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Gradient discretization of Hybrid Dimensional Darcy Flows in Fractured Porous Media with discontinuous pressures at the matrix fracture interfaces

K. Brenner ^{*}, J. Hennicker ^{*†}, R. Masson ^{*}, P. Samier [†]

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Abstract

We investigate the discretization of Darcy flow through fractured porous media on general meshes. We consider a hybrid dimensional model, invoking a complex network of planar fractures. The model accounts for matrix-fracture interactions and fractures acting either as drains or as barriers, i.e. we have to deal with pressure discontinuities at matrix-fracture interfaces. The numerical analysis is performed in the general framework of gradient discretizations which is extended to the model under consideration. Two families of schemes namely the Vertex Approximate Gradient scheme (VAG) and the Hybrid Finite Volume scheme (HFV) are detailed and shown to satisfy the gradient scheme framework, which yields, in particular, convergence. Numerical tests confirm the theoretical results. Gradient Discretization; Darcy Flow, Discrete Fracture Networks, Finite Volume

1 Introduction

This work deals with the discretization of Darcy flows in fractured porous media for which the fractures are modeled as interfaces of codimension one. In this framework, the $d - 1$ dimensional flow in the fractures is coupled with the d dimensional flow in the matrix leading to the so called, hybrid dimensional Darcy flow model. We consider the case for which the pressure can be discontinuous at the matrix fracture interfaces in order to account for fractures acting either as drains or as barriers as described in [10], [12] and [3]. In this paper, we will study the family of models described in [12] and [3].

It is also assumed in the following that the pressure is continuous at the fracture intersections. This corresponds to a ratio between the permeability at the fracture intersection and the width of the fracture assumed to be high compared with the ratio between the tangential permeability of each fracture and its length. We refer to [14] for a more general reduced model taking into account discontinuous pressures at fracture intersections in dimension $d = 2$.

The discretization of such hybrid dimensional Darcy flow model has been the object of several works. In [10], [11], [3] a cell-centered Finite Volume scheme using a Two Point Flux Approximation (TPFA) is proposed assuming the orthogonality of the mesh and isotropic permeability fields. Cell-centered Finite Volume schemes have been extended to general meshes

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and anisotropic permeability fields using MultiPoint Flux Approximations (MPFA) in [13], [16], and [2]. In [12], a Mixed Finite Element (MFE) method is proposed and a MFE discretization adapted to non-matching fracture and matrix meshes is studied in [6]. More recently the Hybrid Finite Volume (HFV) scheme, introduced in [8], has been extended in [27] for the non matching discretization of two reduced fault models. Also a Mimetic Finite Difference (MFD) scheme is used in [1] in the matrix domain coupled with a TPFA scheme in the fracture network. Discretizations of the related reduced model [28] assuming a continuous pressure at the matrix fracture interfaces have been proposed in [28] using a MFE method, in [20] using a Control Volume Finite Element method (CVFE), in [19] using the HFV scheme, and in [19, 5] using an extension of the Vertex Approximate Gradient (VAG) scheme introduced in [7].

In terms of convergence analysis, the case of continuous pressure models at the matrix fracture interfaces [28] is studied in [19] for a general fracture network but the current state of the art for the discontinuous pressure models at the matrix fracture interfaces is still limited to rather simple geometries. Let us recall that the family of models introduced in [12] and [3] depends on a quadrature parameter denoted by $\xi \in [\frac{1}{2}, 1]$ for the approximate integration in the width of the fractures. Existing convergence analysis for such models cover the case of one non immersed fracture separating the domain into two subdomains using a MFE discretization in [12] or a non matching MFE discretization in [6] for the range $\xi \in (\frac{1}{2}, 1]$. In [3], the case of one fully immersed fracture in dimension $d = 2$ using a TPFA discretization is analysed for the full range of parameters $\xi \in [\frac{1}{2}, 1]$.

The main goal of this paper is to study the discretizations of such models and their convergence properties by extension of the gradient scheme framework. The gradient scheme framework has been introduced in [7], [22], [21] to analyse the convergence of numerical methods for linear and nonlinear second order diffusion problems. As shown in [22], this framework accounts for various conforming and non conforming discretizations such as Finite Element methods, Mixed and Mixed Hybrid Finite Element methods, and some Finite Volume schemes like symmetric MPFA, the VAG schemes [7], and the HFV schemes [8].

Our extension of the gradient scheme framework to the hybrid dimensional Darcy flow model will account for general fracture networks including fully, partially and non immersed fractures as well as fracture intersections in a 3D surrounding matrix domain. Each individual fracture will be assumed to be planar. The framework will cover the range of parameters $\xi \in (\frac{1}{2}, 1]$ excluding the value $\xi = \frac{1}{2}$ in order to allow for a primal variational formulation.

Two examples of gradient discretizations will be provided, namely the extension of the VAG and HFV schemes defined in [7] and [8] to the family of hybrid dimensional Darcy flow models. In both cases, it is assumed that the fracture network is conforming to the mesh in the sense that it is defined as a collection of faces of the mesh. The mesh is assumed to be polyhedral with possibly non planar faces for the VAG scheme and planar faces for the HFV scheme. Two versions of the VAG scheme will be studied, the first corresponding to the conforming \mathbb{P}_1 finite element on a tetrahedral submesh, and the second to a finite volume scheme using lumping for the source terms as well as for the matrix fracture fluxes. The VAG scheme has the advantage to lead to a sparse discretization on tetrahedral or mainly tetrahedral meshes. It will be compared to the HFV discretization using face and fracture edge unknowns in addition to the cell unknowns. Note that the HFV scheme of [8] has been generalized in [23] as the family of Hybrid Mimetic Mixed methods which encompasses the family of MFD schemes [24]. In this article, we will focus without restriction on the particular case presented in [8] for the sake of simplicity.

In section 2 we introduce the geometry of the matrix and fracture domains and present

the strong and weak formulation of the model. Section 3 is devoted to the introduction of the general framework of gradient discretizations and the derivation of the error estimate 3.3. In section 4 we define and investigate the families of VAG and HFV discretizations. Having in mind applications to multi-phase flow, we also present a Finite Volume formulation involving conservative fluxes, which applies for both schemes. In section 5, the VAG and HFV schemes are compared in terms of accuracy and CPU efficiency for both Cartesian and tetrahedral meshes on heterogeneous isotropic and anisotropic media using a family of analytical solutions.

2 Hybrid dimensional Darcy Flow Model in Fractured Porous Media

2.1 Geometry and Function Spaces

Let Ω denote a bounded domain of \mathbb{R}^d , $d = 2, 3$ assumed to be polyhedral for $d = 3$ and polygonal for $d = 2$. To fix ideas the dimension will be fixed to $d = 3$ when it needs to be specified, for instance in the naming of the geometrical objects or for the space discretization in the next section. The adaptations to the case $d = 2$ are straightforward.

Let $\bar{\Gamma} = \bigcup_{i \in I} \bar{\Gamma}_i$ and its interior $\Gamma = \bar{\Gamma} \setminus \partial \bar{\Gamma}$ denote the network of fractures $\Gamma_i \subset \Omega$, $i \in I$, such that each Γ_i is a planar polygonal simply connected open domain included in a plane \mathcal{P}_i of \mathbb{R}^d . It is assumed that the angles of Γ_i are strictly smaller than 2π , and that $\Gamma_i \cap \bar{\Gamma}_j = \emptyset$ for all $i \neq j$.

For all $i \in I$, let us set $\Sigma_i = \partial \Gamma_i$, with \mathbf{n}_{Σ_i} as unit vector in \mathcal{P}_i , normal to Σ_i and outward to Γ_i . Further $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j$, $j \in I \setminus \{i\}$, $\Sigma_{i,0} = \Sigma_i \cap \partial \Omega$, $\Sigma_{i,N} = \Sigma_i \setminus (\bigcup_{j \in I \setminus \{i\}} \Sigma_{i,j} \cup \Sigma_{i,0})$, $\Sigma = \bigcup_{(i,j) \in I \times I, i \neq j} (\Sigma_{i,j} \setminus \Sigma_{i,0})$ and $\Sigma_0 = \bigcup_{i \in I} \Sigma_{i,0}$. It is assumed that $\Sigma_{i,0} = \bar{\Gamma}_i \cap \partial \Omega$.

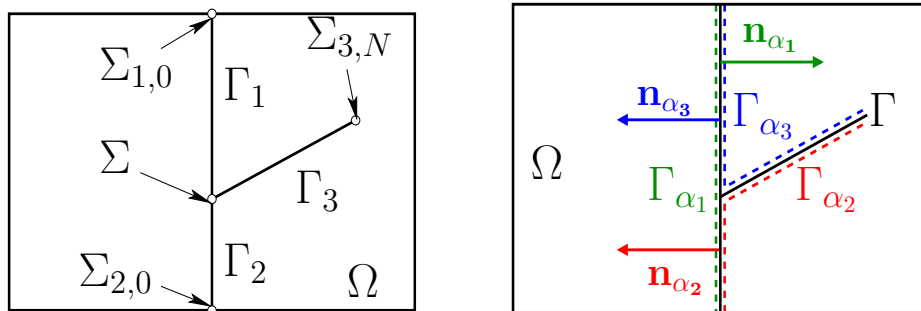


Figure 1: Example of a 2D domain Ω and 3 intersecting fractures Γ_i , $i = 1, 2, 3$. We might define the fracture plane orientations by $\alpha^+(1) = \alpha_1$, $\alpha^-(1) = \alpha_3$ for Γ_1 , $\alpha^+(2) = \alpha_1$, $\alpha^-(2) = \alpha_2$ for Γ_2 , and $\alpha^+(3) = \alpha_3$, $\alpha^-(3) = \alpha_2$ for Γ_3 .

We will denote by $d\tau(\mathbf{x})$ the $d - 1$ dimensional Lebesgue measure on Γ . On the fracture network Γ , we define the function space $L^2(\Gamma) = \{v = (v_i)_{i \in I}, v_i \in L^2(\Gamma_i), i \in I\}$, endowed with the norm $\|v\|_{L^2(\Gamma)} = (\sum_{i \in I} \|v_i\|_{L^2(\Gamma_i)}^2)^{\frac{1}{2}}$ and its subspace $H^1(\Gamma)$ consisting of functions $v = (v_i)_{i \in I}$ such that $v_i \in H^1(\Gamma_i)$, $i \in I$ with continuous traces at the fracture intersections $\Sigma_{i,j}$, $j \in I \setminus \{i\}$. The space $H^1(\Gamma)$ is endowed with the norm $\|v\|_{H^1(\Gamma)} = (\sum_{i \in I} \|v_i\|_{H^1(\Gamma_i)}^2)^{\frac{1}{2}}$. We also define its subspace with vanishing traces on Σ_0 , which we denote by $H_{\Sigma_0}^1(\Gamma)$.

On $\Omega \setminus \bar{\Gamma}$, the gradient operator from $H^1(\Omega \setminus \bar{\Gamma})$ to $L^2(\Omega)^d$ is denoted by ∇ . On the fracture network Γ , the tangential gradient, acting from $H^1(\Gamma)$ to $L^2(\Gamma)^{d-1}$, is denoted by ∇_τ , and such

that

$$\nabla_\tau v = (\nabla_{\tau_i} v_i)_{i \in I},$$

where, for each $i \in I$, the tangential gradient ∇_{τ_i} is defined from $H^1(\Gamma_i)$ to $L^2(\Gamma_i)^{d-1}$ by fixing a reference Cartesian coordinate system of the plane \mathcal{P}_i containing Γ_i . We also denote by div_{τ_i} the divergence operator from $H_{\text{div}}(\Gamma_i)$ to $L^2(\Gamma_i)$.

We assume that there exists a finite family $(\Gamma_\alpha)_{\alpha \in \chi}$ such that for all $\alpha \in \chi$ holds: $\Gamma_\alpha \subset \Gamma$ and there exists a lipschitz domain $\omega_\alpha \subset \Omega \setminus \bar{\Gamma}$, such that $\Gamma_\alpha = \partial\omega_\alpha \cap \Gamma$. For $\alpha \in \chi$ and an appropriate choice of $I_\alpha \subset I$ we assume that $\bar{\Gamma}_\alpha = \bigcup_{i \in I_\alpha} \bar{\Gamma}_i$. Furthermore should hold $\bar{\Gamma} = \bigcup_{\alpha \in \chi} \bar{\Gamma}_\alpha$. We also assume that each $\Gamma_i \subset \Gamma$ is contained in Γ_α for exactly two $\alpha \in \chi$ and that we can define a unique mapping $i \mapsto (\alpha^+(i), \alpha^-(i))$ from I to $\chi \times \chi$, such that $\Gamma_i \subset \Gamma_{\alpha^+(i)} \cap \Gamma_{\alpha^-(i)}$ and $\alpha^+(i) \neq \alpha^-(i)$ (cf. figure 1). For all $i \in I$, $\alpha^\pm(i)$ defines the two sides of the fracture Γ_i in $\Omega \setminus \bar{\Gamma}$ and we can introduce the corresponding unit normal vectors $n_{\alpha^\pm(i)}$ at Γ_i outward to $\omega_{\alpha^\pm(i)}$, such that $\mathbf{n}_{\alpha^+(i)} + \mathbf{n}_{\alpha^-(i)} = 0$. We therefore obtain for $\alpha \in \chi$ and a.e. $\mathbf{x} \in \Gamma_\alpha$ a unique unit normal vector $\mathbf{n}_\alpha(\mathbf{x})$ outward to ω_α . A simple choice of $(\Gamma_\alpha)_{\alpha \in \chi}$ is given by both sides of each fracture $i \in I$ but more general choices are also possible such as for example the one exhibited in figure 1.

Then, for $\alpha \in \chi$, we can define the trace operator on Γ_α :

$$\gamma_\alpha : H^1(\Omega \setminus \bar{\Gamma}) \rightarrow L^2(\Gamma_\alpha),$$

and the normal trace operator on Γ_α outward to the side α :

$$\gamma_{\mathbf{n},\alpha} : H_{\text{div}}(\Omega \setminus \bar{\Gamma}) \rightarrow \mathcal{D}'(\Gamma_\alpha).$$

We now define the hybrid dimensional function spaces that will be used as variational spaces for the Darcy flow model in the next subsection:

$$V = H^1(\Omega \setminus \bar{\Gamma}) \times H^1(\Gamma),$$

and its subspace

$$V^0 = H_{\partial\Omega}^1(\Omega \setminus \bar{\Gamma}) \times H_{\Sigma_0}^1(\Gamma),$$

where (with $\gamma_{\partial\Omega} : H^1(\Omega \setminus \bar{\Gamma}) \rightarrow L^2(\partial\Omega)$ denoting the trace operator on $\partial\Omega$)

$$H_{\partial\Omega}^1(\Omega \setminus \bar{\Gamma}) = \{v \in H^1(\Omega \setminus \bar{\Gamma}) \mid \gamma_{\partial\Omega} v = 0 \text{ on } \partial\Omega\},$$

as well as

$$W = W_m \times W_f,$$

where

$$\begin{aligned} W_m &= \{\mathbf{q}_m \in H_{\text{div}}(\Omega \setminus \bar{\Gamma}) \mid \gamma_{\mathbf{n},\alpha} \mathbf{q}_m \in L^2(\Gamma_\alpha) \text{ for all } \alpha \in \chi\} \text{ and} \\ W_f &= \{\mathbf{q}_f = (\mathbf{q}_{f,i})_{i \in I} \mid \mathbf{q}_{f,i} \in H_{\text{div}}(\Gamma_i) \text{ for all } i \in I \\ &\text{and } \sum_{i \in \Gamma} \int_{\Gamma_i} (\nabla_\tau v \cdot \mathbf{q}_{f,i} + v \cdot \text{div}_{\tau_i} \mathbf{q}_{f,i}) d\tau(\mathbf{x}) = 0 \text{ for all } v \in H_{\Sigma_0}^1(\Gamma)\}. \end{aligned}$$

On V , we define the positive semidefinite, symmetric bilinear form

$$\begin{aligned} ((u_m, u_f), (v_m, v_f))_V &= \int_\Omega \nabla u_m \cdot \nabla v_m d\mathbf{x} + \int_\Gamma \nabla_\tau u_f \cdot \nabla_\tau v_f d\tau(\mathbf{x}) \\ &\quad + \sum_{\alpha \in \chi} \int_{\Gamma_\alpha} (\gamma_\alpha u_m - u_f)(\gamma_\alpha v_m - v_f) d\tau(\mathbf{x}) \end{aligned}$$

for $(u_m, u_f), (v_m, v_f) \in V$, which induces the seminorm $|(v_m, v_f)|_V$. Note that $(\cdot, \cdot)_V$ is a scalar product and $|\cdot|_V$ is a norm on V^0 , denoted by $\|\cdot\|_{V^0}$ in the following.

We define for all $(\mathbf{p}_m, \mathbf{p}_f), (\mathbf{q}_m, \mathbf{q}_f) \in W$ the scalar product

$$\begin{aligned} ((\mathbf{p}_m, \mathbf{p}_f), (\mathbf{q}_m, \mathbf{q}_f))_W &= \int_{\Omega} \mathbf{p}_m \mathbf{q}_m \, d\mathbf{x} + \int_{\Omega} \operatorname{div} \mathbf{p}_m \cdot \operatorname{div} \mathbf{q}_m \, d\mathbf{x} \\ &\quad + \int_{\Gamma} \mathbf{p}_f \mathbf{q}_f \, d\tau(\mathbf{x}) + \int_{\Gamma} \operatorname{div}_{\tau} \mathbf{p}_f \cdot \operatorname{div}_{\tau} \mathbf{q}_f \, d\tau(\mathbf{x}) \\ &\quad + \sum_{\alpha \in \mathcal{X}} \int_{\Gamma_{\alpha}} (\gamma_{\mathbf{n}, \alpha} \mathbf{p}_m \cdot \gamma_{\mathbf{n}, \alpha} \mathbf{q}_m) \, d\tau(\mathbf{x}), \end{aligned}$$

which induces the norm $\|(\mathbf{q}_m, \mathbf{q}_f)\|_W$, and where we have used the notation $\operatorname{div}_{\tau} \mathbf{p}_f = \operatorname{div}_{\tau_i} \mathbf{p}_{f,i}$ on Γ_i for all $i \in I$ and $\mathbf{p}_f = (\mathbf{p}_{f,i})_{i \in I} \in W_f$.

Using similar arguments as in the proof of [15], example II.3.4, one can prove the following Poincaré type inequality.

Proposition 2.1 *The norm $\|\cdot\|_{V^0}$ satisfies the following inequality*

$$\|v_m\|_{H^1(\Omega \setminus \bar{\Gamma})} + \|v_f\|_{H^1(\Gamma)} \leq \mathcal{C}_P \|(v_m, v_f)\|_{V^0}, \quad (1)$$

for all $(v_m, v_f) \in V^0$.

Proof We apply the ideas of the proof of [15], example II.3.4 and assume that the statement of the proposition is not true. Then we can define a sequence $(v_l)_{l \in \mathbb{N}}$ in V^0 , such that

$$\|v_l\|_{H^1} = 1 \quad \text{and} \quad \|v_l\|_{V^0} < \frac{1}{l}, \quad (2)$$

where, for this proof, $\|\cdot\|_{H^1} = \|\cdot\|_{H^1(\Omega \setminus \bar{\Gamma})} + \|\cdot\|_{H^1(\Gamma)}$. The imbedding

$$(V^0, \|\cdot\|_{H^1}) \hookrightarrow \left(L^2(\Omega) \times L^2(\Gamma), \|\cdot\|_{L^2(\Omega)} + \|\cdot\|_{L^2(\Gamma)} \right)$$

is compact, provided that $\Omega \setminus \bar{\Gamma}$ has the cone property (see [18], theorem 6.2). Thus, there is a subsequence $(v_{\mu})_{\mu}$ of $(v_l)_{l \in \mathbb{N}}$ and $v \in L^2(\Omega) \times L^2(\Gamma)$, such that

$$v_{\mu} \longrightarrow v \quad \text{in } L^2(\Omega) \times L^2(\Gamma).$$

On the other hand it follows from (2) that

$$\begin{aligned} \nabla v_{m_{\mu}} &\longrightarrow 0 && \text{in } L^2(\Omega) \\ \nabla_{\tau} v_{f_{\mu}} &\longrightarrow 0 && \text{in } L^2(\Gamma). \end{aligned}$$

Since $(V^0, \|\cdot\|_{H^1})$ is complete, we have

$$v_{\mu} \longrightarrow v \quad \text{in } V^0,$$

with

$$\|v\|_{V^0} = \lim_{\mu \rightarrow \infty} \|v_{\mu}\|_{V^0} = 0.$$

Since $\|\cdot\|_{V^0}$ is a norm on V^0 , we have $v = 0 \in V^0$, but $\|v\| = 1$, which is a contradiction. \square

Remark 2.1 *With the precedent proof it is readily seen that inequality (1) holds for all functions $v \in V$ whose trace vanishes on a subset of $\partial(\Omega \setminus \bar{\Gamma})$ with positive surface measure. The requirement is that v has to be in a closed subspace of $(V, \|\cdot\|_{H^1})$ for which $\|\cdot\|_{V^0}$ is a well defined norm.*

The convergence analysis presented in section 4 requires some results on the density of smooth subspaces of V and W , which we state below.

- Definition 2.1**
1. C_Ω^∞ is defined as the subspace of functions in $C_b^\infty(\Omega \setminus \bar{\Gamma})$ vanishing on a neighbourhood of the boundary $\partial\Omega$, where $C_b^\infty(\Omega \setminus \bar{\Gamma}) \subset C^\infty(\Omega \setminus \bar{\Gamma})$ is the set of functions φ , such that for all $\mathbf{x} \in \Omega$ there exists $r > 0$, such that for all connected components ω of $\{\mathbf{x} + \mathbf{y} \in \mathbb{R}^d \mid |\mathbf{y}| < r\} \cap (\Omega \setminus \bar{\Gamma})$ one has $\varphi \in C^\infty(\bar{\omega})$.
 2. $C_\Gamma^\infty = \gamma_\Gamma(C_0^\infty(\Omega))$ is defined as the image of $C_0^\infty(\Omega)$ of the trace operator $\gamma_\Gamma: H_0^1(\Omega) \rightarrow L^2(\Gamma)$.
 3. $C_{W_m}^\infty = C_b^\infty(\Omega \setminus \bar{\Gamma})^d$.
 4. $C_{W_f}^\infty = \{\mathbf{q}_f = (\mathbf{q}_{f,i})_{i \in I} \mid \mathbf{q}_{f,i} \in C^\infty(\bar{\Gamma}_i)^{d-1}, \sum_{i \in I} \mathbf{q}_{f,i} \cdot \mathbf{n}_{\Sigma_i} = 0 \text{ on } \Sigma, \mathbf{q}_{f,i} \cdot \mathbf{n}_{\Sigma_i} = 0 \text{ on } \Sigma_{i,N}, i \in I\}$.

Let us first state the following Lemma that will be used to prove the density of $C_{W_m}^\infty \times C_{W_f}^\infty$ in W .

Lemma 2.1 *Let $v_m \in L^2(\Omega)$, $v_f \in L^2(\Gamma)$, $G \in L^2(\Omega)^d$, $H \in L^2(\Gamma)^{d-1}$ and $J_\alpha \in L^2(\Gamma_\alpha)$, $\alpha \in \chi$ such that*

$$\int_\Omega (G \cdot \mathbf{q}_m + v_m \operatorname{div} \mathbf{q}_m) d\mathbf{x} + \int_\Gamma (H \cdot \mathbf{q}_f + v_f \operatorname{div}_\tau \mathbf{q}_f) d\tau(\mathbf{x}) + \sum_{\alpha \in \chi} \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m d\tau(\mathbf{x}) (J_\alpha - v_f) = 0 \quad (3)$$

for all $(\mathbf{q}_m, \mathbf{q}_f) \in C_{W_m}^\infty \times C_{W_f}^\infty$. Then holds $(v_m, v_f) \in V^0$, $(G, H) = (\nabla v_m, \nabla_\tau v_f)$ and $J_\alpha = v_f - \gamma_\alpha v_m$ for $\alpha \in \chi$.

Proof Firstly, for all $\mathbf{q}_m \in C_0^\infty(\Omega \setminus \bar{\Gamma})^d$, we have

$$\int_\Omega (G \cdot \mathbf{q}_m + v_m \operatorname{div} \mathbf{q}_m) d\mathbf{x} = 0$$

and therefore $v_m \in H^1(\Omega \setminus \bar{\Gamma})$ and $\nabla v_m = G$.

For a.e. $\mathbf{x} \in \partial\Omega$, there exists an open planar domain $\omega \subset\subset \partial\Omega \setminus \partial\Gamma$ containing \mathbf{x} such that for all $f \in C_0^\infty(\omega)$ there exists $\mathbf{q}_m \in C_{W_m}^\infty$ with

$$\gamma_{\mathbf{n},\partial\Omega} \mathbf{q}_m = \begin{cases} f & \text{on } \omega, \\ 0 & \text{on } \partial\Omega \setminus \omega, \end{cases}$$

$$\gamma_{\mathbf{n},\alpha} \mathbf{q}_m = 0 \quad \text{on } \Gamma_\alpha, \alpha \in \chi,$$

where $\gamma_{\mathbf{n},\partial\Omega}$ denotes the normal trace operator on the boundary of Ω . From (3), taking $\mathbf{q}_f = 0$, we obtain

$$0 = \int_\Omega (\nabla v_m \cdot \mathbf{q}_m + v_m \operatorname{div} \mathbf{q}_m) d\mathbf{x} = \int_{\partial\Omega} \gamma_{\partial\Omega} v_m \gamma_{\mathbf{n},\partial\Omega} \mathbf{q}_m d\tau(\mathbf{x}) = \int_\omega \gamma_{\partial\Omega} v_m f d\tau(\mathbf{x}).$$

where $\gamma_{\partial\Omega}$ denotes the trace operator on the boundary of Ω . We deduce $\gamma_{\partial\Omega}v_m = 0$ a.e. on $\partial\Omega \setminus \partial\Gamma$. Hence $v_m \in H_{\partial\Omega}^1(\Omega \setminus \bar{\Gamma})$.

Further, for a.e. $\mathbf{x} \in \Gamma_\alpha$ there exists an open planar domain $\omega_\alpha \subset\subset \Gamma_\alpha$ containing \mathbf{x} such that for all $g \in C_0^\infty(\omega_\alpha)$ there exists $\mathbf{q}_m \in C_{W_m}^\infty$ with

$$\begin{aligned}\gamma_{\mathbf{n},\alpha}\mathbf{q}_m &= \begin{cases} g & \text{on } \omega_\alpha, \\ 0 & \text{on } \Gamma_\alpha \setminus \omega_\alpha, \end{cases} \\ \gamma_{\mathbf{n},\beta}\mathbf{q}_m &= 0 \quad \text{on } \Gamma_\beta, \text{ for } \beta \neq \alpha, \\ \gamma_{\mathbf{n},\partial\Omega}\mathbf{q}_m &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

From (3) we obtain

$$\begin{aligned}0 &= \int_{\Omega} (\nabla v_m \cdot \mathbf{q}_m + v_m \operatorname{div} \mathbf{q}_m) d\mathbf{x} + \sum_{\alpha \in \chi} \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m (J_\alpha - v_f) d\tau(\mathbf{x}) \\ &= \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m (J_\alpha - v_f + \gamma_\alpha v_m) d\tau(\mathbf{x}) = \int_{\omega_\alpha} g (J_\alpha - v_f + \gamma_\alpha v_m) d\tau(\mathbf{x}).\end{aligned}$$

We deduce $J_\alpha = v_f - \gamma_\alpha v_m$ a.e. on Γ_α , $\alpha \in \chi$.

Next, for all $\mathbf{q}_f \in C_0^\infty(\Gamma_i)^{d-1}$, $i \in I$, we have from (3)

$$\int_{\Gamma_i} (H \cdot \mathbf{q}_f + v_f \operatorname{div} \mathbf{q}_f) d\tau(\mathbf{x}) = 0$$

and therefore $v_f|_{\Gamma_i} \in H^1(\Gamma_i)$ for $i \in I$ and $\nabla_{\tau_i} v_f|_{\Gamma_i} = H|_{\Gamma_i}$.

Let $i, j \in I$, $i \neq j$. For a.e. $\mathbf{x} \in \Sigma_{i,j} \setminus \Sigma_{i,0}$ there exists an open interval $c_{ij} \subset\subset \Sigma_{i,j} \setminus \Sigma_{i,0}$ containing \mathbf{x} such that for all $h \in C_0^\infty(c_{ij})$ there exists $s \in C_{W_f}^\infty$ with

$$\begin{aligned}\gamma_{\mathbf{n}_{\Sigma_i}} s &= h = -\gamma_{\mathbf{n}_{\Sigma_j}} s \quad \text{on } c_{ij}, \\ \gamma_{\mathbf{n}_{\Sigma_k}} s &= 0 \quad \text{on } \Sigma_k \setminus c_{ij}, \quad k \in I.\end{aligned}$$

From (3) we obtain

$$0 = \int_{\Gamma} (\nabla_{\tau} v_f \cdot s + v_f \operatorname{div}_{\tau} s) d\tau(\mathbf{x}) = \int_{c_{ij}} (\gamma_{\Sigma_i} v_f - \gamma_{\Sigma_j} v_f) \gamma_{\mathbf{n}_{\Sigma_i}} s d\sigma(\mathbf{x}),$$

$d\sigma(\mathbf{x})$ denoting the $d-2$ dimensional Lebesgue measure on Σ . We deduce $\gamma_{\Sigma_i} v_f = \gamma_{\Sigma_j} v_f$ a.e. on $\Sigma_{i,j} \setminus \Sigma_{i,0}$, $i, j \in I$, $i \neq j$. The proof of $\gamma_{\Sigma_0} v_f = 0$ a.e. on Σ_0 goes analogously. Hence $v_f \in H_{\Sigma_0}^1(\Gamma)$. \square

Proposition 2.2 $C_\Omega^\infty \times C_\Gamma^\infty$ is dense in V^0 .

Proof Firstly, note that we have

$$\begin{aligned}\frac{1}{\sqrt{2}} \left(\|\nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_{\tau} u_f\|_{L^2(\Gamma)^{d-1}} \right) &\leq \|(u_m, u_f)\|_{V^0} \\ &\leq C(\Omega, \Gamma) \cdot \left(\|\nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_{\tau} u_f\|_{L^2(\Gamma)^{d-1}} \right),\end{aligned}$$

i.e. $\|\cdot\|_{V^0}$ is equivalent to the standard norm $\|\nabla \cdot\|_{L^2(\Omega)^d} + \|\nabla_{\tau} \cdot\|_{L^2(\Gamma)^{d-1}}$ on V^0 . The density of C_Ω^∞ in $H_{\partial\Omega}^1(\Omega \setminus \bar{\Gamma})$ being a classical result, we are concerned to prove the density of C_Γ^∞ in $H_{\Sigma_0}^1(\Gamma)$ in the following. Since $H_{\Sigma_0}^1(\Gamma) \subset \gamma_\Gamma(H_0^1(\Omega))$, we can define $\tilde{V}^0 = \gamma_\Gamma^{-1}(H_{\Sigma_0}^1(\Gamma)) \subset H_0^1(\Omega)$. In Proposition 2 of [19] it is shown that $C_0^\infty(\Omega)$ is dense in $(\tilde{V}^0, \|\nabla \cdot\|_{L^2(\Omega)^d} + \|\nabla_{\tau} \gamma_\Gamma \cdot\|_{L^2(\Gamma)^{d-1}})$. Hence C_Γ^∞ is dense in $(H_{\Sigma_0}^1(\Gamma), \|\nabla_{\tau} \cdot\|_{L^2(\Gamma)^{d-1}})$. \square

Proposition 2.3 $C_{W_m}^\infty \times C_{W_f}^\infty$ is dense in W .

Proof Since W_f is a closed subspace of the Hilbert space $\prod_{i \in I} H_{\text{div}}(\Gamma_i)$, any linear form $l \in W'_f$ is the restriction to W_f of a linear form still denoted by l in $\prod_{i \in I} H_{\text{div}}(\Gamma_i)'$. Then, for some $f \in L^2(\Gamma)$ and $\mathbf{g} \in L^2(\Gamma)^{d-1}$ holds

$$\langle l, \mathbf{q}_f \rangle = \sum_{i \in I} \int_{\Gamma_i} (\mathbf{g} \cdot \mathbf{q}_f + f \cdot \text{div}_\tau \mathbf{q}_f) d\tau(\mathbf{x}),$$

for all $\mathbf{q}_f \in W_f$. Let us assume now that $\langle l, \boldsymbol{\varphi} \rangle = 0$ for all $\boldsymbol{\varphi} \in C_{W_f}^\infty$. Corresponding to Lemma 2.1 holds $f \in H_{\Sigma_0}^1(\Gamma)$. From the definition of W_f we conclude that $\langle l, \mathbf{q}_f \rangle = 0$ for all $\mathbf{q}_f \in W_f$.

Let now $l \in W'_m$. Then there exist $f \in L^2(\Omega)$, $\mathbf{g} \in L^2(\Omega)^d$ and $h_\alpha \in L^2(\Gamma_\alpha)$ ($\alpha \in \chi$), such that

$$\langle l, \mathbf{q}_m \rangle = \int_{\Omega} (\mathbf{g} \cdot \mathbf{q}_m + f \cdot \text{div} \mathbf{q}_m) d\mathbf{x} + \sum_{\alpha \in \chi} \int_{\Gamma_\alpha} h_\alpha \gamma_{\mathbf{n}, \alpha} \mathbf{q}_m d\tau(\mathbf{x}),$$

for all $\mathbf{q}_m \in W_m$. Furthermore, let us assume that $\langle l, \boldsymbol{\varphi} \rangle = 0$ for all $\boldsymbol{\varphi} \in C_{W_m}^\infty$. From Lemma 2.1 we deduce that $f \in H_{\partial\Omega}^1(\Omega \setminus \bar{\Gamma})$, that $\mathbf{g} = \nabla f$ and that $h_\alpha = \gamma_\alpha f$ ($\alpha \in \chi$). Using this, we conclude, again by the rule of partial integration, that $\langle l, \mathbf{q}_m \rangle = 0$ for all $\mathbf{q}_m \in W_m$. \square

2.2 Single Phase Darcy Flow Model

2.2.1 Strong formulation

In the matrix domain $\Omega \setminus \bar{\Gamma}$, let us denote by $\Lambda_m \in L^\infty(\Omega)^{d \times d}$ the permeability tensor such that there exist $\bar{\lambda}_m \geq \underline{\lambda}_m > 0$ with

$$\underline{\lambda}_m |\zeta|^2 \leq (\Lambda_m(\mathbf{x})\zeta, \zeta) \leq \bar{\lambda}_m |\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^d, \mathbf{x} \in \Omega,$$

Analogously, in the fracture network Γ , we denote by $\Lambda_f \in L^\infty(\Gamma)^{(d-1) \times (d-1)}$ the tangential permeability tensor, and assume that there exist $\bar{\lambda}_f \geq \underline{\lambda}_f > 0$, such that holds

$$\underline{\lambda}_f |\zeta|^2 \leq (\Lambda_f(\mathbf{x})\zeta, \zeta) \leq \bar{\lambda}_f |\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^{d-1}, \mathbf{x} \in \Gamma.$$

At the fracture network Γ , we introduce the orthonormal system $(\boldsymbol{\tau}_1(\mathbf{x}), \boldsymbol{\tau}_2(\mathbf{x}), \mathbf{n}(\mathbf{x}))$, defined a.e. on Γ . Inside the fractures, the normal direction is assumed to be a permeability principal direction. The normal permeability $\lambda_{f, \mathbf{n}} \in L^\infty(\Gamma)$ is such that $\underline{\lambda}_{f, \mathbf{n}} \leq \lambda_{f, \mathbf{n}}(\mathbf{x}) \leq \bar{\lambda}_{f, \mathbf{n}}$ for a.e. $\mathbf{x} \in \Gamma$ with $0 < \underline{\lambda}_{f, \mathbf{n}} \leq \bar{\lambda}_{f, \mathbf{n}}$. We also denote by $d_f \in L^\infty(\Gamma)$ the width of the fractures assumed to be such that there exist $\bar{d}_f \geq \underline{d}_f > 0$ with

$$\underline{d}_f \leq d_f(\mathbf{x}) \leq \bar{d}_f$$

for a.e. $\mathbf{x} \in \Gamma$. Let us define the weighted Lebesgue $d - 1$ dimensional measure on Γ by $d\tau_f(\mathbf{x}) = d_f(\mathbf{x})d\tau(\mathbf{x})$. We consider the source terms $h_m \in L^2(\Omega)$ (resp. $h_f \in L^2(\Gamma)$) in the matrix domain $\Omega \setminus \bar{\Gamma}$ (resp. in the fracture network Γ). The half normal transmissibility in the fracture network is denoted by $T_f = \frac{2\lambda_{f, \mathbf{n}}}{d_f}$.

Given $\xi \in (\frac{1}{2}, 1]$, the PDEs model writes: find $(u_m, u_f) \in V^0$, $(\mathbf{q}_m, \mathbf{q}_f) \in W$ such that:

$$\left\{ \begin{array}{ll} \text{div}(\mathbf{q}_m) = h_m & \text{on } \Omega \setminus \bar{\Gamma}, \\ \mathbf{q}_m = -\Lambda_m \nabla u_m & \text{on } \Omega \setminus \bar{\Gamma}, \\ \gamma_{\mathbf{n}, \alpha^\pm(i)} \mathbf{q}_m = \frac{T_f}{2\xi - 1} (\xi \gamma_{\alpha^\pm(i)} u_m + (1 - \xi) \gamma_{\alpha^\mp(i)} u_m - u_f) & \text{on } \Gamma_i, i \in I, \\ \text{div}_{\tau_i}(\mathbf{q}_f) - \gamma_{\mathbf{n}, \alpha^+(i)} \mathbf{q}_m - \gamma_{\mathbf{n}, \alpha^-(i)} \mathbf{q}_m = d_f h_f & \text{on } \Gamma_i, i \in I \\ \mathbf{q}_f = -d_f \Lambda_f \nabla_\tau u_f & \text{on } \Gamma, \end{array} \right. \quad (4)$$

2.2.2 Weak formulation

The hybrid dimensional weak formulation amounts to find $(u_m, u_f) \in V^0$ satisfying the following variational equality for all $(v_m, v_f) \in V^0$:

$$\begin{aligned} & \int_{\Omega} \Lambda_m \nabla u_m \cdot \nabla v_m d\mathbf{x} + \int_{\Gamma} \Lambda_f \nabla_{\tau} u_f \cdot \nabla_{\tau} v_f d\tau_f(\mathbf{x}) \\ & + \sum_{i \in I} \int_{\Gamma_i} \frac{T_f}{2\xi - 1} \sum_{(\alpha, \beta) \in \{(\alpha^{\pm}(i), \alpha^{\mp}(i))\}} \left(\xi \gamma_{\alpha} u_m + (1 - \xi) \gamma_{\beta} u_m - u_f \right) \left(\gamma_{\alpha} v_m - v_f \right) d\tau(\mathbf{x}) \\ & - \int_{\Omega} h_m v_m d\mathbf{x} - \int_{\Gamma} h_f v_f d\tau_f(\mathbf{x}) = 0. \end{aligned} \quad (5)$$

The following proposition states the well posedness of the variational formulation (5).

Proposition 2.4 *For all $\xi \in (\frac{1}{2}, 1]$, the variational problem (5) has a unique solution $(u_m, u_f) \in V^0$ which satisfies the a priori estimate*

$$\|(u_m, u_f)\|_{V^0} \leq C \left(\|h_m\|_{L^2(\Omega)} + \|h_f\|_{L^2(\Gamma)} \right),$$

with C depending only on ξ , \mathcal{C}_P , $\underline{\lambda}_m$, $\underline{\lambda}_f$, \underline{d}_f , \bar{d}_f , and $\underline{\lambda}_{f,n}$. In addition $(\mathbf{q}_m, \mathbf{q}_f) = -(\Lambda_m \nabla u_m, d_f \Lambda_f \nabla_{\tau} u_f)$ belongs to W .

Proof Using that for all $\xi \in (\frac{1}{2}, 1]$ and for all $(a, b) \in \mathbb{R}^2$ one has

$$a^2 + b^2 \leq (\xi a + (1 - \xi)b)a + (\xi b + (1 - \xi)a)b \leq \frac{1}{2\xi - 1}(a^2 + b^2),$$

the Lax-Milgram Theorem applies, which ensures the statement of the proposition. \square

3 Gradient Discretization of the Hybrid Dimensional Model

3.1 Gradient Scheme Framework

A gradient discretization \mathcal{D} of hybrid dimensional Darcy flow models is defined by a vector space of degrees of freedom $X_{\mathcal{D}} = X_{\mathcal{D}_m} \times X_{\mathcal{D}_f}$, its subspace satisfying ad hoc homogeneous boundary conditions $X_{\mathcal{D}}^0 = X_{\mathcal{D}_m}^0 \times X_{\mathcal{D}_f}^0$, and the following gradient and reconstruction operators:

- Gradient operator on the matrix domain: $\nabla_{\mathcal{D}_m} : X_{\mathcal{D}_m} \rightarrow L^2(\Omega)^d$
- Gradient operator on the fracture network: $\nabla_{\mathcal{D}_f} : X_{\mathcal{D}_f} \rightarrow L^2(\Gamma)^{d-1}$
- A function reconstruction operator on the matrix domain:
 $\Pi_{\mathcal{D}_m} : X_{\mathcal{D}_m} \rightarrow L^2(\Omega)$
- Two function reconstruction operators on the fracture network:
 $\Pi_{\mathcal{D}_f} : X_{\mathcal{D}_f} \rightarrow L^2(\Gamma)$ and $\tilde{\Pi}_{\mathcal{D}_f} : X_{\mathcal{D}_f} \rightarrow L^2(\Gamma)$
- Reconstruction operators of the trace on Γ_{α} for $\alpha \in \chi$:
 $\Pi_{\mathcal{D}_m}^{\alpha} : X_{\mathcal{D}_m} \rightarrow L^2(\Gamma_{\alpha})$.

The space $X_{\mathcal{D}}$ is endowed with the seminorm

$$\|(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\|_{\mathcal{D}} = \left(\|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)^d}^2 + \|\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}}^2 + \sum_{\alpha \in \mathcal{X}} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)}^2 \right)^{\frac{1}{2}},$$

which is assumed to define a norm on $X_{\mathcal{D}}^0$.

The following properties of gradient discretizations are crucial for the convergence analysis of the corresponding numerical schemes:

Coercivity: Let \mathcal{D} be a gradient discretization and

$$\mathcal{C}_{\mathcal{D}} = \max_{0 \neq (v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0} \frac{\|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)}}{\|(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\|_{\mathcal{D}}}.$$

A sequence $(\mathcal{D}^l)_{l \in \mathbb{N}}$ of gradient discretizations is said to be coercive, if there exists $\bar{\mathcal{C}}_P > 0$ such that $\mathcal{C}_{\mathcal{D}^l} \leq \bar{\mathcal{C}}_P$ for all $l \in \mathbb{N}$.

Consistency: Let \mathcal{D} be a gradient discretization. For $u = (u_m, u_f) \in V^0$ and $v_{\mathcal{D}} = (v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$ let us define

$$\begin{aligned} s(v_{\mathcal{D}}, u) &= \|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} - \nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} - \nabla_{\tau} u_f\|_{L^2(\Gamma)^{d-1}} \\ &\quad + \|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} - u_m\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} \\ &\quad + \|\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} + \sum_{\alpha \in \mathcal{X}} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \gamma_{\alpha} u_m\|_{L^2(\Gamma_{\alpha})}. \end{aligned}$$

and $\mathcal{S}_{\mathcal{D}}(u) = \min_{v_{\mathcal{D}} \in X_{\mathcal{D}}^0} s(v_{\mathcal{D}}, u)$. A sequence $(\mathcal{D}^l)_{l \in \mathbb{N}}$ of gradient discretizations is said to be consistent, if for all $u = (u_m, u_f) \in V^0$ holds

$$\lim_{l \rightarrow \infty} \mathcal{S}_{\mathcal{D}^l}(u) = 0.$$

Limit Conformity: Let \mathcal{D} be a gradient discretization. For all $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$, $v_{\mathcal{D}} = (v_{\mathcal{D}_m}, v_{\mathcal{D}_f})$ we define

$$\begin{aligned} w(v_{\mathcal{D}}, \mathbf{q}) &= \int_{\Omega} \left(\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} \cdot \mathbf{q}_m + (\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m}) \operatorname{div} \mathbf{q}_m \right) dx \\ &\quad + \int_{\Gamma} \left(\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} \cdot \mathbf{q}_f + (\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f}) \operatorname{div}_{\tau} \mathbf{q}_f \right) d\tau(\mathbf{x}) \\ &\quad + \sum_{\alpha \in \mathcal{X}} \int_{\Gamma_{\alpha}} \gamma_{\mathbf{n}, \alpha} \mathbf{q}_m \left(\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} \right) d\tau(\mathbf{x}) \end{aligned}$$

and $\mathcal{W}_{\mathcal{D}}(\mathbf{q}) = \max_{0 \neq v_{\mathcal{D}} \in X_{\mathcal{D}}^0} \frac{1}{\|v_{\mathcal{D}}\|_{\mathcal{D}}} |w(v_{\mathcal{D}}, \mathbf{q})|$. A sequence $(\mathcal{D}^l)_{l \in \mathbb{N}}$ of gradient discretizations is said to be limit conforming, if for all $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$ holds

$$\lim_{l \rightarrow \infty} \mathcal{W}_{\mathcal{D}^l}(\mathbf{q}) = 0.$$

Lemma 3.1 *Let $(\mathcal{D}^l)_{l \in \mathbb{N}} = (X_{\mathcal{D}^l}^0, \Pi_{\mathcal{D}_m}^l, \Pi_{\mathcal{D}_f}^l, \tilde{\Pi}_{\mathcal{D}_f}^l, (\Pi_{\mathcal{D}_m}^{\alpha})_{\alpha \in \mathcal{X}}, \nabla_{\mathcal{D}_m}^l, \nabla_{\mathcal{D}_f}^l)_{l \in \mathbb{N}}$ and $(\bar{\mathcal{D}}^l)_{l \in \mathbb{N}} = (X_{\mathcal{D}^l}^0, \bar{\Pi}_{\mathcal{D}_m}^l, \bar{\Pi}_{\mathcal{D}_f}^l, \tilde{\bar{\Pi}}_{\mathcal{D}_f}^l, (\bar{\Pi}_{\mathcal{D}_m}^{\alpha})_{\alpha \in \mathcal{X}}, \nabla_{\mathcal{D}_m}^l, \nabla_{\mathcal{D}_f}^l)_{l \in \mathbb{N}}$ be two sequences of gradient discretizations of (5) and let us assume that $(\mathcal{D}^l)_{l \in \mathbb{N}}$ is coercive, consistent and limit conforming. Let*

us furthermore assume that the sequence $(\zeta_{\mathcal{D}^l, \bar{\mathcal{D}}^l})_{l \in \mathbb{N}}$, defined by

$$\begin{aligned} \zeta_{\mathcal{D}^l, \bar{\mathcal{D}}^l} := & \max_{0 \neq v_{\mathcal{D}^l} \in X_{\mathcal{D}^l}^0} \left(\frac{1}{\|v_{\mathcal{D}^l}\|_{\mathcal{D}^l}} \cdot \left(\|\Pi_{\mathcal{D}_m^l} v_{\mathcal{D}_m^l} - \bar{\Pi}_{\mathcal{D}_m^l} v_{\mathcal{D}_m^l}\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l} - \bar{\Pi}_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l}\|_{L^2(\Gamma)} \right. \right. \\ & \left. \left. + \|\tilde{\Pi}_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l} - \tilde{\bar{\Pi}}_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l}\|_{L^2(\Gamma)} + \sum_{\alpha \in \mathcal{X}} \|\Pi_{\mathcal{D}_m^l}^\alpha v_{\mathcal{D}_m^l} - \bar{\Pi}_{\mathcal{D}_m^l}^\alpha v_{\mathcal{D}_m^l}\|_{L^2(\Gamma_\alpha)} \right) \right), \end{aligned}$$

satisfies

$$\lim_{l \rightarrow \infty} \zeta_{\mathcal{D}^l, \bar{\mathcal{D}}^l} = 0 \quad (6)$$

and that there is a constant $C \in \mathbb{R}$ independent of l such that

$$\sum_{\alpha \in \mathcal{X}} \|\Pi_{\mathcal{D}_m^l}^\alpha v_{\mathcal{D}_m^l} - \tilde{\bar{\Pi}}_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l}\|_{L^2(\Gamma_\alpha)} \leq C \cdot \sum_{\alpha \in \mathcal{X}} \|\bar{\Pi}_{\mathcal{D}_m^l}^\alpha v_{\mathcal{D}_m^l} - \tilde{\bar{\Pi}}_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l}\|_{L^2(\Gamma_\alpha)} \quad (7)$$

for all $v_{\mathcal{D}^l} \in X_{\mathcal{D}^l}^0$, $l \in \mathbb{N}$. Then $(\bar{\mathcal{D}}^l)_{l \in \mathbb{N}}$ is coercive, consistent and limit conforming.

Proof Coercivity: $(\bar{\mathcal{D}}^l)_{l \in \mathbb{N}}$ is coercive, since for all $l \in \mathbb{N}$ we have (with $\mathcal{D} = \mathcal{D}^l$, $\bar{\mathcal{D}} = \bar{\mathcal{D}}^l$)

$$\|\bar{\Pi}_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} + \|\bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} \leq (\zeta_{\mathcal{D}, \bar{\mathcal{D}}} + \mathcal{C}_{\mathcal{D}}) \|v_{\mathcal{D}}\|_{\mathcal{D}} \leq \max(1, C) (\zeta_{\mathcal{D}, \bar{\mathcal{D}}} + \mathcal{C}_{\mathcal{D}}) \|v_{\mathcal{D}}\|_{\bar{\mathcal{D}}}$$

and since $\max(1, C) \cdot (\zeta_{\mathcal{D}^l, \bar{\mathcal{D}}^l} + \mathcal{C}_{\mathcal{D}^l})$ is uniformly bounded. In the last inequality we have used that $\|v_{\mathcal{D}}\|_{\mathcal{D}} \leq \max(1, C) \|v_{\mathcal{D}}\|_{\bar{\mathcal{D}}}$, which follows from (7).

Consistency: Let $l \in \mathbb{N}$ be fixed and $\mathcal{D} = \mathcal{D}^l$, $\bar{\mathcal{D}} = \bar{\mathcal{D}}^l$. We first choose, for a given $u = (u_m, u_f) \in V^0$, a $\underline{v}_{\mathcal{D}} \in X_{\mathcal{D}}^0$, such that $s_{\mathcal{D}}(\underline{v}_{\mathcal{D}}, u) = \mathcal{S}_{\mathcal{D}}(u)$. Using the inequality

$$\begin{aligned} s_{\bar{\mathcal{D}}}(\underline{v}_{\mathcal{D}}, u) & \leq s_{\mathcal{D}}(\underline{v}_{\mathcal{D}}, u) + \|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} \\ & \quad + \|\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \tilde{\bar{\Pi}}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} + \sum_{\alpha \in \mathcal{X}} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}\|_{L^2(\Gamma_\alpha)}, \end{aligned}$$

which holds for all $v_{\mathcal{D}} \in X_{\mathcal{D}}$, we obtain

$$\mathcal{S}_{\bar{\mathcal{D}}}(u) \leq \mathcal{S}_{\mathcal{D}}(u) + \zeta_{\mathcal{D}, \bar{\mathcal{D}}} \|\underline{v}_{\mathcal{D}}\|_{\mathcal{D}}.$$

Moreover

$$\|\underline{v}_{\mathcal{D}}\|_{\mathcal{D}} \leq \mathcal{S}_{\mathcal{D}}(u) + \|\nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_\tau u_f\|_{L^2(\Gamma)^{d-1}} + \sum_{\alpha \in \mathcal{X}} \|(\gamma_\alpha u_m - u_f)\|_{L^2(\Gamma_\alpha)},$$

which implies that $\|\underline{v}_{\mathcal{D}^l}\|_{\mathcal{D}^l}$ is uniformly bounded and therefore $\mathcal{S}_{\bar{\mathcal{D}}^l}(u) \rightarrow 0$ as $l \rightarrow \infty$.

Limit Conformity: Let again $l \in \mathbb{N}$ be fixed and $\mathcal{D} = \mathcal{D}^l$, $\bar{\mathcal{D}} = \bar{\mathcal{D}}^l$. For given $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$ and $v_{\mathcal{D}} \in X_{\mathcal{D}}^0$ we calculate

$$\begin{aligned} w_{\bar{\mathcal{D}}}(v_{\mathcal{D}}, \mathbf{q}) & \leq w_{\mathcal{D}}(v_{\mathcal{D}}, \mathbf{q}) + \|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} \cdot \|\operatorname{div} \mathbf{q}_m\|_{L^2(\Omega)} \\ & \quad + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} \cdot \|\operatorname{div}_\tau \mathbf{q}_f\|_{L^2(\Gamma)} + \sum_{\alpha \in \mathcal{X}} \left(\|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)} \right. \\ & \quad \left. + \|\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \tilde{\bar{\Pi}}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)} + \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}\|_{L^2(\Gamma_\alpha)} \right) \cdot \|\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m\|_{L^2(\Gamma_\alpha)} \\ & \leq w_{\mathcal{D}}(v_{\mathcal{D}}, \mathbf{q}) + \zeta_{\mathcal{D}, \bar{\mathcal{D}}} \cdot \|v_{\mathcal{D}}\|_{\mathcal{D}} \cdot \left(\|\operatorname{div} \mathbf{q}_m\|_{L^2(\Omega)} + \|\operatorname{div}_\tau \mathbf{q}_f\|_{L^2(\Gamma)} + \sum_{\alpha \in \mathcal{X}} \|\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m\|_{L^2(\Gamma_\alpha)} \right). \end{aligned}$$

Taking (7) into account, we derive

$$\mathcal{W}_{\overline{\mathcal{D}}}(\mathbf{q}_m, \mathbf{q}_f) \leq \max(1, C) \cdot \sup_{0 \neq v_{\mathcal{D}} \in X_{\mathcal{D}}} \frac{w_{\overline{\mathcal{D}}}(v_{\mathcal{D}}, \mathbf{q})}{\|v_{\mathcal{D}}\|_{\mathcal{D}}} \leq \max(1, C) \cdot (\mathcal{W}_{\mathcal{D}}(\mathbf{q}_m, \mathbf{q}_f) + \zeta_{\mathcal{D}, \overline{\mathcal{D}}} \|\mathbf{q}\|_W).$$

Therefore $\mathcal{W}_{\overline{\mathcal{D}}^l}(\mathbf{q}_m, \mathbf{q}_f)$ tends to zero as l goes to infinity. \square

Proposition 3.1 (Regularity at the Limit) *Let $(\mathcal{D}^l)_{l \in \mathbb{N}}$ be a coercive and limit conforming sequence of gradient discretizations and let $(v_{\mathcal{D}_m^l}, v_{\mathcal{D}_f^l})_{l \in \mathbb{N}}$ be a uniformly bounded sequence in $X_{\mathcal{D}^l}^0$. Then, there exist $(v_m, v_f) \in V^0$ and a subsequence still denoted by $(v_{\mathcal{D}_m^l}, v_{\mathcal{D}_f^l})_{l \in \mathbb{N}}$ such that*

$$\left\{ \begin{array}{l} \Pi_{\mathcal{D}_m^l} v_{\mathcal{D}_m^l} \rightharpoonup v_m \quad \text{in } L^2(\Omega), \\ \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m^l} \rightharpoonup \nabla v_m \quad \text{in } L^2(\Omega)^d, \\ \Pi_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l} \rightharpoonup v_f \quad \text{in } L^2(\Gamma), \\ \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f^l} \rightharpoonup \nabla_{\tau} v_f \quad \text{in } L^2(\Gamma)^{d-1}, \\ \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f^l} - \Pi_{\mathcal{D}_m}^{\alpha} v_{\mathcal{D}_m^l} \rightharpoonup v_f - \gamma_{\alpha} v_m \quad \text{in } L^2(\Gamma_{\alpha}), \text{ for all } \alpha \in \chi. \end{array} \right.$$

Proof By definition of the norm of $X_{\mathcal{D}^l}^0$ and by coercivity, $\Pi_{\mathcal{D}_m^l} v_{\mathcal{D}_m^l}$, $\Pi_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l}$, $\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m^l}$, $\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f^l}$ and $(\Pi_{\mathcal{D}_m}^{\alpha} u_{\mathcal{D}_m^l} - \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f^l})$, $\alpha \in \chi$, are uniformly bounded in L^2 (for $l \rightarrow \infty$). Therefore there exist $v_m \in L^2(\Omega)$, $v_f \in L^2(\Gamma)$, $G \in L^2(\Omega)^d$, $H \in L^2(\Gamma)^{d-1}$ and $J_{\alpha} \in L^2(\Gamma_{\alpha})$, $\alpha \in \chi$, and a subsequence still denoted by $(v_{\mathcal{D}_m^l}, v_{\mathcal{D}_f^l})_{l \in \mathbb{N}}$ such that

$$\begin{aligned} \Pi_{\mathcal{D}_m^l} v_{\mathcal{D}_m^l} &\rightharpoonup v_m && \text{in } L^2(\Omega), \\ \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m^l} &\rightharpoonup G && \text{in } L^2(\Omega)^d, \\ \Pi_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l} &\rightharpoonup v_f && \text{in } L^2(\Gamma), \\ \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f^l} &\rightharpoonup H && \text{in } L^2(\Gamma)^{d-1}, \\ \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f^l} - \Pi_{\mathcal{D}_m}^{\alpha} v_{\mathcal{D}_m^l} &\rightharpoonup J_{\alpha} && \text{in } L^2(\Gamma_{\alpha}), \text{ for } \alpha \in \chi. \end{aligned}$$

Using limit conformity we obtain (by letting $l \rightarrow \infty$)

$$\int_{\Omega} (G \cdot \mathbf{q}_m + v_m \operatorname{div} \mathbf{q}_m) \, d\mathbf{x} + \int_{\Gamma} (H \cdot \mathbf{q}_f + v_f \operatorname{div}_{\tau} \mathbf{q}_f) \, d\tau(\mathbf{x}) + \sum_{\alpha \in \chi} \int_{\Gamma_{\alpha}} \gamma_{\mathbf{n}, \alpha} \mathbf{q}_m (J_{\alpha} - v_f) \, d\tau(\mathbf{x}) = 0 \quad (8)$$

for all $(\mathbf{q}_m, \mathbf{q}_f) \in C_{W_m}^{\infty} \times C_{W_f}^{\infty}$. The statement of the proposition follows now from Lemma 2.1. \square

Corollary 3.1 *Let $(\mathcal{D}^l)_{l \in \mathbb{N}}$ be a sequence of gradient discretizations, assumed to be limit conforming against regular test functions $(\mathbf{q}_m, \mathbf{q}_f) \in C_{W_m}^{\infty} \times C_{W_f}^{\infty}$ and let $(v_{\mathcal{D}_m^l}, v_{\mathcal{D}_f^l})_{l \in \mathbb{N}}$ be a uniformly bounded sequence in $X_{\mathcal{D}^l}^0$, such that $\Pi_{\mathcal{D}_m^l} v_{\mathcal{D}_m^l}$ and $\Pi_{\mathcal{D}_f^l} v_{\mathcal{D}_f^l}$ are uniformly bounded in L^2 (for $l \rightarrow \infty$). Then holds the conclusion of Proposition 3.1.*

3.2 Application to (5)

The non conforming discrete variational formulation of the model problem is defined by: find $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$ such that

$$\begin{aligned} &\int_{\Omega} \Lambda_m \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \cdot \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} \, d\mathbf{x} + \int_{\Gamma} \Lambda_f \nabla_{\mathcal{D}_f} u_{\mathcal{D}_f} \cdot \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} \, d\tau_f(\mathbf{x}) + \sum_{i \in I} \int_{\Gamma_i} \frac{T_f}{2\xi - 1} \\ &\quad \sum_{(\alpha, \beta) \in \{(\alpha^{\pm}(i), \alpha^{\mp}(i))\}} \left(\xi \Pi_{\mathcal{D}_m}^{\alpha} u_{\mathcal{D}_m} + (1 - \xi) \Pi_{\mathcal{D}_m}^{\beta} u_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} \right) \left(\Pi_{\mathcal{D}_m}^{\alpha} v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} \right) \, d\tau(\mathbf{x}) \quad (9) \\ &- \int_{\Omega} h_m \Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} \, d\mathbf{x} - \int_{\Gamma} h_f \Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} \, d\tau_f(\mathbf{x}) = 0, \end{aligned}$$

for all $(v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$.

Proposition 3.2 *Let $\xi \in (\frac{1}{2}, 1]$ and \mathcal{D} be a gradient discretization, then (9) has a unique solution $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$ satisfying the a priori estimate*

$$\|(u_{\mathcal{D}_m}, u_{\mathcal{D}_f})\|_{\mathcal{D}} \leq C \left(\|h_m\|_{L^2(\Omega)} + \|h_f\|_{L^2(\Gamma)} \right)$$

with C depending only on ξ , $\mathcal{C}_{\mathcal{D}}$, $\underline{\lambda}_m$, $\underline{\lambda}_f$, \underline{d}_f , \bar{d}_f , and $\underline{\lambda}_{f,n}$.

Proof The Lax-Milgram Theorem applies, which ensures this result. \square

The main theoretical result for gradient schemes is stated by the following proposition:

Proposition 3.3 (Error Estimate) *Let $u = (u_m, u_f) \in V^0$, $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$ be the solution of (4). Let $\xi \in (\frac{1}{2}, 1]$, \mathcal{D} be a gradient discretization and $u_{\mathcal{D}} = (u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$ be the solution of (9). Then, there exists $C_0 > 0$ depending only on ξ , $\mathcal{C}_{\mathcal{D}}$, $\underline{\lambda}_m$, $\underline{\lambda}_f$, $\bar{\lambda}_m$, $\bar{\lambda}_f$, \underline{d}_f , \bar{d}_f , $\underline{\lambda}_{f,n}$, and $\bar{\lambda}_{f,n}$ such that one has the following error estimate:*

$$\begin{aligned} & \|\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} - u_m\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} + \|\tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} \\ & + \sum_{\alpha \in \chi} \|\Pi_{\mathcal{D}_m}^{\alpha} u_{\mathcal{D}_m} - \gamma_{\alpha} u_m\|_{L^2(\Gamma_{\alpha})} + \|\nabla u_m - \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)^d} + \|\nabla_{\tau} u_f - \nabla_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}} \\ & \leq C_0 (\mathcal{S}_{\mathcal{D}}(u_m, u_f) + \mathcal{W}_{\mathcal{D}}(\mathbf{q}_m, \mathbf{q}_f)). \end{aligned}$$

Proof From the definition of $\mathcal{W}_{\mathcal{D}}$, and using the definitions (4) of the solution u, \mathbf{q} and (9) of the discrete solution $u_{\mathcal{D}}$, it holds for all $(v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in V^0$

$$\begin{aligned} & \|(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\|_{\mathcal{D}} \cdot \mathcal{W}_{\mathcal{D}}(\mathbf{q}_m, \mathbf{q}_f) \\ & \geq \left| \int_{\Omega} \left(\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} \cdot \mathbf{q}_m + (\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m}) h_m \right) dx + \int_{\Gamma} \left(\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} \cdot \mathbf{q}_f + (\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f}) d_f h_f \right) d\tau(\mathbf{x}) \right. \\ & + \left. \sum_{i \in I} \int_{\Gamma_i} \frac{T_f}{2\xi - 1} \sum_{(\alpha, \beta) \in \{(\alpha^{\pm}(i), \alpha^{\mp}(i))\}} \left(\xi \gamma_{\alpha} u_m + (1 - \xi) \gamma_{\beta} u_m - u_f \right) \left(\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \Pi_{\mathcal{D}_m}^{\alpha} v_{\mathcal{D}_m} \right) d\tau(\mathbf{x}) \right| \\ & = \left| \int_{\Omega} \left(\Lambda_m \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} \cdot (\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} - \nabla u_m) \right) dx + \int_{\Gamma} \left(\Lambda_f \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} \cdot (\nabla_{\mathcal{D}_f} u_{\mathcal{D}_f} - \nabla_{\tau} u_f) \right) d\tau_f(\mathbf{x}) \right. \\ & + \left. \sum_{i \in I} \int_{\Gamma_i} \frac{T_f}{2\xi - 1} \sum_{(\alpha, \beta) \in \{(\alpha^{\pm}(i), \alpha^{\mp}(i))\}} \left(\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \Pi_{\mathcal{D}_m}^{\alpha} v_{\mathcal{D}_m} \right) \right. \\ & \quad \left. \times \left(\xi \gamma_{\alpha} u_m + (1 - \xi) \gamma_{\beta} u_m - u_f - \xi \Pi_{\mathcal{D}_m}^{\alpha} u_{\mathcal{D}_m} - (1 - \xi) \Pi_{\mathcal{D}_m}^{\beta} u_{\mathcal{D}_m} + \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} \right) d\tau(\mathbf{x}) \right| \end{aligned} \tag{10}$$

Let us choose $w_{\mathcal{D}} = (w_{\mathcal{D}_m}, w_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$, s.t. $s(w_{\mathcal{D}}, u) = \mathcal{S}_{\mathcal{D}}(u)$ and set $(v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) = u_{\mathcal{D}} - w_{\mathcal{D}}$ in (10). Then holds

$$\begin{aligned} & \|\nabla u_m - \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)^d} + \|\nabla_{\tau} u_f - \nabla_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}} \\ & + \sum_{\alpha \in \chi} \|\Pi_{\mathcal{D}_m}^{\alpha} u_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} - \gamma_{\alpha} u_m + u_f\|_{L^2(\Gamma_{\alpha})} \leq C \cdot (\mathcal{S}_{\mathcal{D}}(u_m, u_f) + \mathcal{W}_{\mathcal{D}}(\mathbf{q}_m, \mathbf{q}_f)), \end{aligned}$$

with a constant $C > 0$ depending only on ξ , $\underline{\lambda}_m$, $\underline{\lambda}_f$, $\bar{\lambda}_m$, $\bar{\lambda}_f$, \underline{d}_f , \bar{d}_f , $\underline{\lambda}_{f,n}$, and $\bar{\lambda}_{f,n}$. Taking coercivity into account leads to the statement of the proposition. \square

4 Two Examples of Gradient Schemes

Following [7], we consider generalised polyhedral meshes of Ω . Let \mathcal{M} be the set of cells that are disjoint open subsets of Ω such that $\bigcup_{K \in \mathcal{M}} \bar{K} = \bar{\Omega}$. For all $K \in \mathcal{M}$, \mathbf{x}_K denotes the so-called ‘‘center’’ of the cell K under the assumption that K is star-shaped with respect to \mathbf{x}_K . Let \mathcal{F} denote the set of faces of the mesh. The faces are not assumed to be planar for the VAG discretization, hence the term ‘‘generalised polyhedral cells’’, but they need to be planar for the HFV discretization. We denote by \mathcal{V} the set of vertices of the mesh. Let \mathcal{V}_K , \mathcal{F}_K , \mathcal{V}_σ respectively denote the set of the vertices of $K \in \mathcal{M}$, faces of K , and vertices of $\sigma \in \mathcal{F}$. For any face $\sigma \in \mathcal{F}_K$, we have $\mathcal{V}_\sigma \subset \mathcal{V}_K$. Let \mathcal{M}_s (resp. \mathcal{F}_s) denote the set of the cells (resp. faces) sharing the vertex $s \in \mathcal{V}$. The set of edges of the mesh is denoted by \mathcal{E} and \mathcal{E}_σ denotes the set of edges of the face $\sigma \in \mathcal{F}$. Let \mathcal{F}_e denote the set of faces sharing the edge $e \in \mathcal{E}$, and \mathcal{M}_σ denote the set of cells sharing the face $\sigma \in \mathcal{F}$. We denote by \mathcal{F}_{ext} the subset of faces $\sigma \in \mathcal{F}$ such that \mathcal{M}_σ has only one element, and we set $\mathcal{E}_{ext} = \bigcup_{\sigma \in \mathcal{F}_{ext}} \mathcal{E}_\sigma$, and $\mathcal{V}_{ext} = \bigcup_{\sigma \in \mathcal{F}_{ext}} \mathcal{V}_\sigma$. The mesh is assumed to be conforming in the sense that for all $\sigma \in \mathcal{F} \setminus \mathcal{F}_{ext}$, the set \mathcal{M}_σ contains exactly two cells. It is assumed that for each face $\sigma \in \mathcal{F}$, there exists a so-called ‘‘center’’ of the face \mathbf{x}_σ such that

$$\mathbf{x}_\sigma = \sum_{s \in \mathcal{V}_\sigma} \beta_{\sigma,s} \mathbf{x}_s, \quad \text{with} \quad \sum_{s \in \mathcal{V}_\sigma} \beta_{\sigma,s} = 1,$$

where $\beta_{\sigma,s} \geq 0$ for all $s \in \mathcal{V}_\sigma$. The face σ is assumed to match with the union of the triangles $T_{\sigma,e}$ defined by the face center \mathbf{x}_σ and each of its edge $e \in \mathcal{E}_\sigma$.

The mesh is assumed to be conforming w.r.t. the fracture network Γ in the sense that there exist subsets \mathcal{F}_{Γ_i} , $i \in I$ of \mathcal{F} such that

$$\bar{\Gamma}_i = \bigcup_{\sigma \in \mathcal{F}_{\Gamma_i}} \bar{\sigma}. \quad (11)$$

We will denote by \mathcal{F}_Γ the set of fracture faces $\bigcup_{i \in I} \mathcal{F}_{\Gamma_i}$. Similarly, we will denote by \mathcal{E}_Γ the set of fracture edges $\bigcup_{\sigma \in \mathcal{F}_\Gamma} \mathcal{E}_\sigma$ and by \mathcal{V}_Γ the set of fracture vertices $\bigcup_{\sigma \in \mathcal{F}_\Gamma} \mathcal{V}_\sigma$.

We also define a submesh \mathcal{T} of tetrahedra, where each tetrahedron $D_{K,\sigma,e}$ is the convex hull of the cell center \mathbf{x}_K of K , the face center \mathbf{x}_σ of $\sigma \in \mathcal{F}_K$ and the edge $e \in \mathcal{E}_\sigma$. Similarly we define a triangulation Δ of Γ , such that we have:

$$\mathcal{T} = \bigcup_{K \in \mathcal{F}, \sigma \in \mathcal{F}_K, e \in \mathcal{E}_\sigma} D_{K,\sigma,e} \quad \text{and} \quad \Delta = \bigcup_{\sigma \in \mathcal{F}_\Gamma, e \in \mathcal{E}_\sigma} T_{\sigma,e}.$$

We introduce for $D \in \mathcal{T}$ the diameter h_D of D and set $h_\mathcal{T} = \max_{D \in \mathcal{T}} h_D$. The regularity of our polyhedral mesh will be measured by the shape regularity of the tetrahedral submesh defined by $\theta_\mathcal{T} = \max_{D \in \mathcal{T}} \frac{h_D}{\rho_D}$ where ρ_D is the insphere diameter of $D \in \mathcal{T}$.

The set of matrix \times fracture degrees of freedom is denoted by $dof_{\mathcal{D}_m} \times dof_{\mathcal{D}_f}$. The real vector spaces $X_{\mathcal{D}_m}$ and $X_{\mathcal{D}_f}$ of discrete unknowns in the matrix and in the fracture network respectively are then defined by

$$\begin{aligned} X_{\mathcal{D}_m} &= \text{span}\{\mathbf{e}_\nu \mid \nu \in dof_{\mathcal{D}_m}\} \\ X_{\mathcal{D}_f} &= \text{span}\{\mathbf{e}_\nu \mid \nu \in dof_{\mathcal{D}_f}\}, \end{aligned}$$

where

$$\mathbf{e}_\nu = \begin{cases} (\delta_{\nu\mu})_{\mu \in dof_{\mathcal{D}_m}} & \text{for } \nu \in dof_{\mathcal{D}_m} \\ (\delta_{\nu\mu})_{\mu \in dof_{\mathcal{D}_f}} & \text{for } \nu \in dof_{\mathcal{D}_f}. \end{cases}$$

For $u_{\mathcal{D}_m} \in X_{\mathcal{D}_m}$ and $\nu \in \text{dof}_{\mathcal{D}_m}$ we denote by u_ν the ν th component of $u_{\mathcal{D}_m}$ and likewise for $u_{\mathcal{D}_f} \in X_{\mathcal{D}_f}$ and $\nu \in \text{dof}_{\mathcal{D}_f}$. We also introduce the product of these vector spaces

$$X_{\mathcal{D}} = X_{\mathcal{D}_m} \times X_{\mathcal{D}_f},$$

for which we have $\dim X_{\mathcal{D}} = \#\text{dof}_{\mathcal{D}_m} + \#\text{dof}_{\mathcal{D}_f}$.

To account for our homogeneous boundary conditions on $\partial\Omega$ and Σ_0 we introduce the subsets $\text{dof}_{\text{Dir}_m} \subset \text{dof}_{\mathcal{D}_m}$, and $\text{dof}_{\text{Dir}_f} \subset \text{dof}_{\mathcal{D}_f}$, and we set $\text{dof}_{\text{Dir}} = \text{dof}_{\text{Dir}_m} \times \text{dof}_{\text{Dir}_f}$, and

$$X_{\mathcal{D}}^0 = \{u \in X_{\mathcal{D}} \mid u_\nu = 0 \text{ for all } \nu \in \text{dof}_{\text{Dir}}\}.$$

4.1 Vertex Approximate Gradient Discretization

In this subsection, the VAG discretization introduced in [7] for diffusive problems on heterogeneous anisotropic media is extended to the hybrid dimensional model. We consider the \mathbb{P}_1 finite element construction as well as a finite volume version using lumping both for the source terms and the matrix fracture fluxes.

We first establish an equivalence relation on each \mathcal{M}_s , $s \in \mathcal{V}$, by

$$\begin{aligned} K \equiv_{\mathcal{M}_s} L \iff & \text{there exists } n \in \mathbb{N} \text{ and a sequence } (\sigma_i)_{i=1, \dots, n} \text{ in } \mathcal{F}_s \setminus \mathcal{F}_\Gamma, \\ & \text{such that } K \in \mathcal{M}_{\sigma_1}, L \in \mathcal{M}_{\sigma_n} \text{ and } \mathcal{M}_{\sigma_{i+1}} \cap \mathcal{M}_{\sigma_i} \neq \emptyset \\ & \text{for } i = 1, \dots, n-1. \end{aligned}$$

Let us then denote by $\overline{\mathcal{M}}_s$ the set of all classes of equivalence of \mathcal{M}_s and by \overline{K}_s the element of $\overline{\mathcal{M}}_s$ containing $K \in \mathcal{M}$. Obviously $\overline{\mathcal{M}}_s$ might have more than one element only if $s \in \mathcal{V}_\Gamma$. Then we define (cf. figure 2)

$$\begin{aligned} \text{dof}_{\mathcal{D}_m} &= \mathcal{M} \cup \left\{ K_\sigma \mid \sigma \in \mathcal{F}_\Gamma, K \in \mathcal{M}_\sigma \right\} \cup \left\{ \overline{K}_s \mid s \in \mathcal{V}, \overline{K}_s \in \overline{\mathcal{M}}_s \right\}, \\ \text{dof}_{\mathcal{D}_f} &= \mathcal{F}_\Gamma \cup \mathcal{V}_\Gamma, \\ \text{dof}_{\text{Dir}_m} &:= \left\{ \overline{K}_s \mid s \in \mathcal{V}_{\text{ext}}, \overline{K}_s \in \overline{\mathcal{M}}_s \right\}, \\ \text{dof}_{\text{Dir}_f} &= \mathcal{V}_\Gamma \cap \mathcal{V}_{\text{ext}}. \end{aligned}$$

We thus have

$$\begin{aligned} X_{\mathcal{D}_m} &= \left\{ u_K \mid K \in \mathcal{M} \right\} \cup \left\{ u_{K_\sigma} \mid \sigma \in \mathcal{F}_\Gamma, K \in \mathcal{M}_\sigma \right\} \\ &\quad \cup \left\{ u_{\overline{K}_s} \mid s \in \mathcal{V}, \overline{K}_s \in \overline{\mathcal{M}}_s \right\}, \\ X_{\mathcal{D}_f} &= \left\{ u_\sigma \mid \sigma \in \mathcal{F}_\Gamma \right\} \cup \left\{ u_s \mid s \in \mathcal{V}_\Gamma \right\}. \end{aligned} \tag{12}$$

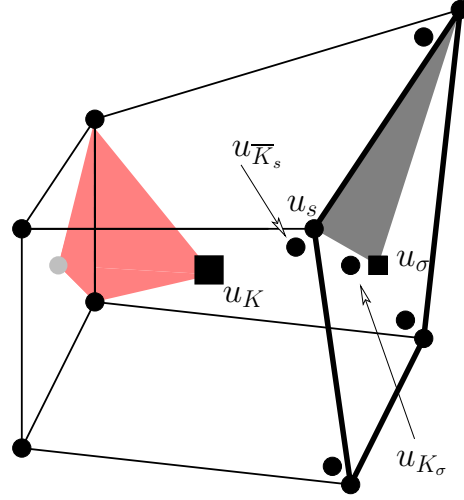
Now we can introduce the piecewise affine interpolators (or reconstruction operators)

$$\Pi_{\mathcal{T}}: X_{\mathcal{D}_m} \longrightarrow H^1(\Omega \setminus \overline{\Gamma}) \quad \text{and} \quad \Pi_{\Delta}: X_{\mathcal{D}_f} \longrightarrow H^1(\Gamma),$$

which act linearly on $X_{\mathcal{D}_m}$ and $X_{\mathcal{D}_f}$, such that $\Pi_{\mathcal{T}} u_{\mathcal{D}_m}$ is affine on each $D_{K,\sigma,e} \in \mathcal{T}$ and satisfies on each cell $K \in \mathcal{M}$

$$\begin{aligned} \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_K) &= u_K, \\ \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_s) &= u_{\overline{K}_s} \quad \forall s \in \mathcal{V}_K, \\ \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_\sigma) &= u_{K_\sigma} \quad \forall \sigma \in \mathcal{F}_K \cap \mathcal{F}_\Gamma, \\ \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_\sigma) &= \sum_{s \in \mathcal{V}_\sigma} \beta_{\sigma,s} u_{\overline{K}_s} \quad \forall \sigma \in \mathcal{F}_K \setminus \mathcal{F}_\Gamma, \end{aligned}$$

Figure 2: Cell K touching a fracture face σ . Illustration of the simplices on which:
 Red: $\nabla_{\mathcal{D}_m}$ is constant.
 Grey: $\nabla_{\mathcal{D}_f}$ is constant.



while $\Pi_{\Delta}u_{\mathcal{D}_f}$ is affine on each $T_{\sigma,e} \in \Delta$ and satisfies for all $\nu \in \text{dof}_{\mathcal{D}_f}$

$$\Pi_{\Delta}u_{\mathcal{D}_f}(\mathbf{x}_{\nu}) = u_{\nu},$$

where $\mathbf{x}_{\nu} \in \bar{\Omega}$ is the grid point associated with the degree of freedom $\nu \in \text{dof}_{\mathcal{D}_m} \cup \text{dof}_{\mathcal{D}_f}$. The discrete gradients on $X_{\mathcal{D}_m}$ and $X_{\mathcal{D}_f}$ are subsequently defined by

$$\nabla_{\mathcal{D}_m} = \nabla \Pi_{\mathcal{T}} \quad \text{and} \quad \nabla_{\mathcal{D}_f} = \nabla_{\tau} \Pi_{\Delta}. \quad (13)$$

We define the VAG-FE scheme's reconstruction operators by

- $\Pi_{\mathcal{D}_m} = \Pi_{\mathcal{T}}$,
 - $\Pi_{\mathcal{D}_f} = \tilde{\Pi}_{\mathcal{D}_f} = \Pi_{\Delta}$,
 - $\Pi_{\mathcal{D}_m}^{\alpha} = \gamma_{\alpha} \Pi_{\mathcal{T}} \quad \text{for all } \alpha \in \chi$.
- (14)

For the family of VAG-CV schemes, reconstruction operators are piecewise constant. We introduce, for any given $K \in \mathcal{M}$, a partition

$$\bar{K} = \bar{w}_K \cup \left(\bigcup_{s \in \mathcal{V}_K \setminus \mathcal{V}_{ext}} \bar{w}_{K, \bar{K}_s} \right) \cup \left(\bigcup_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_{\Gamma}} \bar{w}_{K\sigma} \right).$$

Similarly, we define for any given $\sigma \in \mathcal{F}_{\Gamma}$ a partition

$$\bar{\sigma} = \bar{w}_{\sigma} \cup \left(\bigcup_{s \in \mathcal{V}_{\sigma} \setminus \mathcal{V}_{ext}} \bar{w}_{\sigma, s} \right).$$

With each $s \in \mathcal{V} \setminus \mathcal{V}_{ext}$ and $\bar{K}_s \in \bar{\mathcal{M}}_s$ we associate an open set $\omega_{\bar{K}_s}$, satisfying

$$\omega_{\bar{K}_s} = \bigcup_{K \in \bar{K}_s} \bar{w}_{K, \bar{K}_s}.$$

Similarly, for all $s \in \mathcal{V}_{\Gamma} \setminus \mathcal{V}_{ext}$ we define ω_s by

$$\omega_s = \bigcup_{\sigma \in \mathcal{F}_s \cap \mathcal{F}_{\Gamma}} \bar{w}_{\sigma, s}.$$

We obtain the partitions

$$\bar{\Omega} = \left(\bigcup_{\nu \in \text{dof}_{\mathcal{D}_m} \setminus \text{dof}_{\text{Dir}_m}} \bar{\omega}_\nu \right), \quad \bar{\Gamma} = \left(\bigcup_{\nu \in \text{dof}_{\mathcal{D}_f} \setminus \text{dof}_{\text{Dir}_f}} \bar{\omega}_\nu \right).$$

We also introduce for each $T = T_{\sigma, s, s'} \in \Delta$ a partition $\bar{T} = \bigcup_{i=1}^3 \bar{T}_i$, which we need for the definition of the VAG-CV matrix-fracture interaction operators. We assume that holds $|T_1| = |T_2| = |T_3| = \frac{1}{3}|T|$ in order to preserve the first order convergence of the scheme.

Finally, we need a mapping between the degrees of freedom of the matrix domain, which are situated on one side of the fracture network, and the set of indices χ . For $K_\sigma \in \text{dof}_{\mathcal{D}_m}$ we have the one-element set $\chi(K_\sigma) = \{\alpha \in \chi \mid \mathbf{n}_{K, \sigma} = \mathbf{n}_\alpha \text{ on } \sigma\}$ and therefore the notation $\alpha(K_\sigma) = \alpha \in \chi(K_\sigma)$.

The VAG-CV scheme's reconstruction operators are

$$\begin{aligned} \bullet \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} &= \sum_{\nu \in \text{dof}_{\mathcal{D}_m} \setminus \text{dof}_{\text{Dir}_m}} u_\nu \mathbb{1}_{\omega_\nu}, \\ \bullet \Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} &= \sum_{\nu \in \text{dof}_{\mathcal{D}_f} \setminus \text{dof}_{\text{Dir}_f}} u_\nu \mathbb{1}_{\omega_\nu}, \\ \bullet \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} &= \sum_{T_{\sigma, s, s'} \in \Delta} (u_\sigma \mathbb{1}_{T_1} + u_s \mathbb{1}_{T_2} + u_{s'} \mathbb{1}_{T_3}), \\ \bullet \Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} &= \sum_{T_{\sigma, s, s'} \in \Delta} \sum_{K \in \mathcal{M}_\sigma} (u_{K_\sigma} \mathbb{1}_{T_1} + u_{\bar{K}_s} \mathbb{1}_{T_2} + u_{\bar{K}_{s'}} \mathbb{1}_{T_3}) \delta_{\alpha(K_\sigma), \alpha} \mathbb{1}_{\Gamma_\alpha}. \end{aligned} \tag{15}$$

Remark 4.1 *The VAG-CV scheme leads us to recover fluxes for the matrix-fracture interactions involving degrees of freedom located at the same physical point (see subsection 4.3).*

Proposition 4.1 *Let us consider a sequence of meshes $(\mathcal{M}^l)_{l \in \mathbb{N}}$ and let us assume that the sequence $(\mathcal{T}^l)_{l \in \mathbb{N}}$ of tetrahedral submeshes is shape regular, i.e. $\theta_{\mathcal{T}^l}$ is uniformly bounded. We also assume that $\lim_{l \rightarrow \infty} h_{\mathcal{T}^l} = 0$. Then, the corresponding sequence of gradient discretizations $(\mathcal{D}^l)_{l \in \mathbb{N}}$, defined by (12), (13), (14), is coercive, consistent and limit conforming.*

Proof The VAG-FE scheme's reconstruction operators are conforming, i.e. $V_{\mathcal{D}} \subset V^0$. Therefore we deduce coercivity from Proposition 2.1. Furthermore we have by partial integration $\mathcal{W}_{\mathcal{D}}(\mathbf{q}_m, \mathbf{q}_f) = 0$ for all $(\mathbf{q}_m, \mathbf{q}_f) \in W$. Hence $(\mathcal{D}^l)_{l \in \mathbb{N}}$ is limit conforming.

To prove consistency, we need the following prerequisites. We define the linear mapping $P_{\mathcal{D}_m} : C_\Omega^\infty \rightarrow X_{\mathcal{D}_m}^0$ such that for all $\psi_m \in C_\Omega^\infty$ and any cell $K \in \mathcal{M}$ one has

$$\begin{aligned} (P_{\mathcal{D}_m} \psi_m)_K &= \psi_m(\mathbf{x}_K), \\ (P_{\mathcal{D}_m} \psi_m)_{\bar{K}_s} &= \psi_m(\mathbf{x}_s) \quad \forall s \in \mathcal{V}_K, \\ (P_{\mathcal{D}_m} \psi_m)_{K_\sigma} &= \psi_m(\mathbf{x}_\sigma) \quad \forall \sigma \in \mathcal{F}_K \cap \mathcal{F}_\Gamma. \end{aligned}$$

Likewise, we define the linear mapping $P_{\mathcal{D}_f} : C_\Gamma^\infty \rightarrow X_{\mathcal{D}_f}^0$ such that for all $\psi_f \in C_\Gamma^\infty$ holds $(P_{\mathcal{D}_f} \psi_f)_\nu = \psi_f(\mathbf{x}_\nu)$ for all $\nu \in \text{dof}_{\mathcal{D}_f}$. It follows from the classical Finite Element approximation theory and from the fact that the interpolation $\sum_{s \in \mathcal{V}_\sigma} \beta_{\sigma, s} (P_{\mathcal{D}_m} \psi_m)_{\bar{K}_s}$ at the point \mathbf{x}_σ , $\sigma \in \mathcal{F}_K \setminus \mathcal{F}_\Gamma$ is exact on cellwise affine functions, that for all $(\psi_m, \psi_f) \in C_\Omega^\infty \times C_\Gamma^\infty$ holds

$$\|\Pi_{\mathcal{T}} P_{\mathcal{D}_m} \psi_m - \psi_m\|_{H^1(\Omega \setminus \bar{\Gamma})} + \|\Pi_{\Delta} P_{\mathcal{D}_f} \psi_f - \psi_f\|_{H^1(\Gamma)} \leq C(\psi_m, \psi_f, \theta_{\mathcal{T}}) h_{\mathcal{T}}. \tag{16}$$

The trace inequality implies that for all $v \in H_{\partial\Omega}^1(\Omega \setminus \bar{\Gamma})$ holds

$$\|\gamma_\alpha v\|_{L^2(\Gamma_\alpha)} \leq C(\Omega \setminus \bar{\Gamma}) \|v\|_{H^1(\Omega \setminus \bar{\Gamma})} \quad \text{for } \alpha \in \chi.$$

We can then calculate for $(u_m, u_f) \in C_\Omega^\infty \times C_\Gamma^\infty$:

$$\begin{aligned} \mathcal{S}_{\mathcal{D}}(u_m, u_f) &\leq \sqrt{2} \|\Pi_{\mathcal{T}} P_{\mathcal{D}_m} u_m - u_m\|_{H^1(\Omega \setminus \bar{\Gamma})} + \sum_{\alpha \in \chi} \|\gamma_\alpha (\Pi_{\mathcal{T}} P_{\mathcal{D}_m} u_m - u_m)\|_{L^2(\Gamma_\alpha)} \\ &\quad + \sum_{i \in I} \sqrt{8} \|\Pi_{\Delta} P_{\mathcal{D}_f} u_f - u_f\|_{H^1(\Gamma_i)} \\ &\leq C(\Omega \setminus \bar{\Gamma}, \#\chi, \#I, (u_m, u_f), \theta_{\mathcal{T}}) h_{\mathcal{T}}. \end{aligned}$$

Since $C_\Omega^\infty \times C_\Gamma^\infty$ is dense in V^0 , the sequence of VAG-FE discretisations $(\mathcal{D}_m^l)_{l \in \mathbb{N}}$ is consistent if $h_{\mathcal{T}^l} \rightarrow 0$ and $\theta_{\mathcal{T}^l}$ is bounded for $l \rightarrow \infty$. \square

Proposition 4.2 *Let us consider a sequence of meshes $(\mathcal{M}^l)_{l \in \mathbb{N}}$ and let us assume that the sequence $(\mathcal{T}^l)_{l \in \mathbb{N}}$ of tetrahedral submeshes is shape regular, i.e. $\theta_{\mathcal{T}^l}$ is uniformly bounded. We also assume that $\lim_{l \rightarrow \infty} h_{\mathcal{T}^l} = 0$. Then, any corresponding sequence of gradient discretizations $(\mathcal{D}^l)_{l \in \mathbb{N}}$, defined by (12), (13), (15), is coercive, consistent and limit conforming.*

Proof We combine Lemma 3.1 and Proposition 4.1. Thus, we have to show that the assumptions of Lemma 3.1 are satisfied, where $(\bar{\mathcal{D}}^l)_{l \in \mathbb{N}}$ corresponds to the sequence of VAG-CV gradient discretisations and $(\mathcal{D}^l)_{l \in \mathbb{N}}$ to the corresponding sequence of VAG-FE gradient discretisations.

For the following, we define $\mathcal{F}^\alpha = \bigcup_{i \in I_\alpha} \mathcal{F}_{\Gamma_i}$ and $\mathcal{V}^\alpha = \bigcup_{\sigma \in \mathcal{F}^\alpha} \mathcal{V}_\sigma$. To ease the notation in the proof, we will use, for $\alpha \in \chi$, the uniquely identified mapping $\mu^\alpha: \mathcal{V}^\alpha \cup \mathcal{F}^\alpha \subset \text{dof}_{\mathcal{D}_f} \rightarrow \text{dof}_{\mathcal{D}_m}$, defined by $\mu^\alpha(\sigma) = K_\sigma$ (such that $\chi(K_\sigma) = \{\alpha\}$) and $\mu^\alpha(s) = \bar{K}_s$ (for a cell K such that $K \in \mathcal{M}_\sigma$ with $\sigma \in \mathcal{F}^\alpha \cap \mathcal{F}_s$ and $\chi(K_\sigma) = \{\alpha\}$). Let now $\alpha \in \chi$ be fixed. Since the mesh is conforming with respect to the fracture network, there is for every $\sigma \in \mathcal{F}^\alpha$, $e = ss' \in \mathcal{E}_\sigma$ a $\nu(\sigma, e) \in \{\sigma, s, s'\}$, such that

$$\sup_{\mathbf{x} \in T_{\sigma, e}} |(\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f})(\mathbf{x})| = |(\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f})(\mathbf{x}_{\nu(\sigma, e)})| = |v_{\mu^\alpha(\nu(\sigma, e))} - v_{\nu(\sigma, e)}|.$$

Then we have

$$\begin{aligned} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)}^2 &\leq \sum_{\sigma \in \mathcal{F}^\alpha} \sum_{e \in \mathcal{E}_\sigma} |T_{\sigma, e}| |v_{\mu^\alpha(\nu(\sigma, e))} - v_{\nu(\sigma, e)}|^2 \\ &\leq 3 \|\bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)}^2. \end{aligned}$$

We have to check (6) now. It can be verified that [4], Lemma 3.4 applies to our case, both, in the matrix domain, where face unknowns might occur, as well as in the fracture network, a domain of codimension 1. This means that we can state that there exist constants $C_m(\theta_{\mathcal{T}}), C_f(\theta_{\mathcal{T}}) > 0$, such that

$$\|\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} \leq C_m \cdot h_{\mathcal{T}} \cdot \|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)^d} \quad \text{and} \quad (17)$$

$$\|\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} = \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} \leq C_f \cdot h_\Delta \cdot \|\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}} \quad (18)$$

For the following calculation we take into account [4], Lemmata 3.2 and 3.4. We also use that the mesh is conforming with respect to the fracture network and that for $\sigma \in \mathcal{F}$ and $K \in \mathcal{M}_\sigma$ (or equivalently for $K \in \mathcal{M}$, $\sigma \in \mathcal{F}_K$) holds: h_K is asymptotically equivalent to h_σ and $|K|$ is

asymptotically equivalent to $h_\sigma|\sigma|$, where $h_K := \max_{\mathcal{T} \ni D \subset K} h_D$ and $h_\sigma := \max_{\Delta \ni T \subset \sigma} h_T$. Let $\alpha \in \chi$, $\sigma \in \mathcal{F}^\alpha$ and $K \in \mathcal{M}_\sigma$, such that $\chi(K_\sigma) = \{\alpha\}$. Then we have

$$\begin{aligned} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}\|_{L^2(\sigma)}^2 &= \left\| \sum_{\nu \in \{\sigma\} \cup (\mathcal{V}_\sigma)} v_{\mu^\alpha(\nu)} (\Pi_{\mathcal{D}_f} \mathbf{e}_\nu - \bar{\Pi}_{\mathcal{D}_f} \mathbf{e}_\nu) \right\|_{L^2(\sigma)}^2 \\ &\leq C \cdot |\sigma| \sum_{s \in \mathcal{V}_\sigma} (v_{\mu^\alpha(s)} - v_{\mu^\alpha(\sigma)})^2 \\ &\leq C \cdot \frac{|K|}{|h_K|} \cdot \left(\sum_{s \in \mathcal{V}_K} (v_{\bar{K}_s} - v_K)^2 + \sum_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_T} (v_{K_\sigma} - v_K)^2 \right) \leq C \cdot h_\sigma \cdot \|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(K)}^2. \end{aligned}$$

Therefore

$$\|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}\|_{L^2(\Gamma_\alpha)}^2 \leq \sum_{\sigma \in \mathcal{F}^\alpha} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}\|_{L^2(\sigma)}^2 \leq C \cdot h_\Delta \cdot \|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)}^2. \quad (19)$$

Altogether we obtain

$$\begin{aligned} &\|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} + \|\tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \tilde{\bar{\Pi}}_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)} \\ &\quad + \sum_{\alpha \in \chi} \|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}\|_{L^2(\Gamma_\alpha)} \leq C \cdot (h_\mathcal{T} + h_\Delta + h_\Delta^{\frac{1}{2}}) \cdot \|(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\|_{\mathcal{D}}, \end{aligned}$$

with a constant C depending only on $\#\chi$ and $\theta_\mathcal{T}$. This proves that (6) is satisfied. \square

Corollary 4.1 *The precedent proof shows that $\mathcal{S}_\mathcal{D}(u_m, u_f) = \mathcal{O}(h_\mathcal{T}^{\frac{1}{2}})$ for $(u_m, u_f) \in C_\Omega^\infty \times C_\Gamma^\infty$ and that $\mathcal{W}_\mathcal{D}(\mathbf{q}_m, \mathbf{q}_f) = \mathcal{O}(h_\mathcal{T}^{\frac{1}{2}})$ for $(\mathbf{q}_m, \mathbf{q}_f) \in C_{W_m}^\infty \times C_{W_f}^\infty$. However, we can prove a higher order of convergence, i.e. $\mathcal{W}_\mathcal{D}(\mathbf{q}_m, \mathbf{q}_f) = \mathcal{O}(h_\mathcal{T})$ for $(\mathbf{q}_m, \mathbf{q}_f) \in C_{W_m}^\infty \times C_{W_f}^\infty$ and $\mathcal{S}_\mathcal{D}(u_m, u_f) = \mathcal{O}(h_\mathcal{T})$ for $(u_m, u_f) \in C_\Omega^\infty \times C_\Gamma^\infty$.*

Proof Consistency: Classically, for all $(\varphi_m, \varphi_f) \in C_\Omega^\infty \times C_\Gamma^\infty$, we have the estimate

$$\begin{aligned} &\|\Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi_m - \varphi_m\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_m}^\alpha P_{\mathcal{D}_m} \varphi_m - \gamma_\alpha \varphi_m\|_{L^2(\Gamma_\alpha)} \\ &\quad + \|\Pi_{\mathcal{D}_f} P_{\mathcal{D}_f} \varphi_f - \varphi_f\|_{L^2(\Gamma)} + \|\tilde{\Pi}_{\mathcal{D}_f} P_{\mathcal{D}_f} \varphi_f - \varphi_f\|_{L^2(\Gamma)} \leq cst(\varphi_m, \varphi_f) \cdot h_\mathcal{T}, \end{aligned}$$

while (16) grants that holds

$$\|\nabla_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi_m - \nabla \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}_f} P_{\mathcal{D}_f} \varphi_f - \nabla \varphi\|_{L^2(\Gamma)} \leq cst(\varphi_m, \varphi_f, \theta_\mathcal{T}) h_\mathcal{T}.$$

Taking into account that $C_\Omega^\infty \times C_\Gamma^\infty$ is dense in V , we see that the treated discretisation is consistent with $\mathcal{S}_\mathcal{D}(\varphi_m, \varphi_f) = \mathcal{O}(h_\mathcal{T})$ for $(\varphi_m, \varphi_f) \in C_\Omega^\infty \times C_\Gamma^\infty$.

Limit Conformity: For all $T \in \Delta$ and for all $u_{\mathcal{D}_m} \in X_{\mathcal{D}_m}$ we have that

$$\int_T (\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}) d\tau(\mathbf{x}) = 0.$$

Introducing the linear operator $P : L^2(\Gamma_\alpha) \rightarrow L^2(\Gamma_\alpha)$ such that $P(\varphi) = \frac{1}{|T|} \int_T \varphi d\tau(\mathbf{x})$ on T for all $T \in \Delta$, we first calculate for any $\mathbf{q}_m \in C_{W_m}^\infty$

$$\|\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m - P(\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m)\|_{L^2(\Gamma_\alpha)}^2 = \sum_{\sigma \in \mathcal{F}_\alpha} \sum_{\Delta \ni T \subset \sigma} \|\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m - P(\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m)\|_{L^2(T)}^2 \leq C(\mathbf{q}_m, \theta_\mathcal{T}) \cdot h_\mathcal{T}^2.$$

We proceed:

$$\begin{aligned}
& \left| \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m (\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}) d\tau(\mathbf{x}) \right| \\
&= \left| \int_{\Gamma_\alpha} (\gamma_{\mathbf{n},\alpha} \mathbf{q}_m - P(\gamma_{\mathbf{n},\alpha} \mathbf{q}_m)) (\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}) d\tau(\mathbf{x}) \right| \\
&\leq \|\gamma_{\mathbf{n},\alpha} \mathbf{q}_m - P(\gamma_{\mathbf{n},\alpha} \mathbf{q}_m)\|_{L^2(\Gamma_\alpha)} \|\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}\|_{L^2(\Gamma_\alpha)} \\
&\leq C(\mathbf{q}_m, \theta_{\mathcal{T}}) h_{\mathcal{T}}^{\frac{3}{2}} \|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)}
\end{aligned}$$

for all $\mathbf{q}_m \in C_{W_m}^\infty$, where we have used (19) in the last inequality. We can now conclude by calculating for all for $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in C_{W_m}^\infty \times C_{W_f}^\infty$

$$\begin{aligned}
w_{\bar{\mathcal{D}}}(u_{\mathcal{D}}, \mathbf{q}) &= (w_{\bar{\mathcal{D}}} - w_{\mathcal{D}})(u_{\mathcal{D}}, \mathbf{q}) \\
&= \int_{\Omega} \operatorname{div} \mathbf{q}_m (\bar{\Pi}_{\mathcal{D}_m} - \Pi_{\mathcal{D}_m}) u_{\mathcal{D}_m} d\mathbf{x} + \int_{\Gamma} \operatorname{div}_{\tau} \mathbf{q}_f (\bar{\Pi}_{\mathcal{D}_f} - \Pi_{\mathcal{D}_f}) u_{\mathcal{D}_f} d\tau(\mathbf{x}) \\
&+ \sum_{\alpha \in \mathcal{X}} \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m \left((\tilde{\Pi}_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f}) u_{\mathcal{D}_f} - (\bar{\Pi}_{\mathcal{D}_f} - \Pi_{\mathcal{D}_f}) u_{\mathcal{D}_f} - (\bar{\Pi}_{\mathcal{D}_m}^\alpha - \Pi_{\mathcal{D}_m}^\alpha) u_{\mathcal{D}_m} \right) d\tau(\mathbf{x}) \\
&\leq \|\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)} \cdot \|\operatorname{div} \mathbf{q}_m\|_{L^2(\Omega)} \\
&+ \|\Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)} \cdot \|\operatorname{div}_{\tau} \mathbf{q}_f\|_{L^2(\Gamma)} + \sum_{\alpha \in \mathcal{X}} \left(\|\tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)} \right. \\
&+ \|\tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} - \bar{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma_\alpha)} \cdot \|\gamma_{\mathbf{n},\alpha} \mathbf{q}_m\|_{L^2(\Gamma_\alpha)} \\
&\left. + \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m (\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} - \bar{\Pi}_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}) d\tau(\mathbf{x}) \right) \leq C(\theta_{\mathcal{T}}, \mathbf{q}) \cdot h_{\mathcal{T}} \cdot \|u_{\mathcal{D}}\|_{\mathcal{D}},
\end{aligned}$$

where we have taken into account the conformity of \mathcal{D} in the first equation and (17), (18) in the last inequality. \square

Remark 4.2 *The proofs of Propositions 4.1 and 4.2 show that for solutions $(u_m, u_f) \in V^0$ and $(\mathbf{q}_m, \mathbf{q}_f) \in W$ of (4) such that $u_m \in C^2(\bar{K})$, $u_f \in C^2(\bar{\sigma})$, $\mathbf{q}_m \in (C^1(\bar{K}))^d$, $\mathbf{q}_f \in (C^1(\bar{\sigma}))^{d-1}$ for all $K \in \mathcal{M}$ and all $\sigma \in \Gamma_f$, the VAG schemes are consistent and limit conforming of order 1, and therefore convergent of order 1.*

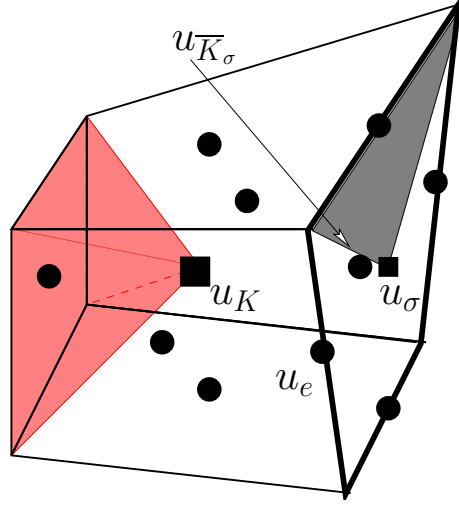
4.2 Hybrid Finite Volume Discretization

In this subsection, the HFV scheme introduced in [8] is extended to the hybrid dimensional Darcy flow model. We assume here that the faces are planar and that \mathbf{x}_σ is the barycenter of σ for all $\sigma \in \mathcal{F}$.

The set of indices $dof_{\mathcal{D}_m} \times dof_{\mathcal{D}_f}$ for the unknowns is defined by (cf. figure 3)

$$\begin{aligned}
dof_{\mathcal{D}_m} &= \mathcal{M} \cup \left(\bigcup_{\sigma \in \mathcal{F}} \bar{\mathcal{M}}_\sigma \right) \\
dof_{\mathcal{D}_f} &= \mathcal{F}_\Gamma \cup \mathcal{E}_\Gamma, \\
dof_{Dir_m} &= \mathcal{F}_{ext}, \\
dof_{Dir_f} &= \mathcal{E}_\Gamma \cap \mathcal{E}_{ext},
\end{aligned}$$

Figure 3: Cell K touching a fracture face σ . Illustration of the polyhedron and polygone on which:
 Red: $\nabla_{\mathcal{D}_m}$ is constant.
 Grey: $\nabla_{\mathcal{D}_f}$ is constant.



where for $\sigma \in \mathcal{F}$ and $K \in \mathcal{M}_\sigma$

$$\bar{K}_\sigma = \begin{cases} \mathcal{M}_\sigma & \text{if } \sigma \in \mathcal{F} \setminus \mathcal{F}_\Gamma \\ \{K\} & \text{if } \sigma \in \mathcal{F}_\Gamma. \end{cases}$$

and $\bar{\mathcal{M}}_\sigma = \{\bar{K}_\sigma \mid K \in \mathcal{M}_\sigma\}$. We thus have

$$\begin{aligned} X_{\mathcal{D}_m} &= \{u_K \mid K \in \mathcal{M}\} \cup \{u_{\bar{K}_\sigma} \mid \sigma \in \mathcal{F}_\Gamma, \bar{K}_\sigma \in \bar{\mathcal{M}}_\sigma\}, \\ X_{\mathcal{D}_f} &= \{u_\sigma \mid \sigma \in \mathcal{F}_\Gamma\} \cup \{u_e \mid e \in \mathcal{E}_\Gamma\}. \end{aligned} \quad (20)$$

The discrete gradients in the matrix (respectively in the fracture domain) are defined in each cell (respectively in each face) by the 3D (respectively 2D) discrete gradients

$$\nabla_{\mathcal{D}_m} \text{ (resp. } \nabla_{\mathcal{D}_f}\text{) as proposed in [8], pp. 8-9.} \quad (21)$$

The function reconstruction operators are piecewise constant on a partition of the cells and of the fracture faces.

These partitions are respectively denoted, for all $K \in \mathcal{M}$, by

$$\bar{K} = \bar{\omega}_K \cup \left(\bigcup_{\sigma \in \mathcal{F}_K \setminus \mathcal{F}_{ext}} \bar{\omega}_{K, \bar{K}_\sigma} \right),$$

and, for all $\sigma \in \mathcal{F}_\Gamma$, by

$$\bar{\sigma} = \bar{\omega}_\sigma \cup \left(\bigcup_{e \in \mathcal{E}_\sigma \setminus \mathcal{E}_{ext}} \bar{\omega}_{\sigma, e} \right).$$

With each $\sigma \in \mathcal{F} \setminus \mathcal{F}_{ext}$ and $\bar{K}_\sigma \in \bar{\mathcal{M}}_\sigma$ we associate an open set $\omega_{\bar{K}_\sigma}$, s.t.

$$\bar{\omega}_{\bar{K}_\sigma} = \bigcup_{K \in \bar{K}_\sigma} \bar{\omega}_{K, \bar{K}_\sigma}.$$

Similarly, for all $e \in \mathcal{E}_\Gamma \setminus \mathcal{E}_{ext}$ we define ω_e by

$$\bar{\omega}_e = \bigcup_{\sigma \in \mathcal{F}_e \cap \mathcal{F}_\Gamma} \bar{\omega}_{\sigma, e}.$$

We obtain the partitions $\bar{\Omega} = \left(\bigcup_{\nu \in \text{dof}_{\mathcal{D}_m} \setminus \text{dof}_{\text{Dir}_m}} \bar{\omega}_\nu \right)$ $\bar{\Gamma} = \left(\bigcup_{\nu \in \text{dof}_{\mathcal{D}_f} \setminus \text{dof}_{\text{Dir}_f}} \bar{\omega}_\nu \right)$.

We also need a mapping between the degrees of freedom of the matrix domain, which are situated on one side of the fracture network, and the set of indices χ . For $\sigma \in \mathcal{F}_\Gamma$ and $\bar{K}_\sigma \in \bar{\mathcal{M}}_\sigma$ holds by definition $\bar{K}_\sigma = \{K\}$ for a $K \in \mathcal{M}_\sigma$ and hence $\mathbf{n}_{\bar{K}_\sigma} = \mathbf{n}_{K,\sigma}$ is well defined. We obtain the one-element set $\chi(\bar{K}_\sigma) = \{\alpha \in \chi \mid \mathbf{n}_{\bar{K}_\sigma} = \mathbf{n}_\alpha \text{ on } \sigma\}$ and therefore the notation $\alpha(\bar{K}_\sigma) = \alpha \in \chi(\bar{K}_\sigma)$.

We define the HFV scheme's reconstruction operators by

$$\begin{aligned}
\bullet \quad \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} &= \sum_{\nu \in \text{dof}_{\mathcal{D}_m} \setminus \text{dof}_{\text{Dir}_m}} u_\nu \mathbb{1}_{\omega_\nu}, \\
\bullet \quad \Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} &= \sum_{\nu \in \text{dof}_{\mathcal{D}_f} \setminus \text{dof}_{\text{Dir}_f}} u_\nu \mathbb{1}_{\omega_\nu}, \\
\bullet \quad \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} &= \sum_{\sigma \in \mathcal{F}_\Gamma} u_\sigma \mathbb{1}_\sigma, \\
\bullet \quad \Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} &= \sum_{\sigma \in \mathcal{F}_\Gamma} \sum_{\bar{K}_\sigma \in \bar{\mathcal{M}}_\sigma} \delta_{\alpha(\bar{K}_\sigma)\alpha} u_{\bar{K}_\sigma} \mathbb{1}_\sigma \quad \text{for all } \alpha \in \chi.
\end{aligned} \tag{22}$$

Proposition 4.3 *Let us consider a sequence of meshes $(\mathcal{M}^l)_{l \in \mathbb{N}}$ and let us assume that the sequence $(\mathcal{T}^l)_{l \in \mathbb{N}}$ of tetrahedral submeshes is shape regular, i.e. $\theta_{\mathcal{T}^l}$ is uniformly bounded. We also assume that $\lim_{l \rightarrow \infty} h_{\mathcal{T}^l} = 0$. Then, any corresponding sequence of gradient discretizations $(\mathcal{D}^l)_{l \in \mathbb{N}}$, defined by (20), (21) and definition (22), is coercive, consistent and limit conforming.*

Proof Let us denote in the following by $\Pi_{\mathcal{M}}$ and $\Pi_{\mathcal{F}} = \tilde{\Pi}_{\mathcal{F}}$ the HFV matrix and fracture reconstruction operators for the special case that $\omega_{\bar{K}_\sigma} = \emptyset = \omega_e$ for all $\bar{K}_\sigma \in \bigcup_{\sigma \in \mathcal{F}} \bar{\mathcal{M}}_\sigma$ and $e \in \mathcal{E}_\Gamma$. We start our numerical analysis for HFV by proving the proposition for these special choices and then use Lemma 3.1 for generalizing the results.

Coercivity: We first prove that limit conformity against regular test functions, as proved below, implies coercivity.

Assume that the sequence of discretizations $(\mathcal{D}^l)_{l \in \mathbb{N}}$ is not coercive. Then we can find a sequence $((u_{\mathcal{D}_m}^l, u_{\mathcal{D}_f}^l))_{l \in \mathbb{N}}$ with $(u_{\mathcal{D}_m}^l, u_{\mathcal{D}_f}^l) \in X_{\mathcal{D}^l}^0$, such that

$$\|\Pi_{\mathcal{D}_m}^l u_{\mathcal{D}_m}^l\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f}^l u_{\mathcal{D}_f}^l\|_{L^2(\Gamma)} = 1 \quad \text{and} \quad \|(u_{\mathcal{D}_m}^l, u_{\mathcal{D}_f}^l)\|_{\mathcal{D}^l} < \frac{1}{l}. \tag{23}$$

Then follows from a compactness result of [21] that there exists a $u = (u_m, u_f) \in L^2(\Omega) \times L^2(\Gamma)$, s.t. up to a subsequence

$$(\Pi_{\mathcal{D}_m}^l u_{\mathcal{D}_m}^l, \Pi_{\mathcal{D}_f}^l u_{\mathcal{D}_f}^l) \longrightarrow (u_m, u_f) \quad \text{in } L^2(\Omega) \times L^2(\Gamma) \quad (\text{for } l \rightarrow \infty)$$

and therefore $\|u_m\|_{L^2(\Omega)} + \|u_f\|_{L^2(\Gamma)} = 1$. On the other hand follows from the discretizations' limit conformity against regular test functions (see below) by Proposition 3.1 and Corollary 3.1 that $(u_m, u_f) \in V^0$ and that up to a subsequence

$$\begin{cases} \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}^l \rightharpoonup \nabla v_m & \text{in } L^2(\Omega)^d, \\ \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f}^l \rightharpoonup \nabla_\tau v_f & \text{in } L^2(\Gamma)^{d-1}, \\ \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}^l - \Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m}^l \rightharpoonup v_f - \gamma_\alpha v_m & \text{in } L^2(\Gamma_\alpha), \text{ for } \alpha \in \chi. \end{cases}$$

Since by construction holds $\|(u_{\mathcal{D}_m^i}, u_{\mathcal{D}_f^i})\|_{\mathcal{D}^i} \rightarrow 0$, we obtain $\|(u_m, u_f)\|_{V^0} = 0$. But $\|\cdot\|_{V^0}$ is a norm on V^0 , which contradicts the fact that $\|u_m\|_{L^2(\Omega)} + \|u_f\|_{L^2(\Gamma)} = 1$.

Consistency: For $(\varphi_m, \varphi_f) \in C_\Omega^\infty \times C_\Gamma^\infty$ let us define the projection $P_{\mathcal{D}_m}\varphi_m \in X_{\mathcal{D}_m}^0$ such that for all cell $K \in \mathcal{M}$ one has

$$\begin{aligned} (P_{\mathcal{D}_m}\varphi_m)_K &= \varphi_m(\mathbf{x}_K), \\ (P_{\mathcal{D}_m}\varphi_m)_{\overline{K}_\sigma} &= \varphi_m(\mathbf{x}_\sigma) \quad \forall \sigma \in \mathcal{F}_K, \end{aligned}$$

and the projection $P_{\mathcal{D}_f}\varphi_f \in X_{\mathcal{D}_f}^0$ such that $(P_{\mathcal{D}_f}\varphi_f)_\nu = \varphi_f(\mathbf{x}_\nu)$ for all $\nu \in \text{dof}_{\mathcal{D}_f}$. Let us set $v_{\mathcal{D}} = (P_{\mathcal{D}_m}\varphi_m, P_{\mathcal{D}_f}\varphi_f)$. Then holds

$$\|v_K - \varphi_m\|_{L^2(K)} \leq C_{\varphi_m} \cdot h_{\mathcal{T}} \cdot |K|^{\frac{1}{2}} \quad \text{for } K \in \mathcal{M},$$

where $C_{\varphi_m} := \max_\Omega \|\nabla \varphi_m\|$. Summing over $K \in \mathcal{M}$ yields

$$\|\Pi_{\mathcal{M}}v_{\mathcal{D}_m} - \varphi_m\|_{L^2(\Omega)} \leq C_{\varphi_m} \cdot h_{\mathcal{T}} \cdot |\Omega|^{\frac{1}{2}}.$$

We also have

$$\|v_{\overline{K}_\sigma} - \gamma_\alpha \varphi_m\|_{L^2(\Gamma_\alpha)} \leq c_{\varphi_m}^\alpha \cdot h_{\mathcal{T}} \cdot |\sigma|^{\frac{1}{2}} \quad \text{for } \sigma \in \mathcal{F}^\alpha, \overline{K}_\sigma \in \overline{\mathcal{M}}_\sigma^\alpha$$

where $c_{\varphi_m}^\alpha := \max_{\Gamma_\alpha} \|\nabla_\tau \gamma_\alpha \varphi_m\|$, from which we obtain

$$\|\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \gamma_\alpha \varphi_m\|_{L^2(\Gamma_\alpha)} \leq c_{\varphi_m}^\alpha \cdot h_{\mathcal{T}} \cdot |\Gamma_\alpha|^{\frac{1}{2}}.$$

Analogously we can derive

$$\|\Pi_{\mathcal{F}}v_{\mathcal{D}_f} - \varphi_f\|_{L^2(\Gamma)} \leq c_{\varphi_f} \cdot h_{\mathcal{T}} \cdot |\Gamma|^{\frac{1}{2}},$$

where $c_{\varphi_f} := \max_\Gamma \|\nabla_\tau \varphi_f\|$. Furthermore, it follows from Lemma 4.3 of [8] that there exists $C > 0$ depending only on $\theta_{\mathcal{T}}$ and φ such that

$$\|\nabla_{\mathcal{D}_m}v_{\mathcal{D}_m} - \nabla\varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}_f}v_{\mathcal{D}_f} - \nabla\varphi\|_{L^2(\Gamma)} \leq Ch_{\mathcal{T}}$$

Taking into account that $C_\Omega^\infty \times C_\Gamma^\infty$ is dense in V^0 , we see that the treated discretisation is consistent.

Limit Conformity: Let $\varphi_m \in C_{W_m}^\infty$ and for all $K \in \mathcal{M}$, $\sigma \in \mathcal{F}_K$ let $\varphi_K := \frac{1}{|K|} \int_K \varphi_m d\mathbf{x}$ and $\varphi_{K,\sigma} := \frac{1}{|\sigma|} \int_\sigma \gamma_{\mathbf{n}_{K,\sigma}} \varphi_m d\tau(\mathbf{x})$. In exactly the same manner as [19], (29)-(31) are proved, we can show that holds

$$A_{\mathcal{D}_m}^{12} u_{\mathcal{D}_m} \leq Ch_{\mathcal{T}} \|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)^d} \quad \text{and} \quad (24)$$

$$\begin{aligned} A_{\mathcal{D}_m}^{11} u_{\mathcal{D}_m} + A_{\mathcal{D}_m}^2 u_{\mathcal{D}_m} - \sum_{\alpha \in \mathcal{X}} \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \varphi_m (\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}) d\tau(\mathbf{x}) \\ = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\sigma| (u_K - u_{\overline{K}_\sigma}) (\varphi_{K,\sigma} - \varphi_K) \cdot \mathbf{n}_{K,\sigma}, \end{aligned} \quad (25)$$

where

$$\begin{aligned}
A_{\mathcal{D}_m}^{11} u_{\mathcal{D}_m} &:= \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\sigma| (u_{\bar{K}_\sigma} - u_K) \boldsymbol{\varphi}_K \cdot \mathbf{n}_{K,\sigma}, \\
A_{\mathcal{D}_m}^{12} u_{\mathcal{D}_m} &:= \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} R_{K,\sigma}(u_{\mathcal{D}_m}) \mathbf{n}_{K,\sigma} \cdot \int_{D_{K,\sigma}} \boldsymbol{\varphi}_m \, d\mathbf{x}, \\
A_{\mathcal{D}_m}^2 u_{\mathcal{D}_m} &:= \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\sigma| u_K \boldsymbol{\varphi}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma} \quad \text{and} \\
A_{\mathcal{D}_m}^{11} u_{\mathcal{D}_m} + A_{\mathcal{D}_m}^{12} u_{\mathcal{D}_m} + A_{\mathcal{D}_m}^2 u_{\mathcal{D}_m} &= \int_{\Omega} \left(\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \cdot \boldsymbol{\varphi}_m + (\Pi_{\mathcal{M}} u_{\mathcal{D}_m}) \operatorname{div}(\boldsymbol{\varphi}_m) \right) d\mathbf{x},
\end{aligned}$$

with the definition of the gradient stabilization term $R_{K,\sigma}(u_{\mathcal{D}_m})$ as in [8], pp. 8-9. Therefore, applying Cauchy-Schwarz inequality to (25), using the regularity of $\boldsymbol{\varphi}_m$, and the estimate (24), we deduce that there exists C depending only on $\boldsymbol{\varphi}_m$, $\theta_{\mathcal{T}}$, such that

$$\int_{\Omega} \left(\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \cdot \boldsymbol{\varphi}_m + (\Pi_{\mathcal{M}} u_{\mathcal{D}_m}) \operatorname{div}(\boldsymbol{\varphi}_m) \right) d\mathbf{x} - \sum_{\alpha \in \mathcal{X}} \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \boldsymbol{\varphi}_m (\Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m}) d\tau(\mathbf{x}) \leq Ch_{\mathcal{T}} \|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)^d}.$$

Taking into account the result [19] (33), i.e. for all $\boldsymbol{\varphi} \in C_{W_f}^\infty$ exists a constant $C > 0$ depending only on $\theta_{\mathcal{T}}$, such that

$$\begin{aligned}
&\left| \int_{\Gamma} \left(\nabla_{\mathcal{D}_f} u_{\mathcal{D}_f} \cdot \boldsymbol{\varphi}_f + (\Pi_{\mathcal{F}} u_{\mathcal{D}_f}) \operatorname{div}(\boldsymbol{\varphi}_f) \right) d\tau(\mathbf{x}) \right| \\
&\leq Ch_{\Delta} \|\nabla_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}},
\end{aligned}$$

we obtain all together

$$w_{\mathcal{D}}(u_{\mathcal{D}}, \mathbf{q}) \leq C \cdot h_{\mathcal{T}} \cdot \|u_{\mathcal{D}}\|_{\mathcal{D}} \quad \text{for all } \mathbf{q} \in C_{W_m}^\infty \times C_{W_f}^\infty.$$

This result is shown above to imply coercivity, which is needed to conclude now.

Finally, using that $C_{W_m}^\infty \times C_{W_f}^\infty$ is dense in W and the coercivity of the scheme, we derive limit conformity on the whole space of test functions.

Generalization to arbitrary HFV discretizations: We want to apply Lemma 3.1. From [8] Lemma 4.1 and [21], it follows that there are positive constants C_m and C_f only depending on $\theta_{\mathcal{T}}$ and d , such that for all $u_{\mathcal{D}} \in X_{\mathcal{D}}$ holds

$$\begin{aligned}
\|\Pi_{\mathcal{M}} u_{\mathcal{D}_m} - \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)}^2 &= \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\omega_{K,\bar{K}_\sigma}| (u_K - u_{\bar{K}_\sigma})^2 \leq C_m \cdot h_{\mathcal{T}}^2 \cdot \|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)^d}^2 \\
\|\Pi_{\mathcal{F}} u_{\mathcal{D}_f} - \Pi_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)}^2 &= \sum_{\sigma \in \mathcal{F}_\Gamma} \sum_{e \in \mathcal{E}_\sigma} |\omega_{\sigma,e}| (u_\sigma - u_e)^2 \leq C_f \cdot h_{\Delta}^2 \cdot \|\nabla_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}}^2.
\end{aligned}$$

The remaining conditions of Lemma 3.1 are trivially satisfied, from what follows the statement of the proposition. \square

Remark 4.3 *The precedent proof shows that for solutions $(u_m, u_f) \in V^0$ and $(\mathbf{q}_m, \mathbf{q}_f) \in W$ of (4) such that $u_m \in C^2(\bar{K})$, $u_f \in C^2(\bar{\sigma})$, $\mathbf{q}_m \in (C^1(\bar{K}))^d$, $\mathbf{q}_f \in (C^1(\bar{\sigma}))^{d-1}$ for all $K \in \mathcal{M}$ and all $\sigma \in \Gamma_f$, the HFV schemes are consistent and limit conforming of order 1, and therefore convergent of order 1.*

4.3 Finite Volume Formulation for VAG and HFV Schemes

For $K \in \mathcal{M}$ let

$$dof_K = \begin{cases} \{\bar{K}_s, s \in \mathcal{V}_K\} \cup \{K_\sigma, \sigma \in \mathcal{F}_K \cap \mathcal{F}_\Gamma\} & \text{for VAG,} \\ \{\bar{K}_\sigma, \sigma \in \mathcal{F}_K\} & \text{for HFV.} \end{cases}$$

Analogously, in the fracture domain, for $\sigma \in \mathcal{F}_\Gamma$ let

$$dof_\sigma = \begin{cases} \mathcal{V}_\sigma & \text{for VAG,} \\ \mathcal{E}_\sigma & \text{for HFV.} \end{cases}$$

Then, for any $\nu \in dof_K$ the discrete *matrix-matrix*-fluxes are defined as

$$F_{K\nu}(u_{\mathcal{D}_m}) = \sum_{\nu' \in dof_K} \left(\int_K \Lambda_m \nabla_{\mathcal{D}_m} \mathbf{e}_\nu \nabla_{\mathcal{D}_m} \mathbf{e}_{\nu'} d\mathbf{x} \right) (u_K - u_{\nu'}).$$

such that $\int_\Omega \Lambda_m \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} d\mathbf{x} = \sum_{K \in \mathcal{M}} \sum_{\nu \in dof_K} F_{K\nu}(u_{\mathcal{D}_m})(v_K - v_\nu)$. For all $\nu \in dof_\sigma$ the discrete *fracture-fracture*-fluxes are defined as

$$F_{\sigma\nu}(u_{\mathcal{D}_f}) = \sum_{\nu' \in dof_\sigma} \left(\int_\sigma \Lambda_f \nabla_{\mathcal{D}_f} \mathbf{e}_\nu \nabla_{\mathcal{D}_f} \mathbf{e}_{\nu'} d\tau_f(\mathbf{x}) \right) (u_\sigma - u_{\nu'}),$$

such that $\int_\Gamma \Lambda_f \nabla_{\mathcal{D}_f} u_{\mathcal{D}_f} \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} d\tau_f(\mathbf{x}) = \sum_{\sigma \in \mathcal{F}_\Gamma} \sum_{\nu \in dof_\sigma} F_{\sigma\nu}(u_{\mathcal{D}_f})(v_\sigma - v_\nu)$. To take interactions of the matrix and the fracture domain into account we introduce the set of *matrix-fracture (mf)* connectivities

$$\mathcal{C} = \{(\nu_m, \nu_f) \mid \nu_m \in dof_{\mathcal{D}_m}^\Gamma, \nu_f \in dof_{\mathcal{D}_f} \text{ s.t. } \mathbf{x}_{\nu_m} = \mathbf{x}_{\nu_f}\}$$

with $dof_{\mathcal{D}_m}^\Gamma = \{\nu \in dof_{\mathcal{D}_m} \mid \mathbf{x}_\nu \in \bar{\Gamma}\}$. The *mf*-fluxes are built such that

$$\begin{aligned} a_{\mathcal{D}_{mf}}((u_{\mathcal{D}_m}, u_{\mathcal{D}_f}), (v_{\mathcal{D}_m}, v_{\mathcal{D}_f})) &= \sum_{(\nu_m, \nu_f) \in \mathcal{C}} F_{\nu_m \nu_f}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f})(v_{\nu_m} - v_{\nu_f}) \\ &= \sum_{i \in I} \int_{\Gamma_i} \frac{T_f}{2\xi - 1} \sum_{\substack{(\alpha, \beta) \in \\ \{(\alpha^\pm(i), \alpha^\mp(i))\}}} \left(\xi \Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} + (1 - \xi) \Pi_{\mathcal{D}_m}^\beta u_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} \right) \left(\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} \right) d\tau(\mathbf{x}), \end{aligned}$$

for all $(v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in X_{\mathcal{D}}$. For all $\sigma \in \mathcal{F}_\Gamma$ and $K \in \mathcal{M}_\sigma$, let us denote by $\alpha(K, \sigma)$ the unique $\alpha \in \chi$ such that $\sigma \in \mathcal{F}_\alpha$ and $\mathbf{n}_\alpha = \mathbf{n}_{K, \sigma}$. Let us also set for all $\sigma \in \mathcal{F}_\Gamma$, $(\chi \times \chi)_\sigma = \{(\alpha(K, \sigma), \alpha(L, \sigma)), (\alpha(L, \sigma), \alpha(K, \sigma))\}$ with $\mathcal{M}_\sigma = \{K, L\}$. Then, holds

$$\begin{aligned} a_{\mathcal{D}_{mf}}((u_{\mathcal{D}_m}, u_{\mathcal{D}_f}), (v_{\mathcal{D}_m}, v_{\mathcal{D}_f})) &= \\ &= \sum_{\sigma \in \mathcal{F}_\Gamma} \sum_{(\alpha, \beta) \in (\chi \times \chi)_\sigma} \int_\sigma \frac{T_f}{2\xi - 1} \left(\xi \Pi_{\mathcal{D}_m}^\alpha u_{\mathcal{D}_m} + (1 - \xi) \Pi_{\mathcal{D}_m}^\beta u_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} \right) \left(\Pi_{\mathcal{D}_m}^\alpha v_{\mathcal{D}_m} - \tilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} \right) d\tau(\mathbf{x}). \end{aligned}$$

For all $\sigma \in \mathcal{F}_\Gamma$, $K \in \mathcal{M}_\sigma$ and $\mathbf{x} \in \sigma$, let us notice that, for the VAG scheme, one has $\Pi_{\mathcal{D}_m}^{\alpha(K, \sigma)} \mathbf{e}_{K_\sigma}(\mathbf{x}) = \tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_\sigma(\mathbf{x})$, and $\Pi_{\mathcal{D}_m}^{\alpha(K, \sigma)} \mathbf{e}_{\bar{K}_s}(\mathbf{x}) = \tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s(\mathbf{x})$ for all $s \in \mathcal{V}_\sigma$, and for the HFV scheme, one has $\Pi_{\mathcal{D}_m}^{\alpha(K, \sigma)} \mathbf{e}_{\bar{K}_\sigma}(\mathbf{x}) = \tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_\sigma(\mathbf{x}) = 1|_\sigma$. It result after some computations that the VAG matrix fracture fluxes are defined by

$$\begin{aligned} F_{K_\sigma \sigma}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) &= \sum_{s \in \mathcal{V}_\sigma} \left(\int_\sigma \frac{T_f}{2\xi - 1} (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_\sigma) (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s) d\tau(\mathbf{x}) \right) \left(\xi u_{\bar{K}_s} + (1 - \xi) u_{L_s} - u_s \right) \\ &+ \left(\int_\sigma \frac{T_f}{2\xi - 1} (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_\sigma)^2 d\tau(\mathbf{x}) \right) \left(\xi u_{K_\sigma} + (1 - \xi) u_{L_\sigma} - u_\sigma \right), \end{aligned}$$

for all $\sigma \in \mathcal{F}_\Gamma$, $\mathcal{M}_\sigma = \{K, L\}$, and by

$$\begin{aligned}
F_{\overline{Q}_s s}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = & \sum_{\sigma \in (\cup_{Q \in \overline{Q}_s} \mathcal{F}_Q) \cap \mathcal{F}_s \cap \mathcal{F}_\Gamma} \sum_{K \in \mathcal{M}_\sigma \cap \overline{Q}_s, L \in \mathcal{M}_\sigma \setminus \{K\}} \left\{ \right. \\
& \left(\int_\sigma \frac{T_f}{2\xi - 1} (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s)^2 d\tau(\mathbf{x}) \right) \left(\xi u_{\overline{K}_s} + (1 - \xi) u_{\overline{L}_s} - u_s \right) \\
& + \sum_{s' \in \mathcal{V}_\sigma \mid ss' \in \mathcal{E}_\sigma} \left(\int_\sigma \frac{T_f}{2\xi - 1} (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_{s'}) (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s) d\tau(\mathbf{x}) \right) \left(\xi u_{\overline{K}_{s'}} + (1 - \xi) u_{\overline{L}_{s'}} - u_{s'} \right) \\
& \left. + \left(\int_\sigma \frac{T_f}{2\xi - 1} (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_\sigma) (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s) d\tau(\mathbf{x}) \right) \left(\xi u_{K_\sigma} + (1 - \xi) u_{L_\sigma} - u_\sigma \right) \right\},
\end{aligned}$$

for all $s \in \mathcal{V}_\Gamma$, $\overline{Q}_s \in \overline{\mathcal{M}}_s$. Similarly the HFV matrix fracture fluxes are defined by

$$F_{\overline{K}_\sigma \sigma}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = \frac{1}{2\xi - 1} \left(\int_\sigma T_f(\mathbf{x}) d\tau(\mathbf{x}) \right) \left(\xi u_{K_\sigma} + (1 - \xi) u_{L_\sigma} - u_\sigma \right),$$

for all $\sigma \in \mathcal{F}_\Gamma$, $\mathcal{M}_\sigma = \{K, L\}$.

We observe that for the VAG-CV scheme (since $\int_\sigma T_f(\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_{s'}) (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s) d\tau(\mathbf{x}) = 0$ for $s \neq s'$ and $\int_\sigma T_f(\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_\sigma) (\tilde{\Pi}_{\mathcal{D}_f} \mathbf{e}_s) d\tau(\mathbf{x}) = 0$) as well as for the HFV scheme, the fluxes $F_{\nu_m \nu_f}$ only involves the d.o.f. located at the point $\mathbf{x}_{\nu_m} = \mathbf{x}_{\nu_f}$.

The discrete source terms are defined by

$$H_\nu = \begin{cases} \int_\Omega h_m \Pi_{\mathcal{D}_m} \mathbf{e}_\nu d\mathbf{x} & \text{for } \nu \in \text{dof}_{\mathcal{D}_m}, \\ \int_\Gamma h_f \Pi_{\mathcal{D}_f} \mathbf{e}_\nu d\tau_f(\mathbf{x}) & \text{for } \nu \in \text{dof}_{\mathcal{D}_f}. \end{cases}$$

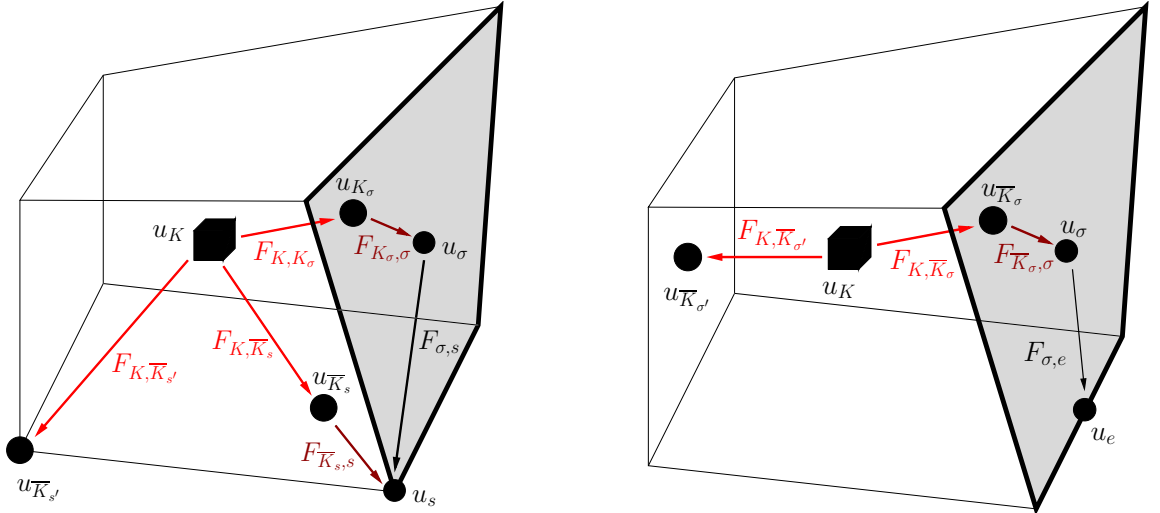


Figure 4: mm -fluxes (red), mf -fluxes (dark red) and ff -fluxes (black) for VAG (left) and HFV (right) on a 3D cell touching a fracture

The following Finite Volume formulation of (5) is equivalent to the discrete variational

formulation (9): find $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X_{\mathcal{D}}^0$ such that

$$\left\{ \begin{array}{l} \text{for all } K \in \mathcal{M} : \sum_{\nu \in \text{dof}_K} F_{K\nu}(u_{\mathcal{D}_m}) = H_K \\ \text{for all } \sigma \in \mathcal{F}_{\Gamma} : \sum_{\nu \in \text{dof}_{\sigma}} F_{\sigma\nu}(u_{\mathcal{D}_f}) - \sum_{\substack{\nu_m \in \text{dof}_{\mathcal{D}_m} \\ \text{s.t. } (\nu_m, \sigma) \in \mathcal{C}}} F_{\nu_m\sigma}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = H_{\sigma} \\ \text{for all } \nu_m \in \text{dof}_{\mathcal{D}_m} \setminus (\mathcal{M} \cup \text{dof}_{\text{Dir}_m}) : \\ \quad - \sum_{K \in \mathcal{M}_{\nu_m}} F_{K\nu_m}(u_{\mathcal{D}_m}) + \sum_{\substack{\nu_f \in \text{dof}_{\mathcal{D}_f} \\ \text{s.t. } (\nu_m, \nu_f) \in \mathcal{C}}} F_{\nu_m\nu_f}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = H_{\nu_m} \\ \text{for all } \nu_f \in \text{dof}_{\mathcal{D}_f} \setminus (\mathcal{F}_{\Gamma} \cup \text{dof}_{\text{Dir}_f}) : \\ \quad - \sum_{\sigma \in \mathcal{F}_{\Gamma, \nu_f}} F_{\sigma\nu_f}(u_{\mathcal{D}_f}) - \sum_{\substack{\nu_m \in \text{dof}_{\mathcal{D}_m} \\ \text{s.t. } (\nu_m, \nu_f) \in \mathcal{C}}} F_{\nu_m\nu_f}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = H_{\nu_f}. \end{array} \right.$$

Here, \mathcal{M}_{ν_m} stands for the set of indices $\{K \in \mathcal{M} \mid \nu_m \in \text{dof}_K\}$ and $\mathcal{F}_{\Gamma, \nu_f}$ stands for the set $\{\sigma \in \mathcal{F}_{\Gamma} \mid \nu_f \in \text{dof}_{\sigma}\}$.

It is important to note that, using the equation in each cell, the cell unknowns u_K , $K \in \mathcal{M}$, can be eliminated without fill-in.

5 Numerical Results

The objective of this numerical section is to compare the VAG-FE, VAG-CV, and the HFV schemes in terms of accuracy and CPU efficiency for both Cartesian and tetrahedral meshes on heterogeneous isotropic and anisotropic media. For that purpose a family of analytical solutions is built for the fixed value of the parameter $\xi = 1$. We refer to [12], [3], [2] for a comparison of the solutions obtained with different values of the parameter $\xi \in [\frac{1}{2}, 1]$ with the solution obtained with a 3D representation of the fractures.

Table 1 exhibits for the Cartesian and tetrahedral meshes, as well as for both the VAG and HFV schemes, the number of degrees of freedom (Nb dof), the number of d.o.f. after elimination of the cell and Dirichlet unknowns (nb dof el.), and the number of nonzero element in the linear system after elimination without any fill-in of the cell unknowns (Nb Jac).

In all test cases, the linear system obtained after elimination of the cell unknowns is solved using the GMRes iterative solver with the stopping criteria 10^{-10} . The GMRes solver is preconditioned by ILUT [25], [26] using the thresholding parameter 10^{-4} chosen small enough in such a way that all the linear systems can be solved for both schemes and for all meshes. In tables 2 and 3, we report the number of GMRes iterations *Iter* and the CPU time taking into account the elimination of the cell unknowns, the ILUT factorization, the GMRes iterations, and the computation of the cell values.

We ran the program on a 2,6 GHz Intel Core i5 processor with 8 GB 1600 MHz DDR3 memory.

5.1 A class of analytical solutions

We consider a 3-dimensional open, bounded, simply connected domain $\Omega = (-0.5, 0.5)^3$ with four intersecting fractures $\Gamma_{12} = \{(x, y, z) \in \Omega \mid x = 0, y > 0\}$, $\Gamma_{23} = \{(x, y, z) \in \Omega \mid y = 0, x >$

$0\}$, $\Gamma_{34} = \{(x, y, z) \in \Omega \mid x = 0, y < 0\}$ and $\Gamma_{14} = \{(x, y, z) \in \Omega \mid y = 0, x < 0\}$. We also introduce the piecewise disjoint, connex subspaces of Ω , $\Omega_1 = \{(x, y, z) \in \Omega \mid y > 0, x < 0\}$, $\Omega_2 = \{(x, y, z) \in \Omega \mid y > 0, x > 0\}$, $\Omega_3 = \{(x, y, z) \in \Omega \mid y < 0, x > 0\}$ and $\Omega_4 = \{(x, y, z) \in \Omega \mid y < 0, x < 0\}$.

Derivation: For $(u_m, u_f) \in V$, we denote $u_m(x, y, z) = u_i(x, y, z)$ on Ω_i , $i = 1, \dots, 4$ and $u_f(x, y, z) = u_{ij}(y, z)$ on Γ_{ij} , $ij \in J$, where we have introduced $J = \{12, 23, 34, 14\}$. We assume that a solution of the discontinuous pressure model writes in the fracture network $u_{ij}(y, z) = \alpha_f(z) + \beta_{ij}(z)\gamma_{ij}(y)$, $ij \in J$ and in the matrix domain

$$\begin{cases} u_1(x, y, z) = \alpha_1(z)u_{12}(y, z)u_{14}(x, z) \\ u_2(x, y, z) = \alpha_2(z)u_{12}(y, z)u_{23}(x, z) \\ u_3(x, y, z) = \alpha_3(z)u_{34}(y, z)u_{23}(x, z) \\ u_4(x, y, z) = \alpha_4(z)u_{34}(y, z)u_{14}(x, z). \end{cases}$$

On γ_{ij} , $ij \in J$ we assume $\gamma_{ij}(0) = 0$, such that the continuity of u_f is well established at the fracture-fracture intersection, as well as $\gamma'_{ij}(0) = 1$, to ease the following calculations. For $i = 1, \dots, 4$ let $K_i = \Lambda_m|_{\Omega_i}$ and for $ij \in J$ let $T_{ij} = T_f|_{\Gamma_{ij}}$. From the conditions $\gamma_{\mathbf{n}, \alpha} \mathbf{q}_m = T_f(\gamma_\alpha u_m - u_f)$ on Γ_α , $\alpha \in \chi$, we then get, after some effort in computation,

$$\begin{aligned} \alpha_1(z) &= \left(\alpha_f(z) - \frac{K_{1y}\beta_{12}(z)}{T_{14}} \right)^{-1}, & \alpha_2(z) &= \left(\alpha_f(z) - \frac{K_{1y}K_{2x}K_{3y}K_{4x}\beta_{12}(z)}{K_{1x}K_{3x}K_{4y}T_{23}} \right)^{-1}, \\ \alpha_3(z) &= \left(\alpha_f(z) - \frac{K_{1y}K_{3y}K_{4x}T_{12}\beta_{12}(z)}{K_{1x}K_{4y}T_{23}T_{34}} \right)^{-1}, & \alpha_4(z) &= \left(\alpha_f(z) - \frac{K_{1y}K_{4x}T_{12}\beta_{12}(z)}{K_{1x}T_{14}T_{34}} \right)^{-1}, \\ \beta_{23}(z) &= \frac{K_{1y}K_{3y}K_{4x}T_{12}}{K_{1x}K_{3x}K_{4y}T_{23}}\beta_{12}(z), & \beta_{34}(z) &= -\frac{K_{1y}K_{4x}T_{12}}{K_{1x}K_{4y}T_{34}}\beta_{12}(z) \\ \beta_{14}(z) &= -\frac{K_{1y}T_{12}}{K_{1x}T_{14}}\beta_{12}(z), & \frac{K_{1y}K_{2x}K_{3y}K_{4x}}{K_{1x}K_{2y}K_{3x}K_{4y}} &= 1. \end{aligned} \tag{26}$$

Obviously, we have taken α_f and β_{12} as degrees of freedom, here. However, these functions must be chosen in such a way that $\frac{1}{\alpha_i(z)} \neq 0$ for $i = 1, \dots, 4$.

Remark 5.1 *We would like to explicitly calculate the jump at the matrix-fracture interfaces for this class of solutions. At Γ_{ij} we have*

$$\begin{aligned} u_i(0, y, z) - u_j(0, y, z) &= (\alpha_i(z) - \alpha_j(z)) \cdot \alpha_f(z) \cdot u_{ij}(y, z), & \text{for } ij \in \{12, 34\} \\ u_i(x, 0, z) - u_j(x, 0, z) &= (\alpha_i(z) - \alpha_j(z)) \cdot \alpha_f(z) \cdot u_{ij}(x, z), & \text{for } ij \in \{23, 14\}. \end{aligned}$$

From (26), we observe, that the pressure becomes continuous at the matrix-fracture interfaces, as the T_{ij} tend to ∞ uniformly.

Remark 5.2 *In order to obtain solutions with discontinuities at the matrix-fracture interfaces, we had to omit the constraint of flux conservation at fracture-fracture intersections.*

5.2 Test Case

We define a solution by setting $\alpha_f(z) = e^{\sin(\pi z)}$, $\beta_{12}(z) = -1$, $\gamma_{12}(y) = \cos(2\pi y) + y - 1$, $\gamma_{23}(x) = x$, $\gamma_{34}(y) = -e^{\cos(\pi y)} + y + e$, $\gamma_{14}(x) = \frac{\sin(\pi x)}{\pi}$. The parameters we used for the different test cases are

- Isotropic Heterogeneous Permeability:

$$\begin{aligned}
K_{1x} &= K_{1y} = K_{1z} = 1, & K_{2x} &= K_{2y} = K_{2z} = 100, \\
K_{3x} &= K_{3y} = K_{3z} = 3, & K_{4x} &= K_{4y} = K_{4z} = 40, \\
T_{12} &= 1, & T_{23} &= 0.2, & T_{34} &= 100, & T_{14} &= 10, \\
K_{12} &= 1, & K_{23} &= 2, & K_{34} &= 3, & K_{14} &= 10.
\end{aligned}$$

- Anisotropic Heterogeneous Permeability:

$$\begin{aligned}
K_{1x} &= K_{1z} = 1, & K_{1y} &= 50, & K_{2x} &= K_{2z} = 2, & K_{2y} &= 100, \\
K_{3y} &= K_{3z} = 3, & K_{3x} &= 30, & K_{4z} &= 4, & K_{4x} &= K_{4y} = 40, \\
T_{12} &= T_{23} = T_{34} = T_{14} = 1, \\
K_{12} &= K_{23} = K_{34} = K_{14} = 1.
\end{aligned}$$

In the following figures we plot the normalized L^2 norms of the errors, which are calculated as follows:

- normalized error of the solution: $err_{sol} = \frac{\|\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} - u_m\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)}}{\|u_m\|_{L^2(\Omega)} + \|u_f\|_{L^2(\Gamma)}}$
- normalized error of the gradient: $err_{grad} = \frac{\|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} - \nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_{\mathcal{D}_f} u_{\mathcal{D}_f} - \nabla_{\tau} u_f\|_{L^2(\Gamma)^{d-1}}}{\|\nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_{\tau} u_f\|_{L^2(\Gamma)^{d-1}}}$

In the following tables is additionally found the normalized error of the jump: $err_{jump} = \frac{\sum_{\alpha \in \chi} \|\Pi_{\mathcal{D}_m}^{\alpha} u_{\mathcal{D}_m} - \Pi_{\mathcal{D}_f}^{\alpha} u_{\mathcal{D}_f} - \gamma_{\alpha} u_m + u_f\|_{L^2(\Gamma_{\alpha})}}{\sum_{\alpha \in \chi} \|\gamma_{\alpha} u_m + u_f\|_{L^2(\Gamma_{\alpha})}}$.

		VAG			HFV		
<i>Hexahedral Meshes</i>							
Key	Nb Cells	Nb dof	Nb dof el.	Nb Jac	Nb dof	Nb dof el.	Nb Jac
1	512	1949	1437	31253	2776	2264	20696
2	4096	11701	7605	178845	19248	15152	150320
3	32768	79205	46437	1154861	142432	109664	1141856
4	262144	578245	316101	8152653	1093824	831680	8892608
5	2097152	4408709	2311557	60910733	8569216	6472064	70173056
<i>Tetrahedral Meshes</i>							
6	1337	2514	1177	18729	4943	3606	22642
7	10706	15765	5059	81741	35520	24814	164246
8	100782	131204	30422	492158	317367	216585	1474817
9	220106	279281	59175	956659	685718	465612	3190244
10	428538	533442	104904	1694008	1324614	896076	6167300
11	2027449	2452416	424967	6818299	6193783	4166334	28862986

Table 1: **Key** defines the mesh reference; **Nb Cells** is the number of cells of the mesh; **Nb dof** is the number of discrete unknowns; **Nb dof el.** is the number of discrete unknowns after elimination of cell unknowns; **Nb Jac** refers to the number of non-zero Jacobian entries after elimination of the cell unknowns and equations.

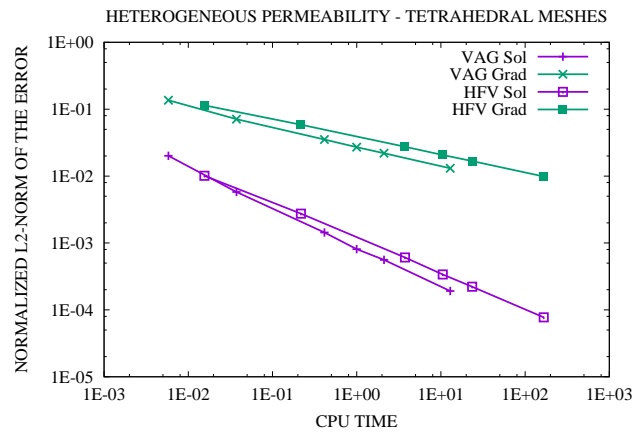
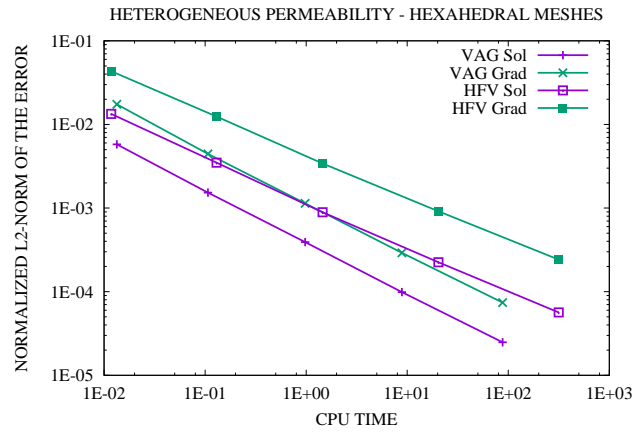


Figure 5: Heterogeneous Permeability: Comparison of VAG-FE and HFV on hexahedral and tetrahedral meshes.

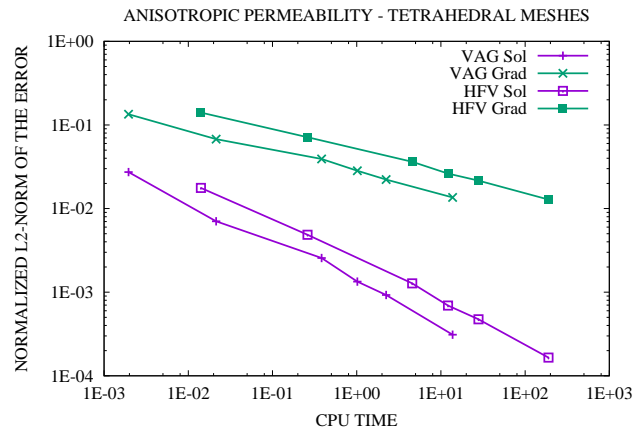
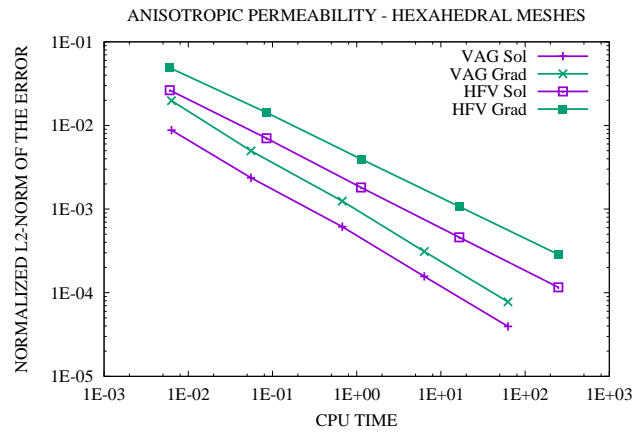


Figure 6: Anisotropic Permeability: Comparison of VAG-FE and HFV on hexahedral and tetrahedral meshes.

Heterogeneous Permeability: <i>VAG</i>								
<i>Hexahedral Meshes</i>								
Key	Iter	CPU	err_{sol}	err_{grad}	err_{jump}	α_{sol}	α_{grad}	α_{jump}
1	8	1.34E-2	5.78E-3	1.74E-2	8.99E-3	1.92	1.97	1.83
2	12	0.11	1.53E-3	4.44E-3	2.53E-3	1.92	1.97	1.83
3	22	0.98	3.92E-4	1.14E-3	6.72E-4	1.97	1.96	1.91
4	41	8.86	9.89E-5	2.91E-4	1.73E-4	1.99	1.97	1.96
5	79	87.91	2.48E-5	7.40E-5	4.40E-5	1.99	1.98	1.98
<i>Tetrahedral Meshes</i>								
6	7	5.82E-3	2.01E-2	0.14	2.25E-2	1.80	0.94	1.68
7	10	3.73E-2	5.78E-3	7.09E-2	7.03E-3	1.80	0.94	1.68
8	20	0.41	1.44E-3	3.52E-2	1.81E-3	1.86	0.94	1.82
9	26	1.00	8.11E-4	2.71E-2	1.06E-3	2.20	1.01	2.06
10	32	2.11	5.60E-4	2.19E-2	7.36E-4	1.67	0.95	1.62
11	53	12.92	1.92E-4	1.31E-2	2.58E-4	2.07	1.00	2.03

Heterogeneous Permeability: <i>HFV</i>								
<i>Hexahedral Meshes</i>								
Key	Iter	CPU	err_{sol}	err_{grad}	err_{jump}	α_{sol}	α_{grad}	α_{jump}
1	11	1.18E-2	1.34E-2	4.3E-2	2.15E-2	1.94	1.80	1.98
2	19	0.13	3.49E-3	1.24E-2	5.44E-3	1.94	1.80	1.98
3	35	1.45	8.91E-4	3.41E-3	1.38E-3	1.97	1.86	1.98
4	73	20.36	2.25E-4	9.15E-4	3.47E-4	1.99	1.90	1.99
5	141	315.38	5.65E-5	2.42E-4	8.69E-5	1.99	1.92	2.00
<i>Tetrahedral Meshes</i>								
6	12	1.56E-2	1.01E-2	0.11	1.74E-2	1.88	0.96	1.73
7	21	0.22	2.74E-3	5.87E-2	5.24E-3	1.88	0.96	1.73
8	43	3.75	6.07E-4	2.75E-2	1.17E-3	2.02	1.02	2.00
9	60	10.51	3.38E-4	2.07E-2	6.62E-4	2.25	1.08	2.20
10	73	23.52	2.22E-4	1.68E-2	4.37E-4	1.90	0.94	1.87
11	119	166.46	7.73E-5	9.87E-3	1.58E-4	2.03	1.02	1.96

Table 2: Isotropic test case. **Key** refers to the mesh defined in table 1; **Iter** is the number of solver iterations; **CPU** refers to the solver CPU time in seconds; err_{sol} , err_{grad} , err_{jump} are the respective L^2 -errors as defined above; α_{sol} , α_{grad} , α_{jump} are the orders of convergence of the solution, of the gradient and of the jump, respectively.

Anisotropic Permeability: <i>VAG</i>								
<i>Hexahedral Meshes</i>								
Key	Iter	CPU	<i>err_{sol}</i>	<i>err_{grad}</i>	<i>err_{jump}</i>	α_{sol}	α_{grad}	α_{jump}
1	7	6.32E-3	8.78E-3	1.98E-2	8.69E-3	1.89	1.99	1.89
2	9	5.56E-2	2.37E-3	4.97E-3	2.34E-3	1.89	1.99	1.89
3	14	0.67	6.15E-4	1.24E-3	6.06E-4	1.95	2.00	1.95
4	26	6.35	2.28E-4	1.57E-4	3.11E-4	1.97	2.00	1.97
5	47	62.65	3.95E-5	7.78E-5	3.89E-5	1.99	2.00	1.99
<i>Tetrahedral Meshes</i>								
6	7	1.95E-3	2.73E-2	0.13	2.70E-2	1.95	0.99	1.95
7	8	2.14E-2	7.05E-3	6.76E-2	6.98E-3	1.95	0.99	1.95
8	15	0.38	2.56E-3	3.92E-2	2.53E-3	1.35	0.73	1.36
9	21	1.02	1.34E-3	2.84E-2	1.32E-3	2.49	1.24	2.49
10	25	2.24	9.26E-4	2.22E-2	9.14E-4	1.66	1.10	1.67
11	41	13.78	3.10E-4	1.36E-2	3.07E-4	2.11	0.95	2.11

Anisotropic Permeability: <i>HFV</i>								
<i>Hexahedral Meshes</i>								
Key	Iter	CPU	<i>err_{sol}</i>	<i>err_{grad}</i>	<i>err_{jump}</i>	α_{sol}	α_{grad}	α_{jump}
1	9	6.02E-3	2.64E-2	4.89E-2	3.35E-2	1.91	1.78	2.01
2	16	8.48E-2	7.02E-3	1.43E-2	8.30E-3	1.91	1.78	2.01
3	29	1.13	1.81E-3	3.96E-3	2.07E-3	1.95	1.85	2.00
4	55	16.55	4.60E-4	1.07E-3	5.19E-4	1.98	1.89	2.00
5	108	248.20	1.16E-4	2.86E-4	1.30E-4	1.99	1.91	2.00
<i>Tetrahedral Meshes</i>								
6	10	1.41E-2	1.77E-2	0.14	1.79E-2	1.86	0.98	1.91
7	19	0.26	4.86E-3	7.13E-2	4.75E-3	1.86	0.98	1.91
8	37	4.56	1.28E-3	3.63E-2	1.21E-3	1.79	0.90	1.83
9	47	12.16	6.92E-4	2.62E-2	6.66E-4	2.35	1.25	2.28
10	63	27.96	4.75E-4	2.16E-2	4.68E-4	1.69	0.88	1.59
11	105	189.66	1.65E-4	1.28E-2	1.58E-4	2.04	1.00	2.09

Anisotropic Permeability: <i>VAG Lump</i>								
<i>Hexahedral Meshes</i>								
Key	Iter	CPU	<i>err_{sol}</i>	<i>err_{grad}</i>	<i>err_{jump}</i>	α_{sol}	α_{grad}	α_{jump}
1	7	3.90E-3	9.09E-3	2.01E-2	9.06E-3	1.89	1.99	1.89
2	9	5.15E-2	2.46E-3	5.06E-3	2.44E-3	1.89	1.99	1.89
3	15	0.66	6.37E-4	1.27E-3	6.34E-4	1.95	2.00	1.95
4	26	6.39	1.62E-4	3.17E-4	1.61E-4	1.97	2.00	1.97
5	47	62.19	4.09E-5	7.93E-5	4.07E-5	1.99	2.00	1.99
<i>Tetrahedral Meshes</i>								
6	7	2.11E-3	2.75E-2	0.13	2.73E-2	1.95	0.99	1.94
7	8	2.00E-2	7.14E-3	6.76E-2	7.10E-3	1.95	0.99	1.94
8	15	0.38	2.60E-3	3.92E-2	2.58E-3	1.35	0.73	1.35
9	21	1.02	1.36E-3	2.84E-2	1.35E-3	2.48	1.24	2.49
10	25	2.24	9.40E-4	2.22E-2	9.33E-4	1.66	1.10	1.67
11	41	13.91	3.15E-4	1.36E-2	3.13E-4	2.11	0.95	2.11

Table 3: Anisotropic test case. **Key** refers to the mesh defined in table 1; **Iter** is the number of solver iterations; **CPU** refers to the solver CPU time in seconds; *err_{sol}*, *err_{grad}*, *err_{jump}* are the respective L^2 -errors as defined above; α_{sol} , α_{grad} , α_{jump} are the orders of convergence w.r.t. $\#\mathcal{M}^{-\frac{1}{3}}$ of the solution, of the gradient and of the jump, respectively.

The test case shows that, on cartesian grids, we obtain, as classically expected, convergence of order 2 for both, the solution and its gradient. For tetrahedral grids, we obtain convergence of order 2 for the solution and convergence of order 1 for its gradient. We observe that the VAG scheme is more efficient than the HFV scheme and this observation gets more obvious with increasing anisotropy. Comparing the precision of the discrete solution (and its gradient) for VAG and HFV on a given mesh, we see that on hexahedral meshes, the advantage is on the side of VAG, whereas on tetrahedral meshes HFV is more precise (but much more expensive). On a given mesh, HFV is usually (see [19]) more accurate than VAG both for tetrahedral and hexahedral meshes. This is not the case for our test cases on Cartesian meshes maybe due to the higher number for VAG than for HFV of d.o.f. at the interfaces Γ_α on the matrix side. It is also important to notice that there is literally no difference between VAG with finite element respectively lumped mf -fluxes concerning accuracy and convergence rate.

6 Conclusion

In this work, we extended the framework of gradient schemes (see [7]) to the model problem (4) of stationary Darcy flow through fractured porous media and gave numerical analysis results for this general framework.

The model problem (an extension to a network of fractures of a PDE model presented in [10], [12] and [3]) takes heterogeneities and anisotropy of the porous medium into account and involves a complex network of planar fractures, which might act either as barriers or as drains.

We also extended the VAG and HFV schemes to our model, where fractures acting as barriers force us to allow for pressure jumps across the fracture network. We developed two versions of VAG schemes, the conforming finite element version and the non-conforming control volume version, the latter particularly adapted for the treatment of material interfaces (cf. [9]). We showed, furthermore, that both versions of VAG schemes, as well as the proposed non-conforming HFV schemes, are incorporated by the gradient scheme's framework. Then, we applied the results for gradient schemes on VAG and HFV to obtain convergence, and, in particular, convergence of order 1 for "piecewise regular" solutions.

For implementation purposes and in view of the application to multi-phase flow, we also proposed a uniform Finite Volume formulation for VAG and HFV schemes. The numerical experiments on a family of analytical solutions show that the VAG scheme offers a better compromise between accuracy and CPU time than the HFV scheme especially for anisotropic problems.

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