Interpolating sequences and Carleson measures in the Hardy-Sobolev spaces of the ball in C^n.

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Abstract

In this work we study Hardy Sobolev spaces in the ball of $\mathbb{C}^n$ with respect to interpolating sequences and Carleson measures.

We compare them with the classical Hardy spaces of the ball and we stress analogies and differences.

Contents

1 Introduction.

We shall work with the Hardy-Sobolev spaces H^p_s. For $1 \leq p < \infty$ and $s \in \mathbb{R}$, H^p_s is the space of holomorphic functions in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ such that the following expression is finite

$$\|f\|^p_{s,p} := \sup_{r<1} \int_{\partial\mathbb{B}} |(I+R)^s f(rz)|^p \, d\sigma(z),$$

where I is the identity, $d\sigma$ is the Lebesgue measure on $\partial\mathbb{B}$ and R is the radial derivative

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

For $s \in \mathbb{N}$, this norm is equivalent to

$$\|f\|^p_{H^p_s} = \max_{0\leq j\leq s} \int_{\partial\mathbb{B}} \left|R^j f(z)\right|^p d\sigma(z).$$

This means that $R^j f \in H^p(\mathbb{B}),\ j = 0, ...,\ s$.

We shall prove estimates only in the case $s \in \mathbb{N}$ but the spaces H^p_s form an interpolating scale with respect to the parameter s, see in section 2, hence, in some cases, this allows to extend the results to the case $s \in \mathbb{R}_+$.

If $sp > n$ the functions in H^p_s are continuous up to the boundary $\partial\mathbb{B}$ hence the results we are interested in are essentially trivial, so we shall restrict ourselves to the case $s \leq n/p$.

For $s = 0$ the Hardy Sobolev spaces H^p_0 are the classical Hardy spaces $H^p(\mathbb{B})$ of the unit ball $\mathbb{B}$ and a natural question is to study what remains true from classical Hardy spaces $H^p(\mathbb{B})$ to Hardy Sobolev H^p_s.

An important notion is that of Carleson measure.

Definition 1.1 *The* **measure** μ *in* $\mathbb{B}$ *is* **Carleson** *for* H^p_s, $\mu \in C_{s,p}$, *if we have the embedding*

$$\forall f \in H^p_s,\ \int_{\mathbb{B}} |f|^p \, d\mu \leq C\|f\|^p_{H^p_s}.$$

Carleson measures where introduced by Carleson [12] in his work on interpolating sequences.

We have the following table concerning the known results about Carleson measures :

$H^p(\mathbb{D})$	$H^p(\mathbb{B}) = H^p_0(\mathbb{B})$	$H^p_s(\mathbb{B})$
Characterized geometrically by L. Carleson [12]	Characterized geometrically by L. Hörmander [17]	Studied by C. Cascante & J. Ortega [13] ; characterized for $n-1 \leq ps \leq n$. For $p = 2$, any s characterized by A. Volberg & B. Wick [25]
Same for all p	Same for all p	Depending on p

Definition 1.2 *The* **multipliers algebra** $\mathcal{M}^p_s$ *of* H^p_s *is the algebra of functions* m *on* $\mathbb{B}$ *such that*

$$\forall h \in H^p_s,\ mh \in H^p_s.$$

The norm of a multiplier is its norm as an operator from H^p_s *into* H^p_s.

We have the following table of already known results, where C.C. means a certain Carleson condition:

$H^p(\mathbb{D})$	$H^p(\mathbb{B})$	$H^p_s(\mathbb{B})$
$\mathcal{M}^p_0(\mathbb{D}) = H^\infty(\mathbb{D}),\ \forall p$	$\mathcal{M}^p_0(\mathbb{B}) = H^\infty(\mathbb{B}),\ \forall p$	$\mathcal{M}^p_s = H^\infty(\mathbb{B}) \cap C.C.$ characterized for $n-1 \leq ps \leq n$ and for $p=2$ by Volberg & Wick [25] Depending on p

Now we shall deal with sequences of points in the ball $\mathbb{B}$ and to state results we shall need several definitions, most of them being well known.

Let p' be the conjugate exponent for p, $\dfrac{1}{p}+\dfrac{1}{p'}=1$; the Hilbert space H^2_s is equipped with the reproducing kernels :

$$\forall a \in \mathbb{B},\ k_a(z) = \frac{1}{(1-\bar{a}\cdot z)^{n-2s}},\ \|k_a\|_{H^p_s} \simeq (1-|a|^2)^{s-n/p'} \tag{1.1}$$

i.e. $\forall a \in \mathbb{B},\ \forall f \in H^p_s,\ f(a) = \langle f, k_a\rangle$, where $\langle\cdot,\cdot\rangle$ is the scalar product of the Hilbert space H^2_s. In the case $s = n/2$ there is a log in k_a.

Definition 1.3 *The sequence S is Carleson in $H^p_s(\mathbb{B})$, if the associated measure*

$$\nu_S := \sum_{a\in S} \|k_{s,a}\|^{-p}_{H^{p'}_s}\delta_a$$

is Carleson for $H^p_s(\mathbb{B})$.

Definition 1.4 *Let $p>1$, the* **sequence** *S of points in $\mathbb{B}$ is* **interpolating** *in $H^p_s(\mathbb{B})$, if there is a $C = C_p > 0$ such that*

$$\forall \lambda \in \ell^p(S),\ \exists f \in H^p_s(\mathbb{B}) :: \forall a \in S,\ f(a) = \lambda_a \|k_a\|_{H^{p'}_s} = \lambda_a (1-|a|^2)^{s-n/p},\ \|f\|_{H^p_s} \leq C\|\lambda\|_p,$$

where p' is the conjugate exponent for p, $\dfrac{1}{p}+\dfrac{1}{p'}=1$.

If $p=1$, we take the limiting case in the above definition : S is IS for H^1_s, if there is a $C>0$ such that $\forall \lambda \in \ell^1(S),\ \exists f \in H^1_s(\mathbb{B}) :: \forall a \in S,\ f(a) = \lambda_a(1-|a|^2)^{s-n},\ \|f\|_{H^1_s} \leq C\|\lambda\|_{\ell^1}$.

Definition 1.5 *The* **sequence** *S of points in $\mathbb{B}$ is* **interpolating** *in the multipliers algebra $\mathcal{M}^p_s$ of $H^p_s(\mathbb{B})$ if there is a $C>0$ such that*

$$\forall \lambda \in \ell^\infty(S),\ \exists m \in \mathcal{M}^p_s :: \forall a \in S,\ m(a) = \lambda_a \text{ and } \|m\|_{\mathcal{M}^p_s} \leq C\|\lambda\|_\infty.$$

Definition 1.6 *Let S be an interpolating sequence in $\mathcal{M}^p_s$; we say that S has a* **bounded linear extension operator, BLEO***, if there is a a bounded linear operator $E : \ell^\infty(S) \to \mathcal{M}^p_s$ and a $C>0$ such that*

$$\forall \lambda \in \ell^\infty(S),\ E(\lambda) \in \mathcal{M}^p_s,\ \|E(\lambda)\|_{\mathcal{M}^p_s} \leq C\|\lambda\|_\infty\ :\ \forall a \in S,\ E(\lambda)(a) = \lambda_a.$$

We have the table of results on interpolating sequences, where A.R.S. means Arcozzi, Rochberg and Sawyer [9].

$H^\infty(\mathbb{D})$	$H^\infty(\mathbb{B})$	$\mathcal{M}_s^p(\mathbb{B})$
IS characterized by L. Carleson	No characterisation	Characterized for $p=2$ and $n-1<2s\leq n$ by A.R.S. and the Pick property
ISM $\Rightarrow$ BLEO by P. Beurling [11]	ISM $\Rightarrow$ BLEO by A. Bernard [10]	ISM $\Rightarrow$ BLEO by E. A. here

In the case of the classical Hardy spaces H^p , whose multiplier algebra is $H^\infty(\mathbb{B})$, we know([4] theorem 5, p. 712 and the lines following it) that if S is interpolating for $H^\infty(\mathbb{B})$ then S is interpolating for $H^p(\mathbb{B})$; this is still true in the case of Hardy Sobolev spaces.

Theorem 1.7 *Let S be an interpolating sequence for the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ then S is also an interpolating sequence for H_s^p, with a bounded linear extension operator.*

In the classical case $s=0$ i.e. $\mathcal{M}_0^p=H^\infty(\mathbb{B})$, $H_0^p=H^p(\mathbb{B})$, N. Varopoulos [24] proved that S interpolating in $H^\infty(\mathbb{B})$ implies that S is Carleson in $\mathbb{B}$ and P. Thomas [22] (see also [7]) proved that S interpolating in $H^p(\mathbb{B})$ implies that S is Carleson in $\mathbb{B}$. The next results generalise this fact to $\mathcal{M}_s^p$ for $p\leq 2$ and any real values of $s\in[0,n/p]$.

Theorem 1.8 *Let S be an interpolating sequence for $\mathcal{M}_s^p$ with $p\leq 2$, then S is Carleson $H_s^p(\mathbb{B})$.*

And

Corollary 1.9 *Let S be an interpolating sequence for H_s^2 with $n-2s\leq 1$, i.e. $\mathcal{M}_s^2$ is a Pick algebra, then S is Carleson for H_r^2, $\forall r\leq s$.*

Because $\mathcal{M}_s^2$ is an operators algebra in a Hilbert space, then we know [3] that the union S of two interpolating sequences in $\mathcal{M}_s^2$ is still interpolating in $\mathcal{M}_s^2$ if S is separated. This generalises a theorem of Varopoulos [23] done for uniform algebras.

We prove the analogous result but we shall have to use a more precise notion of separation, see section 7.

Theorem 1.10 *Let $s\in\mathbb{N}\cap[0,n/p]$ and S_1 and S_2 be two completely separated interpolating sequences in $\mathcal{M}_s^p$ then $S:=S_1\cup S_2$ is still an interpolating sequence in $\mathcal{M}_s^p$.*

In $\mathbb{C}^n$, $n\geq 2$ we know [2] that the union S of two interpolating sequences in $H^2(\mathbb{B})$ is not in general an interpolating sequence even if S is separated, so the next result is in complete opposition to this fact.

Corollary 1.11 *If $\mathcal{M}_s^2$ is a Pick algebra, i.e. if $s\geq\dfrac{n-1}{2}$, and S, S' are two interpolating sequences for H_s^2 such that $S\cup S'$ is separated then $S\cup S'$ is still interpolating for H_s^2.*

We shall need the following notion.

Definition 1.12 *The* **sequence** S *of points in* $\mathbb{B}$ *is* **dual bounded** *(or minimal, or weakly interpolating) in* $H^p_s(\mathbb{B})$ *if there is a bounded sequence* $\{\rho_a\}_{a\in S} \subset H^p_s$ *such that*

$$\forall a, b \in S,\ \rho_a(b) = \delta_{ab}\|k_a\|_{H^{p'}_s}.$$

Clearly if S is interpolating for H^p_s then it is dual bounded in H^p_s.

This notion characterizes interpolating sequences for the classical Hardy spaces in the unit disc $\mathbb{D}$; the question is open for the Hardy spaces $H^p(\mathbb{B})$ in the ball in $\mathbb{C}^n \geq 2$. Nevertheless we know [6] that if $S \subset \mathbb{B}$ is dual bounded for $H^p(\mathbb{B})$ then it is interpolating for $H^q(\mathbb{B}),\ \forall q < p$.

The next results generalise only partially this result and we get an analogous result to theorem 6.1 in [5].

Definition 1.13 *We shall say that* S *is a* H^p_s *weighted interpolating sequence for the weight* $w = \{w_a\}_{a\in S}$ *if*

$$\forall \lambda \in \ell^p(S),\ \exists f \in H^p_s :: \forall a \in S,\ f(a) = \lambda_a w_a \|k_a\|_{H^{p'}_s}.$$

Theorem 1.14 *Let* S *be a sequence of points in* $\mathbb{B}$ *such that, with* $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$, *and* $p \leq 2$,

- S *is dual bounded in* H^p_s.
- S *is Carleson in* $H^q_s(\mathbb{B})$.

Then S *is a* H^r_s *weighted interpolating sequence for the weight* $\{(1-|a|^2)^s\}_{a\in S}$ *with the bounded linear extension property.*

This work was exposed in Oberwolfach workshop "Hilbert Modules and Complex Geometry " in April 2014, and also in the conference in honor of A. Bonami in June 2014 in Orleans, France. This is an improved version of these talks.
This work is presented the following way.

In the next section we study the basics of Hardy Sobolev spaces H^p_s : they make an interpolating scale with respect to s, p ; they have the same type and cotype than L^p spaces.

In section 3 we start the study of the multipliers algebra $\mathcal{M}^p_s$ of H^p_s. We prove that $\mathcal{M}^p_s$ is invariant by the automorphisms of the ball.

In the following section we study Carleson measures and Carleson sequences.

In the following section we study links between p interpolating sequences of vectors in a general Banach space B and Carleson measures and basic sequence in ℓ^p. We study also algebras of operators on B which diagonalize along a sequence of vectors in B. Application to H^p_s are done.

Then, in the next harmonic analysis section, we develop a very useful method due to S. Drury [15], for the union of two Sidon sets, to fit Hardy Sobolev spaces.

The following section contains the results on interpolating sequences of points for the multipliers algebra $\mathcal{M}^p_s$.

In section 8 we study the notion of dual boundedness in the framework of Hardy Sobolev spaces.

Finally in the appendix we put technical lemmas to ease the reading of section 7.

In the sequel we shall deal only with finite sequences of points $S \subset \mathbb{B}$ but with estimates not depending of the number of points in S. The results for infinite sequences is then got by a normal family argument.

2 Hardy Sobolev spaces.

By a result of J. Ortega and J. Fabrega [19], corollary 3.4, (see also E. Ligocka [18]), we have that the Hardy Sobolev spaces H^p_s form a interpolating scale with respect to s and p. This means that for $1 < p_0, p_1 < \infty,\ 0 \leq s_0, s_1,\ 0 < \theta < 1$ and $\dfrac{1}{p} = \dfrac{(1-\theta)}{p_0} + \dfrac{\theta}{p_1},\ s = (1-\theta)s_0 + \theta s_1$, we have

$$(H^{p_0}_{s_0}, H^{p_1}_{s_1})_{[\theta]} = H^p_s. \tag{2.2}$$

We shall use this result in relation with the Banach spaces interpolation method. In particular we shall prove results essentially when s is an integer and, by use of it, we shall get the same results for s real.

2.1 Similarity between H^p_s and H^p.

Definition 2.1 *Let S be a sequence, we set $\epsilon := \{\epsilon_a,\ a \in S\} \in \mathcal{R}(S)$, a Rademacher sequence,where the random variables ϵ_a are Bernouilli independent and such that $P(\epsilon_a = 1) = P(\epsilon_a = -1) = 1/2$.*

Let $f(\epsilon, z) \in H^p_s$ for any value of the random variable $\epsilon \in \mathcal{R}(S)$, then :

Lemma 2.2 *We have*
$$\|\mathbb{E}(f)\|^p_{H^p_s} \lesssim \mathbb{E}(\|f\|^p_{H^p_s}).$$

Proof.
We can take as an equivalent norm in H^p_s the sum of the H^p norms of the R^k derivatives, i.e. with $p < \infty$,
$$\|f\|^p_{H^p_s} \simeq \sum_{k=0}^{s} \left\|R^k(f)\right\|^p_{H^p}.$$
Hence, because $\mathbb{E}$ is linear, we have
$$R^k\mathbb{E}(f) = \mathbb{E}(R^k f).$$
On the other hand
$$|\mathbb{E}(g)|^p \leq (\mathbb{E}(|g|))^p \leq \mathbb{E}(|g|^p)$$
hence
$$\|\mathbb{E}(g)\|^p_{H^p} = \int_{\partial\mathbb{B}} |\mathbb{E}(g)|^p\, d\sigma \leq \int_{\partial\mathbb{B}} \mathbb{E}(|g|^p) d\sigma = \mathbb{E}(\int_{\partial\mathbb{B}} |g|^p\, d\sigma) = \mathbb{E}(\|g\|^p_{H^p}).$$
So applying this with $g = R^k f$ we get
$$\left\|R^k(\mathbb{E}(f))\right\|^p_{H^p} = \left\|\mathbb{E}(R^k(f))\right\|^p_{H^p} \leq \mathbb{E}(\left\|R^k(f)\right\|^p_{H^p}),$$

and

$$\|\mathbb{E}(f)\|_{H_s^p}^p \lesssim \mathbb{E}(\|f\|_{H_s^p}^p).$$ ■

Proposition 2.3 *The spaces H_s^p have, for any $s \in \mathbb{N}$, the same type as H^p.*

Proof.
We can prove it by use of the fact that the Sobolev spaces W_k^p have this property by [14], but because the problem is on the boundary of the ball which is not isotropic with respect to the derivatives, we shall prove it directly.

So let $p \leq 2$ we want to prove that H_s^p is of type p, which means, with $\epsilon \in \mathcal{R}(1,...,N)$ a Rademacher sequence and $\mathbb{E}$ the expectation,

$$(\mathbb{E}(\left\|\sum_{j=1}^N \epsilon_j f_j\right\|_{H_s^p}^2))^{1/2} \leq T_p(\sum_{j=1}^N \|f_j\|_{H_s^p}^p)^{1/p}.$$

We can take as a norm in H_s^p the sum of the H^p norms of the R^k derivatives, hence, because $\mathbb{E}$ is linear, it suffices to have

$$(\mathbb{E}(\left\|\sum_{j=1}^N \epsilon_j R^k(f_j)\right\|_{H^p}^2))^{1/2} \leq T_p(\sum_{j=1}^N \|f_j\|_{H_s^p}^p)^{1/p}.$$

But H^p being a subspace of $L^p(\partial\mathbb{B})$, it is already of type p hence

$$(\mathbb{E}(\left\|\sum_{j=1}^N \epsilon_j R^k(f_j)\right\|_{H^p}^2)^{1/2} \leq T_p(\sum_{j=1}^N \left\|R^k(f_j)\right\|_{H^p}^p)^{1/p},$$

So, because $f \in H_s^p$ implies

$$\forall k \leq s,\ R^k(f) \in H^p,\ \left\|R^k(f)\right\|_{H^p} \leq \|f\|_{H_s^p},$$

we get

$$(\mathbb{E}(\left\|\sum_{j=1}^N \epsilon_j R^k(f_j)\right\|_{H^p}^2))^{1/2} \leq T_p(\sum_{j=1}^N \|f_j\|_{H_s^p}^p)^{1/p},$$

and, adding a finite number of terms, we get

$$(\mathbb{E}(\left\|\sum_{j=1}^N \epsilon_j f_j\right\|_{H_s^p}^2))^{1/2} \leq (s+1)T_p(\sum_{j=1}^N \|f_j\|_{H_s^p}^p)^{1/p}.$$

If $p > 2$, then the dual space of H_s^p is $H_s^{p'}$ with $p' < 2$ hence $H_s^{p'}$ is of type p' ; this implies that the dual of $H_s^{p'}$, namely H_s^p is of cotype p. ■

Using it we get the following theorem.

Theorem 2.4 *The spaces H_s^p have, for any $s \in \mathbb{R}_+$, the same type as H^p.*

Proof.
Fix $N \in \mathbb{N}$ and consider the space $(H_s^p)^N$ with the following ℓ^p norm :

$$\forall f = (f_1,...,f_N) \in (H_s^p)^N,\ \|f\|_p := (\sum_{j=1}^N \|f_j\|_{H_s^p}^p)^{1/p}.$$

Consider the linear operator $T\ :\ \mathcal{R}(1,...,N)\times(H_s^p)^N \to H_s^p$ defined by

$$\forall f = (f_1, ..., f_N) \in (H_s^p)^N,\ T_N(\epsilon, f) := \sum_{j=1}^{N} \epsilon_j f_j \in H_s^p.$$

To say that H_s^p is of type p means that, for any $N \geq 1$,

$$(\mathbb{E}(\|T_N(\epsilon, f)\|_{H_s^p}^2))^{1/2} \leq C\|f\|_p,$$

i.e. the linear operator T_N is bounded from $F_s := (H_s^p)^N$ equipped with the norm $\|\cdot\|_p$ to $L^2(\Omega, H_s^p)$, the space $L^2(\Omega, \mathcal{A}, P)$ with value in H_s^p. Because the H_s^p form an interpolating scale with respect to the parameter $s \in \mathbb{R}_+$, we have the same for the spaces F_s and $L^2(\Omega, H_s^p)$.

Fix $p \leq 2$ and $s \in \mathbb{N}$; by the proposition 2.3 we have that T_N is bounded from F_s to $L^2(\Omega, H_s^p)$, and from F_0 to $L^2(\Omega, H_0^p)$, the constant being independent of $N \in \mathbb{N}$, hence by interpolation T_N is bounded from F_r to $L^2(\Omega, H_r^p)$, for any $0 \leq r \leq s$, with a constant independent of $N \in \mathbb{N}$. This proves that H_r^p is of type p for any real $r \in [0, s]$. By duality as in proposition 2.3 we have that for $p > 2$, H_s^p is of cotype p. ■

Up to a constant, we have the Young inequalities.

Proposition 2.5 *We have, with* $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$,

$$\forall f \in H_s^p,\ \forall g \in H_s^q,\ fg \in H_s^r \text{ and } \|fg\|_{H_s^r} \leq C_s\|f\|_{H_s^p}\|g\|_{H_s^q}.$$

Proof.

We have to compute the H^p norm, for $j = 0, ..., s$, of, by Leibnitz formula,

$$R^j(fg) = \sum_{k=0}^{j} C_j^k R^k(f) R^{(j-k)}(g). \tag{2.3}$$

By Minkowski inequality it is enough to control the norm of

$$R^k(f)R^{(j-k)}(g).$$

But by Young inequality

$$\left\|R^k(f)R^{(j-k)}(g)\right\|_{H^r} \leq \left\|R^k(f)\right\|_{H^p}\left\|R^{(j-k)}(g)\right\|_{H^q}.$$

Now $f \in H_s^p$ implies

$$\forall k \leq s,\ R^k(f) \in H^p,\ \left\|R^k(f)\right\|_{H^p} \leq \|f\|_{H_s^p}.$$

The same $g \in H_s^q$ implies

$$\forall k \leq s,\ R^k(g) \in H^p,\ \left\|R^k(g)\right\|_{H^q} \leq \|g\|_{H_s^q}.$$

So

$$\forall j = 0, ..., s,\ \forall k \leq j,\ \left\|R^k(f)R^{(j-k)}(g)\right\|_{H^r} \leq \|f\|_{H_s^p}\|g\|_{H_s^q}.$$

Because we have a finite number of terms in (2.3) we get the existence of a constant $C_s > 0$ such that

$$\|fg\|_{H_s^r} \leq C_s\|f\|_{H_s^p}\|g\|_{H_s^q},$$

which proves the proposition. ■

3 The multipliers algebra of H_s^p.

Recall that the multipliers algebra $\mathcal{M}_s^p$ of H_s^p is the algebra of functions m on $\mathbb{B}$ such that $\forall h \in H_s^p,\ mh \in H_s^p$, and its norm is its norm as an operator from H_s^p into H_s^p.

As an easy corollary of the interpolating result (2.2), we get the following theorem.

Theorem 3.1 *We have the embedding :*
$\mathcal{M}_s^p \subset \mathcal{M}_r^p$, *for* $1 < p < \infty$ *and* $0 \leq r \leq s$, *with* $\forall m \in \mathcal{M}_s^p,\ \|m\|_{\mathcal{M}_r^p} \leq \|m\|_{\mathcal{M}_s^p}$.

Proof.
Let $m \in \mathcal{M}_s^p$ then m is also in $\mathcal{M}_0^p = H^\infty(\mathbb{B})$ which means that m is a bounded operator on H_s^p and on H_0^p. Hence m is bounded on H_r^p for any $r \in [0, s]$, by Banach spaces interpolation (2.2), which means that $m \in \mathcal{M}_r^p$. Moreover we have $\|m\|_{H^\infty(\mathbb{B})} \leq \|m\|_{\mathcal{M}_s^p}$ hence $\|m\|_{\mathcal{M}_r^p} \leq \|m\|_{\mathcal{M}_s^p}$. ■

3.1 Invariance by automorphisms.

Let $e_a(z) := \dfrac{(1-|a|^2)^{\rho/2}}{(1-\bar{a}\cdot z)^\rho},\ \rho := n - 2s$, the normalized reproducing kernel for the point $a \in \mathbb{B}$ in H_s^2.

We shall show the following theorem which is true for any $s \in \mathbb{R}_+$.

Theorem 3.2 *Let* φ *be an automorphism of the ball* $\mathbb{B}$; *for any* $a \in \mathbb{B}$, *there is a number* $\eta(\varphi, a)$ *of modulus one such that, setting* $U(\varphi)e_a := \eta(\varphi, a)e_{\varphi(a)}$, $U(\varphi)$ *extends as an unitary representation of* $\mathrm{Aut}(\mathbb{B})$ *in* $\mathcal{L}(H_s^2)$.

Proof.
We shall adapt the proof of theorem 2 p. 35 in [3]. We know that $\mathrm{Aut}(\mathbb{B})$ is isomorphic to $U(n,1)$ the group of isometries for the sesquilinear form of $\mathbb{C}^{n+1}$:

$$(z,w) := \sum_{j=1}^{n} z_j\bar{w}_j - z_0\bar{w}_0.$$

Let $T \in U(n,1)$; in the canonical basis of $\mathbb{C}^{n+1}$ its matrix $[T]$ can be written by blocs :

$$[T] = \begin{bmatrix} A & B \\ C & D \end{bmatrix};$$

where A is a $n\times n$ matrix, B is $n\times 1$, C is $1\times n$ and D is 1×1. The automorphism associated to T is then

$$\forall z \in \mathbb{B},\ \varphi(z) := \frac{AZ+B}{CZ+D}, \text{ where } Z = (z_1, ...,\ z_n).$$

If α, β are two vectors in $\mathbb{C}^n$, we denote by $\alpha\cdot\bar{\beta}$ their scalar product ; the scalar product in H_s^2 is still denoted by $\langle\cdot,\cdot\rangle$.
We have

$$\left\langle e_{\varphi(a)},\ e_{\varphi(b)}\right\rangle = \frac{(1-|\varphi(a)|^2)^{\rho/2}(1-|\varphi(b)|^2)^{\rho/2}}{(1-\varphi(a)\cdot\bar{\varphi}(b))^\rho}.$$

But

$$1-\varphi(a)\cdot\overline{\varphi(b)} = 1 - \frac{Aa+B}{Ca+D}\cdot\overline{\left(\frac{Ab+B}{Cb+D},\right)} =$$
$$= \frac{1}{(Ca+D)\overline{(Cb+D)}}[(Ca+D)\overline{(Cb+D)} - (Aa+B)(\overline{Ab+B})].$$

Let (X,t) and (Y,v) two elements in $\mathbb{C}^{n+1}$ and set $\alpha = T(X,t),\ \beta = T(Y,v)$ we get

$$(\alpha,\beta)=(AX+Bt)(\overline{AY+Bv})-(CX+Dt)(\overline{CY+Dv})=X\cdot\bar{Y}-t\bar{v},$$
because T let $(\cdot,\cdot)$ invariant.

Back to the inhomogeneous coordinates $a=X/t,\ b=Y/v$ we get
$$(Ca+D)\overline{(Cb+D)}-(Aa+B)(\overline{Ab+B})=1-a\cdot\bar{b},$$
hence, putting it in $\left\langle e_{\varphi(a)},\ e_{\varphi(b)}\right\rangle$ we get
$$\left\langle e_{\varphi(a)},\ e_{\varphi(b)}\right\rangle=\frac{(Ca+D)^{\rho}}{|Ca+D|^{\rho}}\times\frac{(\overline{Cb+D})^{\rho}}{|Cb+D|^{\rho}}\langle e_a,e_b\rangle.$$
The linear combinations of $\{e_c,\ c\in\mathbb{B}\}$ being dense in H^2_s, we define on them the operator $U(\varphi)$ by
$$U(\varphi)e_a:=\eta(\varphi,a)e_{\varphi(a)},$$
where $\eta(\varphi,a):=\dfrac{(\overline{Ca+D})^{\rho}}{|Ca+D|^{\rho}}$ is of modulus 1.

The previous computation gives
$$\langle U(\varphi)e_a,U(\varphi)e_b\rangle=\langle e_a,e_b\rangle$$
hence $U(\varphi)$ is unitary. Moreover $U(\varphi)$ is a representation of $\mathrm{Aut}(\mathbb{B})$. To see this we have to show that :
$$\forall\psi,\varphi\in\mathrm{Aut}(\mathbb{B}),\ \forall a\in\mathbb{B},\ \eta(\psi\circ\varphi,\ a)=\eta(\psi,\varphi(a))\times\eta(\varphi,a).$$
Setting
$$\varphi(a):=\frac{Aa+B}{Ca+D},\ \psi(b):=\frac{Ab+B}{Cb+D},$$
the computation is easy. ■

Remark 3.3 *We can use equivalently the following identities (Theorem 2.2.2 p. 26 in [21])*
$$1-\varphi(a)\cdot\overline{\varphi(b)}=\frac{(1-|\mu|^2)}{(1-\bar{\mu}\cdot a)}\times\frac{(1-\bar{b}\cdot a)}{(1-\mu\cdot\bar{b})},$$
$$1-|\varphi(a)|^2=\frac{(1-|\mu|^2)(1-|a|^2)}{|(1-\bar{a}\cdot z)|^2},$$
where $\varphi(z)=\varphi_\mu(z)$ *is the automorphism exchanging* μ *and* 0. *In any case we get*
$$\eta(\varphi_\mu,a)=\frac{(1-\bar{\mu}\cdot a)^{\rho}}{|1-\bar{\mu}\cdot a|^{\rho}}.$$

Corollary 3.4 *The space of multipliers* $\mathcal{M}^2_s$ *of* H^2_s *is invariant by* $\mathrm{Aut}(\mathbb{B})$.

Proof.
Let $m\in\mathcal{M}^2_s$ then we have
$$\forall a\in\mathbb{B},\ m^*k_a=\bar{m}(a)k_a$$
because
$$\forall h\in H^2_s,\ \langle h,m^*k_a\rangle=\langle mh,k_a\rangle=m(a)h(a)=m(a)\langle h,k_a\rangle.$$
Hence
$$m^*U(\varphi)e_a=m^*(\eta(\varphi,a)e_{\varphi(a)})=\eta(\varphi,a)m^*e_{\varphi(a)}=\eta(\varphi,a)\overline{m(\varphi(a))}e_{\varphi(a)}.$$
So
$$U^{-1}(\varphi)m^*U(\varphi)e_a=U^{-1}(\varphi)(\eta(\varphi,a)\overline{m(\varphi(a))}e_{\varphi(a)})=\eta(\varphi,a)\overline{m(\varphi(a))}U^{-1}(\varphi)e_{\varphi(a)}.\qquad(3.4)$$

But from $U(\varphi)e_a := \eta(\varphi, a)e_{\varphi(a)}$, we get

$$e_a = U^{-1}Ue_a = \eta U^{-1}e_{\varphi(a)} \Rightarrow U^{-1}e_{\varphi(a)} = \eta^{-1}e_a$$

and putting this in (3.4) we get

$$U^{-1}(\varphi)m^*U(\varphi)e_a = \eta\overline{m(\varphi(a))}\eta^{-1}e_a = \overline{m(\varphi(a))}e_a = (m\circ\varphi)^*e_a.$$

So by the density of the linear combinations of the $\{e_a,\ a \in \mathbb{B}\}$ we get

$$(m\circ\varphi)^* = U^{-1}(\varphi)m^*U(\varphi).$$

Now because $U(\varphi)$ is unitary on H^2_s we have

$$\|(m\circ\varphi)^*\|_{\mathcal{L}(H^2_s)} = \|m^*\|_{\mathcal{L}(H^2_s)} \Rightarrow \|m\circ\varphi\|_{\mathcal{M}^2_s} = \|m\|_{\mathcal{M}^2_s}.$$ ■

4 Carleson measures and Carleson sequences.

Let $Q(\zeta, h) := \{z \in \bar{\mathbb{B}},\ \left|1 - \bar{\zeta}z\right| < h\}$ be the "pseudo ball" centered at $\zeta \in \partial\mathbb{B}$ and of radius $h > 0$.

We shall use the following well known lemma.

Lemma 4.1 *If μ is a Carleson measure for H^p_s, then $\mu(Q(\zeta, h)) \lesssim h^{n-sp} = |Q(\zeta, h) \cap \partial\mathbb{B}|^{1-p\frac{s}{n}}$.*

Proof.

Because μ is a Carleson measure for H^p_s, we have $\displaystyle\int_{\mathbb{B}} |k_a(z)|^p\, d\mu \lesssim \|k_a\|^p_{s,p}$;

recall that $k_a(z) = \dfrac{1}{(1-\bar{a}z)^\rho}$ with $\rho = n - 2s$, then we get, with

$$Q_a := Q(\frac{a}{|a|}, 1-|a|) \iff \{z \in \mathbb{B} :: |1 - \bar{a}\cdot z| < h\},\ h := (1-|a|),$$

that

$$\int_{Q_a} \left|\frac{1}{(1-\bar{a}z)^\rho}\right|^p d\mu \le \int_{\mathbb{B}} |k_a(z)|^p\, d\mu \lesssim \|k_a\|^p_{s,p}\ ;$$

hence

$$\frac{1}{h^{\rho p}}\mu(Q_a) \lesssim \|k_a\|^p_{s,p} \simeq (1-|a|^2)^{-\rho p - sp + n} \Rightarrow \mu(Q_a) \lesssim h^{n-sp}.$$ ■

Let us recall the definitions of Carleson sequences.

Definition 4.2 *The sequence S is Carleson in $H^p_s(\mathbb{B})$, if the associated measure*

$$\nu_S := \sum_{a\in S} (1-|a|^2)^{n-sp}\delta_a$$

is Carleson for $H^p_s(\mathbb{B})$.

At this point we notice that the coefficients of the measure ν_S depend on the parameter s.

Lemma 4.3 *Let S be sequence in $\mathbb{B}$ which is Carleson for H^p_s and for $H^p_0 = H^p$ then S is Carleson for H^p_r, $0 \le r \le s$.*

Proof.

Consider the linear operator

$$T\ :\ H^p_r \to \ell^p(w_r),\ Tf := \{f(a)\}_{a\in S}$$

with the weight $w_r(a) := (1-|a|^2)^{n-pr}$. Because S is Carleson H^p_s we have that T is bounded from H^p_s to $\ell^p(w_s)$, i.e.

$$\sum_{a\in S}(1-|a|^2)^{n-ps}\,|f(a)|^p \lesssim \|f\|_{H^p_s}.$$

The same for $s=0$, i.e.

$$\sum_{a\in S}(1-|a|^2)^{n}\,|f(a)|^p \lesssim \|f\|_{H^p}$$

hence, because $w_\zeta(a) = (1-|a|^2)^{n-p\zeta}$ is holomorphic in the strip $0 \le \Re\zeta \le s$ and the scale of $\{H^p_s\}_{s>0}$ forms an interpolating scale by the interpolating result (2.2), we have that T is bounded from H^p_r to $\ell^p(w_r)$ which means exactly that S is Carleson for $H^p_r,\ 0 \le r \le s$. ■

If μ is a Carleson measure for H^p_s, then it is a Carleson measure for $H^p_r,\ \forall r \ge s$, simply because $\|f\|_{H^p_s} \le \|f\|_{H^p_r}$. For the Carleson sequences, this goes the opposite way.

Theorem 4.4 *If the sequence S is Carleson in $H^p_s(\mathbb{B})$, then S is Carleson in H^p_r for all $r,\ 0 \le r \le s$.*

Proof.
We first show that the measure $\mu := \sum_{a\in S}(1-|a|^2)^n\delta_a$ is Carleson V^1, i.e. that

$$\sum_{a\in S\cap Q(\zeta,h)}(1-|a|^2)^n \lesssim h^n.$$

For this we have that ν_S Carleson in $H^p_s(\mathbb{B})$ implies that ν_S is finite, just using lemma 4.1 with $Q_a \supset \mathbb{B}$. So we have $\sum_{a\in S}(1-|a|^2)^{n-sp} \le C$. Now still with lemma 4.1 we get

$$\sum_{a\in S\cap Q(\zeta,h)}(1-|a|^2)^{n-sp} = \mu(Q(\zeta,h)) \lesssim h^{n-sp}.$$

But $a \in Q(\zeta,h) \Rightarrow (1-|a|^2) < h$ hence, with

$$(1-|a|^2)^n = (1-|a|^2)^{sp}(1-|a|^2)^{n-sp} \le h^{sp}(1-|a|^2)^{n-sp}$$

we get

$$\sum_{a\in S\cap Q(\zeta,h)}(1-|a|^2)^n \le h^{sp}\sum_{a\in S\cap Q(\zeta,h)}(1-|a|^2)^{n-sp} = h^{sp}\mu(Q(\zeta,h)) \lesssim h^n.$$

This is valid for all $Q(\zeta,h)$ so we get that the measure $\mu := \sum_{a\in S}(1-|a|^2)^n\delta_a$ is Carleson V^1, or, equivalently Carleson $H^p := H^p_0$.

Now we apply lemma 4.3 to end the proof of the theorem. ■

5 General results

We shall establish a link between Carleson sequences and sequences like canonical basis of ℓ^p. Let B be a Banach space, B' its dual.

Definition 5.1 *We say that the sequence of bounded vectors $\{e_a\}_{a\in S}$ in B is equivalent to a canonical basis of ℓ^p if*

$$\exists B_p > 0,\ \forall \lambda \in \ell^p(S),\ \frac{1}{B_p}\|\lambda\|_{\ell^p} \leq \left\|\sum_{a\in S}\lambda_a e_a\right\|_B \leq B_p C\|\lambda\|_{\ell^p}.$$

Definition 5.2 *We say that the sequence of bounded vectors $\{e_a\}_{a\in S}$ in B is p interpolating if*
$\exists I_p > 0,\ \forall \mu \in \ell^{p'}(S),\ \exists h \in B',\ \|h\|_{B'} \leq I_p\|\mu\|_{\ell^{p'}} :: \forall a \in S,\ \langle h, e_a\rangle = \mu_a.$

Definition 5.3 *We say that the sequence of bounded vectors $\{e_a\}_{a\in S}$ in B is dual bounded if*
$\exists C > 0,\ \exists\{\rho_a\}_{a\in S} \subset B' :: \forall a \in S,\ \|f_a\|_{B'} \leq C,\ \langle \rho_a, e_b\rangle = \delta_{ab}.$

Clearly if $\{e_a\}_{a\in S}$ is p interpolating then it is dual bounded : just interpolate the basic sequence of $\ell^p(S)$.

Definition 5.4 *We say that the sequence of bounded vectors $\{e_a\}_{a\in S}$ in B is p Carleson if*
$\exists C_p > 0,\ \forall h \in B',\ \sum_{a\in S}|\langle h, e_a\rangle|^{p'} \leq C_p^{p'}\|h\|_{B'}^{p'}.$

We have :

Lemma 5.5 *Let $\{e_a\}_{a\in S}$ be a sequence in B of bounded vectors, then the following assertions are equivalent :*
(i) $\{e_a\}_{a\in S}$ is p Carleson in B.
(ii) $\{e_a\}_{a\in S}$ verifies $\exists C > 0,\ \forall \lambda \in \ell^p(S),\ \left\|\sum_{a\in S}\lambda_a e_a\right\|_B \leq C\|\lambda\|_{\ell^p(S)}.$

Proof.
Suppose that $\{e_a\}_{a\in S}$ verifies the *(i)* of the lemma, then using the duality $B - B'$ we have
$$\forall \lambda \in \ell^p(S),\ \forall h \in B',\ \left|\sum_{a\in S}\lambda_a\langle e_a, h\rangle\right| \leq C\|\lambda\|_{\ell^p}\|h\|_{B'}.$$
By the duality $\ell^p - \ell^{p'}$ we get then
$$\sum_{a\in S}|\langle e_a, h\rangle|^{p'} \leq C^{p'}\|h\|_{B'}^{p'},$$
which is the definition of $\{e_a\}_{a\in S}$ p Carleson in B.
Suppose now that $\{e_a\}_{a\in S}$ verifies the *(ii)* of the lemma, this means
$$\sum_{a\in S}|\langle e_a, h\rangle|^{p'} \leq C^{p'}\|h\|_{H_s^{p'}}^{p'}$$
which leads by the duality $\ell^p - \ell^{p'}$ to
$$\forall \lambda \in \ell^p(S),\quad \left|\sum_{a\in S}\lambda_a\langle e_a, h\rangle\right| \leq C\|\lambda\|_{\ell^p}\|h\|_{H_s^{p'}}$$
and with the duality $B - B'$ to the *(i)* of the lemma. ■

Theorem 5.6 *Let $\{e_a\}_{a\in S}$ be a p interpolating sequence for B and suppose moreover that $\{e_a\}_{a\in S}$ is p Carleson for B then $\{e_a\}_{a\in S}$ makes a system equivalent to a canonical basis in ℓ^p.*

Proof.
We have to show that

$$\forall \lambda \in \ell^p,\ \left\| \sum_{a\in S} \lambda_a e_a \right\|_B \simeq \|\lambda\|_{\ell^p}.$$

We have

$$\left\| \sum_{a\in S} \lambda_a e_a \right\|_B = \sup_{\ h\in B',\ \|h\|\leq 1} \left| \sum_{a\in S} \lambda_a \langle e_a, h\rangle \right|$$

but by Hölder

$$\left| \sum_{a\in S} \lambda_a \langle e_a, h\rangle \right| \leq \|\lambda\|_{\ell^p} (\sum_{a\in S} |\langle e_a, h\rangle|^{p'})^{1/p'}$$

and because S is p Carleson we have

$$(\sum_{a\in S} |\langle e_a, h\rangle|^{p'})^{1/p'} \leq C_p \|h\|_{B'},$$

hence

$$\left\| \sum_{a\in S} \lambda_a e_a \right\|_B \leq C_p \|\lambda\|_{\ell^p}.$$

For the other direction we still have

$$\left\| \sum_{a\in S} \lambda_a e_a \right\|_B = \sup_{\ h\in B',\ \|h\|\leq 1} \left| \sum_{a\in S} \lambda_a \langle e_a, h\rangle \right|$$

but, because $\{e_a\}_{a\in S}$ is p interpolating, we can find a $h \in B'$ such that

$$\forall a \in S,\ \langle h, e_a\rangle = \mu_a,\ \|h\|_{B'} \leq I_p \|\mu\|_{\ell^{p'}}.$$

So we get

$$\left\| \sum_{a\in S} \lambda_a e_a \right\|_B \geq \frac{1}{I_p} \left| \sum_{a\in S} \lambda_a \langle e_a, h\rangle \right| = \frac{1}{I_p} \left| \sum_{a\in S} \lambda_a \bar{\mu}_a \right|,$$

and we choose μ such that $\|\mu\|_{\ell^{p'}} = 1$ and $\sum_{a\in S} \lambda_a \bar{\mu}_a = \|\lambda\|_{\ell^p}$.

So we get

$$\left\| \sum_{a\in S} \lambda_a e_a \right\|_B \geq \frac{1}{I_p} \|\lambda\|_{\ell^p}.$$

■

Theorem 5.7 *Let $\{e_a\}_{a\in S}$ makes a system equivalent to the canonical basis in ℓ^p and suppose moreover that :*

$$P_S\ :\ \varphi \in B \to P_S\varphi := \sum_{a\in S} \langle \varphi, \rho_a\rangle e_a$$

is bounded, then $\{e_a\}_{a\in S}$ is p Carleson and p interpolating with a bounded linear extension operator.

Proof.
Because $\{e_a\}_{a\in S}$ makes a system equivalent to a canonical basis in ℓ^p means

$$\forall \lambda \in \ell^p,\ \left\| \sum_{a\in S} \lambda_a e_a \right\|_B \simeq \|\lambda\|_{\ell^p} \tag{5.5}$$

we have in particular that

$$\left\| \sum_{a\in S} \lambda_a e_a \right\|_B \leq C_p \|\lambda\|_{\ell^p} \tag{5.6}$$

which, by lemma 5.5 gives that $\{e_a\}_{a\in S}$ is p Carleson in B.

Suppose first that S is finite, then there is a dual system $\{\rho_a\}_{a\in S}$ in B'. Set

$$\forall \mu \in \ell^{p'}(S),\ h := \sum_{a\in S} \mu_a \rho_a\ ;$$

we have $\langle h, e_b\rangle = \sum_{a\in S} \mu_a \langle \rho_a, e_b\rangle = \mu_b$

hence h interpolates μ. It remains to control its norm. We have

$$P_S\varphi = \sum_{a\in S} \langle \varphi, \rho_a\rangle e_a,\ \|P_S\varphi\|_B \leq C\|\varphi\|_B,$$

and by use of (5.5) we get

$$\|P_S\varphi\|_B \geq \frac{1}{B_p}(\sum_{a\in S} |\langle \varphi, \rho_a\rangle|^p)^{1/p}$$

hence

$$(\sum_{a\in S} |\langle \varphi, \rho_a\rangle|^p)^{1/p} \leq B_p C\|\varphi\|_B \tag{5.7}$$

which means that $\{\rho_a\}_{a\in S}$ is p' Carleson.

Now let us estimate the norm of h

$$\|h\|_{B'} = \left\| \sum_{a\in S} \mu_a\rho_a \right\|_{B'} = \sup_{\varphi\in B,\ \|\varphi\|\leq 1} \left| \sum_{a\in S} \mu_a \langle \rho_a, \varphi\rangle \right|$$

but

$$\left| \sum_{a\in S} \mu_a \langle \rho_a, \varphi\rangle \right| \leq \|\mu\|_{\ell^{p'}} (\sum_{a\in S} |\langle \rho_a, \varphi\rangle|^p)^{1/p}$$

and by (5.7) we get

$$\left| \sum_{a\in S} \mu_a \langle \rho_a, \varphi\rangle \right| \leq \|\mu\|_{\ell^{p'}} B_p C\|\varphi\|_B$$

so we have

$$\|h\|_{B'} \leq B_p C\|\mu\|_{\ell^{p'}}.$$

The bounded linear extension operator is then

$$\mu \in \ell^{p'} \to E(\mu) := \sum_{a\in S} \mu_a\rho_a,\ \|E(\cdot)\|_{\ell^{p'}\to B'} \leq B_p C.$$

Hence we prove the theorem. ■

Remark 5.8 *The fact that P_S is bounded implies that $E_S := \mathrm{Span}(e_a,\ a\in S)$ is complemented in B. Just set :*

$$\forall \varphi \in B,\ \varphi_1 := P_S\varphi \in E_S,\ \varphi_2 := \varphi - \varphi_1.$$

Lemma 5.9 *Let $\{e_a\}_{a\in S}$ be dual bounded and such that $\{\rho_a\}_{a\in S}$ is p' Carleson, then $\{e_a\}_{a\in S}$ is p interpolating with a bounded linear extension operator.*

Proof.
Because $\{\rho_a\}_{a\in S}$ is p' Carleson we have

$$\forall \varphi \in B \subset B'',\ (\sum_{a\in S} |\langle \rho_a, \varphi\rangle|^p)^{1/p} \leq C_{p'} \|\varphi\|_B. \tag{5.8}$$

Now take $\mu \in \ell^{p'}$ and set $h := \sum_{a\in S} \mu_a \rho_a$ we have

$$\|h\|_{B'} = \left\| \sum_{a\in S} \mu_a \rho_a \right\|_{B'} = \sup_{\ \varphi\in B,\ \|\varphi\|\leq 1} \left| \sum_{a\in S} \mu_a \langle \rho_a, \varphi\rangle \right|$$

but

$$\left| \sum_{a\in S} \mu_a \langle \rho_a, \varphi\rangle \right| \leq \|\mu\|_{\ell^{p'}} (\sum_{a\in S} |\langle \rho_a, \varphi\rangle|^p)^{1/p}$$

and by (5.8) we get

$$\left| \sum_{a\in S} \mu_a \langle \rho_a, \varphi\rangle \right| \leq \|\mu\|_{\ell^{p'}} B_p C \|\varphi\|_B$$

so we have

$$\|h\|_{B'} \leq B_p C \|\mu\|_{\ell^{p'}}.$$

The bounded linear extension operator is then

$$\mu \in \ell^{p'} \to E(\mu) := \sum_{a\in S} \mu_a \rho_a,\ \|E(\cdot)\|_{\ell^{p'}\to B'} \leq B_p C.$$

Hence we prove the lemma. ■

5.1 Diagonalizing operators algebras.

Let B be a Banach space and $\{e_a\}_{a\in S}$ be a sequence of bounded vectors in B ; we shall work with operators M such that $M\ :\ B \to B$ is bounded and

$$\forall a \in S,\ Me_a = m_a e_a.$$

Let $\mathcal{A}$ be a commutative algebra of operators on B diagonalizing on $E := \text{Span}\{e_a,\ a \in S\}$, with the norm inherited from $\mathcal{L}(B)$; we shall extend our definition of interpolation to this context.

Definition 5.10 *We say that the sequence of bounded vectors $\{e_a\}_{a\in S}$ in B is interpolating for $\mathcal{A}$ if*

$$\exists A > 0,\ \forall \lambda \in \ell^\infty(S),\ \exists M \in \mathcal{A},\ \|M\|_{\mathcal{L}(E)} \leq A\|\lambda\|_{\ell^\infty} :: \forall a \in S,\ Me_a = \lambda_a e_a.$$

The first general result is in the special case of Hilbert spaces.

Theorem 5.11 *Let H be a Hilbert space, $\{e_a\}_{a\in S}$ be a sequence of normalized vectors in H.*

If $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{A}$ then $\{e_a\}_{a\in S}$ is equivalent to a basic sequence in $\ell^2(S)$.

If $\{e_a\}_{a\in S}$ is equivalent to a basic sequence in $\ell^2(S)$, set $E := \text{Span}\{e_a,\ a \in S\}$ and $\mathcal{D}$ the algebra of operators in $\mathcal{L}(E)$ diagonalizing in $\{e_a\}_{a\in S}$, then $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{D}$.

This theorem was proved in [3], (Proposition 3, p. 17) en route to a characterisation of interpolating sequences in the spectrum of a commutative algebra of operators in $\mathcal{L}(H)$. I shall reprove it here for the reader's convenience.

Proof.
Suppose that $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{A}$, and take $\epsilon \in \mathcal{R}(S)$ a Rademacher sequence. Then $\epsilon \in \ell^\infty(S)$ hence there is an operator $M_\epsilon \in \mathcal{A}$ such that

$$M_\epsilon e_a = \epsilon_a e_a,\ \|M_\epsilon\|_{\mathcal{L}(H)} \leq A.$$

Now consider $h := \sum_{a\in S} h_a e_a \in E \subset H$ we have

$$M_\epsilon h = \sum_{a\in S} \epsilon_a h_a e_a, \text{ and } \|M_\epsilon h\|_H \leq A\|h\|_H$$

so

$$A^2\|h\|_H^2 \geq \mathbb{E}(\|M_\epsilon h\|_H^2) = \sum_{a\in S} |h_a|^2 \|e_a\|_H^2,$$

because the ϵ_a are independent and of mean 0. So we get, the e_a being normalized,

$$\sum_{a\in S} |h_a|^2 \leq A^2\|h\|^2.$$

Because $\epsilon_a^2 = 1$, we get $M_\epsilon M_\epsilon = I_d$ on E, hence by the boundedness of M_ϵ,

$$\forall h \in E,\ h = M_\epsilon(M_\epsilon h) \Rightarrow \|h\|_H \leq A\|M_\epsilon h\|_H$$

hence taking again expectation

$$\|h\|_H^2 \leq A^2\mathbb{E}(\|M_\epsilon h\|^2) = A^2 \sum_{a\in S} |h_a|^2.$$

So we proved

$$\frac{1}{A^2}\sum_{a\in S} |h_a|^2 \leq \|h\|_H^2 \leq A^2 \sum_{a\in S} |h_a|^2,$$

which means that $\{e_a\}_{a\in S}$ is equivalent to a basic sequence in $\ell^2(S)$.

Now suppose that $\{e_a\}_{a\in S}$ is equivalent to a basic sequence in $\ell^2(S)$. This means (see for instance [3]) that there is a bounded operator Q in $\mathcal{L}(E)$, with Q^{-1} also bounded, and an orthonormal system $\{\eta_a\}_{a\in S}$ in E such that

$$\forall a \in S,\ Q\eta_a = e_a.$$

Let $\lambda \in \ell^\infty(S)$ then the diagonal operator $T_\lambda \eta_a := \lambda_a \eta_a$ is bounded on E with $\|T_\lambda\| \leq \|\lambda\|_\infty$. Now set

$$R_\lambda := QT_\lambda Q^{-1},$$

then we get

$$\forall a \in S,\ R_\lambda e_a = QT_\lambda \eta_a = Q\lambda_a \eta_a = \lambda_a e_a$$

hence $R \in \mathcal{D}$ and

$$\|R_\lambda\|_{\mathcal{L}(E)} \leq \|Q\|_{\mathcal{L}(E)}\|Q^{-1}\|_{\mathcal{L}(E)}\|T_\lambda\|_{\mathcal{L}(E)} \leq C\|\lambda\|_\infty$$

hence $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{D}$. ■
Now we shall need a definition.

Definition 5.12 *We shall say that the algebra $\mathcal{A}$ separates the points $\{e_a\}_{a\in S}$ if*

$$\exists C > 0,\ \forall a, b \neq a \in S,\ \exists M_{ab} \in \mathcal{A} :: M_{ab}e_a = e_a,\ M_{ab}e_b = 0 \text{ and } \|M_{ab}\|_{\mathcal{A}} \leq C.$$

Then we have the following remark.

Remark 5.13 *Suppose that $\mathcal{A}$ separates $\{e_a\}_{a\in S}$, this implies easily that, for any finite set S and any $\lambda \in \ell^\infty(S)$ there is a $M \in \mathcal{A}$ such that $\forall a \in S,\ M(a)e_a = \lambda_a e_a$. Hence if there is $C > 0$ such that there is a $M' \in \mathcal{A}$ with $M'_{|E} = M_{|E}$ and $\|M'\|_{\mathcal{A}} \leq C\|M\|_{\mathcal{L}(E)}$ then, as a corollary of theorem 5.11,*

we get that if $\{e_a\}_{a\in S}$ is equivalent to a basic sequence in $\ell^2(S)$, then $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{A}$. We say that $\mathcal{A}$ is a Pick algebra if this property is true for $\mathcal{A}$. This is very well studied in the nice book by Agler and McCarthy [1].

We shall generalise this result to p interpolating sequences.
Let B be a Banach space and $\{e_a\}_{a\in S}$ be a sequence of normalized vectors in B.
Recall that the Banach B is of type p if

$$\exists T_p > 0 :: \forall N \in \mathbb{N},\ \epsilon \in \mathcal{R}(\{1,...,N\}),\ \forall f_1,...,f_N \in B,\ \mathbb{E}(\left\|\sum_{j=1}^N \epsilon_j f_j\right\|^2)^{1/2} \leq T_p(\sum_{j=1}^N \|f_j\|_B^p)^{1/p}.$$

Theorem 5.14 *If $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{A}$:*
if B is of type $p > 1$ then $\{e_a\}_{a\in S}$ is p Carleson.
if B' is of type $p' > 1$, then there is a dual sequence $\{\rho_a\}_{a\in S} \subset B'$ to $\{e_a\}_{a\in S}$ and $\{\rho_a\}_{a\in S}$ is p' Carleson, hence $\{e_a\}_{a\in S}$ is p interpolating for B with a bounded linear extension operator ;

Proof.
Because $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{A}$ we have

$$\forall a \in S,\ \exists M_a \in \mathcal{A} :: M_a e_b = \delta_{ab} e_b,\ \|M_a\|_{\mathcal{A}} \leq A.$$

Now fix $a \in S$ and take $h \in B'$ such that $\langle h, e_a\rangle = 1$. This h exists by Hahn Banach with norm 1 and

$$\langle M_a^* h,\ e_b\rangle = \langle h,\ M_a e_b\rangle = \delta_{ab}\langle h, e_b\rangle = \delta_{ab}.$$

So, setting $\rho_a := M_a^* h$, we get $\rho_a \in B'$, $\langle \rho_a, e_b\rangle = \delta_{ab}$ and $\|\rho_a\|_{B'} \leq A\|h\|_{B'} \leq A$. Doing the same for any $a \in S$ we get that $\{\rho_a\}_{a\in S} \subset B'$ exists hence $\{e_a\}_{a\in S}$ is dual bounded.

Now as above, take $\epsilon \in \mathcal{R}(S)$. Then $\epsilon \in \ell^\infty(S)$ hence there is an operator $M_\epsilon \in \mathcal{A}$ such that

$$M_\epsilon e_a = \epsilon_a e_a,\ \|M_\epsilon\|_{\mathcal{L}(B)} \leq A.$$

By duality, $M_\epsilon^* : B' \to B'$ is such that $M_\epsilon^* \rho_a = \epsilon_a \rho_a$, and $\|M_\epsilon^*\|_{\mathcal{L}(B')} \leq A$; so let

$$\forall \mu \in \ell^{p'},\ h := \sum_{a\in S} \mu_a \rho_a.$$

We have

$$\|M_\epsilon^* h\|_{B'} = \left\|\sum_{a\in S} \epsilon_a \mu_a \rho_a\right\|_{B'}.$$

Using $M_\epsilon M_\epsilon = I_d$, we get

$$\|h\|_{B'} \leq A\|M_\epsilon^* h\|_{B'} = A\left\|\sum_{a\in S} \epsilon_a \mu_a \rho_a\right\|_{B'},$$

hence, taking expectation,

$$\|h\|_{B'} \leq A\mathbb{E}(\left\|\sum_{a\in S} \epsilon_a \mu_a \rho_a\right\|_{B'})$$

so B' of type p' means $\mathbb{E}(\left\|\sum_{a\in S} \epsilon_a \mu_a \rho_a\right\|_{B'}^2)^{1/2} \leq T_{p'}(\sum_{a\in S} |\mu_a|^{p'} \|\rho_a\|_{B'}^{p'})^{1/p'}$, hence

$$\mathbb{E}(\left\|\sum_{a\in S} \epsilon_a \mu_a \rho_a\right\|_{B'}) \leq \mathbb{E}(\left\|\sum_{a\in S} \epsilon_a \mu_a \rho_a\right\|_{B'}^2)^{1/2} \leq T_{p'}(\sum_{a\in S} |\mu_a|^{p'} \|\rho_a\|_{B'}^{p'})^{1/p'},$$

so

$$\|h\|_{B'} \le AT_{p'}(\sum_{a\in S} |\mu_a|^{p'} \|\rho_a\|_{B'}^{p'})^{1/p'} = AT_{p'}\|\mu\|_{\ell^{p'}},$$

which prove that $\{\rho_a\}_{a\in S}$ is p' Carleson hence applying lemma 5.9 we get that $\{e_a\}_{a\in S}$ is p interpolating for B with a bounded linear extension operator.

To get the second part, set $\varphi := \sum_{a\in S} \lambda_a e_a$ and use again

$$\varphi = M_\epsilon(M_\epsilon\varphi) \Rightarrow \|\varphi\|_B \le A\|M_\epsilon\varphi\|_B = A\left\|\sum_{a\in S} \epsilon_a\lambda_a e_a\right\|_{B'},$$

hence, taking expectation,

$\|\varphi\|_B \le A\mathbb{E}(\left\|\sum_{a\in S} \epsilon_a\lambda_a e_a\right\|_B)$

so if B is of type p then again

$$\|\varphi\|_B \le AT_p(\sum_{a\in S} |\lambda_a|^p \|e_a\|_B^p)^{1/p} = AT_p\|\lambda\|_{\ell^p},$$

which prove that $\{e_a\}_{a\in S}$ is p Carleson. ■

5.2 Application to Hardy Sobolev spaces.

Let H_s^p be the Hardy Sobolev space and $\mathcal{M}_s^p$ its multipliers algebra ; let also $S \subset \mathbb{B}$ be a finite sequence of points in $\mathbb{B}$.

Set, for $a \in \mathbb{B}$, $e_a := \dfrac{k_a}{\|k_a\|_{H_s^p}}$ the normalized reproducing kernel in H_s^p for functions in $H_s^{p'}$.

Then we have that

$$\forall m \in \mathcal{M}_s^p,\ \forall a \in \mathbb{B},\ m^*k_a = \overline{m(a)}k_a \Rightarrow m^*e_a = \overline{m(a)}e_a,$$

because

$$\forall h \in H_s^p,\ \langle h, m^*k_a\rangle = \langle mh, k_a\rangle = m(a)h(a) = m(a)\langle h, k_a\rangle.$$

So we have that the adjoint of elements in $\mathcal{M}_s^p$ make an algebra diagonalizing in $\{e_a\}_{a\in S}$ so we can apply the previous results with the diagonalizing algebra $\mathcal{A} := \{m^*,\ m \in \mathcal{M}_s^p\}$ operating on $H_s^{p'}$.

The first thing to know is that H_s^p has the same type and cotype than L^p. We prove it directly in theorem 2.4.

So we have $H_s^p,\ \forall s \in \mathbb{R}_+$, is of type min $(2,p)$ and of cotype max $(2,p)$, hence we can apply theorem 5.14 to get directly, for all real values of $s \in [0, n/p]$,

Theorem 5.15 *If $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{M}_s^p$ then $\{e_a\}_{a\in S}$ is dual bounded and*
because H_s^p is of type min $(2,p)$ *then $\{e_a\}_{a\in S}$ is* min $(2,p)$ *Carleson ;*
because $H_s^{p'}$ of type min $(2,p')$ *then $\{\rho_a\}_{a\in S}$ is* min $(2,p')$ *Carleson, hence $\{e_a\}_{a\in S}$ is p interpolating for H_s^p with a bounded linear extension operator provided that $p \ge 2$.*

In fact we shall prove later on a better result by use of harmonic analysis for the last case : we shall get rid of the condition $p \ge 2$. Nevertheless we have, in the special case $p = 2$, as an application of theorem 5.11, for all real values of $s \in [0, n/p]$, :

Theorem 5.16 *Let $\{e_a\}_{a\in S}$ be a sequence of normalized vectors in H_s^2 ; if $\{e_a\}_{a\in S}$ is interpolating for $\mathcal{M}_s^2$ then $\{e_a\}_{a\in S}$ is equivalent to a basic sequence in $\ell^2(S)$. If $\mathcal{M}_s^2$ is a Pick algebra, i.e. if*

$s \geq \dfrac{n-1}{2}$, *then* $\{e_a\}_{a\in S}$ *equivalent to a basic sequence in* $\ell^2(S)$ *implies that* $\{e_a\}_{a\in S}$ *is interpolating for* $\mathcal{M}_s^2$.

6 Harmonic analysis.

Let S be an interpolating sequence for the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ and recall that the interpolating constant for S is the smallest number $C = C(S)$ such that

$$\forall \lambda \in l^\infty(S),\ \exists m \in \mathcal{M}_s^p :: \forall a \in S,\ m(a) = \lambda_a \text{ and } \|m\|_{\mathcal{M}_s^p} \leq C\|\lambda\|_{l^\infty}.$$

We have easily $\mathcal{M}_s^p \subset H^\infty(\mathbb{B})$ with $\forall m \in \mathcal{M}_s^p,\ \|m\|_{H^\infty(\mathbb{B})} \leq \|m\|_{\mathcal{M}_s^p}$.
We shall develop here a very useful feature introduced by S. Drury [15]. Consider a finite sequence in $\mathbb{B}$ with interpolating constant $C(S)$.
Set $N = \#S \in \mathbb{N},\ S := \{a_1, ..., a_N\} \subset \mathbb{B}$ and $\theta := \exp\frac{2i\pi}{N}$. S interpolating in $\mathcal{M}_s^p$ implies that

$$\forall j = 1, ..., N,\ \exists \beta(j,z) \in \mathcal{M}_s^p :: \forall k = 1, ..., N,\ \beta(j, a_k) = \theta^{jk}$$

and $\forall j = 1, ..., N,\ \|\beta(j,\cdot)\|_{\mathcal{M}_s^p} \leq C(S)$.

Let

$$\gamma(l,z) := \tfrac{1}{N}\textstyle\sum_{j=1}^N \theta^{-jl}\beta(j,z) \in \mathcal{M}_s^p,\ \|\gamma(l,\cdot)\|_{\mathcal{M}_s^p} \leq C(S).$$

this is the Fourier transform, on the group of n^{th} roots of unity, of the function $\beta(\cdot, z)$, i.e.

$$\gamma(l,z) = \hat{\beta}(l,z),$$

the parameter $z \in \mathbb{B}$ being fixed.

We have

$$\gamma(l, a_k) = \frac{1}{N}\sum_{j=1}^N \theta^{-jl}\beta(j, a_k) = \delta_{lk}. \tag{6.9}$$

Hence the $\gamma(l,\cdot)$ make a dual bounded sequence for S, with a norm in $\mathcal{M}_s^p$ bounded by $C(S)$.
We have by Plancherel on this group

$$\sum_{l=1}^N |\gamma(l,z)|^2 = \frac{1}{N}\sum_{j=1}^N |\beta(j,z)|^2. \tag{6.10}$$

Multiplying on both side by $|h|^2$ with $h \in H_s^p(\mathbb{B})$, we get

$$\sum_{l=1}^N |\gamma(l,z)h(z)|^2 = \frac{1}{N}\sum_{j=1}^N |\beta(j,z)h(z)|^2$$

and applying R^j on both sides, recalling R^j operates only on the holomorphic part,

$$\sum_{l=1}^N \bar{\gamma}(l,\cdot)\bar{h}R^j(\gamma(l,\cdot)h) = \frac{1}{N}\sum_{l=1}^N \bar{\beta}(l,\cdot)\bar{h}R^j(\beta(l,\cdot)h)$$

and again $\bar{R}^j$ on both sides

$$\sum_{l=1}^N \left|R^j(\gamma(l,\cdot)h)\right|^2 = \frac{1}{N}\sum_{l=1}^N \left|R^j(\beta(l,\cdot)h)\right|^2. \tag{6.11}$$

Lemma 6.1 *Let* $Q_l(k,z) := \underbrace{\beta * \cdots * \beta(k,z)}_{l \text{ times}}$ *then* $\|Q_l(k,\cdot)\|_{\mathcal{M}^p_s} \leq C(S)^l$ *and hence* $\|Q_l(k,\cdot)\|_{H^\infty(\mathbb{B})} \leq \|Q_l(k,\cdot)\|_{\mathcal{M}^p_s} \leq C(S)^l$.

Proof.
Let $Q_2(k,z) := \beta * \beta(k,z) = \frac{1}{N}\sum_{j=1}^N \beta(j,z)\beta(k-j,z)$ then, because $\mathcal{M}^p_s$ is an Banach algebra, we have

$$\|\beta(j,\cdot)\beta(j-k,\cdot)\|_{\mathcal{M}^p_s} \leq \|\beta(j,\cdot)\|_{\mathcal{M}^p_s}\|\beta(j-k,\cdot)\|_{\mathcal{M}^p_s} \leq C(S)^{2.}$$

Hence by induction we get the lemma. ■

Lemma 6.2 *We have*

$$\sum_{k=1}^N \left|R^j(\gamma(k,\cdot)^l h)\right|^2 = \frac{1}{N}\sum_{k=1}^N \left|R^j(\underbrace{\beta * \beta * \cdots * \beta(k,\cdot)}_{l \text{ times}} h)\right|^2 .$$

Proof.
We have

$$\gamma(k,\cdot)^l = \underbrace{\beta * \beta * \widehat{\cdots * \beta(k,\cdot)}}_{l \text{ times}}$$

hence by Plancherel

$$\sum_{k=1}^N \left|\gamma(k,z)^l\right|^2 = \frac{1}{N}\sum_{k=1}^N \left|\beta * \cdots * \beta\right|^2(k,z)$$

and by lemma 6.1, because the $\mathcal{M}^p_s$ norm is bigger than the $H^\infty(\mathbb{B})$ one,

$$\forall z \in \mathbb{B},\ \sum_{k=1}^N \left|\gamma(k,z)^l\right|^2 \leq C(S)^{2l}. \tag{6.12}$$

Multiplying by $|h|^2$ on both sides, we get

$$\sum_{k=1}^N \left|\gamma(k,\cdot)^l h\right|^2 = \frac{1}{N}\sum_{k=1}^N \left|\beta * \cdots * \beta(k,\cdot)h\right|^2$$

and taking R^j derivatives, which operate only on the holomorphic part

$$\sum_{k=1}^N R^j(\gamma(k,\cdot)^l h)\overline{\gamma(k,\cdot)^l h} = \frac{1}{N}\sum_{k=1}^N R^j(\beta * \cdots * \beta(k,\cdot)h)\overline{\beta * \cdots * \beta(k,\cdot)h}\ ;$$

Now we take $\bar{R}^j$ derivatives on both sides to get the lemma. ■

Let $S := \{a_1, ..., a_N\} \subset \mathbb{B}$ be a finite sequence in $\mathbb{B}$ then we have built the functions $\{\gamma(l,z)\}_{l=1,...,N} \subset \mathcal{M}^p_s$ such that

$$\forall k,l = 1, ..., N,\ \gamma(l,a_k) = \delta_{lk} \text{ and } \|\gamma(l,\cdot)\|_{\mathcal{M}^p_s} \leq C(S)$$

where $C(S)$ is the interpolating constant of the sequence S. Now on we shall also use the notation $\forall a \in S,\ \gamma_a(z) := \gamma(l,z)$ if $a = a_l$ and we call $\{\gamma_a\}_{a\in S}$ the canonical dual sequence for S in $\mathcal{M}^p_s$.

The following proposition will be very useful for the sequel.

Proposition 6.3 *Let $\{\gamma_a\}_{a\in S}$ be the canonical dual sequence for S in $\mathcal{M}_s^p$ then we have*

$$\forall l \geq 1,\ \forall z \in \mathbb{B},\ \sum_{a\in S} |\gamma_a(z)|^{2l} \leq C(S)^{2l},$$

where $C(S)$ is the interpolating constant for S.

Proof.
This is just inequality (6.12) with the new notations ■

7 Interpolating sequences in the multipliers algebra.

We shall generalise theorem 5.15 valid for $p \geq 2$ to all values of $p \geq 1$, but here s must be an integer.

Theorem 7.1 *Let S be an interpolating sequence for $\mathcal{M}_s^p$ and γ_a its canonical dual sequence, then, with e_a the normalised reproducing kernel for the point $a \in \mathbb{B}$ in H_s^p,*

$$\forall \lambda \in l^p(S),\ f := \sum_{a\in S} \lambda_a \gamma_a^l e_a \in H_s^p(\mathbb{B}),\ \|f\|_{s,p} \lesssim \|\lambda\|_p.$$

This means that S is interpolating for H_s^p with the bounded extension property.

Proof.
As usual S is finite hence the series is well defined and we have

$$\forall b \in S,\ f(b) = \lambda_b e_b(b) = \lambda_b \|k_b\|_{H_s^{p'}}$$

because by lemma 8.3 :

$$e_b(z) := \frac{k_b(z)}{\|k_b\|_{H_s^p}} \Rightarrow e_b(b) := \frac{k_b(b)}{\|k_b\|_{H_s^p}} = \frac{(1-|b|^2)^{2s-n}}{(1-|b|^2)^{s-n/p'}} = \|k_b\|_{H_s^{p'}}.$$

This means that f interpolates the right values. So it remains to show that $f \in H_s^p(\mathbb{B}),\ \|f\|_{H_s^p} \leq C\|\lambda\|_p$.

So we have to show that

$$\forall j \leq s,\ \left\|R^j f\right\|_{H^p} \leq C\|\lambda\|_{\ell^p}.$$

Fix $j \leq s$ then

$$R^j(f) = \sum_{a\in S} \lambda_a R^j(\gamma_a^l e_a).$$

By the exclusion proposition 9.3 with $l > s$, hence $m := \min(j,l) = j$, we get

$$R^j(\gamma_a^l e_a) = \sum_{q=0}^{j} A_q \gamma^{l-q} R^j(\gamma^q e_a),$$

because we have at most s terms, it is enough to control sums like :

$$T_1 := \sum_{a\in S} \lambda_a \gamma_a^{l-q} R^j(\gamma_a^q e_a).$$

By Hölder we get

$$|T_1|^p \leq (\sum_{a\in S} |\lambda_a|^p \left|R^j(\gamma_a^q e_a)\right|^p)(\sum_{a\in S} |\gamma_a|^{(l-q)p'})^{p/p'}.$$

Now we have by proposition 6.3, provided that $(l-q)p' \geq 2$,

$$\forall z \in \mathbb{B},\ \sum_{a\in S} |\gamma_a|^{(l-q)p'} \leq C(S)^{(l-q)p'}$$

hence

$$\forall z \in \mathbb{B},\ |T_1|^p \leq C(S)^{(l-q)p'}(\sum_{a\in S} |\lambda_a|^p \left|R^j(\gamma_a^q e_a)\right|^p).$$

So integrating

$$\forall r<1,\ \int_{\partial\mathbb{B}} |T_1(r\zeta)|^p\, d\sigma(\zeta) \leq C(S)^{(l-q)p'} \int_{\partial\mathbb{B}} \sum_{a\in S} |\lambda_a|^p \left|R^j(\gamma_a^q e_a)(r\zeta)\right|^p d\sigma(\zeta),$$

hence

$$\forall r<1,\ \int_{\partial\mathbb{B}} |T_1(r\zeta)|^p\, d\sigma(\zeta) \leq C(S)^{(l-q)p'} \sum_{a\in S} |\lambda_a|^p \int_{\partial\mathbb{B}} \left|R^j(\gamma_a^q e_a)(r\zeta)\right|^p d\sigma(\zeta). \tag{7.13}$$

But we have

$$\gamma_a \in \mathcal{M}_s^p \Rightarrow \gamma_a^q \in \mathcal{M}_s^p \text{ with } \|\gamma_a^q\|_{\mathcal{M}_s^p} \leq \|\gamma_a\|_{\mathcal{M}_s^p}^q,$$

because $\mathcal{M}_s^p$ is a Banach algebra, so

$$\forall j \leq s,\ \left\|R^j(\gamma_a^q e_a)\right\|_{H^p} \leq \|\gamma_a^q\|_{\mathcal{M}_s^p} \|e_a\|_{H_s^p} \leq \|\gamma_a\|_{\mathcal{M}_s^p}^q,$$

because e_a is normalised in H_s^p.
So replacing in (7.13) we get

$$\forall r<1,\ \int_{\partial\mathbb{B}} |T_1(r\zeta)|^p\, d\sigma(\zeta) \leq C(S)^{(l-q)p'} \sum_{a\in S} |\lambda_a|^p \|\gamma_a\|_{\mathcal{M}_s^p}^q.$$

But S being interpolating we get

$$\|\gamma_a\|_{\mathcal{M}_s^p}^q \leq C(S)^q$$

so finally

$$\forall r<1,\ \int_{\partial\mathbb{B}} |T_1(r\zeta)|^p\, d\sigma(\zeta) \leq C(S)^q C(S)^{(l-q)p'} \|\lambda\|_{\ell^p(S)}^p.$$

Adding these s set of sums we get, because $p' \geq 1$,

$$\left\|R^j f\right\|_{H^p} \leq s(\max\ A_q) C(S)^{lp'/p} \|\lambda\|_{\ell^p(S)}$$

and we are done. ■

Now we shall improve theorem 5.15, for all real values of $s \in [0, n/p]$,

Theorem 7.2 *Let S be interpolating for $\mathcal{M}_s^p$ and suppose that $p \leq 2$, then S is Carleson in $H_r^p,\ \forall r \leq s$.*

Proof.
We know, by theorem 5.15, that if S is interpolating for $\mathcal{M}_s^p$ and if $p \leq 2$, then S is Carleson H_s^p. hence we apply theorem 4.4 to get the result. ■

Arcozzi, Rochberg and Sawyer in [8] proved, in particular, that if S is interpolating in $B^p = B_0^p$, where B_σ^p is a Besov space of the ball $\mathbb{B}$, then we have that S is Carleson for B^p. In the case $p = 2,\ B^2 = H_{n/2}^2$, we have a better result.

Corollary 7.3 *Let S be an interpolating sequence for H_s^2 with $n - 2s \leq 1$, then S is Carleson for $H_r^2,\ \forall r \leq s$.*

Proof.
We know that $H_s^2 = B_\sigma^2$ where B_σ^2 is the Besov space of the ball $\mathbb{B}$ and where $\sigma = \dfrac{n}{2} - s$. We know by [1] that for $\sigma \leq 1/2,\ B_\sigma^2$ has Pick kernels hence S interpolating for $H_s^2 = B_\sigma^2$ implies S interpolating for $\mathcal{M}_s^2$ so we can apply theorem 7.2 to get the result. ■

7.1 Union of separated interpolating sequences.

In the case $s = 0,\ \mathcal{M}_0^p = H^\infty(\mathbb{B})$, the union S of two interpolating sequences in $H^\infty(\mathbb{B})$ is still interpolating in $H^\infty(\mathbb{B})$ if S is separated by a theorem of Varopoulos [23]. We shall generalise this fact in the next results.

Definition 7.4 *A sequence S is separated in $\mathcal{M}_s^p$ if there is a $c_S > 0$ such that*
$\forall a, b \neq a \in S,\ \exists m_{a,b} \in \mathcal{M}_s^p :: m_{a,b}(a) = 1,\ m_{a,b}(b) = 0,\ \|m_{a,b}\|_{\mathcal{M}_s^p} \leq c_S.$

Definition 7.5 *A sequence S is strongly separated in $\mathcal{M}_s^p$ if there is a $c_S > 0$ such that*
$\forall a, b \neq a \in S,\ \exists m_{a,b} \in \mathcal{M}_s^p :: m_{a,b}(a) = 1,\ m_{a,b}(b) = 0,$
and

$$\forall h \in H_s^p,\ \forall a \in S,\ \exists H \in H_s^p,\ \|H\|_{H_s^p} \leq c_S\|h\|_{H_s^p} :: \forall b \in S,\ b \neq a,\ \forall j \leq s,\ \left|R^j(m_{a,b}h)\right| \leq \left|R^j(H)\right|.$$

Clearly the strong separation in $\mathcal{M}_s^p$ implies the separation in $\mathcal{M}_s^p$.

Definition 7.6 *The sequences $S_1,\ S_2$ are completely separated in $\mathcal{M}_s^p$ if there is a $c_A > 0$ such that*
$\forall a \in S_1,\ \forall b \in S_2,\ \exists m_{a,b} \in \mathcal{M}_s^p :: m_{a,b}(a) = 1,\ m_{a,b}(b) = 0$
and
$\forall h \in H_s^p,\ \exists H \in H_s^p,\ \|H\|_{H_s^p} \leq c_A\|h\|_{H_s^p} :: \forall a \in S_1, b \in S_2,\ \forall j \leq s,\ \left|R^j(m_{a,b}h)\right| \leq \left|R^j(H)\right|.$

This time the vector H does not depend on a nor on b.

Theorem 7.7 *Let S_1 and S_2 be two interpolating sequences in $\mathcal{M}_s^p,\ s \in \mathbb{N} \cap [0, n/p]$, then $S := S_1 \cup S_2$ is an interpolating sequence in $\mathcal{M}_s^p$ if and only if S_1 and S_2 are completely separated.*

Proof.
Suppose first that $S := S_1 \cup S_2$ is an interpolating sequence in $\mathcal{M}_s^p$ and take $\forall a \in S_1,\ \lambda_a = 1,\ \forall b \in S_2,\ \lambda_b = 0$. Then $\lambda \in \ell^\infty(S)$ hence there is function $m \in \mathcal{M}_s^p$ such that
$\forall a \in S,\ m(a) = \lambda_a$, i.e. $\forall a \in S_1,\ m(a) = 1,\ \forall b \in S_2,\ m(b) = 0.$
Now we choose $\forall a \in S_1,\ \forall b \in S_2,\ m_{a,b} := m$ which works and if we set $\forall h \in H_s^p,\ H := mh$ then we are done with $c_A := \|m\|_{\mathcal{M}_s^p}$, proving that the complete separation is necessary to have $S := S_1 \cup S_2$ interpolating.

Now we suppose we have the complete separation. As usual we suppose S_1, S_2 finite and we set $\{\gamma_a\}_{a\in S_1}$ the canonical dual sequence for S_1 in $\mathcal{M}_s^p$ and $\{\Gamma_b\}_{b\in S_2}$ the canonical dual sequence for S_2 in $\mathcal{M}_s^p$ and we want estimates not depending on the number of points in S_1 and in S_2.
Take $b \in S_2$, then by hypothesis we have
$\forall a \in S_1,\ \exists m_{a,b}(z) \in \mathcal{M}_s^p\ ::\ m_{a,b}(a) = 1,\ m_{a,b}(b) = 0.$
We set, $m_b := \sum_{a\in S_1} \gamma_a^l m_{a,b}$. Then we have $\forall a \in S_1,\ m_b(a) = 1$ and $m_b(b) = 0$.
Because S_1 and S_2 are finite, the functions m_b are in $\mathcal{M}_s^p$ and they verify
$$\forall a \in S_1,\ \forall b \in S_2,\ m_b(z) = \begin{cases} 1 & if\ z = a \\ 0 & if\ z = b \end{cases}.$$
Now we shall glue them by setting

$$m := \sum_{b\in S_2} \Gamma_b^l(1-m_b).$$

We have

$m(a) = 0$ if $a \in S_1$ and $m(b) = 1$ if $b \in S_2$,

hence if $m \in \mathcal{M}_s^p$ with a norm depending only on the constants of interpolation of S_1 and S_2 and of the complete separation, then we shall be done because then :

$\forall \lambda^1 \in \ell^\infty(S_1),\ \forall \lambda^2 \in \ell^\infty(S_2),\ \exists m_j \in \mathcal{M}_s^p,\ \forall c \in S_j,\ m_j(c) = \lambda_c^j,\ j = 1, 2$;

now we set, with $m_j,\ j = 1, 2$ as above,

$M := (1-m)m_1 + mm_2 \in \mathcal{M}_s^p$

because $\mathcal{M}_s^p$ is an algebra, and we get

$\forall a \in S_1,\ M(a) = (1-m(a))m_1(a) + m(a)m_2(a) = m_1(a) = \lambda_a^1$

and

$\forall b \in S_2,\ M(b) = (1-m(b))m_1(b) + m(b)m_2(b) = m_2(b) = \lambda_b^2,$

hence M interpolates the sequence (λ^1, λ^2) on $S_1 \cup S_2$.

In order to have $m \in \mathcal{M}_s^p$, we have to show that

$\forall h \in H_s^p,\ \forall j \le s,\ R^j(mh) \in H^p$ with control of the norms.

We start the same way we did with the linear extension :

$$R^j(mh) = R^j(\sum_{b\in S_2} \Gamma_b^l(1-m_b)h) = R^j(\sum_{b\in S_2} \Gamma_b^l(1 - \sum_{a\in S_1} \gamma_a^l(z)m_{ab}(z))h)$$

so we have two terms

$$T_1 = \sum_{b\in S_2} R^j(\Gamma_b^l h)$$

and

$$T = \sum_{b\in S_2} R^j(\Gamma_b^l(\sum_{a\in S_1} \gamma_a^l(z)m_{ab}(z))h).$$

For T_1 we are exactly in the situation of the linear extension with $\lambda_b = 1,\ \forall b \in S_2$ so we get

$$T_1 \in \mathcal{M}_s^p,\ \|T_1\|_{\mathcal{M}_s^p} \le C(S_2)^l \max_{\ j=0,\dots,s} \sum_{q=0}^{j} |A_q|.$$

Now for T this is more delicate. First we set $h_{ab} := m_{ab}h \in H_s^p$ so we have

$$T = \sum_{b\in S_2} R^j(\Gamma_b^l(\sum_{a\in S_1} \gamma_a^l h_{ab})) = \sum_{a\in S_1,\ b\in S_2} R^j(\Gamma_b^l\gamma_a^l h_{ab}).$$

We have to exit the converging factors $(\gamma_a\Gamma_b)^{l-q}$ by the exclusion proposition 9.3 :

$$R^j(\Gamma_b^l\gamma_a^l h_{ab}) = \sum_{q=0}^{j} A_q(\gamma_a\Gamma_b)^{l-q}R^j((\gamma_a\Gamma_b)^q h_{ab}).$$

Because s is fixed and $j \le s$, we have only less than s terms in the sum and the constants A_q are bounded, hence, up to a finite sum, it is enough to control terms of the forms

$$T_2 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q}\,|\Gamma_b|^{l-q}\left|R^j((\gamma_a\Gamma_b)^q h_{ab})\right|.$$

By the Leibnitz formula we get

$$R^j((\gamma_a\Gamma_b)^q h_{ab}) = \sum_{k=0}^{j} C_j^k R^k((\gamma_a\Gamma_b)^q)R^{j-k}(h_{ab}).$$

But the complete separation assumption gives the domination :

$\left|R^{j-k}(h_{ab})\right| \leq \left|R^{j-k}(H)\right|$ with $H \in H^p_s$, $\|H\|_{H^p_s} \leq C_{S1}\|h\|_{H^p_s}$, and H independent of a, b.
So again up to finite number of terms and bounded constants, we are lead to control terms of the form

$$T_3 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} \left|R^k((\gamma_a\Gamma_b)^q)\right| \left|R^{j-k}(H)\right|.$$

Let $H' := R^{j-k}(H)$, still independent of a and b, we have $H' \in H^p_{s-j+k}$ so

$$T_3 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} \left|R^k((\gamma_a\Gamma_b)^q)\right| |H'|.$$

Now the inclusion lemma 9.5 gives

$$R^k((\gamma_a\Gamma_b)^q)H' = \sum_{m=0}^{k} A_{km} R^m((\gamma_a\Gamma_b)^q R^{k-m}(H'))$$

so again it is enough to deal with terms of the form

$$T_4 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} \left|R^m((\gamma_a\Gamma_b)^q R^{k-m}(H'))\right|.$$

But $H' := R^{j-k}(H)$ hence $H'' := R^{k-m}(H') = R^{j-m}(H)$ with $H \in H^p_s$, so $H'' \in H^p_{s-j+m}$, with $\|H''\|_{H^p_{s-j+m}} \leq C_4\|H\|_{H^p_s}$, still independent of a and b. So we have

$$T_4 = \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} |R^m(\gamma_a^q(\Gamma_b^q H''))|.$$

By the Leibnitz formula again we get

$$R^m(\gamma_a^q(\Gamma_b^q H'')) = \sum_{k=0}^{m} C_k^m R^k(\gamma_a^q) R^{m-k}(\Gamma_b^q H'')$$

hence by the finiteness of the number of terms, it is enough to control terms of the form

$$T_5 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} \left|R^k(\gamma_a^q)\right| \left|R^{m-k}(\Gamma_b^q H'')\right|.$$

But the sequence S_2 is interpolating for $\mathcal{M}^p_s$ hence, still by theorem 3.1 we have that S_2 is interpolating for $\mathcal{M}^p_r$, $\forall r \leq s$ so we can apply the domination lemma 9.4 from the appendix to $R^{m-k}(\Gamma_b^q H'')$:

$$\left|R^{m-k}(\Gamma_b^q H'')\right| \leq \frac{1}{N_2} \sum_{\mu=1}^{N_2} R^{m-k}(H_\mu)$$

with $H_\mu \in H^p_{s-j+m}$, $\|H_\mu\|_{H^p_{s-j+m}} \leq C(S_2)^q \|H''\|_{H^p_{s-j+m}}$ and H_μ independent of a and b.
Because of the $\dfrac{1}{N}$ it is enough to control uniformly in μ terms of the form

$$T_6 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} \left|R^k(\gamma_a^q)\right| \left|R^{m-k}(H_\mu)\right|.$$

We use the inclusion lemma 9.5 to get

$$R^k(\gamma_a^q)R^{m-k}(H_\mu) = \sum_{r=0}^{k} A_{kr} R^r(\gamma_a^q R^{k-r}(R^{m-k}(H_\mu))) = \sum_{r=0}^{k} A_{kr} R^r(\gamma_a^q R^{m-r}(H_\mu)).$$

So it remains to control terms of the form

$$T_7 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} \left|R^r(\gamma_a^q R^{m-r}(H_\mu))\right|.$$

Set $V_\mu := R^{m-r}(H_\mu)$; because $H_\mu \in H^p_{s-j+m}$, we have that $V_\mu \in H^p_{s-j+r}$ with control of its norm, $\|V_\mu\|_{H^p_{s-j+m}} \leq C_7\|H_\mu\|_{H^p_{s-j+m}}$.
So we have

$$T_7 = \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} |R^r(\gamma_a^q V_\mu)|.$$

Now we shall use that S_1 is interpolating for $\mathcal{M}_s^p$ hence, still by theorem 3.1 we have that S_1 is interpolating for $\mathcal{M}_r^p$, $\forall r \leq s$ so we can apply the domination lemma 9.4 :

$$|R^r(\gamma_a^q V_\mu)| \leq \frac{1}{N_1} \sum_{\nu=1}^{N_1} |R^r(H_{\nu\mu})|$$

with $1 \leq \nu \leq N$, $\|H_{\nu\mu}\|_{H^p_{s-j+r}} \leq C(S_1)^q \|V_\mu\|_{H^p_{s-j+r}}$ and $H_{\nu\mu}$ not depending on $a \in S_1$ nor on $b \in S_2$.

So, because of the $\frac{1}{N_1}$ we need to control uniformly in ν, terms of the form

$$T_8 := \sum_{a\in S_1, b\in S_2} |\gamma_a|^{l-q} |\Gamma_b|^{l-q} |R^r(H_{\nu\mu})|.$$

But now we use proposition 6.3 which tells us for $l - q \geq 2$:

$$\sum_{a\in S_1} |\gamma_a|^{l-q} \leq C(S_1)^{l-q}$$

and the same for S_2

$$\sum_{b\in S_2} |\Gamma_b|^{l-q} \leq C(S_2)^{l-q}.$$

Hence porting in T_8

$$T_8 \leq (C(S_1)C(S_2))^{l-q} |R^r(H_{\nu\mu})|.$$

Now taking the H^p norm we get

$$\|T_8\|_{H^p} \leq (C(S_1)C(S_2))^{l-q} \|R^r(H_{\nu\mu})\|_{H^p}$$

but recall that

$$\|H_{\nu\mu}\|_{H^p_{s-j+r}} \leq C(S_1)^q \|V_\mu\|_{H^p_{s-j+r}} \Rightarrow \|R^r(H_{\nu\mu})\|_{H^p} \leq C(S_1)^q \|V_\mu\|_{H^p_{s-j+r}}$$

and

$$\|V_\mu\|_{H^p_{s-j+m}} \leq C_7 \|H_\mu\|_{H^p_{s-j+m}}$$

and

$$\|H_\mu\|_{H^p_{s-j+m}} \leq C(S_2) \|H''\|_{H^p_{s-j+m}}$$

and

$$\|H''\|_{H^p_{s-j+m}} \leq C_4 \|H\|_{H^p_s}$$

and

$$\|H\|_{H^p_s} \leq C_A \|h\|_{H^p_s},$$

so concatenating we get

$$\|T_8\|_{H^p} \leq C_A C_4 C_7 (C(S_1)C(S_2))^l \|h\|_{H^p_s}$$

and the proof is complete. ■

8 Dual boundedness and interpolating sequences in H^p_s.

The Sobolev embedding theorem gives, in $\mathbb{R}^n$,

$$f \in W^p_s(\mathbb{R}^n) \Rightarrow f \in L^q(\mathbb{R}^n),\ \frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

Here we are on the manifold $\partial\mathbb{B}$ which is of dimension $2n-1$, and with complex tangential derivatives of order $2s$ and normal conjugate derivative of order s.

Thanks to Folland and Stein [16], theorem 2, which we iterate and which we apply with $\alpha = 0$ or by use of Romanovskii [20], theorem 7, we have a Sobolev anisotropic embedding in Heisenberg group, which is also a representation of the boundary of the ball $\mathbb{B}$:

$$f \in H_s^p(\mathbb{B}) \Rightarrow f \in H^q(\mathbb{B}),\ \frac{1}{q} = \frac{1}{p} - \frac{s}{n}. \tag{8.14}$$

Theorem 8.1 *Let $S \subset \mathbb{B}$ be a dual bounded sequence for $H_s^p(\mathbb{B})$, then S is dual bounded for $H^q(\mathbb{B})$ with* $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$.

Proof.
Saying S dual bounded in $H_s^p(\mathbb{B})$ means, with $k_{s,a}$ the reproducing kernel for $H_s^2(\mathbb{B})$,

$$\exists C > 0,\ \forall a \in S,\ \exists \rho_a \in H_s^p(\mathbb{B}) :: \rho_a(b) = \delta_{a,b} \|k_{s,a}\|_{s,p},\ \|\rho_a\|_{s,p} \leq C.$$

But by use of anisotropic Sobolev embeddings (8.14) we have, with $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$,

$$\exists C > 0,\ f \in H_s^p(\mathbb{B}) \Rightarrow f \in H^q(\mathbb{B}),\ \|f\|_q \leq C \|f\|_{s,p}.$$

On the other hand we have

$$\|k_{s,a}\|_{s,p} = (1 - |a|^2)^{s - \frac{n}{p'}},$$

hence with $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, we get

$$\|k_{0,a}\|_{q'} = (1 - |a|^2)^{-\frac{n}{q}} = (1 - |a|^2)^{-n(\frac{1}{p} - \frac{s}{n})} = (1 - |a|^2)^{s - \frac{n}{p}} = \|k_{s,a}\|_{s,p'}.$$

So we have a dual sequence for S in $H^q(\mathbb{B})$, namely $\{\rho_a\}_{a \in S}$ itself, doing

$$\exists C > 0,\ \forall a \in S,\ \exists \rho_a \in H^q(\mathbb{B}) :: \rho_a(b) \simeq \delta_{a,b} \|k_a\|_q,\ \|\rho_a\|_q \leq C,$$

which means that S is dual bounded in $H^q(\mathbb{B})$. ■

S interpolating for $H_s^p(\mathbb{B})$ means

$$\forall \lambda \in l^p(S),\ \exists f \in H_s^p(\mathbb{B}) :: \forall a \in S,\ f(a) = \lambda_a \|k_a\|_{s,p'}$$

so we have $f \in H^q(\mathbb{B}),\ \|f\|_q \leq C \|f\|_{s,p}$ such that

$$\forall a \in S,\ f(a) = \lambda_a \|k_a\|_q$$

hence we interpolate $l^p(S)$ sequences in $H^q(\mathbb{B})$ for $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, but *not* $\ell^q(S)$ sequences so this is *not* the $H^q(\mathbb{B})$ interpolation !

Corollary 8.2 *Let $S \subset \mathbb{B}$ be a dual bounded sequence for $H_s^p(\mathbb{B})$, then S is Carleson for $H^p(\mathbb{B})$.*

Proof.
This is exactly the result in [7], because S is dual bounded in $H^q(\mathbb{B})$ hence Carleson for all $H^r(\mathbb{B})$. ■

The first structural hypothesis (see [5]) is true for these spaces :

Lemma 8.3 *we have*

$$\forall r > 1,\ k_a(a) = \|k_a\|_{H_s^2}^2 = \|k_a\|_{H_s^r} \|k_a\|_{H_s^{r'}}.$$

Proof.
We have

$$\|k_a\|_{H^p_s} = (1-|a|^2)^{s-n/p'}$$

and

$$\|k_a\|_{H^r_s}\|k_a\|_{H^{r'}_s} = (1-|a|^2)^{s-n/r'}(1-|a|^2)^{s-n/r} = (1-|a|^2)^{2s-n}$$

hence

$$\|k_a\|_{H^2_s} = (1-|a|^2)^{s-n/2} \Rightarrow \|k_a\|^2_{H^2_s} = (1-|a|^2)^{2s-n}$$

which proves the lemma. ■

Recall that we shall say that S is a H^p_s weighted interpolating sequence for the weight $w = \{w_a\}_{a\in S}$ if

$$\forall \lambda \in \ell^p(S),\ \exists f \in H^p_s :: \forall a \in S,\ f(a) = \lambda_a w_a \|k_a\|_{H^{p'}_s}.$$

By use of lemma 8.3 we get

Theorem 8.4 *Let $p > 1$ and suppose that S is dual bounded in H^p_s, then S is a H^1_s weighted interpolating sequence for the weight $\{(1-|a|^2)^s\}_{a\in S}$ with the bounded linear extension property.*

Proof.
Consider the dual sequence $\{\rho_a\}_{a\in S}$ in H^p_s, given by the dual boundedness, it verifies

$$\exists C > 0,\ \forall a \in S,\ \|\rho_a\|_{H^p_s} \le C,\ \forall b \in S,\ \rho_a(b) = \delta_{ab}\|k_a\|_{H^{p'}_s}.$$

Let, for $\lambda \in \ell^1(S)$,

$$h := \sum_{a\in S} \lambda_a \rho_a \frac{k_a}{\|k_a\|_{H^{p'}_s}},$$

we have

$$h(a) = \lambda_a k_a(a) = \lambda_a (1-|a|^2)^{2s-n}$$

which is the right value. As its norm we get

$$\|h\|_{H^1_s} \le \sum_{a\in S} |\lambda_a| \left\| \rho_a \frac{k_a}{\|k_a\|_{H^{p'}_s}} \right\|_{H^1_s} \le C\|\lambda\|_{\ell^1},$$

because, using proposition 2.5 we get

$$\left\| \rho_a \frac{k_a}{\|k_a\|_{H^{p'}_s}} \right\|_{H^1_s} \le C_s \|\rho_a\|_{H^p_s} \left\| \frac{k_a}{\|k_a\|_{H^{p'}_s}} \right\|_{H^{p'}_s} \le C_s \|\rho_a\|_{H^p_s} \le C_s C.$$ ■

For the second structural hypothesis we have

Lemma 8.5 *Let $p, r \in [1,\infty]$ and q such that $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$ then we have*

$$\|k_a\|_{H^{r'}_s} \simeq (1-|a|^2)^{-s} \|k_a\|_{H^{p'}_s} \|k_a\|_{H^{q'}_s}.$$

Proof.
We have

$$\|k_a\|_{H^{r'}_s} \simeq (1-|a|^2)^{s-n/r}$$

and

$$\|k_a\|_{H^{p'}_s}\|k_a\|_{H^{q'}_s} \simeq (1-|a|^2)^{s-n/p}(1-|a|^2)^{s-n/q} = (1-|a|^2)^{2s-n/r}$$

hence

$$\|k_a\|_{H^{r'}_s} \simeq (1-|a|^2)^{-s} \|k_a\|_{H^{p'}_s} \|k_a\|_{H^{q'}_s}$$

which proves the lemma. ■

Now we are in position to get an analogous result to theorem 6.1 in [5] by an analogous proof.

Theorem 8.6 *Let S be a sequence of points in $\mathbb{B}$ such that, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $p \leq 2$,*

- *S is dual bounded in H^p_s.*
- *S is Carleson in $H^q_s(\mathbb{B})$.*

Then S is a H^r_s weighted interpolating sequence for the weight $\{(1-|a|^2)^s\}_{a\in S}$ with the bounded linear extension property.

Proof.

Consider the dual sequence $\{\rho_a\}_{a\in S}$ in H^p_s, given by the hypothesis, it verifies

$$\exists C > 0,\ \forall a \in S,\ \|\rho_a\|_{H^p_s} \leq C,\ \forall b \in S,\ \rho_a(b) = \delta_{ab}\|k_a\|_{H^{p'}_s}.$$

Now we set

$$h(z) := \sum_{a\in S} \lambda_a \rho_a (1-|a|^2)^s \frac{k_a(z)}{\|k_a\|_{H^{p'}_s}\|k_a\|_{H^r_s}}.$$

We get

$$\forall a \in S,\ h(a) = \lambda_a (1-|a|^2)^s \rho_a(a) \frac{k_a(a)}{\|k_a\|_{H^{p'}_s}\|k_a\|_{H^r_s}}$$

and using the first structural hypothesis, lemma 8.3, we get

$$h(a) = \lambda_a (1-|a|^2)^s \|k_a\|_{H^{r'}_s},$$

hence h interpolates the correct values.

Clearly h is linear in λ, and it remains to estimate the norm of h.

Proof of the estimates.

In order to do this, we proceed as in [5] :

let $\{\epsilon_a\}_{a\in S} \in \mathcal{R}(S)$ be a Rademacher sequence of random variables, we set, with $\forall a \in S,\ \lambda_a = \mu_a\nu_a,\ \mu \in l^p,\ \nu \in \ell^q$ explicitly :

$$\mu_a := \frac{\lambda_a}{|\lambda_a|^\alpha},\ \nu_a := |\lambda_a|^\alpha \text{ with } \alpha = \frac{r}{q}\ ;$$

then we get $\lambda_a = \mu_a\nu_a,\ \mu \in l^p,\ \nu \in \ell^q$ and $\|\mu\|^p_{\ell^p} = \|\nu\|^q_{\ell^q} = \|\lambda\|^r_{\ell^r} \Rightarrow \|\lambda\|_{\ell^r} = \|\nu\|_{\ell^q}\|\mu\|_{\ell^q}$.

Now set

$$f(\epsilon, z) := \sum_{a\in S} \mu_a \epsilon_a \rho_a(z)\ ;\ g(\epsilon, z) := \sum_{a\in S} \nu_a (1-|a|^2)^s \epsilon_a \frac{k_a(z)}{\|k_a\|_{H^{p'}_s}\|k_a\|_{H^r_s}}.$$

We have $\mathbb{E}(fg) = h$ hence

$$\|h\|_{H^r_s} = \|\mathbb{E}(fg)\|_{H^r_s}.$$

By lemma 2.2 we get $\|\mathbb{E}(fg)\|^r_{H^r_s} \leq \mathbb{E}(\|fg\|^r_{H^r_s})$ and by proposition 2.5 we get $\|fg\|_{H^r_s} \leq C_s\|f\|_{H^p_s}\|g\|_{H^q_s}$, so

$$\|\mathbb{E}(fg)\|^r_{H^r_s} \leq \mathbb{E}(\|fg\|^r_{H^r_s}) \leq C^r_s \mathbb{E}(\|f\|^r_{H^p_s}\|g\|^r_{H^q_s}). \tag{8.15}$$

Set $\gamma_a := \dfrac{(1-|a|^2)^s\|k_a\|_{H^q_s}}{\|k_a\|_{H^{p'}_s}\|k_a\|_{H^r_s}}$ we have $g(\epsilon, z) := \displaystyle\sum_{a\in S} \nu_a\epsilon_a\gamma_a \frac{k_a(z)}{\|k_a\|_{H^q_s}}$.

Because S is q Carleson by assumption we get

$$\exists C > 0 :: \forall \epsilon,\ \|g\|_{H_s^q}^q \leq c_q^q \sum_{a\in S} |\nu_a|^q \gamma_a^q.$$

Let us compute γ_a :

$$\gamma_a := \frac{(1-|a|^2)^s \|k_a\|_{H_s^q}}{\|k_a\|_{H_s^{p'}} \|k_a\|_{H_s^r}} = \frac{(1-|a|^2)^s (1-|a|^2)^{s-n/q'}}{(1-|a|^2)^{s-n/p}(1-|a|^2)^{s-n/r'}} = 1$$

because $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$. We see here that the weight $(1-|a|^2)^s$ compensates the second structural hypothesis, which is given by lemma 8.5.

So we get

$$\|g\|_{H_s^q} \leq c_q \|\nu\|_{\ell^q(S)}.$$

Putting this in (8.15) we get

$$\|h\|_{H_s^r}^r = \|\mathbb{E}(fg)\|_{H_s^r}^r \leq C_s^r \mathbb{E}(\|f\|_{H_s^p}^r \|g\|_{H_s^q}^r) \leq C_s^r c_q^r \|\nu\|_{\ell^q(S)}^r \mathbb{E}(\|f\|_{H_s^p}^r). \tag{8.16}$$

Now we use that $p \leq 2$ to get, because $r < p \leq 2$,

$$\mathbb{E}(\|f\|_{H_s^p}^r) \leq (\mathbb{E}(\|f\|_{H_s^p}^2))^{r/2}$$

and H_s^p is of type p so, with $f = \sum_{a\in S} \mu_a \epsilon_a \rho_a(z)$, we get

$$(\mathbb{E}(\|f\|_{H_s^p}^2))^{1/2} \leq T_p (\sum_{a\in S} |\mu_a|^p \|\rho_a\|_{H_s^p}^p)^{1/p}$$

hence, because $\forall a \in S,\ \|\rho_a\|_{H_s^p} \leq C$, we get

$$(\mathbb{E}(\|f\|_{H_s^p}^2))^{1/2} \leq T_p C \|\mu\|_{\ell^q}.$$

Putting this in (8.16) we get

$$\|\mathbb{E}(fg)\|_{H_s^r} \leq C_s c_q T_p C \|\nu\|_{\ell^q(w_s)} \|\mu\|_{\ell^q}.$$

Hence finally

$$\|h\|_{H_s^r} = \|\mathbb{E}(fg)\|_{H_s^r} \leq C_s c_q T_p C \|\nu\|_{\ell^q} \|\mu\|_{\ell^q}$$

which proves the theorem because $\|\lambda\|_{\ell^r} = \|\nu\|_{\ell^q} \|\mu\|_{\ell^q}$. ■

9 Appendix.

9.1 Technical lemmas.

With the notations of section 6 and 7, let $S = \{a_1, ..., a_N\}$, fix $a \in S$ and set $\gamma = \gamma_a$ to ease the notations. Also if $f \in H_s^p$, we set $f^{(j)} := R^j f$.

Lemma 9.1 *We have, with* $m := \min(l, j),\ \forall j, l \in \mathbb{N}$,

$$R^j(\gamma^l h) = \gamma^l F_{0,j}(z) + l\gamma^{l-1} F_{1,j}(z) + \cdots + l(l-1)\cdots(l-m+1)\gamma^{l-m} F_{m,j}(z) \tag{9.17}$$

where the functions $F_{k,j}(z)$ *do not depend on* l *for* $k \leq m$.

Proof.

By induction on j. For $j = 1$ we have :

$$R(\gamma^l h) = \gamma^l h^{(1)} + l\gamma^{l-1}\gamma^{(1)} h,$$

hence

$$\forall l \geq 1,\ F_{0,1} = h^{(1)},\ F_{1,1} = \gamma^{(1)}h,$$

so (9.17) is true.

Suppose that (9.17) is true for j and let us see for $j+1$.

Suppose that $l > j$, we have

$$R^{j+1}(\gamma^l h) = R(R^j(\gamma^l h)) = \gamma^l R(F_{0,j}) + l\gamma^{l-1}(\gamma^{(1)}F_{0,j}(z) + R(F_{1,j})) + \cdots +$$
$$+l(l-1)\cdots(l-k+1)\gamma^{l-k}(\gamma^{(1)}F_{k-1,j}(z) + R(F_{k,j})) + ... +$$
$$+l(l-1)\cdots(l-j+1)\gamma^{l-j}(\gamma^{(1)}F_{j-1,j}(z)+R(F_{j,j}))+l(l-1)\cdots(l-j)\gamma^{l-j-1}(\gamma^{(1)}F_{j,j}).$$

Hence we set

$$F_{0,j+1} := R(F_{0,j}) = h^{(j+1)},$$

and

$$\forall k,\ 1 \leq k \leq m,\ F_{k,j+1} := \gamma^{(1)}F_{k-1,j}(z) + R(F_{k,j})$$

and the last one

$$F_{j+1,j+1} := \gamma^{(1)}F_{j,j}(z).$$

If $l = j$: the formula (9.17) read

$$R^j(\gamma^l h) = \gamma^l F_{0,j}(z) + l\gamma^{l-1}F_{1,j}(z) + \cdots + l!F_{l,j}(z)$$

hence we get

$$R^{j+1}(\gamma^l h) = R(\gamma^l F_{0,j}(z) + l\gamma^{l-1}F_{1,j}(z) + \cdots + l!\gamma F_{l-1,j}) + \ l!R(F_{l,j}).$$

So again

$$F_{0,j+1} = R(F_{0,j}),$$

and

$$\forall k,\ 1 \leq k \leq l-1,\ F_{k,j+1} = \gamma^{(1)}F_{k-1,j}(z) + R(F_{k,j})$$

but

$$F_{l,j+1} = \gamma^{(1)}F_{l-1,j}$$

and

$$F_{l+1,j+1} = R(F_{l,j})$$

which is formula (9.17) with $m = l = \min\,(j+1, l)$.

If $l < j$: by use of formula (9.17) with $m = l = \min\,(j, l)$ we get

$$R^{j+1}(\gamma^l h) = R(\gamma^l F_{0,j} + l\gamma^{l-1}F_{1,j} + \cdots + l!F_{l,j})$$

hence again

$$F_{0,j+1} = R(F_{0,j}),$$

and

$$\forall k,\ 1 \leq k \leq l-1,\ F_{k,j+1} = \gamma^{(1)}F_{k-1,j}(z) + R(F_{k,j})$$

and

$$F_{l,j+1} := \gamma^{(1)}F_{l-1,j} + R(F_{l,j}).$$

Clearly the $F_{k,j+1}$ do not depend on l, for $k \leq m$, because the $F_{k,j}$ do not. ■

Lemma 9.2 *We have, with α_m constants independent of γ and of h :*

$$\forall k \leq j,\ F_{k,j} = \alpha_k R^j(\gamma^k h) + \alpha_{k-1}\gamma R^j(\gamma^{k-1}h) + \cdots + \alpha_1\gamma^{k-1}R^j(\gamma h) + \alpha_0\gamma^k h^{(j)}. \tag{9.18}$$

Proof.

To get $F_{1,j}$ we take $l = 1$ in (9.17) so we get

$$R^j(\gamma h) = \gamma h^{(j)} + F_{1,j} \Rightarrow F_{1,j} = R^j(\gamma h) - \gamma h^{(j)}.$$

So it is true for $k = 1$ and any $l \geq 1$ because $F_{1,j}$ is independent of l.

Suppose it is true up to k ; let us see for $k+1$.

We choose $l = k+1,\ j \geq l$ in (9.17), we get

$$R^j(\gamma^{k+1}h) = \gamma^{k+1}h^{(j)} + (k+1)\gamma^k F_{1,j}(z) + \cdots + (k+1)!\gamma F_{k,j} + (k+1)!F_{k+1,j},$$

hence

$$(k+1)!F_{k+1,j} = R^j(\gamma^{k+1}h) - \gamma^{k+1}h^{(j)} - (k+1)\gamma^k F_{1,j}(z) - \cdots - (k+1)!\gamma F_{k,j}$$

and assuming the decomposition (9.18) for all the $F_{m,j},\ m \leq k$, we get that the formula is true for $k+1$. ■

Proposition 9.3 *(Exclusion) We have, with* $m := \min(l, j)$,

$$\forall j, l \in \mathbb{N},\ R^j(\gamma^l h) = \sum_{q=0}^{m} A_q \gamma^{l-q} R^j(\gamma^q h),$$

where the A_q *are constants independent of* γ *and* h.

Proof.

This is trivial if $l \leq j$, just take $A_q = 0$ for $q < l$ and $A_l = 1$. Now take $l > j+1$.

From lemma 9.1 we get

$$\forall j, l \in \mathbb{N},\ R^j(\gamma^l h) = \gamma^l F_{0,j}(z) + l\gamma^{l-1}F_{1,j}(z) + \cdots + l(l-1)\cdots(l-m+1)\gamma^{l-m}F_{m,j}(z)$$

and with lemma 9.2 we replace the functions $F_{k,j}$ to get what we want

$$\forall j, l \in \mathbb{N},\ R^j(\gamma^l h) = \sum_{q=0}^{m} A_q \gamma^{l-q} R^j(\gamma^q h).$$ ■

Lemma 9.4 *(Domination) Let* $S = \{a_m\}_{m=1,\ldots,N}$ *be an interpolating sequence in* $\mathbb{B}$ *for* H^p_s, *of interpolating constant* $C(S)$, *and* $\{\gamma_a\}_{a\in S}$ *its canonical dual sequence. Then*

$$\forall l \in \mathbb{N},\ \forall j \leq s,\ \forall h \in H^p_s,\ \forall q \leq N,\ \exists H_q \in H^p_s :: \forall a \in S,\ \ \left|R^j(\gamma_a^l h)\right| \leq \frac{1}{N}\sum_{q=1}^{N}\left|R^j(H_q)\right|.$$

So H_q *depends on* l, j *and* h, *but* **not** *on* a *and we have* $1 \leq q \leq N,\ \|H_q\|_{H^p_s} \leq C(S)^l \|h\|_{H^p_s}$.

Proof.

We have by definition of γ_a

$$\gamma_{a_m}(z) := \frac{1}{N}\sum_{q=1}^{N}\theta^{-qm}\beta(q,z) \in \mathcal{M}^p_s,\ \|\beta(q,\cdot)\|_{\mathcal{M}^p_s} \leq C(S).$$

By lemma 6.1 with $Q_l(k,z) := \underbrace{\beta * \cdots * \beta(k,z)}_{l\ times}$ and $\|Q_l(k,\cdot)\|_{\mathcal{M}^p_s} \leq C(S)^l$,

$$\gamma_{a_m}(z)^l = \widehat{Q_l(m,z)} = \frac{1}{N}\sum_{q=1}^{N}\theta^{-qm}Q_l(q,z)$$

so

$$\gamma_{a_m}(z)^l h = \frac{1}{N}\sum_{q=1}^{N}\theta^{-qm}Q_l(q,z)h$$

and

$$R^j(\gamma^l_{a_m}h) = R^j(\frac{1}{N}\sum_{q=1}^{N}\theta^{-qm}Q_l(q,z)h) = \frac{1}{N}\sum_{q=1}^{N}\theta^{-qm}R^j(Q_l(q,z)h).$$

So

$$\left|R^j(\gamma_a^l h)\right| \leq \frac{1}{N}\sum_{q=1}^{N}\left|R^j(Q_l(q,z)h)\right|,$$

hence setting

$$\forall z \in \mathbb{B},\ H_q(z) := Q_l(q,z)h(z)$$

we have that H_q is independent of $a \in S$ and

$$\|H_q\|_{H^p_s} \leq \|Q(q,\cdot)\|_{\mathcal{M}^p_s}\|h\|_{H^p_s} \leq C(S)^l\|h\|_{H^p_s}.$$

This ends the proof of the lemma. ■

Lemma 9.5 *(Inclusion) Let $\gamma \in \mathcal{M}^p_s$ and $h \in H^p_s$, then there are constants A_q such that*

$$\forall j,l,\ R^j(\gamma^l)h = \sum_{q=0}^{j} A_{j,q}R^q(\gamma^l R^{j-q}(h)).$$

Proof.

By induction on j. For $j = 1$ we have $R(\gamma^l h) = R(\gamma^l)h + \gamma^l R(h)$ hence

$$R(\gamma^l)h = R(\gamma^l h) - \gamma^l R(h),$$

so it is true. Suppose it is true for any $q < j$ then we have

$$R^j(\gamma^l h) = \sum_{q=0}^{j} C^q_j R^q(\gamma^l)R^{j-q}(h) = R^j(\gamma^l)h + \sum_{q=0}^{j-1} C^q_j R^q(\gamma^l)R^{j-q}(h)$$

hence

$$R^j(\gamma^l)h = R^j(\gamma^l h) - \sum_{q=0}^{j-1} C^q_j R^q(\gamma^l)R^{j-q}(h). \tag{9.19}$$

Now because $q < j$ we have, with $k := R^{j-q}(h)$,

$$R^q(\gamma^l)k = \sum_{m=0}^{q} A_{q,m}R^m(\gamma^l R^{q-m}(k))$$

hence

$$R^q(\gamma^l)k = \sum_{m=0}^{q} A_{q,m}R^m(\gamma^l R^{q-m}(R^{j-q}(h))) = \sum_{m=0}^{q} A_{q,m}R^m(\gamma^l R^{j-m}(h)).$$

Replacing in (9.19) we get the lemma. ■

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