widehat{X})dx = \int_M \nu_o(\partial\widehat{X})dx = \int_M dx = \mathrm{Vol}(M). \qquad \square$$

Finally the argument at the end of Section 2 of [LW10] can be used to show the following:

Theorem 2.5. *With the notation above, if Y is a continuous vector field on X_M which is C^1 along the leaves $X \times \{\xi\}$ so that $\|Y\|_g$ and $\mathrm{div}^{\mathcal{W}} Y$ are in $L^1(X_M, d\nu)$ then*

$$\int \mathrm{div}^{\mathcal{W}} Y \, d\nu = \delta_\Gamma \int \langle Y, \nabla^{\mathcal{W}} \xi \rangle \, d\nu.$$

Remark 2.6. Since $\left\|\nabla^{\mathcal{W}} \xi\right\| \leqslant 1$ almost everywhere we see that

$$\int \left|\langle Y, \nabla^{\mathcal{W}} \xi \rangle\right| d\nu \leqslant \int \|Y\|_g \, d\nu < \infty.$$

Thus the right hand side of the equation in Theorem 2.5 is well defined.

2.4. **A special Busemann function.**

Proposition 2.7. *Suppose (X, g) is a complete simply connected Riemannian manifold with $\mathrm{Ric} \geqslant -(n-1)$ and bounded sectional curvature. Assume $\Gamma \leqslant \mathrm{Isom}(X)$ is a discrete group that acts properly and freely on X so that $M = \Gamma \backslash X$ has finite volume (with respect to the Riemannian volume form). If $\delta_\Gamma = n - 1$ then there exists $\xi_0 \in \partial\widehat{X}$ so that $\Delta\xi_0 \equiv n - 1$.*

For the rest of the subsection assume (X, g) and $\Gamma \leqslant \mathrm{Isom}(X, g)$ satisfy the hypothesis of Proposition 2.7.

Let $p_t(x, y)$ be the heat kernel on X. By Theorem 4 in [CLY81], we have that, for any $t > 0$ there exists $C = C(t) \geqslant 1$, such that

$$p_t(x, y) \leqslant C e^{\frac{-d(x,y)^2}{C}}.$$

On the space $\widehat{X} = X \times \partial\widehat{X}$ define the function

$$F_t(x, \xi) := \int_X p_t(x, y)\xi(y)dy.$$

Because of the estimate on $p_t(x, y)$ above F_t is well defined. In Appendix A we will use standard facts about the heat kernel to prove the following:

Proposition 2.8. *With the notation above,*

1. *For any $t > 0$ and $\xi \in \partial\widehat{X}$, the function $x \to F_t(x, \xi)$ is C^∞.*
2. *For any $t > 0$, the functions $(x, \xi) \to \nabla_x F_t(x, \xi)$ and $(x, \xi) \to \Delta_x F_t(x, \xi)$ are continuous.*
3. *For any $t > 0$ and $\xi \in \partial\widehat{X}$,*
$$\|\nabla_x F_t(x, \xi)\| \leqslant e^{(n-1)t}.$$
4. *For any $t > 0$ and $\xi \in \partial\widehat{X}$,*
$$\Delta_x F_t(x, \xi) \leqslant n - 1.$$

Now, let $\widetilde{Y}_t(x, \xi) = \nabla_x F_t(x, \xi)$. Then $\widetilde{Y}_t$ descends to a continuous vector field Y_t on X_M which is C^∞ along the leaves $X \times \{\xi\}$.

Lemma 2.9. *Suppose M is a complete Riemannian manifold and $x_0 \in M$, then there exists $C > 0$ so that for any $r > 4$ there is a C^∞ function $\varphi_r : M \to \mathbb{R}$ such that*

1. *$0 \leqslant \varphi_r \leqslant 1$ on M,*
2. *$\varphi_r \equiv 1$ on $B_r(x_0)$,*
3. *$\varphi_r \equiv 0$ on $M \setminus B_{2r}(x_0)$,*
4. *$\|\nabla\varphi_r\| \leqslant C/r$ on M.*

Proof. Pick a smooth function $f\colon [0,\infty) \to \mathbb{R}$ such that $0 \leqslant f \leqslant 1$, $f \equiv 1$ on $[0,1]$, and $f \equiv 0$ on $[2,\infty)$. Let $C_1 = \max\{|f'(t)|\}$. Next, let $g\colon [-1/3, 4/3] \to [0,1]$ be a C^∞ function with $g \equiv 0$ on $[-1/3, 1/3]$ and $g \equiv 1$ on $[2/3, 4/3]$. Let $C_2 = \max\{|g'(t)|\}$. We claim that $C = 2C_1C_2$ satisfies the conclusion of the lemma.

Fix $r > 0$ and define the function $\phi : M \to \mathbb{R}$ by

$$\phi(x) = f(d(x, x_0)/r).$$

Then ϕ is C_1/r-Lipschitz. Then, we can approximate ϕ by a C^∞ function, $\theta\colon X \to \mathbb{R}$, so that $|\phi - \theta| < 1/r$ and θ is $2C_1/r$-Lipschitz (see, for instance, [AFLMR07]). Finally, define

$$\varphi_r(x) := g(\theta(x)).$$

Then $0 \leqslant \varphi_r \leqslant 1$ on N by construction. Moreover, if $x \in B_r(x_0)$, we have that $\phi(x) = 1$ and so, $\theta(x) \in [1-1/r, 1+1/r] \subset [2/3, 4/3]$. Thus, $\varphi_r(x) = 1$. Similarly, if $x \in M \setminus B_{2r}(x_0)$ then $\varphi_r(x) = 0$. Finally, we see that φ_r is $2C_1C_2/r$-Lipschitz. □

Next, let $\varphi_r\colon M \to \mathbb{R}$ be as in the above lemma for some $x_0 \in M$. Then, define $\widetilde{f}_r\colon X \times \partial\widehat{X} \to \mathbb{R}$ by $\widetilde{f}_r(x,\xi) = \varphi_r(\pi'(x))$, where $\pi' : X \to M$ is the universal cover map. Since $\widetilde{f}_r$ is Γ-invariant, it descends to a continuous function $f_r\colon X_M \to \mathbb{R}$ which is C^∞ along the leaves $X \times \{\xi\}$.

Let $\widetilde{x}_0 \in X$ be a preimage of $x_0 \in M$. For $r > 0$, let $K_r \subset X_M$ be the image of $B_r(\widetilde{x}_0) \times \partial\widehat{X}$ under the map

$$\pi\colon X \times \partial\widehat{X} \to X_M.$$

Lemma 2.10. *K_r is compact, $f_r \equiv 1$ on K_r, and $f_r \equiv 0$ on $X_M \setminus K_{2r}$.*

Proof. Clearly, K_r is compact by definition. Notice that $(x,\xi) \in \pi^{-1}(K_r)$ if and only if $x \in \cup_{\gamma\in\Gamma} B_r(\gamma\widetilde{x}_0)$. Thus, if $(x,\xi) \in \pi^{-1}(K_r)$ then $\widetilde{f}_r(x,\xi) \equiv 1$, and, if $(x,\xi) \notin \pi^{-1}(K_{2r})$ then $\widetilde{f}_r \equiv 0$. □

Lemma 2.11. *For any $r > 0$ and $t > 0$,*

$$\|f_r Y_t\| \in L^1(X_M, d\nu)$$

and

$$\operatorname{div}^{\mathcal{W}}(f_r Y_t) \in L^1(X_M, d\nu).$$

Proof. Since $\|f_r Y_t\| \leqslant e^{(n-1)t}$, the first assertion is obvious. Now

$$\operatorname{div}^{\mathcal{W}}(f_r Y_t) = f_r \operatorname{div}^{\mathcal{W}} Y_t + \left\langle \nabla^{\mathcal{W}} f_r, Y_t \right\rangle,$$

so

$$\int_{X_M} \left|\operatorname{div}^{\mathcal{W}} f_r Y_t\right| d\nu \leqslant \int_{X_M} f_r \left|\operatorname{div}^{\mathcal{W}} Y_t\right| d\nu + \frac{Ce^{(n-1)t}}{r}\nu(X_M).$$

However, the support of f_r is compact in X_M and the map $(x,\xi) \to \Delta_x F(x,\xi)$ is continuous. Thus, $\left|\operatorname{div}^{\mathcal{W}} Y_t\right|$ is bounded on the support of f_r. Hence,

$$\int_{X_M} f_r \left|\operatorname{div}^{\mathcal{W}} Y_t\right| d\nu < +\infty.$$ □

Lemma 2.12. *For any $t > 0$,*

$$\operatorname{div}^{\mathcal{W}} Y_t \in L^1(X_M, d\nu),$$

and

$$\int_{X_M \setminus K_{2r}} \left|\operatorname{div}^{\mathcal{W}} Y_t\right| d\nu \leqslant \left(2n - 2 + \frac{C}{r}\right) e^{(n-1)t}\nu(X_M \setminus K_r).$$

Proof. For a real number t, let $t^+ = \max\{0, t\}$ and $t^- = \min\{0, t\}$. Then,

$$\int_{X_M} \left|\operatorname{div}^{\mathcal{W}}(Y_t)\right| d\nu = \int_{X_M} \operatorname{div}^{\mathcal{W}}(Y_t)^+ d\nu - \int_{X_M} \operatorname{div}^{\mathcal{W}}(Y_t)^- d\nu$$

and, by Proposition 2.8,

$$\int_{X_M} \operatorname{div}^{\mathcal{W}}(Y_t)^+ d\nu \leqslant (n-1)\nu(X_M).$$

So, it is enough to bound the integral of $\operatorname{div}^{\mathcal{W}}(Y_t)^-$.

By Theorem 2.5,

$$\int_{X_M} \operatorname{div}^{\mathcal{W}}(f_r Y_t) d\nu = (n-1) \int_{X_M} \langle f_r Y_t, \nabla^{\mathcal{W}} \xi \rangle \, d\nu.$$

So, by Proposition 2.8,

$$\left| \int_{X_M} \operatorname{div}^{\mathcal{W}}(f_r Y_t) d\nu \right| \leqslant (n-1) e^{(n-1)t} \nu(X_M).$$

Now,

$$\operatorname{div}^{\mathcal{W}} f_r Y_t = f_r \operatorname{div}^{\mathcal{W}} Y_t + \langle \nabla^{\mathcal{W}} f_r, Y_t \rangle \quad \text{and} \quad \left| \langle \nabla^{\mathcal{W}} f_r, Y_t \rangle \right| \leqslant \frac{C e^{(n-1)t}}{r},$$

so

$$\left| \int_{X_M} f_r \operatorname{div}^{\mathcal{W}}(Y_t) d\nu \right| \leqslant \left(\frac{C}{r} + n - 1 \right) e^{(n-1)t} \nu(X_M).$$

Then

$$\begin{aligned} -\int f_r \operatorname{div}^{\mathcal{W}}(Y_t)^- d\nu &= -\int f_r \operatorname{div}^{\mathcal{W}}(Y_t) d\nu + \int f_r \operatorname{div}^{\mathcal{W}}(Y_t)^+ d\nu \\ &\leqslant -\int f_r \operatorname{div}^{\mathcal{W}}(Y_t) d\nu + (n-1)\nu(X_M). \end{aligned}$$

Which implies that

$$-\int f_r \operatorname{div}^{\mathcal{W}}(Y_t)^- d\nu \leqslant \left(\frac{C}{r} + 2n - 2 \right) e^{(n-1)t} \nu(X_M).$$

Finally $\lim_{r\to\infty} f_r = 1$ and so, by Fatou's Lemma,

$$-\int_{X_M} \operatorname{div}^{\mathcal{W}}(Y_t)^- d\nu \leqslant \liminf_{r\to\infty} -\int f_r \operatorname{div}^{\mathcal{W}}(Y_t)^- d\nu \leqslant (2n-2)\, e^{(n-1)t} \nu(X_M).$$

By the remarks at the start of the proof we then have that $\operatorname{div}^{\mathcal{W}} Y_t \in L^1(X_M, d\nu)$.

To prove the second assertion first observe that, for any $r \in \mathbb{R}$, $|r| = -r + 2r^+$. So,

$$\begin{aligned} \int_{X_M \setminus K_{2r}} \left| \operatorname{div}^{\mathcal{W}} Y_t \right| d\nu &\leqslant \int_{X_M} (1 - f_r) \left| \operatorname{div}^{\mathcal{W}} Y_t \right| d\nu \\ &= -\int_{X_M} (1 - f_r) \operatorname{div}^{\mathcal{W}} Y_t d\nu + 2 \int_{X_M} (1 - f_r)(\operatorname{div}^{\mathcal{W}} Y_t)^+ d\nu. \end{aligned}$$

Now,

$$\int_{X_M} (1 - f_r)(\operatorname{div}^{\mathcal{W}} Y_t)^+ d\nu \leqslant \int_{X_M} (1 - f_r)(n-1) d\nu \leqslant (n-1)\nu(X_M \setminus K_r),$$

and, by Theorem 2.5,

$$\begin{aligned} -\int_{X_M} (1 - f_r) \operatorname{div}^{\mathcal{W}} Y_t d\nu &= -\int_{X_M} \operatorname{div}^{\mathcal{W}} \left((1 - f_r) Y_t \right) d\nu + \int_{X_M} \langle \nabla^{\mathcal{W}} (1 - f_r), Y_t \rangle \, d\nu \\ &\leqslant (n-1) \left| \int_{X_M} \langle (1 - f_r) Y_t, \nabla^{\mathcal{W}} \xi \rangle \, d\nu \right| + \frac{C e^{(n-1)t}}{r} \nu(X_M \setminus K_r) \\ &\leqslant \left(n - 1 + \frac{C}{r} \right) e^{(n-1)t} \nu(X_M \setminus K_r). \end{aligned}$$

Combining the above inequalities establishes the second assertion of the lemma. □

Now, for any $r > 0$, we have

$$\begin{aligned}
&\int_{X_M \setminus K_{2r}} \operatorname{div}^{\mathcal{W}} Y_t d\nu \\
&= \int_{x \in M \setminus B_{2r}(x_0)} \left(\int_{\xi \in \partial \widehat{M}} (\operatorname{div}^{\mathcal{W}} Y_t) e^{-(n-1)\xi(x)} d\nu_o \right) dx \\
&= \int_{x \in M \setminus B_{2r}(x_0)} \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) + (n-1) \langle \nabla^{\mathcal{W}} \xi, Y_t \rangle e^{-(n-1)\xi(x)} d\nu_o \right) dx \\
&= \int_{x \in M \setminus B_{2r}(x_0)} \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) dx + (n-1) \int_{X_M \setminus K_{2r}} \langle \nabla^{\mathcal{W}} \xi, Y_t \rangle d\nu.
\end{aligned}$$

So, by Lemma 2.12, we see that

$$\int_{x \in M \setminus B_{2r}(x_0)} \left| \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) \right| dx \leqslant \left(2n - 2 + \frac{C}{r} \right) e^{(n-1)t} \nu(X_M \setminus K_r) + (n-1) \nu(X_M \setminus K_{2r}). \tag{1}$$

Thanks to Lemma 2.12, we can apply Theorem 2.5 to $Y_t(x, \xi)$. So, redoing the same computations as above, we get

$$(n-1) \int_{X_M} \langle \nabla^{\mathcal{W}} \xi, Y_t \rangle d\nu = \int_{x \in M} \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) dx + (n-1) \int_{X_M} \langle \nabla^{\mathcal{W}} \xi, Y_t \rangle d\nu.$$

and thus

$$\int_{x \in M} \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) dx = 0. \tag{2}$$

We now choose a countable and locally finite open cover $\{U_i\}$ of M such that each U_i is small enough so that $\pi^{-1}(U_i)$ is a disjoint union of open sets all diffeomorphic to U_i.

Let $\{\chi_i\}$ be a partition of unity subordinated to $\{U_i\}$. For each U_i, we choose one connected component of its lift that we denote by $\widetilde{U}_i$ and we write $\widetilde{\chi}_i$ for the lift of χ_i to $\widetilde{U}_i$.

Now by equation (1) above, for any $\varepsilon > 0$, there exists $r > 0$ so that, for all $0 \leqslant t \leqslant 1$, we have

$$\int_{x \in M \setminus B_{2r}(x_0)} \left| \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) \right| dx \leqslant \varepsilon.$$

Let

$$\mathcal{J} := \{ j \in \mathbb{N} : U_j \cap B_{2r}(x_0) \neq \emptyset \}.$$

Because the cover $M = \bigcup U_i$ is locally finite, we see that $\mathcal{J}$ is a finite subset of $\mathbb{N}$. Moreover by Equation 2

$$\begin{aligned}
&\sum_{j \in \mathcal{J}} \int_{x \in M} \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) \chi_j(x) dx \\
&\quad = - \int_{x \in M} \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) \sum_{j \notin \mathcal{J}} \chi_j(x) dx \\
&\quad \geqslant - \int_{x \in M \setminus B_{2r}(0)} \left| \left(\int_{\xi \in \partial \widehat{M}} \operatorname{div}^{\mathcal{W}} \left(Y_t e^{-(n-1)\xi(x)} \right) d\nu_o \right) \right| dx \\
&\quad \geqslant -\varepsilon.
\end{aligned}$$

Then, since the sum is finite,

$$\begin{aligned}
&\sum_{j\in\mathcal{J}}\int_{x\in M}\left(\int_{\xi\in\partial\widehat{M}}\mathrm{div}^{\mathcal{W}}\left(Y_te^{-(n-1)\xi(x)}\right)\chi_j(x)d\nu_o(\xi)\right)dx\\
&=\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}\mathrm{div}^{\mathcal{W}}\left(Y_te^{-(n-1)\xi(x)}\right)\widetilde{\chi}_jdx\right)d\nu_o(\xi)\\
&=\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}\mathrm{div}^{\mathcal{W}}\left(Y_te^{-(n-1)\xi(x)}\widetilde{\chi}_j\right)dx-\int_{x\in\widetilde{U}_j}\langle Y_t,\nabla\widetilde{\chi}_i\rangle e^{-(n-1)\xi(x)}dx\right)d\nu_o(\xi)\\
&=-\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_i}\langle Y_t,\nabla\widetilde{\chi}_j\rangle e^{-(n-1)\xi(x)}dx\right)d\nu_o(\xi).
\end{aligned}$$

Since the sum is finite, one can send $t\to 0$ to obtain

$$\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}\langle\nabla\xi,\nabla\widetilde{\chi}_j\rangle e^{-(n-1)\xi(x)}dx\right)d\nu_o(\xi)\leqslant\varepsilon.$$

By integration by parts, we have

$$\begin{aligned}
\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}e^{-(n-1)\xi(x)}\Delta\widetilde{\chi}_jdx\right)d\nu_o(\xi)&=-\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}\langle\nabla e^{-(n-1)\xi(x)},\nabla\widetilde{\chi}_j\rangle dx\right)d\nu_o(\xi)\\
&=(n-1)\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}\langle\nabla\xi,\nabla\widetilde{\chi}_j\rangle e^{-(n-1)\xi(x)}dx\right)d\nu_o(\xi).
\end{aligned}$$

So,

$$\sum_{j\in\mathcal{J}}\int_{\xi\in\partial\widehat{M}}\left(\int_{x\in\widetilde{U}_j}e^{-(n-1)\xi(x)}\Delta\widetilde{\chi}_jdx\right)d\nu_o(\xi)\leqslant\frac{\varepsilon}{n-1}.$$

By [LW10, Proposition 4] (that is still true in our context), $\Delta e^{-(n-1)\xi}\geqslant 0$ in the sense of distribution. Hence, for all $\xi\in\partial\widehat{M}$ and all i,

$$\int_{x\in\widetilde{U}_j}e^{-(n-1)\xi(x)}\Delta\widetilde{\chi}_jdx\geqslant 0.$$

So, we conclude that for all $j\in\mathcal{J}$, and for ν_o-a.e. $\xi\in\partial\widehat{M}$

$$\int_{x\in\widetilde{U}_j}e^{-(n-1)\xi(x)}\Delta\widetilde{\chi}_jdx\leqslant\frac{\varepsilon}{n-1}.$$

Since ε was arbitrarily small, we deduce that, for all $j\in\mathbb{N}$ and ν_o-a.e. $\xi\in\partial\widehat{M}$,

$$\int_{x\in\widetilde{U}_j}e^{-(n-1)\xi(x)}\Delta\widetilde{\chi}_jdx=0.$$

In the argument above, one can replace $\widetilde{U}_j$ by $g\cdot\widetilde{U}_j$ and $\widetilde{\chi}_j$ by $g\cdot\widetilde{\chi}_j$ for any $g\in\Gamma$. Since Γ is countable, we conclude that for any $g\in\Gamma$, any j, and ν_o-a.e. $\xi\in\partial\widehat{M}$

$$\int_{x\in g\cdot\widetilde{U}_i}e^{-(n-1)\xi(x)}\Delta(g\cdot\widetilde{\chi}_j)dx=0.$$

One can now conclude that, for ν_o-a.e. $\xi\in\partial\widehat{M}$, $\Delta e^{-(n-1)\xi(x)}=0$ in the sense of distribution in the same way as in [LW10, p.472], which concludes the proof of Proposition 2.7.

2.5. **Final steps.**

Proposition 2.13. *Suppose (X, g) is a complete simply connected Riemannian manifold with* $\mathrm{Ric} \geqslant -(n-1)$ *and $\Gamma \leqslant \mathrm{Isom}(X)$ is a discrete group that acts properly discontinuously on X so that $M = \Gamma\backslash X$ has finite volume (with respect to the Riemannian volume form).*

If there exists $\xi_0 \in \partial\widehat{X}$ so that $\Delta\xi_0 \equiv n-1$ then X is isometric to the real hyperbolic n-space.

Proof. By the proof of Theorem 6 in [Wan08] if there exists some $\xi_1 \in \partial\widehat{X}$ so that $\Delta\xi_1 \equiv n-1$ and $\xi_1 \neq \xi_0$ then X is isometric to the real hyperbolic n-space. So, suppose for a contradiction, that we have

$$\{\xi_0\} = \{\xi \in \partial\widehat{X} : \Delta\xi \equiv n-1\}.$$

Since

$$\Delta(\gamma \cdot \xi)(x) = (\Delta\xi)(\gamma^{-1}x)$$

we see that $\gamma \cdot \xi_0$ also has constant Laplacian equal to $n-1$. Thus $\gamma \cdot \xi_0 = \xi_0$ for all $\gamma \in \Gamma$.

Now if $\gamma \in \Gamma$ we see that

$$\mathrm{diff}(\gamma)_{\gamma^{-1}x}\nabla\xi_0(\gamma^{-1}x) = \nabla\Big(\xi_0(\gamma^{-1}x)\Big) = \nabla\Big(\xi_0(\gamma^{-1}x) - \xi_0(\gamma^{-1}o)\Big) = \nabla(\gamma \cdot \xi_0)(x) = \nabla\xi_0(x).$$

Thus, $\nabla\xi_0(x)$ is a Γ-invariant vector field, and therefore descends to a vector field V on M.

Now, $\operatorname{div} V = n-1$ since $\operatorname{div}\nabla\xi_0 = \Delta\xi_0 \equiv n-1$, and moreover $\|V\| \leqslant 1$. But, since M has finite volume, there cannot exists a vector field V with $\|V\|, \operatorname{div} V \in L^1(M)$ and $\operatorname{div} V > 0$ (see for instance [Kar81]). □

Putting together Proposition 2.7 and Proposition 2.13 finishes the proof of Theorem D.

Remark 2.14. Note that Proposition 2.13 is not necessary for the proof of Theorem D. Indeed, since we assume bounded curvature, we can replace Proposition 2.13 by [LW10, Theorem 6]. We however included that result since it shows that the bounded curvature assumption can be replaced by the finite volume assumption. In particular, we want to emphasize that the bounded curvature assumption is only used in order to get the heat kernel estimates needed for Proposition 2.7.

3. Entropy rigidity for Hilbert metrics

We begin by observing that the Blaschke metric has bounded sectional curvature. For the definition and some properties of the Blaschke metric, we refer to [Lof01, BH13].

Lemma 3.1. *Let Ω be a proper convex open set in $\mathbb{P}(\mathbb{R}^{n+1})$. There exists a universal constant C_n, depending only on the dimension such that the sectional curvature of the Blaschke metric on Ω is bounded above by C_n and below by $-C_n$.*

Proof. Benzécri [Ben60] proved that the action of $\mathrm{PGL}_{n+1}(\mathbb{R})$ on the set of pointed proper convex open sets $\mathcal{E} := \{(x, \Omega),\ x \in \Omega\}$ is cocompact, so all we have to show is that the functions that, to an element of $\mathcal{E}$ associates the maximum and minimum of the sectional curvature of the Blaschke metric at x, is $\mathrm{PGL}_{n+1}(\mathbb{R})$ are invariant and continuous. The invariance is clear from the definition of the Blaschke metric, and the continuity follows from Corollary 3.3 in [BH13]. □

We next prove Theorem B from the introduction:

Theorem 3.2. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set and there exists a discrete group Γ which preserves Ω and acts properly, freely, and with cofinite volume. Then $\delta_\Gamma \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

Proof. Let B_Ω be the Blaschke metric on Ω. Then

(1) Γ acts by isometries on (Ω, B_Ω) and the action is proper and free,
(2) B_Ω has bounded sectional curvature by Lemma 3.1,
(3) B_Ω has Ricci curvature bounded below by $-(n-1)$ by a result of Calabi [Cal72],
(4) by Theorem 1.6, $\delta_\Gamma(\Omega, B_\Omega) = n-1$,
(5) by [BH13, Proposition 2.6], $\Gamma\backslash\Omega$ has finite volume with respect to the Riemannian volume form induced by B_Ω.

Thus, the Blaschke metric satisfies all of the assumptions of Theorem D, so (Ω, B_Ω) is isometric to the real hyperbolic space. Hence, by definition of the Blaschke metric, (Ω, H_Ω) is the Klein–Beltrami model of the hyperbolic space (see [Lof01, Theorem 1]). □

Since $\delta_\Gamma(\Omega, H_\Omega) = h_{vol}(\Omega, H_\Omega, \mu_B)$ when Γ acts co-compactly on Ω we immediately deduce Theorem A from the introduction:

Corollary 3.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set and there exists a discrete group $\Gamma \leqslant \mathrm{Aut}(\Omega)$ which acts properly, freely, and cocompactly. Then $h_{vol}(\Omega, H_\Omega, \mu_B) \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

In order to prove Corollary C from the introduction, we first show the following:

Proposition 3.4. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a proper convex open set and there exists a discrete group Γ which preserves Ω and acts properly, freely, and with cofinite volume. Then Ω is strictly convex if and only if $\partial\Omega$ is C^1.*

Proof. Recall that Ω is strictly convex if and only if $\partial\Omega^*$ is of class C^1. Now, by [CLT15, Corollary 6.7], the quotient Ω/Γ is of finite volume if and only if the dual quotient Ω^*/Γ^* is also of finite volume. Hence, we only have to show that if Ω is strictly convex and Ω/Γ is of finite volume then $\partial\Omega$ is of class C^1.

Suppose that Ω is strictly convex. We want to use [CLT15, Theorem 0.15] to conclude that $\partial\Omega$ is of class C^1. In order to apply that theorem, we need to prove that Ω/Γ is topologically tame and that the holonomy of each boundary component is parabolic.

Fix a real $\varepsilon > 0$ strictly less than the Margulis constant (see [CM13, Théorème 1] or [CLT15, Theorem 0.1]). For every $x \in \Omega$, let $\Gamma_\varepsilon(x)$ be the group generated by the elements of Γ that moves x at a distance less than ε. By the classification of isometries of proper strictly convex open set, the group $\Gamma_\varepsilon(x)$, if not trivial, is either hyperbolic (in which case it corresponds to a Margulis tube), or parabolic, and in that case it fixes a unique point $p \in \partial\Omega$ (see [CLT15, CM14a, Mar14]).

Since Ω/Γ is of finite volume, the thick part of Ω/Γ is compact (see [CLT15, Theorem 0.8] or [CM14a, Lemma 8.5]) and connected component of the thin part correspond to disjoint maximal parabolic subgroups of Γ. Hence, the action of each maximal parabolic subgroup Γ_p on $\partial\Omega \smallsetminus \{p\}$ is cocompact. So, by [CLT15, Theorem 5.6], each parabolic fix point p is in fact C^1. Therefore, [CLT15, Theorem 0.5] (or [CM14a, Corollaire 7.18]) show that Γ_p is conjugated to a maximal parabolic subgroup of $\mathrm{SO}_{n,1}(\mathbb{R})$. We can thus apply [CLT15, Theorem 0.15] to conclude that $\partial\Omega$ is of class C^1. □

Finally, we can prove Corollary C from the introduction:

Corollary 3.5. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ is a proper convex open set which is either strictly convex or has C^1 boundary and there exists a discrete group Γ which preserves Ω and acts properly, freely, and with cofinite volume. Then $h_{vol}(\Omega, H_\Omega, \mu_B) \leqslant n-1$ with equality if and only if Ω is projectively isomorphic to $\mathcal{B}$ (and in particular (Ω, H_Ω) is isometric to $\mathbb{H}^n$).*

Proof. By Proposition 3.4, $\partial\Omega$ is C^1 and Ω is strictly convex. Thus by [CM14b, Théorème 9.2]

$$h_{vol}(\Omega, H_\Omega, \mu_B) = \delta_\Gamma(\Omega, H_\Omega).$$

So the corollary follows from Theorem B. □

Appendix A. Proof of Proposition 2.8

For the rest of the section suppose that (X, g) is a complete non-compact simply connected Riemannian manifold with $\mathrm{Ric} \geqslant -(n-1)$ and bounded sectional curvature.

For a function $f\colon X \to \mathbb{R}$ define the function $P_t(f)\colon X \to \mathbb{R}$ by

$$P_t(f)(x) = \int_X p_t(x, y) f(y) dy.$$

Lemma A.1. [CLY81, Theorem 4 and Theorem 6] *With the notation above, for any $T > 0$, there exists $C > 0$ so that*

$$p_t(x, y) \leqslant C t^{-\frac{n}{2}} \exp\left(\frac{-d(x, y)^2}{Ct}\right)$$

and

$$\|\nabla_x p_t(x,y)\| \leqslant Ct^{-\frac{n+1}{2}} \exp\left(\frac{-d(x,y)^2}{Ct}\right)$$

for all $t \in (0,T]$ *and* $x, y \in X$.

Proof of Proposition 2.8. Recall that $F_t(x,\xi) = P_t(\xi)(x)$. We claim that for any $\xi \in \partial\widehat{X}$

$$(\partial_t - \Delta_x)F_t(x,\xi) = 0$$

in the sense of distributions. Once this is established part (1) and part (2) follow from standard regularity results (see for instance [Gri09, Theorem 7.4]).

Let $\phi \in C_c^\infty(X \times \mathbb{R}_+)$. By Lemma A.1, $\Delta_x(\phi(x,t))p_t(x,y)\xi(y)$ and $\partial_t(\phi(x,t))p_t(x,y)\xi(y)$ are in $L^1(X \times X \times \mathbb{R}_+, dxdydt)$. Then, using Fubini and the fact that $\partial_t p_t(x,y) = \Delta_x p_t(x,y)$, we obtain

$$\begin{aligned}\int_{X\times\mathbb{R}_+} \Delta_x\phi(x,t)P_t(\xi)(x)dxdt &= \int_X \left(\int_{X\times\mathbb{R}_+} \Delta_x\phi(x,t)p_t(x,y)dxdt\right)\xi(y)dy \\ &= \int_X \left(\int_{X\times\mathbb{R}_+} \phi(x,t)\Delta_x p_t(x,y)dxdt\right)\xi(y)dy = \int_X \left(\int_{X\times\mathbb{R}_+} \phi(x,t)\partial_t p_t(x,y)dxdt\right)\xi(y)dy \\ &= -\int_X \left(\int_{X\times\mathbb{R}_+} \partial_t\phi(x,t)p_t(x,y)dxdt\right)\xi(y)dy = -\int_{X\times\mathbb{R}_+} \partial_t\phi(x,t)P_t(\xi)(x)dxdt.\end{aligned}$$

Thus

$$(\partial_t - \Delta_x)F_t(x,\xi) = 0$$

in the sense of distributions. So part (1) and (2) are established.

Now, by [BÉ84], since $\mathrm{Ric} \geqslant -(n-1)$, if $f \in C_c^\infty(X)$ then

$$\|\nabla P_t(f)\|_\infty \leqslant e^{(n-1)t}\|\nabla f\|_\infty.$$

Moreover, for any $\xi \in \partial\widehat{X}$, there exists a sequence $f_m \in C_c^\infty(X)$ so that f_m converges to ξ locally uniformly and $\|\nabla f_m\|_\infty \to 1$ (see, for instance, [AFLMR07]). Hence, each $P_t(f_m)$ is $e^{(n-1)t}\|\nabla f_m\|_\infty$-Lipschitz. Moreover, by Lemma A.1 and the dominated convergence theorem, $P_t(f_m)(x) \to P_t(\xi)(x)$ for all $x \in X$. Thus, $P_t(\xi)$ is $e^{(n-1)t}$-Lipschitz and

$$\|\nabla_x F_t(x,\xi)\| \leqslant e^{(n-1)t}.$$

Now, for any $\phi \in C_c^\infty(X)$, we have

$$\begin{aligned}\int_X \Delta_x\phi(x)P_t(\xi)(x)dx &= \int_X \left(\int_X \phi(x)\Delta_x p_t(x,y)dx\right)\xi(y)dy \\ &= \int_X \left(\int_X \phi(x)\Delta_y p_t(x,y)dx\right)\xi(y)dy = \int_X \Delta_y P_t(\phi)(y)\xi(y)dy.\end{aligned}$$

For $r > 0$, let $\varphi_r : X \to \mathbb{R}$ be as in Lemma 2.9. Then

$$\begin{aligned}\int_X \Delta_y P_t(\phi)(y)\xi(y)dy &= \int_X \Delta_y\Big(\varphi_r(y)P_t(\phi)(y)\Big)\xi(y)dy + \int_X \Delta_y\Big((1-\varphi_r)(y)P_t(\phi)(y)\Big)\xi(y)dy \\ &\leqslant (n-1)\int_X \varphi_r(y)P_t(\phi)(y)dy + \int_X \Delta_y\Big((1-\varphi_r)(y)P_t(\phi)(y)\Big)\xi(y)dy.\end{aligned}$$

Using the dominated convergence theorem once again, we have

$$\lim_{r\to\infty}\int_X \varphi_r(y)P_t(\phi)(y)dy = \int_X P_t(\phi)(y)dy = \int_X \phi(x)dx.$$

Moreover, since integration by parts holds for Lipschitz functions,

$$\begin{aligned}\int_X \Delta_y\Big((1-\varphi_r)(y)P_t(\phi)(y)\Big)\xi(y)dy &= -\int_X \Big\langle \nabla_y\Big((1-\varphi_r)(y)P_t(\phi)(y)\Big), \nabla_y\xi(y)\Big\rangle dy \\ &\leqslant \frac{C}{r}\int_X P_t(\phi)(y)dy + \int_{X\setminus B_r(o)} \|\nabla P_t(\phi)(y)\|dy.\end{aligned}$$

Now,

$$\nabla P_t(\phi)(y) = \int_X \nabla_y p_t(x,y)\phi(x)dx,$$

and so, by Lemma A.1,

$$\|\nabla P_t(\phi)(y)\| \in L^1(X, dy).$$

Thus,

$$\lim_{r\to\infty} \int_{X\setminus B_r(o)} \|\nabla P_t(\phi)(y)\|\, dy = 0.$$

Which implies that

$$\lim_{r\to\infty} \int_X \Delta_y\Big((1-\varphi_r)(y)P_t(\phi)(y)\Big)\xi(y)dy = 0,$$

and thus

$$\int_X \Delta_x\phi(x)P_t(\xi)(x)dx \leqslant (n-1)\int_X \phi(x)dx.$$

Since $\xi \in \partial\widehat{X}$ and $\phi \in C_c^\infty(X)$ were arbitrary, we see that

$$\Delta_x F_t(x,\xi) \leqslant n-1. \qquad \square$$

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