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# THE ULTIMATE RANK OF TROPICAL MATRICES

PIERRE GUILLON, ZUR IZHAKIAN,  
JEAN MAIRESSE, AND GLENN MERLET

**ABSTRACT.** A tropical matrix is a matrix defined over the max-plus semiring. For such matrices, there exist several non-coinciding notions of rank: the row rank, the column rank, the Schein/Barvinok rank, the Kapranov rank, or the tropical rank, among others. In the present paper, we show that there exists a natural notion of ultimate rank for the powers of a tropical matrix, which does not depend on the underlying notion of rank. Furthermore, we provide a simple formula for the ultimate rank of a matrix which can therefore be computed in polynomial time. Then we turn our attention to finitely generated semigroups of matrices, for which our notion of ultimate rank is generalized naturally. We provide both combinatorial and geometric characterizations of semigroups having maximal ultimate rank. As a byproduct, we obtain a polynomial algorithm to decide if the ultimate rank of a finitely generated semigroup is maximal.

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## 1. Introduction

Tropical matrices are matrices defined over the so-called “max-plus” or “tropical” semiring, that is,  $(\mathbb{R} \cup \{-\infty\}, \max, +)$ . Tropical algebra is a rapidly growing area and surveys with different viewpoints are available, e.g. [3, 9, 16].

The tropical semiring is a “weak” algebraic structure in which inverses do not exist for the max operation. Tropical matrices essentially correspond to weighted

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digraphs; as such, the interplay between algebra and graph theory, and combinatorial ideas, often have helped to bypass the lack of basic algebraic methods in the tropical framework.

Yet, negation is still absent and the elementary process of Gauss elimination is not feasible over the tropical setting; as a consequence, familiar algebraic concepts, such as dependence and spanning, basically do not agree here. This drawback has led to various notions of matrix ranks (the row rank, the column rank, the Schein/Barvinok rank, the Kapranov rank, or the tropical rank, among others, cf. Definition 3.7) which, as one would expect, essentially do not coincide. Over the years, much effort has been invested in the study of the different types of ranks and the relations among them, see for instance [2, 14]. However the use of these ranks for applications, especially for computation, is rather cumbersome.

In the present paper we introduce a canonical notion, termed ultimate rank, that in a sense unifies the known ranks of tropical matrices. The ultimate rank of a matrix is defined as the minimal rank over the closure of the semigroup generated by the matrix, and the minimum is proven not to depend on the underlying notion of rank. The ultimate rank of the matrix  $A$  depends only on the so-called critical graph (see Definition 4.1), with the following simple and explicit formula (Theorem 5.2):

$$\text{urk}(A) = \sum_{C \in \mathfrak{C}} \text{cyc}(C),$$

where  $\mathfrak{C}$  is the set of strongly connected components of the critical graph of  $A$ , and where  $\text{cyc}(C)$  is the cyclicity of  $C$ . Therefore, unlike some of the known ranks, the ultimate rank can be computed in polynomial time-complexity,  $O(n^3)$  for an  $n \times n$  matrix (Corollary 5.4). The proof of the formula relies on the *ultimate expansion* of tropical matrices [27].

The next step is to generalize the notion of ultimate rank to a finitely generated semigroup of tropical matrices, defined as the minimal rank over all the matrices in the closure of the semigroup. Determining the ultimate rank of a given matrix semigroup is then the obvious question. Unfortunately, the algorithmic computability of the ultimate rank is an open question. We settle the case of maximal ultimate rank. Indeed, we provide two different characterizations, one combinatorial and one geometric, for finitely generated matrix semigroups of maximal ultimate rank. The combinatorial characterization (Theorem 6.4) is provided in terms of the structure of the associated graphs of the generators. The geometric characterization (Theorem 6.5) is the existence of a common (tropical) eigenvector for all the generators which lies in their fundamental cells. Having these characterizations, we show that the problem of determining whether a semigroup of  $n \times n$  matrices has maximal ultimate rank is decidable in time complexity  $O(|\mathcal{I}|n^3)$ , where  $\mathcal{I}$  is the set of generators.

A by-product of the present paper is to provide new insight on matrices and semigroups of matrices with maximal tropical rank. Matrices with maximal tropical rank are also known as *non-singular* or *strongly regular* matrices. They have been extensively studied in the literature, see for instance [2, 6, 7, 8, 10, 14, 17, 18, 19, 21, 25, 26]. They play a key role in solving linear systems of tropical equations, in studying the optimal assignment problem, or in the theory of discrete event systems.

## 2. Preliminaries

The *max-plus semiring* or *tropical semiring*  $(\mathbb{T}, \vee, +)$  is the set  $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$  equipped with the binary operations  $(x, y) \mapsto \max(x, y) = x \vee y$  and  $(x, y) \mapsto x + y$ . In this structure, the “additive” operation is the maximum while the “multiplicative” operation is the usual sum; their respective identity elements are  $-\infty$  and 0.

Classical algebraic objects, such as vectors and matrices, have max-plus analogues. The set  $\mathbb{T}^n$  of  $n$ -dimensional *vectors* is a semimodule over the semiring  $\mathbb{T}$ . The set  $\mathbb{T}^{n \times m}$  of  $n \times m$  *matrices* is a semimodule over  $\mathbb{T}$ . A vector  $\mathbf{v}$  having entries  $v_i \in \mathbb{T}$ , resp. a matrix  $A$  having entries  $A_{ij} \in \mathbb{T}$ , is written as  $\mathbf{v} = (v_i)$ , resp.  $A = (A_{ij})$ . The matrix with only  $-\infty$  entries, written  $(-\infty)$  is the *null* matrix (of a given size).

The product of matrices is defined according to the semiring structure and is denoted by using the special symbol  $\odot$ , that is, for matrices  $A, B$  of compatible sizes,  $A \odot B$  is defined by

$$(A \odot B)_{ij} = \bigvee_k (A_{ik} + B_{kj}).$$

The pair  $(\mathbb{T}^{n \times n}, \odot)$  forms a monoid.

It is convenient to use the following notations. As usual, we write  $AB$  for the product  $A \odot B$ . For matrices  $A, B$  of the same size, we define  $A \vee B$  by  $(A \vee B)_{ij} = A_{ij} \vee B_{ij}$  and write  $A \geq B$  if  $A_{ij} \geq B_{ij}$  for every  $i, j$  (that is  $A \vee B = A$ ). The product  $\lambda \odot A$  of a matrix  $A$  and a scalar  $\lambda \in \mathbb{T}$  is defined by  $(\lambda \odot A)_{ij} = \lambda + A_{ij}$ .

**Definition 2.1.** *Two matrices  $A$  and  $B$  of the same size are said to be (tropically) equivalent if there exists  $\lambda$  in  $\mathbb{R}$  such that  $B = \lambda \odot A$ . Let  $\mathbb{P}\mathbb{T}^{n \times m}$  be the (tropical) projective matrix set obtained by identifying equivalent matrices in  $\mathbb{T}^{n \times m}$ . The quotient map is denoted by  $\pi : \mathbb{T}^{n \times m} \rightarrow \mathbb{P}\mathbb{T}^{n \times m}$ .*

Graph theory provides an important tool in the study of tropical matrices, established via the following correspondence.

**Definition 2.2.** *The graph of a matrix  $A \in \mathbb{T}^{n \times n}$ , denoted by  $\mathcal{G}(A)$ , is the weighted directed graph with nodes  $\{1, \dots, n\}$  and arcs  $(i, j)$  whenever  $A_{ij} \neq -\infty$ . The weight of arc  $(i, j)$  is  $A_{ij}$ .*

Powers of a max-plus matrix  $A$  can be interpreted combinatorially in terms of directed paths in  $\mathcal{G}(A)$ . Let a *walk* in  $\mathcal{G}(A)$  be a finite sequence of nodes  $p = (i_0, i_1, \dots, i_\ell)$  such that  $(i_k, i_{k+1})$  is an arc of  $\mathcal{G}(A)$  for any  $k$ . Its *length* is  $\ell$  and its *weight* is 0 if  $\ell = 0$  and  $A_{i_0 i_1} + \dots + A_{i_{\ell-1} i_\ell}$  otherwise. For  $k \in \mathbb{N}$ , the entry  $(A^k)_{ij}$  of  $A^k$  is equal to the maximal weight of a walk of length  $k$  from  $i$  to  $j$  in  $\mathcal{G}(A)$ .

In this paper, we consider only directed graphs, and use the classical terminology of graph theory. A walk from  $i$  to itself is called a *circuit*, and a circuit of length 1 is called a *loop*. Moreover, a circuit is called *simple* if it contains each node at most once, except for the first and last one that are the same. A graph is *strongly connected* if there is a walk from each vertex in the graph to every other vertex. A *strongly connected component* (written *scc*, for short) of a graph is a subgraph which is strongly connected and maximal with respect to inclusion. A scc is *trivial* if it consists of one node with no loop. A graph is *completely reducible* if it is a disjoint union of scc. The *cyclicity* of a strongly connected graph is the gcd of the lengths of its circuits. For a completely reducible graph, the *cyclicity* is the lcm of the cyclicities of the scc.

A matrix  $A$  is called *irreducible* if its associated graph  $\mathcal{G}(A)$  is strongly connected; otherwise  $A$  is called *reducible*.

### 3. The different ranks of a matrix

There exist several relevant notions of rank for matrices over the tropical semi-ring, see [14] and [2] for extensive accounts. In Definition 3.7 below, we recall the main ones. To this aim, we first review some necessary concepts.

**Definition 3.1.**

(1) The (tropical) permanent of a matrix  $A \in \mathbb{T}^{n \times n}$  is defined by:

$$\text{per}(A) = \bigvee_{\sigma \in \mathfrak{S}_n} A_{1\sigma(1)} + \cdots + A_{n\sigma(n)}, \quad (3.1)$$

where  $\mathfrak{S}_n$  is the set of all the permutations on  $\{1, \dots, n\}$ .

(2) A matrix  $A \in \mathbb{T}^{n \times n}$  is called (tropically) singular if  $A = (-\infty) \in \mathbb{T}^{1 \times 1}$  or if there exist at least two different permutations that reach the maximum in  $\text{per}(A)$ , that is,

$$\text{per}(A) = \sum_i A_{i\sigma(i)} = \sum_i A_{i\tau(i)},$$

for some  $\sigma \neq \tau$  in  $\mathfrak{S}_n$ . Otherwise  $A$  is called (tropically) non-singular.

If  $A$  is non-singular, we denote by  $\tau_A$  the unique permutation that reaches the maximum in (3.1).

Non-singularity is equivalent to *strong regularity* in the sense of [7]. In the sequel, we will use the following result.

**Proposition 3.2.** Given  $A, B \in \mathbb{T}^{n \times n}$  if  $AB$  is non-singular then  $A$  and  $B$  are non-singular, and:

$$\tau_{AB} = \tau_B \circ \tau_A, \quad \text{per}(AB) = \text{per}(A) + \text{per}(B).$$

This is proven in [25, Proposition 3.4].

**Definition 3.3.** A family of vectors  $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{T}^n$  is linearly independent (in the Gondran-Minoux sense) if for all disjoint  $I, J \subset \{1, \dots, r\}$ , and for all  $\alpha_i \in \mathbb{T}, i \in I \cup J$ , with  $(\alpha_i)_i \neq (-\infty, \dots, -\infty)$ , we have  $\bigvee_{i \in I} (\alpha_i \odot \mathbf{x}_i) \neq \bigvee_{j \in J} (\alpha_j \odot \mathbf{x}_j)$ .

**Definition 3.4.**

(1) The set  $X \subset \mathbb{T}^n$  is tropically convex if:  $\forall \mathbf{u}, \mathbf{v} \in X, \forall \lambda, \mu \in \mathbb{T}$ ,

$$(\lambda \odot \mathbf{u}) \vee (\mu \odot \mathbf{v}) = ((\lambda, \dots, \lambda) + \mathbf{u}) \vee ((\mu, \dots, \mu) + \mathbf{v}) \in X. \quad (3.2)$$

(2) The tropically convex set generated by a finite family  $\mathcal{I}$  of vectors of  $\mathbb{T}^n$ , that is the set

$$\left\{ \mathbf{y} \in \mathbb{T}^n \mid \exists (\alpha_{\mathbf{s}})_{\mathbf{s} \in \mathcal{I}}, \alpha_{\mathbf{s}} \in \mathbb{T}, \mathbf{y} = \bigvee_{\mathbf{s} \in \mathcal{I}} (\alpha_{\mathbf{s}} \odot \mathbf{s}) \right\},$$

is called the (tropical) convex hull of  $\mathcal{I}$ .

A tropically convex set is invariant by translations along the direction  $(1, \dots, 1)$ . Therefore, it can be identified with its image in the projective space  $\mathbb{P}\mathbb{T}^n$ . Next result is proven in [29].

**Proposition 3.5.** Every finitely generated tropically convex set has a projectively unique generating set which is minimal for inclusion.

**Definition 3.6.** The weak dimension of a finitely generated tropically convex subset  $X$  of  $\mathbb{T}^n$  is the cardinality of the minimal generating set of  $X$ .

The topological dimension of a subset  $X$  of  $\mathbb{T}^n$  is the largest  $k$  for which there exists an affine space  $K \subset \mathbb{R}^n$  of dimension  $k$ , such that  $X \cap K$  has a non-empty relative interior in  $K$ .

Below, we apply this notion of topological dimension to finitely generated tropically convex subsets of  $\mathbb{T}^n$ , for which it coincides with the other classical notions of dimensions, such as the Hausdorff dimension or the Lebesgue covering dimension.

Consider the tropically convex set of Figure 1. The topological dimension is 2 since the set is the union of three infinite strips. The weak dimension is 3 since the convex hull of any two finite vectors is at most the union of two infinite strips.

In general, for a given  $\mathbb{T}^n$ ,  $n \geq 3$ , the weak dimension is unbounded whereas the topological dimension is bounded by  $n$ .

Let us define the main notions of rank for tropical matrices.

**Definition 3.7.** Consider a matrix  $A \in \mathbb{T}^{n \times m}$ .

**Tropical rank:** Let  $\text{rk}_{\text{tr}}(A)$  be the maximal  $r$  for which there exists an  $r \times r$  non-singular submatrix of  $A$ .

**Symmetrized rank:** Define  $\det^+(A)$  (resp.  $\det^-(A)$ ) as in (3.1) but with  $\sigma$  ranging over permutations of even (resp. odd) sign. The symmetrized rank  $\text{rk}_{\text{sym}}$  is the maximal  $r$  such that  $A$  has an  $r \times r$  submatrix  $B$  for which  $\det^+(A) \neq \det^-(A)$ .

**Gondran-Minoux rank:** The Gondran-Minoux row rank  $\text{rk}_{\text{G-M};\text{rw}}(A)$  is the maximal  $r$  such that  $A$  has  $r$  independent rows. The Gondran-Minoux column rank  $\text{rk}_{\text{G-M};\text{cl}}(A)$  is defined similarly with respect to the columns.

**Kapranov rank:** The Kapranov rank  $\text{rk}_{\text{Kp}}(A)$  is defined for instance in [14, Def. 1.2 and 3.2].

**Schein/Barvinok rank:** Let  $\text{rk}_{\text{S/B}}(A)$  be the minimal  $r$  such that:  $\exists B \in \mathbb{T}^{n \times r}, C \in \mathbb{T}^{r \times m}, A = BC \in \mathbb{T}^{n \times m}$ .

**Row rank:** Let  $\text{rk}_{\text{rw}}(A)$  be the weak dimension of the convex hull of the row vectors of  $A$ .

**Column rank:** Let  $\text{rk}_{\text{cl}}(A)$  be the weak dimension of the convex hull of the column vectors of  $A$ .

Two equivalent matrices have the same rank for any of the above notions. Therefore, the different notions of ranks can be viewed as being defined on  $\mathbb{P}\mathbb{T}^{n \times m}$ .

By convention, all the ranks of the null matrix  $(-\infty)$  are set to 0. For a matrix  $A \neq (-\infty)$ , we check that  $\text{rk}_*(A) > 0$  for any of the above notion of rank.

None of the above notions coincide [2, Section 8]. The following relations have been established, see [2, Theorem 8.6] for (3.3) and [14, Theorem 1.4] for (3.4):

$$\text{rk}_{\text{tr}}(A) \leq \text{rk}_{\text{sym}}(A) \leq \left\{ \begin{array}{l} \text{rk}_{\text{G-M};\text{rw}}(A) \\ \text{rk}_{\text{G-M};\text{cl}}(A) \end{array} \right\} \leq \text{rk}_{\text{S/B}}(A) \leq \left\{ \begin{array}{l} \text{rk}_{\text{rw}}(A) \\ \text{rk}_{\text{cl}}(A) \end{array} \right\} \quad (3.3)$$

and

$$\text{rk}_{\text{tr}}(A) \leq \text{rk}_{\text{Kp}}(A) \leq \text{rk}_{\text{S/B}}(A). \quad (3.4)$$

The extremal values of the ranks are of specific interest.

**Rank 0:** Any of the ranks is 0 iff the matrix is null.

**Rank 1:** It can easily be checked that  $\text{rk}_{\text{tr}}(A) = 1$  iff  $A$  is non-null and all the non-null rows are tropically equivalent. Thus, rank 1 occurs (or not) simultaneously for all ranks.

**Maximal rank:** Consider  $A \in \mathbb{T}^{n \times n}$ . According to (3.3) and (3.4), we have:  $\text{rk}_{\text{tr}}(A) = n \Rightarrow \text{rk}_*(A) = n$ , for any of the above notions of rank. It corresponds to the case of a non-singular matrix.

Below, we focus on the extremal ranks given by (3.3) and (3.4), that is, the column rank, the row rank, and the tropical rank. It turns out that these three notions of rank are related to the dimension of the ‘‘image’’ of the matrix. It follows

directly from the definition for the column and row ranks, and from the next result for the tropical rank.

**Proposition 3.8.** *Consider a matrix  $A \in \mathbb{T}^{n \times n}$  such that:  $\forall i, \exists j, A_{ij} \neq -\infty$ . Then  $\text{rk}_{\text{tr}}(A)$  is the topological dimension of the convex hull of the column (resp. row) vectors of  $A$ .*

Proposition 3.8 appears in [14, Theorem 4.2] where the proof is carried out for matrices in  $\mathbb{R}^{n \times n}$  but can be easily adapted. See also [8, Theorems 3.3 and 4.1].

The following examples show that these 3 notions of rank do not coincide.

**Example 3.9.** *Consider the matrices:*

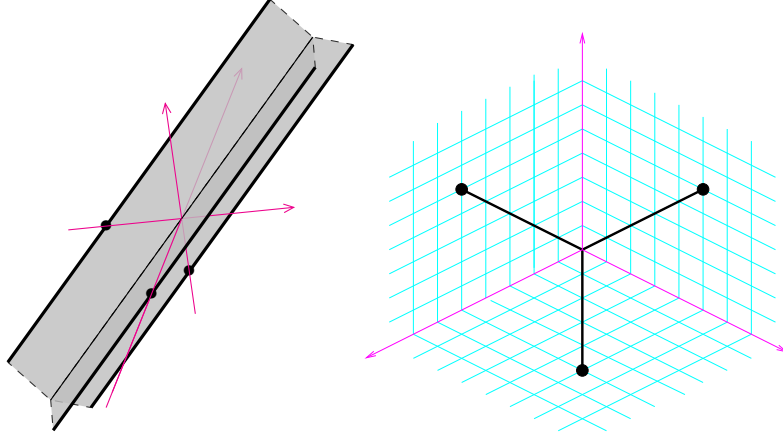
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for which we have:

$$\begin{aligned} \text{rk}_{\text{tr}}(A) &= 2, & \text{rk}_{\text{rw}}(A) &= 3, & \text{rk}_{\text{cl}}(A) &= 3, \\ \text{rk}_{\text{tr}}(B) &= 3, & \text{rk}_{\text{rw}}(B) &= 3, & \text{rk}_{\text{cl}}(B) &= 4. \end{aligned}$$

Let  $\text{Im}(A)$  be the convex hull of the columns of  $A$  (i.e. the image set of the mapping:  $\mathbb{T}^3 \rightarrow \mathbb{T}^3, \mathbf{x} \mapsto A \odot \mathbf{x}$ ). In Figure 1, we have represented the set  $\text{Im}(A)$  on the left, and the projective set  $\pi(\text{Im}(A))$  on the right (represented by orthogonal projection on the plane orthogonal to the direction  $(1, 1, 1)$ ).

FIGURE 1. The tropically convex set  $\text{Im}(A)$  (on the left) and its projective image  $\pi(\text{Im}(A))$  (on the right).



*Complexity.* Computing the row, resp. column, rank of a matrix in  $\mathbb{T}^{n \times n}$  has time-complexity  $O(n^3)$ , see for instance [9, Chapter 3.4]. Computing the tropical rank of a matrix whose entries take only two possible values is NP-complete, see [21, Theorem 13]. On the other hand, determining if a matrix in  $\mathbb{T}^{n \times n}$  has tropical rank  $n$  (i.e., is non-singular) can be done with an algorithm of time-complexity  $O(n^3)$ , see [10].

#### 4. Max-plus spectral theory

The results in this section are classical, and are to be used in the proofs in §5. Observe that the tropical rank, via the definition of the permanent, involves the maximal total weight of the decompositions of  $\mathcal{G}(A)$  into cycles. Therefore, it should come as no surprise that it connects to the notions below.

**Definition 4.1.** For a matrix  $A \in \mathbb{T}^{n \times n}$ , define:

$$\rho(A) = \bigvee_{j \leq n} \bigvee_{i_1, \dots, i_j} \frac{A_{i_1 i_2} + A_{i_2 i_3} + \dots + A_{i_j i_1}}{j}, \quad (4.1)$$

Observe that  $\rho(A)$  is the maximal mean weight of the (simple) circuits of  $\mathcal{G}(A)$ .

A circuit of  $\mathcal{G}(A)$  is called critical if its mean weight is  $\rho(A)$ . The critical graph of  $A$ , denoted by  $\mathcal{G}_c(A)$ , is the union of all the critical circuits of  $\mathcal{G}(A)$ .

If  $\mathcal{G}(A)$  is acyclic then  $\rho(A) = -\infty$  and  $\mathcal{G}_c(A)$  is empty.

**Definition 4.2.** For  $A \in \mathbb{T}^{n \times n}$ , if  $\lambda \in \mathbb{T}$  and  $\mathbf{u} \in \mathbb{T}^n \setminus \{(-\infty)\}$  are such that:

$$A \odot \mathbf{u} = \lambda \odot \mathbf{u},$$

then  $\lambda$  is called a (tropical) eigenvalue and  $\mathbf{u}$  is a (tropical) eigenvector associated to  $\lambda$ . The set of such eigenvectors is the (tropical) eigenspace associated to  $\lambda$ .

Observe that the set of eigenvectors associated with a given eigenvalue is tropically convex.

The next statement is a version of the celebrated max-plus spectral theorem.

**Theorem 4.3.** Consider a matrix  $A \in \mathbb{T}^{n \times n}$ . We have:

- (1)  $\rho(A)$  is the maximal eigenvalue of  $A$ ;
- (2) if  $\rho(A) \neq -\infty$ , the weak dimension of the eigenspace of  $A$  associated to  $\rho(A)$  is equal to the number of scc of  $\mathcal{G}_c(A)$ ;
- (3) if  $A$  is irreducible, or, more generally, if each scc of  $\mathcal{G}(A)$  contains a critical node, then  $\rho(A)$  is the unique eigenvalue.

A full proof can be found for instance in [3, Chapter 3] or [9, Chapter 4].

Recall that a square matrix  $A$  is *torsion* if:  $\exists k, c > 0$ ,  $A^{k+c} = A^k$ . The next result completes Theorem 4.3.

**Theorem 4.4.** Consider a matrix  $A \in \mathbb{T}^{n \times n}$ . There exist  $N \geq 0$ , a torsion matrix  $T$  of  $\mathbb{T}^{n \times n}$ , and a sequence  $(B_k)_k$  of matrices of  $\mathbb{T}^{n \times n}$  with  $\lim_k B_k = (-\infty)$ , such that:

$$\forall k \geq N, \quad A^k = k\rho(A) \odot (T^k \vee B_k). \quad (4.2)$$

Furthermore,  $(-\rho(A)) \odot A$  itself is torsion (that is, we can get rid of  $B_k$  in (4.2)) iff each scc of  $\mathcal{G}(A)$  contains a critical node.

Theorem 4.4 is a direct consequence of the so-called *ultimate expansion* of [27, Theorem 5.6]. The second part of the statement is an extension of the cyclicity Theorem [11], see for instance [15, Lemma 4.4].

#### 5. The ultimate rank of a matrix

**5.1. Statements.** For matrices  $A$  and  $B$  of compatible sizes, we have:

$$\begin{aligned} \text{rk}_{\text{tr}}(AB) &\leq \text{rk}_{\text{tr}}(A), & \text{rk}_{\text{tr}}(AB) &\leq \text{rk}_{\text{tr}}(B), \\ \text{rk}_{\text{rw}}(AB) &\leq \text{rk}_{\text{rw}}(A), & \text{rk}_{\text{cl}}(AB) &\leq \text{rk}_{\text{cl}}(B). \end{aligned}$$

It is straightforward to check these inequalities for  $\text{rk}_{\text{cl}}$  and  $\text{rk}_{\text{rw}}$ ; the case of  $\text{rk}_{\text{tr}}$  is proven in [2, Theorem 9.4].



In particular,  $\text{rk}_\star(A^k)$  does not increase when  $k$  increases, for  $\star \in \{\text{col}, \text{row}, \text{tr}\}$ . It is worth looking at the limit value. Let us start with an example.

**Example 5.1.** *Consider*

$$A = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -\infty & -\infty & -1 & -1 \\ -\infty & -\infty & -1 & -1 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -\infty & -\infty & -k & -k \\ -\infty & -\infty & -k & -k \end{bmatrix},$$

for which we have  $\lim_k \text{rk}_{\text{cl}}(A^k) = \text{rk}_{\text{cl}}(A) = 4$  while  $\lim_k \text{rk}_{\text{tr}}(A^k) = \text{rk}_{\text{tr}}(A) = 3$ .

Now, observe that the sequence  $(A^k)_{k \rightarrow +\infty}$  converges to the matrix

$$B = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{bmatrix}.$$

For the limit matrix, the different ranks coincide:  $\text{rk}_{\text{cl}}(B) = \text{rk}_{\text{rw}}(B) = \text{rk}_{\text{tr}}(B) = 2$ .

The projective semigroup generated by a matrix  $A \in \mathbb{T}^{n \times n}$  is denoted by

$$\pi\langle A \rangle = \{\pi(A^k), k \in \mathbb{N}\},$$

and its closure is denoted by  $\overline{\pi\langle A \rangle}$ . For instance, in Example 5.1, we have  $\overline{\pi\langle A \rangle} = \{\pi(A^k), k \in \mathbb{N}\} \cup \{\pi(B)\}$ .

**Theorem 5.2.** *Consider a matrix  $A \in \mathbb{T}^{n \times n}$ . For any notion of rank (denoted  $\star$ ) of Definition 3.7, the value of  $\min_{B \in \overline{\pi\langle A \rangle}} \text{rk}_\star(B)$  is the same. This common value is called the ultimate rank of  $A$  and is denoted by  $\text{urk}(A)$ . Furthermore, we have:*

$$\text{urk}(A) = \sum_{C \in \mathfrak{C}} \text{cyc}(C), \quad (5.1)$$

where  $\mathfrak{C}$  is the set of scc of  $\mathcal{G}_c(A)$  and  $\text{cyc}(C)$  stands for the cyclicity of  $C$ .

Theorem 5.2 is original. However, related results appear in the litterature. In [2, 25.7: Fact 4] it is stated that all the ranks coincide for a matrix  $A$  which is von Neumann regular (*i.e.*, such that  $\exists X, AXA = A$ ). A proof of this statement appears in [20, Corollary 1.3]. In [25, Lemma 3.3] it is proven that  $\text{rk}_{\text{tr}}(A)$  is the number of scc of  $\mathcal{G}_c(A)$  if  $A \in \mathbb{R}^{n \times n}$  and  $A$  is idempotent (*i.e.*,  $A^2 = A$ ).

If  $A$  is irreducible or, more generally, if each scc of  $\mathcal{G}(A)$  contains a critical node, then the projective semigroup  $\pi\langle A \rangle$  is finite (cf. Theorem 4.4). In this case,  $\overline{\pi\langle A \rangle} = \pi\langle A \rangle$ , and we obtain:

$$\text{urk}(A) = \lim_k \text{rk}_\star(A^k), \quad (5.2)$$

for any notion of rank  $\star$ . The equality (5.2) is not true in general as emphasized by Example 5.1.

Theorem 5.2 will be proved in Section 5.2. Before this, we give two corollaries and one example to illustrate the result.

**Corollary 5.3.** *For a matrix  $A \in \mathbb{T}^{n \times n}$ , the following statements are equivalent:*

- (1)  $A$  has maximal ultimate rank, that is,  $\text{urk}(A) = n$ ;
- (2)  $\mathcal{G}_c(A)$  is the disjoint union of simple circuits covering all the nodes  $\{1, \dots, n\}$ ;
- (3)  $\text{per}(A)$  has a unique maximizing permutation and  $\mathcal{G}_c(A)$  is the graph of this permutation.

Furthermore, if the above hold, then  $\text{per}(A) = n\rho(A)$ .

A weak version of Corollary 5.3 appears in [25, Corollary 3.5].

*Proof of Corollary 5.3.* According to (5.1), the ultimate rank is  $n$  if and only if  $\sum_{C \in \mathcal{C}} \text{cyc}(C) = n$ , which happens if and only if  $\mathcal{G}_c(A)$  is a union of disjoint simple circuits covering all the nodes  $\{1, \dots, n\}$ . Hence,  $\mathcal{G}_c(A)$  is the graph of a permutation  $\tau$  of  $\{1, \dots, n\}$ .

For any permutation  $\sigma$  of  $\{1, \dots, n\}$ ,  $w(\sigma) = A_{1\sigma(1)} + \dots + A_{n\sigma(n)}$  is the total weight of the edges following this permutation. Since  $\sigma$  can be decomposed into disjoint simple circuits  $\sigma_1, \dots, \sigma_k$  of lengths  $\ell_1, \dots, \ell_k$  and mean weights  $w_1, \dots, w_k$ , we have  $w(\sigma) = \ell_1 w_1 + \dots + \ell_k w_k \leq (\ell_1 + \dots + \ell_k) \rho(A) = n \rho(A)$ , with equality if and only if all of the circuits of  $\sigma$  are critical circuits of  $\mathcal{G}(A)$ . Consequently,  $\text{per}(A) = n \rho(A)$  and  $\tau = \tau_A$ .  $\square$

**Corollary 5.4.** *The ultimate rank of  $A \in \mathbb{T}^{n \times n}$  can be computed with an algorithm of time-complexity  $O(n^3)$ .*

*Proof of Corollary 5.4.* The value of  $\rho(A)$  can be computed using Karp's formula whose time-complexity is  $O(n^3)$ , see for instance [3, Theorem 2.19]. The critical graph  $\mathcal{G}_c(A)$  can also be computed with time-complexity  $O(n^3)$ , see for instance [1, Section 25.3 - Fact 13]. The cyclicity can be computed using Denardo's algorithm whose time complexity is again  $O(n^3)$ , see [13].  $\square$

**Example 5.5.** *Consider the matrix  $A$  defined in Example 3.9. Recall that  $\text{rk}_{\text{cl}}(A) = \text{rk}_{\text{rw}}(A) = 3$  and  $\text{rk}_{\text{tr}}(A) = 2$ . The critical graph of  $A$  is strongly connected and has cyclicity 1. Applying Formula (5.1), we get  $\text{urk}(A) = 1$ . Let us check this directly. We have:*

$$\forall k \geq 2, \quad A^k = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}.$$

Therefore,  $\text{urk}(A) = \text{rk}_{\text{cl}}(A^2) = \text{rk}_{\text{rw}}(A^2) = \text{rk}_{\text{tr}}(A^2) = 1$ .

**5.2. Proof of Theorem 5.2.** We first prove the result for a matrix  $E$  which is idempotent, that is, satisfies  $E^2 = E$ . In this case,  $\overline{\pi\langle E \rangle} = \{\pi(E)\}$ . So we just have to prove that:  $\text{rk}_{\text{rw}}(E) = \text{rk}_{\text{cl}}(E) = \text{rk}_{\text{tr}}(E)$  equals the sum of the cyclicities of the scc of  $\mathcal{G}_c(E)$ . It is a direct corollary of the next lemma.

**Lemma 5.6.** *Let  $E \in \mathbb{T}^{n \times n}$  be idempotent (i.e.,  $E^2 = E$ ) and let  $r$  denote the number of scc of  $\mathcal{G}_c(E)$ :*

- (1) *The only eigenvalue of  $E$  belonging to  $\mathbb{R}$  is 0; the associated eigenspace is the convex hull of the columns of  $E$ . The column rank, resp. row rank, of  $E$  is  $r$ .*
- (2) *In  $\mathcal{G}_c(E)$ , each node holds a loop. The tropical rank of  $E$  is  $r$ .*

*Proof.* (1). Let  $K$  be the convex hull of the columns of  $E$  and let  $\tilde{K}$  be the eigenspace associated to 0. Since  $E^2 = E$ , we get that  $E \odot E_j = E_j$  for any column  $E_j$ . Thus  $K \subset \tilde{K}$ . Conversely, if  $\mathbf{u} = (u_i)_i$  is an eigenvector of  $E$  associated to some eigenvalue  $\lambda \in \mathbb{R}$ , then

$$\mathbf{u} = -\lambda \odot E \odot \mathbf{u} = \bigvee_j (u_j - \lambda) \odot E_j.$$

Therefore,  $\mathbf{u} \in K$ , which implies both that  $\lambda = 0$  and that  $\tilde{K} \subset K$ . So  $K = \tilde{K}$ . But the weak dimension of  $K$  is the column rank of  $E$  (definition) and the weak dimension of  $\tilde{K}$  is the number of scc of  $\mathcal{G}_c(E)$  (Theorem 4.3-(2)). So  $\text{rk}_{\text{cl}}(E) = r$ . We obtain similarly that  $\text{rk}_{\text{rw}}(E) = r$ .

(2). If  $E$  is null, the statement is true. Otherwise, according to the above,  $\rho(E) = 0$ . Assume we are in this situation. Let  $i$  be a node of  $\mathcal{G}_c(E)$ . By definition,

there is a critical circuit from  $i$  to  $i$  of length  $\ell \geq 1$ , that is,  $E_{ii}^\ell = \ell\rho(E)$ . Since  $E^\ell = E$  and  $\rho(E) = 0$ , we deduce that  $E_{ii} = 0$ : there is a loop around  $i$  in  $\mathcal{G}_c(E)$ .

Now, consider a submatrix  $Q$  of  $E$  obtained by picking up exactly one index in each scc of  $\mathcal{G}_c(E)$  and by restricting the rows and columns to this set of indices. By construction, the maximum mean weight of circuits is 0 in  $E$  and in  $Q$  and  $\text{per}(Q) = \sum_i Q_{ii} = 0$  is attained by the identity permutation. Assume that there exists another permutation realizing the permanent, say mapping node  $i$  to  $j \neq i$ . Then there is a circuit of weight 0 containing both  $i$  and  $j$ , and this circuit was already present in  $\mathcal{G}_c(E)$ , which contradicts the fact that we selected one index per scc. Thus, the matrix  $Q$  is non-singular, and  $\text{rk}_{\text{tr}}(E) \geq r$ . But, according to (3.3),  $\text{rk}_{\text{tr}}(E) \leq \text{rk}_{\text{cl}}(E) = r$ , and we conclude that  $\text{rk}_{\text{tr}}(E) = r$ .  $\square$

We now turn to the proof of Theorem 5.2 in the general case. If  $\mathcal{G}(A)$  is acyclic then Theorem 5.2 is clearly true. So we assume that  $\mathcal{G}(A)$  is not acyclic, or, equivalently, that  $\rho(A) \neq -\infty$ . Observe that  $(-\rho(A)) \odot A$  has the same ranks, critical graph, and cyclicity as  $A$ , but has spectral radius 0. Thus, without loss of generality, we assume that  $\rho(A) = 0$ .

Set  $d = \sum_{C \in \mathfrak{C}} \text{cyc}(C)$ , where  $\mathfrak{C}$  is the set of scc of  $\mathcal{G}_c(A)$  and  $\text{cyc}(C)$  stands for the cyclicity of  $C$  (right-hand side of (5.1)). Since the tropical rank is the minimal one and the column and row ranks are the maximal ones, it is enough to prove that:

- (i) all matrices in  $\overline{\pi\langle A \rangle}$  have tropical rank greater or equal to  $d$ ;
- (ii) there is a matrix  $P \in \overline{\pi\langle A \rangle}$  such that  $\text{rk}_{\text{cl}}(P) = \text{rk}_{\text{rw}}(P) = d$ .

Applying Theorem 4.4, we get that, for  $k$  large enough,  $A^k = T^k \vee B_k$ . Since  $T$  is torsion, the finite entries of  $T^k$  are uniformly bounded, while the entries of  $B_k$  tend to  $-\infty$ . So, for  $k$  large enough, we have:

$$(T^k)_{ij} \neq -\infty \quad \Rightarrow \quad (A^k)_{ij} = (T^k)_{ij}.$$

We deduce that a non-singular submatrix of  $T^k$  corresponds to a non-singular submatrix of  $A^k$ , for  $k$  large enough. This proves that  $\text{rk}_{\text{tr}}(A^k) \geq \text{rk}_{\text{tr}}(T^k)$ , for  $k$  large enough. Let  $T_i, i \in I$ , be the matrices in the periodic part of the ultimately periodic sequence  $(T^k)_k$ . Observe that  $\overline{\pi\langle A \rangle} = \{\pi(A^k), k \in \mathbb{N}\} \cup \{\pi(T_i), i \in I\}$ . So we have:

$$\min_{B \in \overline{\pi\langle A \rangle}} \text{rk}_{\text{tr}}(B) = \min_k \text{rk}_{\text{tr}}(T^k) = \min_{i \in I} \text{rk}_{\text{tr}}(T_i).$$

Since the sequence  $\text{rk}_{\text{tr}}(T^k)$  is non-increasing, we deduce that  $\text{rk}_{\text{tr}}(T_i)$  is the same for all  $i$ . Observe that there exists a matrix  $T_j, j \in I$ , which is idempotent. By Lemma 5.6,  $\text{rk}_{\text{tr}}(T_j)$  is equal to the number of scc of  $\mathcal{G}_c(T_j)$ . Now, the same argument as above shows that  $\mathcal{G}_c(T^k) = \mathcal{G}_c(A^k)$ , for  $k$  large enough. So  $\mathcal{G}_c(T_j) = \mathcal{G}_c(A^\ell)$  for some  $\ell$ , and  $\text{rk}_{\text{tr}}(T_j)$  is equal to the number of scc of  $\mathcal{G}_c(A^\ell)$ . Let  $B$  be the boolean adjacency matrix of  $\mathcal{G}_c(A)$ . Then  $B^\ell$  is the boolean adjacency matrix of  $\mathcal{G}_c(A^\ell) = \mathcal{G}_c(T_j)$ . In particular  $B^\ell$  is idempotent, and the theory of boolean matrices (see e.g. [5, Chapter 3.4]) tells us that the scc of  $\mathcal{G}(B^\ell) = \mathcal{G}_c(A^\ell)$  are exactly the cyclicity classes of  $\mathcal{G}(B) = \mathcal{G}_c(A)$ . We conclude that the number of scc of  $\mathcal{G}_c(A^\ell)$  is  $d$ . So we have  $\text{rk}_{\text{tr}}(T_j) = d$ .

To prove (ii), we notice that, according to Lemma 5.6,  $\text{rk}_{\text{cl}}(T_j) = \text{rk}_{\text{rw}}(T_j) = \text{rk}_{\text{tr}}(T_j) = d$ . It completes the proof.

**5.3. Visualization and ultimate rank.** Visualization is a standard notion already appearing for instance in [12], and recently developed in [28]. We prove a result on the visualization of matrices with maximal ultimate rank, Theorem 5.10-(4), that plays an important role in Section 6.

For any finite vector  $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ , define  $\text{diag}(\mathbf{u}) \in \mathbb{T}^{n \times n}$  by:

$$\text{diag}(\mathbf{u})_{ij} = \begin{cases} u_i & \text{if } i = j, \\ -\infty & \text{if } i \neq j. \end{cases} \quad (5.3)$$

For  $A \in \mathbb{T}^{n \times n}$ , set  $\widehat{A} = \text{diag}(-\mathbf{u})A \text{diag}(\mathbf{u})$ , where  $-\mathbf{u} = (-u_i)_i$ . The entries of  $\widehat{A}$  satisfy:

$$\widehat{A}_{ij} = A_{ij} + u_j - u_i. \quad (5.4)$$

Matrices  $A$  and  $\widehat{A}$  share many properties.

**Lemma 5.7.** *Given a matrix  $A \in \mathbb{T}^{n \times n}$  and a finite vector  $\mathbf{u} \in \mathbb{R}^n$ , set  $\widehat{A} = \text{diag}(-\mathbf{u})A \text{diag}(\mathbf{u})$ , then  $\rho(\widehat{A}) = \rho(A)$ ,  $\mathcal{G}_c(\widehat{A}) = \mathcal{G}_c(A)$ ,  $\text{per}(\widehat{A}) = \text{per}(A)$ , and  $\text{rk}_*(\widehat{A}) = \text{rk}_*(A)$  for all the ranks of Definition 3.7.*

The proof of the lemma is straightforward.

**Definition 5.8.** *A matrix  $A \in \mathbb{T}^{n \times n}$  is said to be visualized (resp. strictly visualized) if:*

$$\begin{aligned} A_{ij} &= \rho(A) && \text{for all } (i, j) \in \mathcal{G}_c(A), \\ A_{ij} &\leq \rho(A) \text{ (resp. } < \rho(A)) && \text{for all } (i, j) \notin \mathcal{G}_c(A). \end{aligned}$$

A finite vector  $\mathbf{u} = (u_i)_i \in \mathbb{R}^n$  is called a (strict) visualization of  $A$  if the matrix  $\text{diag}(-\mathbf{u})A \text{diag}(\mathbf{u})$  is (strictly) visualized.

We now define the fundamental cell introduced in a different form in [8].

**Definition 5.9.** *The fundamental cell of a non-singular matrix  $A \in \mathbb{T}^{n \times n}$  is defined as the set*

$$\mathcal{F}(A) = \{ \mathbf{x} \in \mathbb{R}^n \mid \forall i, \forall j \neq \tau_A(i), \quad A_{ij} + x_j < A_{i\tau_A(i)} + x_{\tau_A(i)} = (A \odot \mathbf{x})_i \}.$$

The fundamental cell of a singular matrix  $A$  is empty,  $\mathcal{F}(A) = \emptyset$ .

Let us mention several properties of the fundamental cell which can be obtained by adapting the results from [8] and [25].

Let  $A$  be non-singular. The fundamental cell  $\mathcal{F}(A)$  is non-empty and its topological dimension is  $n$  since it is an open set. Consider the mapping  $\varphi_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\mathbf{x} \mapsto A \odot \mathbf{x}$ . We have

$$\mathcal{F}(A) = \varphi_A^{-1}(\{ \mathbf{x} \in \mathbb{R}^n, \exists! \mathbf{y} \in \mathbb{R}^n, \mathbf{x} = \varphi_A(\mathbf{y}) \}).$$

(The set  $\{ \mathbf{x} \in \mathbb{R}^n, \exists! \mathbf{y} \in \mathbb{R}^n, \mathbf{x} = \varphi_A(\mathbf{y}) \}$  is defined for any matrix  $A$ , and is non-empty if and only if  $A$  is non-singular.) The restriction of  $\varphi_A$  to the domain  $\mathcal{F}(A)$  is the affine map given by:

$$\begin{aligned} \varphi_A : \mathcal{F}(A) &\longrightarrow \mathbb{R}^n, \\ \mathbf{x} &\longmapsto \mathbf{y}, \quad y_i = A_{i\tau_A(i)} + x_{\tau_A(i)}. \end{aligned}$$

In particular, on the domain  $\mathcal{F}(A)$ , the map  $\varphi_A$  is an isometry for the euclidean distance.

The next theorem summarizes the results that we need about visualizations.

**Theorem 5.10.** *Let  $A \in \mathbb{T}^{n \times n}$  be a matrix with  $\rho(A) \neq -\infty$ .*

- (1) *The visualizations of  $A$  are the vectors  $\mathbf{u} \in \mathbb{R}^n$  such that  $A \odot \mathbf{u} \leq \rho(A) \odot \mathbf{u}$ .*
- (2) *There exists a strict visualization of  $A$ .*

*Assume that  $\mathcal{G}_c(A)$  contains all the nodes  $\{1, \dots, n\}$ .*

- (3) *The visualizations of  $A$  are the eigenvectors (associated to  $\rho(A)$ ).*

*Assume that  $\mathcal{G}_c(A)$  is the disjoint union of circuits covering all the nodes  $\{1, \dots, n\}$  (equivalently, that  $\text{urk}(A) = n$ ).*

(4) *The strict visualizations of  $A$  are the eigenvectors (associated to  $\rho(A)$ ) belonging to the fundamental cell.*

*Proof.* Property (1) is folklore. Let us prove it for completeness sake. First, the vector  $\mathbf{u} \in \mathbb{R}^n$  is a visualization of  $A$  if and only if

$$\forall i, j, \quad A_{ij} + u_j - u_i \leq \rho(A). \quad (5.5)$$

Indeed, assume (5.5) holds and consider a critical cycle  $(i_1, \dots, i_k, i_{k+1} = i_1)$  of  $\mathcal{G}(A)$ , then:

$$\sum_{\ell=1}^k A_{i_\ell i_{\ell+1}} + u_{i_{\ell+1}} - u_{i_\ell} = \sum_{\ell=1}^k A_{i_\ell i_{\ell+1}} = k\rho(A).$$

Thus, (5.5) implies that  $A_{i_\ell i_{\ell+1}} + u_{i_{\ell+1}} - u_{i_\ell} = \rho(A)$  for all  $\ell$ .

Second, we have the following equivalences:

$$\begin{aligned} [\forall i, j, A_{ij} + u_j - u_i \leq \rho(A)] &\iff [\forall i, \max_j (A_{ij} + u_j) \leq \rho(A) + u_i] \\ &\iff [A \odot \mathbf{u} \leq \rho(A) \odot \mathbf{u}]. \end{aligned}$$

This completes the proof of property (1).

Property (2) is proven in [28, Proposition 3.4].

To obtain property (3), it suffices to show that  $[A \odot \mathbf{u} \leq \rho(A) \odot \mathbf{u}] \Rightarrow [A \odot \mathbf{u} = \rho(A) \odot \mathbf{u}]$ . Assume that  $\mathbf{u} \in \mathbb{R}^n$  satisfies  $A \odot \mathbf{u} \leq \rho(A) \odot \mathbf{u}$ , and fix an arbitrary node  $i$ . Since  $\mathcal{G}_c(A)$  contains all the nodes, there exists a critical arc  $(i, j)$ . By property (1), the vector  $\mathbf{u}$  is a visualization of  $A$ . In particular, we have  $A_{ij} + u_j - u_i = \rho(A)$ , see (5.4). Hence,  $(A \odot \mathbf{u})_i \geq A_{ij} + u_j = \rho(A) + u_i$ . So we have proved that  $A \odot \mathbf{u} \geq \rho(A) \odot \mathbf{u}$ . We conclude that  $A \odot \mathbf{u} = \rho(A) \odot \mathbf{u}$ .

Let us show property (4). Assume that  $\text{urk}(A) = n$ , then, the visualizations of  $A$  are its eigenvectors by property (3). Let  $\mathbf{u}$  be an eigenvector. Then  $\mathbf{u}$  is a visualization which is strict if and only if  $A_{ij} + u_j - u_i < \rho(A)$  for  $(i, j) \notin \mathcal{G}_c(A)$ . By Corollary 5.3, we have

$$[(i, j) \notin \mathcal{G}_c(A)] \iff [j \neq \tau_A(i)],$$

where  $\tau_A$  is the unique maximizing permutation of  $A$ . Therefore, the visualization  $\mathbf{u}$  is strict if and only if

$$\forall i, \forall j \neq \tau_A(i), \quad A_{ij} + u_j < \rho(A) + u_i = (A \odot \mathbf{u})_i = A_{i\tau_A(i)} + u_{\tau_A(i)},$$

which is equivalent to saying that  $\mathbf{u}$  belongs to  $\mathcal{F}(A)$ .  $\square$

## 6. The ultimate rank of a semigroup of matrices

**6.1. Statements.** Let us extend the notion of ultimate rank to a semigroup of tropical matrices.

**Definition 6.1.** *Let  $\mathcal{S} \subset \mathbb{T}^{n \times n}$  be a semigroup of tropical matrices. The ultimate rank of  $\mathcal{S}$ , denoted  $\text{urk}(\mathcal{S})$ , is defined by:*

$$\text{urk}(\mathcal{S}) = \min\{\text{rk}_*(A) \mid A \in \overline{\pi(\mathcal{S})}\} = \min\{\text{urk}(A) \mid A \in \overline{\pi(\mathcal{S})}\}.$$

We do not know if the ultimate rank of a given finitely generated semigroup is algorithmically computable. However, we prove a partial result. Indeed, we give two characterizations of the case of maximal ultimate rank, one combinatorial and one geometric. As a by-product, we obtain a polynomial-time algorithm to decide if the ultimate rank is maximal.

**Lemma 6.2.** *Let  $\mathcal{S}$  be a finitely generated semigroup of matrices of  $\mathbb{T}^{n \times n}$ . Assume that  $\forall P \in \mathcal{S}, \text{urk}(P) = n$ . Then  $\pi(\mathcal{S})$  is finite.*

*Proof.* In [15, Theorem 2.1], it is proven that a finitely generated torsion semigroup of  $\mathbb{P}\mathbb{T}^{n \times n}$  is finite (*i.e.*, the Burnside problem has a positive answer in  $\mathbb{P}\mathbb{T}^{n \times n}$ ). Therefore, we only have to prove that  $\pi(\mathcal{S})$  is torsion, *i.e.*, for any  $P \in \mathcal{S}$ , there exist  $k, c > 0$ , such that  $\pi(P^{k+c}) = \pi(P^k)$ .

Since  $\text{urk}(P) = n$ , by Corollary 5.3, all the nodes of  $\mathcal{G}(P)$  are critical. Then, by Theorem 4.4, the matrix  $(-\rho(P)) \odot P$  is torsion, or, equivalently,  $P$  is projectively torsion.  $\square$

Lemma 6.2 enables to obtain a simplified characterization of finitely generated semigroups of maximal ultimate rank.

**Lemma 6.3.** *Let  $\mathcal{S}$  be a finitely generated semigroup of matrices of  $\mathbb{T}^{n \times n}$ . Then*

$$[\text{urk}(\mathcal{S}) = n] \iff [\forall P \in \mathcal{S}, \text{urk}(P) = n].$$

*Proof.* Obviously, we have  $[\text{urk}(\mathcal{S}) = n] \Rightarrow [\forall P \in \mathcal{S}, \text{urk}(P) = n]$ . Conversely, assume that  $\forall P \in \mathcal{S}, \text{urk}(P) = n$ . By Lemma 6.2,  $\pi(\mathcal{S})$  is finite, and, in particular,  $\pi(\mathcal{S}) = \overline{\pi(\mathcal{S})}$ . We conclude that  $\text{urk}(\mathcal{S}) = n$ .  $\square$

Let us state the main results.

**Theorem 6.4** (Combinatorial characterization). *Let  $\mathcal{S} = \langle \mathcal{I} \rangle$  be the semigroup generated by a finite set  $\mathcal{I}$  of matrices in  $\mathbb{T}^{n \times n}$ .*

*For  $A \in \mathcal{I}$ , define  $\tilde{A} \in \mathbb{T}^{n \times n}$  by:  $\forall i, j, \tilde{A}_{ij} = A_{ij} - \rho(A)$ . Set  $M = \bigvee_{A \in \mathcal{I}} \tilde{A}$ . We have  $\text{urk}(\mathcal{S}) = n$  if and only if the following three properties are satisfied:*

- (C1):  $\forall A \in \mathcal{I}, \text{urk}(A) = n$ ;
- (C2):  $\rho(M) = 0$ ;
- (C3):  $\forall A \in \mathcal{I}, \forall i, j \in \{1, \dots, n\}$ ,

$$[(i, j) \in \mathcal{G}_c(M), \tilde{A}_{ij} = M_{ij}] \Rightarrow (i, j) \in \mathcal{G}_c(A).$$

**Theorem 6.5** (Geometric characterization). *Let  $\mathcal{S} = \langle \mathcal{I} \rangle$  be the semigroup generated by a finite set  $\mathcal{I}$  of matrices in  $\mathbb{T}^{n \times n}$ . We have  $\text{urk}(\mathcal{S}) = n$  if and only if the following property holds:*

- (G): *the generators have a common finite eigenvector that belongs to the intersection of their fundamental cells;*

*or, equivalently, if the following two properties hold:*

- (G1):  $\forall A \in \mathcal{I}, \text{urk}(A) = n$ ;
- (G2): *the generators have a common strict visualization.*

The proofs of Theorems 6.4 and 6.5 are given in §6.2. To the best of our knowledge, Theorem 6.4 is completely original. Theorem 6.5 refines [25, Theorem 3.1] which states that:  $[\text{urk}(\mathcal{S}) = n] \Rightarrow$  the matrices in the semigroup have a common eigenvector. The proof of [25, Theorem 3.1] is different from our proof and relies on Kakutani fixed point Theorem.

**Corollary 6.6.** *Let  $\mathcal{S} = \langle \mathcal{I} \rangle$  be the semigroup generated by a finite set  $\mathcal{I}$  of matrices in  $\mathbb{T}^{n \times n}$ . There exists an algorithm of time-complexity  $O(|\mathcal{I}|n^3)$  that decides whether  $\text{urk}(\mathcal{S}) = n$ .*

*Proof.* We use the characterization in Theorem 6.4. Property (C1) can be checked in  $O(|\mathcal{I}|n^3)$  using Corollary 5.4. The matrix  $M$  can be computed in  $O(|\mathcal{I}|n^2)$  and  $\rho(M)$  can be computed in  $O(n^3)$  using Karp's formula, see [3, Theorem 2.19]. So property (C2) can be verified in  $O(|\mathcal{I}|n^2 + n^3)$ . Checking property (C3) requires to compute  $\mathcal{G}_c(M)$  and  $\mathcal{G}_c(A)$  for every  $A \in \mathcal{I}$ , which can be done in  $O(|\mathcal{I}|n^3)$ , see for instance [1, Section 25.3 - Fact 13].  $\square$

**6.2. Proof of Theorem 6.4 and Theorem 6.5.** We carry out the proofs of Theorems 6.4 and 6.5 together. Let  $\mathcal{S} = \langle \mathcal{I} \rangle$  be the semigroup generated by the finite set  $\mathcal{I}$  of matrices in  $\mathbb{T}^{n \times n}$ . For simplicity and without loss of generality, we assume that  $\rho(A) = 0$  for any  $A \in \mathcal{I}$ . (Otherwise, we replace  $A$  by  $-\rho(A) \odot A$ .) Set  $M = \bigvee_{A \in \mathcal{I}} A$ . Set:

- (P1):  $\text{urk}(\mathcal{S}) = n$ ;
- (P2): Properties (C1), (C2), and (C3) in Theorem 6.4;
- (P3): Properties (G1) and (G2) in Theorem 6.5.

The structure of the proof is:

$$(\mathbf{P1}) \Rightarrow (\mathbf{P2}) \Rightarrow (\mathbf{P3}) \Rightarrow (\mathbf{P1}).$$

[(P1)  $\Rightarrow$  (P2)]. Assume (P1) holds. Then, (C1) is obvious. Let us prove (C2), that is  $\rho(M) = 0$ . Observe that, for any  $A \in \mathcal{I}$ , we have  $M \geq A$ , so that  $\rho(M) \geq \rho(A) = 0$ . Now, (C2) follows from the following lemma.

**Lemma 6.7.** *Assume that (C1) holds and that  $\rho(M) > 0$ . Then, there is a product  $P$  of at most  $n$  matrices of  $\mathcal{I}$  for which  $\text{urk}(P) < n$ .*

*Proof.* Since  $\rho(M) > 0$ , there is a simple circuit, say  $(i_1, \dots, i_k, i_{k+1} = i_1)$ , with positive weight  $w$  in  $\mathcal{G}(M)$ . For each  $\ell \in \{1, \dots, k\}$ , choose  $A(\ell) \in \mathcal{I}$  such that  $A(\ell)_{i_\ell i_{\ell+1}} = M_{i_\ell i_{\ell+1}}$ , and define  $P = A(1) \cdots A(k)$ . Observe that  $\rho(P) \geq P_{i_1 i_1} \geq w > 0$ . Assume that  $\text{urk}(P) = n$ . According to Proposition 3.2 and Corollary 5.3, we must have

$$\rho(P) = \frac{1}{n} \text{per}(P) = \frac{1}{n} \sum_i \text{per}(A(i)) = \sum_i \rho(A(i)) = 0.$$

So we have reached a contradiction, showing that  $\text{urk}(P) < n$ .  $\square$

Now let us prove property (C3). We need a preliminary lemma.

**Lemma 6.8.** *Assume that (C1) and (C2) hold. We have:*

- (i)  $\forall P \in \mathcal{S}, \rho(P) = 0$ ;
- (ii) a visualization of  $M$  is a visualization for any  $P$  in  $\mathcal{S}$ .

*Proof.* By Theorem 5.10-(1), a visualization  $\mathbf{v}$  of  $M$  satisfies  $M \odot \mathbf{v} \leq \mathbf{v}$ . Given a matrix  $A \in \mathcal{I}$ , we have

$$A \odot \mathbf{v} \leq M \odot \mathbf{v} \leq \mathbf{v}.$$

Therefore, by Theorem 5.10-(1) again,  $\mathbf{v}$  is a visualization of any generator  $A$ . Now any generator  $A$  has maximal ultimate rank by property (C1). According to Corollary 5.3, all the nodes of  $\mathcal{G}(A)$  are critical. By Theorem 5.10-(3), it implies that  $\mathbf{v}$  is an eigenvector of  $A$ . Since  $\mathbf{v}$  is a common eigenvector of the generators, it is a common eigenvector for all the matrices in the semigroup:  $\forall P \in \mathcal{S}, P \odot \mathbf{v} = \mathbf{v}$ . It implies, by Theorem 5.10-(1), that  $\mathbf{v}$  is a visualization for  $P$ . It also implies that  $\rho(P) \geq 0$ . Since  $P = A_1 \cdots A_\ell$  for some  $\ell$  and  $A_i \in \mathcal{I}$ , we get that  $P \leq M^\ell$ , implying that  $\rho(P) \leq \rho(M^\ell) = 0$ . We conclude that  $\rho(P) = 0$ .  $\square$

From now on, we assume without loss of generality that  $M$  is strictly visualized and that all the matrices of the semigroup are visualized. This is possible for the following reason. Recall that for any  $P \in \mathbb{T}^{n \times n}$  and  $\mathbf{u} \in \mathbb{R}^n$ , the matrices  $P$  and  $\text{diag}(-\mathbf{u}) \odot P \odot \text{diag}(\mathbf{u})$  have the same ranks, critical graph, etc, see Lemma 5.7. Let  $\mathbf{v}$  be a strict visualization of  $M$ , which exists by Theorem 5.10. By Lemma 6.8,  $\mathbf{v}$  is a visualization for the matrices in the semigroup. Now replace  $A$  by  $\text{diag}(-\mathbf{v}) \odot A \odot \text{diag}(\mathbf{v})$  for  $A \in \mathcal{I}$ , and  $M$  by  $\text{diag}(-\mathbf{v}) \odot M \odot \text{diag}(\mathbf{v})$ .

**Lemma 6.9.** *Assume that (C1) and (C2) hold but (C3) does not. Then, there is a product  $P$  of at most  $n$  matrices of  $\mathcal{I}$  such that  $\text{urk}(P) < n$ .*

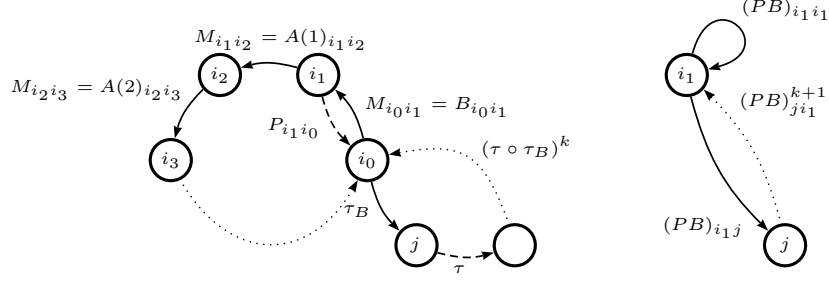


FIGURE 2. Illustration of the proof of Lemma 6.9

*Proof.* Property (C3) does not hold. So there exists  $B \in \mathcal{I}$  and  $(i_0, i_1) \in \mathcal{G}_c(M)$  such that:  $B_{i_0 i_1} = M_{i_0 i_1} = 0$ , but  $(i_0, i_1) \notin \mathcal{G}_c(B)$ . The critical arc  $(i_0, i_1)$  can be completed into a critical circuit  $(i_0, i_1, i_2, \dots, i_\ell, i_{\ell+1} = i_0)$  of  $\mathcal{G}_c(M)$ . For each  $k = 1, \dots, \ell$ , we choose a matrix  $A(k) \in \mathcal{I}$  such that  $A(k)_{i_k i_{k+1}} = M_{i_k i_{k+1}}$ . Set  $P = A(1) \cdots A(\ell)$  and  $\tau = \tau_{A(\ell)} \circ \cdots \circ \tau_{A(1)}$ . (See Figure 2.)

By construction,  $P_{i_1 i_0} = 0$ . We have:

$$(PB)_{i_1 i_1} \geq P_{i_1 i_0} + B_{i_0 i_1} = 0.$$

Since  $PB$  is visualized by assumption and  $\rho(PB) = 0$  by Lemma 6.8-(i), we get  $(PB)_{i_1 i_1} = 0$ . There is a loop around  $i_1$  in  $\mathcal{G}_c(PB)$ . Let us prove that there is another critical circuit going through  $i_1$  in  $\mathcal{G}_c(PB)$ .

Let  $j \neq i_1$  be such that  $(i_0, j) \in \mathcal{G}_c(B)$ . By visualization,  $B_{i_0 j} = 0$ . Therefore,

$$(PB)_{i_1 j} \geq P_{i_1 i_0} + B_{i_0 j} = 0 \Rightarrow (PB)_{i_1 j} = 0.$$

Let  $k \in \mathbb{N}$  be such that  $(\tau_B \circ \tau)^{k+1}(j) = j$ . We have:

$$\tau \circ (\tau_B \circ \tau)^k(j) = \tau_B^{-1}(j) = i_0 \Rightarrow ((PB)^k P)_{j i_0} = 0 \Rightarrow (PB)_{j i_1}^{k+1} = 0.$$

Thus, in  $\mathcal{G}_c(PB)$ , there is a circuit going from  $i_1$  to  $j$  and back to  $i_1$ . Consequently, we have built a product  $PB \in \mathcal{S}$  of at most  $n$  matrices whose critical graph contains two different circuits passing through a node. Then,  $\text{urk}(PB) < n$  by Corollary 5.3.  $\square$

Since **(P1)** holds, property (C3) follows from Lemma 6.9. This completes the proof.

**[(P2)  $\Rightarrow$  (P3)].**

First of all, Property (G1) is the same as Property (C1), so it follows from (P2).

To prove that (G2) holds, let us assume as above that  $M$  is strictly visualized and that the matrices in  $\mathcal{S}$  are visualized. We are going to show that the generators are strictly visualized which will prove (G2).

Consider  $A \in \mathcal{I}$  with  $A_{ij} = 0$  for some  $i, j \in \{1, \dots, n\}$ . Then  $M_{ij} \geq A_{ij} = 0$ . Thus  $M_{ij} = 0$ , and by strict visualization  $(i, j) \in \mathcal{G}_c(M)$ . We conclude by (C3) that  $(i, j) \in \mathcal{G}_c(A)$ . This means precisely that  $A$  is strictly visualized.

The fact that (G1) plus (G2) is equivalent to (G) follows directly from Theorem 5.10-(4).

**[(P3)  $\Rightarrow$  (P1)].**

Let  $\mathbf{u} \in \mathbb{R}^n$  be a common eigenvector of all the generators that lies in the intersection  $\mathcal{W} = \bigcap_{A \in \mathcal{I}} \mathcal{F}(A)$ , which exists according to property (G). Let us show that every  $P \in \mathcal{S}$ , written as a product  $A_1 \cdots A_\ell$ ,  $A_i \in \mathcal{I}$ , is non-singular.



Denote by  $d(\cdot, \cdot)$  the Euclidean distance of  $\mathbb{R}^n$ . Since  $\mathcal{W}$  is a non-empty intersection of finitely many open sets, there exists  $\varepsilon > 0$  such that the ball  $\mathfrak{B} = \{\mathbf{x} \in \mathbb{R}^n \mid d(\mathbf{u}, \mathbf{x}) \leq \varepsilon\} \subset \mathcal{W}$ . We use the notations:  $\varphi_i : \mathbf{x} \mapsto A_i \odot \mathbf{x}$ , and  $\varphi_P : \mathbf{x} \mapsto P \odot \mathbf{x}$ . Recall that  $\varphi_i$  is an affine isometry on  $\mathcal{F}(A_i)$ , see §5.3. Since  $A_i \odot \mathbf{u} = \mathbf{u}$ , we obtain that  $\varphi_i(\mathfrak{B}) = \mathfrak{B}$ . By composition, we get that  $\varphi_P(\mathfrak{B}) = \mathfrak{B}$ . In particular,  $\mathfrak{B}$  is included in the image of  $\varphi_P$ , or, equivalently, in the tropical convex hull of the columns of  $P$ . Since  $\mathfrak{B}$  is of topological dimension  $n$ , we get that the tropical convex hull of the columns of  $P$  is of topological dimension  $n$ . Using Proposition 3.8, we conclude that  $P$  is non-singular.

Therefore, we have proved that all the matrices in  $\mathcal{S}$  have tropical rank  $n$ . In particular, given a matrix  $P$  in  $\mathcal{S}$ , all the products  $P^k$  have tropical rank  $n$ , which implies that the ultimate rank of  $P$  is  $n$ . According to Lemma 6.3, this implies that the ultimate rank of  $\mathcal{S}$  is  $n$ .

**6.3. Examples.** We illustrate Theorems 6.4 and 6.5 using four successive examples. All the matrices in these examples belong to  $\mathbb{T}^{3 \times 3}$  and are non-singular. Their projective fundamental cells belong to  $\mathbb{P}\mathbb{R}^3$ , and are represented (in  $\mathbb{R}^2$ ) by orthogonal projection on the plane orthogonal to the direction  $(1, 1, 1)$ .

Here is a general observation that is useful for the examples below. For  $\mathbf{u} \in \mathbb{R}^n$ , define the matrix  $\text{diag}(\mathbf{u}) \in \mathbb{T}^{n \times n}$  like in (5.3). The fundamental cell of  $\text{diag}(-\mathbf{u})A \text{diag}(\mathbf{u})$  is the translation of the fundamental cell of  $A$  by  $-\mathbf{u}$ .

**Example 6.10.** Consider the matrices:

$$A_1 = \begin{bmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -5 & -2 \\ 1 & 0 & 1 \\ -2 & -5 & 0 \end{bmatrix},$$

where  $A_2 = \text{diag}(-\mathbf{u}) \odot A_1 \odot \text{diag}(\mathbf{u})$  for  $\mathbf{u} = (0, -3, 0)$ . We have  $\text{urk}(A_1) = \text{urk}(A_2) = 3$ . For each matrix, the fundamental cell is the interior of the set of eigenvectors. Here the intersection of the fundamental cells is non-empty, see Figure 3. Applying Theorem 6.5, we conclude that  $\text{urk}\langle A_1, A_2 \rangle = 3$ .

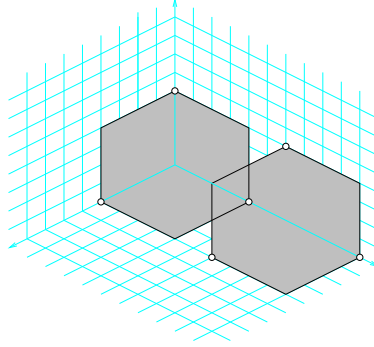


FIGURE 3. In gray, the fundamental cells of  $A_1$  (left) and  $A_2$  (right).

We can also recover the result using Theorem 6.4. We have

$$A_1 \vee A_2 = \begin{bmatrix} 0 & -2 & -2 \\ 1 & 0 & 1 \\ -2 & -2 & 0 \end{bmatrix},$$

for which  $\rho(A_1 \vee A_2) = 0$  and  $\mathcal{G}_c(A_1 \vee A_2) = \mathcal{G}_c(A_1) = \mathcal{G}_c(A_2)$ , so condition (C3) of Theorem 6.4 holds as well.

**Example 6.11.** Consider the matrices:

$$B_1 = \begin{bmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -5 & -1 \\ 1 & 0 & 2 \\ -3 & -6 & 0 \end{bmatrix},$$

where  $B_1 = A_1$  and  $B_2 = \text{diag}(-\mathbf{v}) \odot B_1 \odot \text{diag}(\mathbf{v})$  for  $\mathbf{v} = (0, -3, 1)$ . Here again, we have  $\text{urk}(B_1) = \text{urk}(B_2) = 3$ .

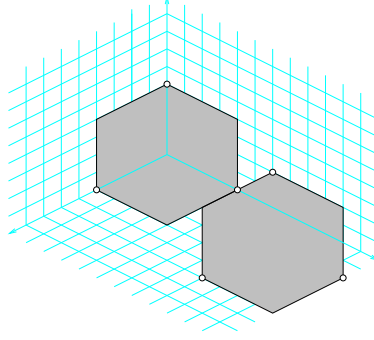


FIGURE 4. In gray, the fundamental cells of  $B_1$  (left) and  $B_2$  (right).

For each matrix, the fundamental cell is again the interior of the set of eigenvectors. But now the intersection of the two fundamental cells is empty, see Figure 4. Then  $\text{urk}\langle B_1, B_2 \rangle < 3$ , by Theorem 6.5, and one can check that indeed  $\text{rk}_{\text{tr}}(B_1 B_2) = 2$ .

Moreover, we can prove that  $\text{urk}\langle B_1, B_2 \rangle = 2$ . Indeed, the intersection of the sets of eigenvectors has topological dimension 2, see Figure 4. The common eigenvectors of  $B_1$  and  $B_2$  are also eigenvectors of any matrix in  $\langle B_1, B_2 \rangle$ . Therefore, a matrix in  $\langle B_1, B_2 \rangle$  has a set of eigenvectors of topological dimension at least 2, hence is not of rank 1.

We can confirm that  $\text{urk}\langle B_1, B_2 \rangle < 3$  using Theorem 6.4. Consider

$$B_1 \vee B_2 = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$$

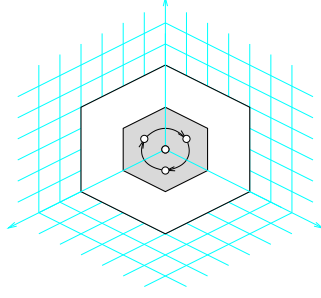
for which  $\rho(B_1 \vee B_2) = 0$ . We have  $(2, 3) \in \mathcal{G}_c(B_1 \vee B_2)$ ,  $(B_2)_{23} = (B_1 \vee B_2)_{23}$ , and  $(2, 3) \notin \mathcal{G}_c(B_2)$ , so condition (C3) of Theorem 6.4 fails.

**Example 6.12.** Consider the matrices:

$$C_1 = \begin{bmatrix} -2 & 0 & -2 \\ -2 & -2 & 0 \\ 0 & -2 & -2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

Observe that  $C_1 \vee C_2 = C_2$ . The conditions of Theorem 6.4 are clearly satisfied, implying that  $\text{urk}\langle C_1, C_2 \rangle = 3$ .

This can also be checked using Theorem 6.5. The fundamental cells of the two matrices have been represented in Figure 5. Each matrix acts as a rotation of angle  $-2\pi/3$  and center  $(0, 0, 0)$  on its respective fundamental cell. In particular,  $(0, 0, 0)$  is the unique eigenvector of both  $C_1$  and  $C_2$ , and it belongs to the intersection of the fundamental cells.

FIGURE 5. The fundamental cells of  $C_1$  (white) and  $C_2$  (gray).

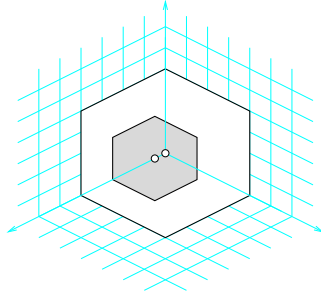
**Example 6.13.** Consider the matrices:

$$D_1 = \begin{bmatrix} -2 & 0 & -2 \\ -2 & -2 & 0 \\ 0 & -2 & -2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -1 & 0.2 & -0.8 \\ -1.2 & -1 & 0 \\ -0.2 & -1 & -1 \end{bmatrix},$$

where  $D_1 = C_1$  and  $D_2 = \text{diag}(-\mathbf{w}) \odot C_2 \odot \text{diag}(\mathbf{w})$  for  $\mathbf{w} = (-0.2, 0, 0)$ . On their fundamental cell, the two matrices act as a rotation of angle  $-2\pi/3$  with respective centers  $(0, 0, 0)$  and  $(0.2, 0, 0)$ . Therefore, they have no common eigenvector, and hence  $\text{urk}\langle D_1, D_2 \rangle < 3$  by Theorem 6.5. Furthermore,

$$D_1 \vee D_2 = \begin{bmatrix} -1 & 0.2 & -0.8 \\ -1.2 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix},$$

with  $\rho(D_1 \vee D_2) = 0.2/3$ , and applying Theorem 6.4, we double-check that we have  $\text{urk}\langle D_1, D_2 \rangle < 3$ .

FIGURE 6. The fundamental cells of  $D_1$  (white) and  $D_2$  (gray).

We check that  $\text{urk}(D_1 D_2 D_1) = 1$  so we have  $\text{urk}\langle D_1, D_2 \rangle = 1$ . (This last result can also be recovered using [23, Theorem 7.3.1].)

**6.4. Projectively bounded semigroups.** In obtaining Theorems 6.4 and 6.5, the assumption that the semigroup of matrices is finitely generated is used only twice. First, to prove Lemmas 6.2 and 6.3, second, to define matrix  $M$  in Theorem 6.4. So we obtain the following extension for free.

**Definition 6.14.** A subset  $\mathcal{I}$  of non-null matrices in  $\mathbb{T}^{n \times n}$  is projectively bounded if  $\sup_{A \in \mathcal{I}} \max_{A_{ij}, A_{k\ell} \in \mathbb{R}} |A_{ij} - A_{k\ell}| < +\infty$ .

Theorems 6.4 and 6.5 hold if we replace in the statements: (i) “finite set  $\mathcal{T}$ ” by “projectively bounded set  $\mathcal{T}$ ”; (ii) “Set  $M = \bigvee_{A \in \mathcal{T}} \tilde{A}$ ” by “Set  $M = \sup_{A \in \mathcal{T}} \tilde{A}$ ” (for Theorem 6.4); (iii) “ $\text{urk}(\mathcal{S}) = n$ ” by “ $\forall P \in \mathcal{S}, \text{urk}(P) = n$ ”.

## 7. Summary and open issues

Next table summarizes the known results for the time-complexity of the two basic questions concerning the rank.

	Deciding if maximal	Computing
$\text{rk}_{\text{tr}}(A)$	Polyn. [10, 7]	NP-hard [21]
$\text{rk}_{\text{cl}}/\text{rk}_{\text{rw}}(A)$	Polyn. [9, §3.4]	Polyn. [9, §3.4]
$\text{urk}(A)$	Polyn., Cor. 5.4	Polyn., Cor. 5.4
$\text{urk}(\mathcal{S})$	Polyn., Cor. 6.6	??

Two points are worth emphasizing. First, computing the ultimate rank of a matrix is of polynomial time-complexity, while computing the initial rank of a matrix could be NP-hard. Second, the general question about the decidability of computing the ultimate rank of a matrix semigroup is still open.

**Problem 7.1.** *How to compute the ultimate rank of a finitely generated matrix semigroup?*

In the present paper, we have proved that the problem of determining whether the ultimate rank is maximal is solvable in polynomial time. Another solvable case is that of semigroups of ultimate rank 0. Indeed, the case of rank 0 is equivalent to mortality, which is known to be decidable and NP-complete [4].

Before approaching the general case of arbitrary ultimate rank, an intermediate important case is that of ultimate rank 1. This case is generic [24], more precisely, if two generators are “chosen randomly”, the probability that the semigroup has ultimate rank 1 is equal to 1. Also some sufficient conditions for ultimate rank 1 are given in [23, Chapter 7]. Yet, the decidability of the question “Is the ultimate rank equal to 1?” is still open.

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