



HAL
open science

Linear electron stability for a bi-kinetic sheath model.

Mehdi Badsì

► **To cite this version:**

Mehdi Badsì. Linear electron stability for a bi-kinetic sheath model.. Journal of Mathematical Analysis and Applications, 2017, 10.1016/j.jmaa.2017.04.055 . hal-01192658

HAL Id: hal-01192658

<https://hal.science/hal-01192658>

Submitted on 18 May 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 **LINEAR ELECTRON STABILITY FOR A BI-KINETIC SHEATH**
2 **MODEL**

3 MEHDI BADSI*

4 **Abstract.** We establish the linear stability of an electron equilibrium for an electrostatic and
5 collisionless plasma in interaction with a wall. The equilibrium we focus on is called in plasma
6 physics a Debye sheath. Specifically, we consider a two species (ions and electrons) Vlasov-Poisson-
7 Ampère system in a bounded and one dimensional geometry. The interaction between the plasma
8 and the wall is modeled by original boundary conditions : On the one hand, ions are absorbed by
9 the wall while electrons are partially re-emitted. On the other hand, the electric field at the wall is
10 induced by the accumulation of charged particles at the wall. These boundary conditions ensure the
11 compatibility with the Maxwell-Ampère equation. A global existence, uniqueness and stability result
12 for the linearized system is proven. The main difficulty lies in the fact that (due to the absorbing
13 boundary conditions) the equilibrium is a discontinuous function of the particle energy, which results
14 in a linearized system that contains a degenerate transport equation at the border.

15 **Key words.** plasma wall interaction, Debye sheath, kinetic equations, Vlasov-Poisson-Ampère
16 system, linear stability, degenerate transport equations

17 **AMS subject classifications.** 68Q25, 68R10, 68U05

18 **1. Introduction.**

19 **1.1. A kinetic model of plasma-wall dynamics: the Vlasov-Poisson-**
20 **Ampère system.** We consider an electrostatic and collisionless plasma consisting of
21 one species of ions and electrons. We use a kinetic approach to model this plasma.
22 To this purpose, we set $\Omega = (0, 1) \times \mathbb{R}$ and denote by $(x, v) \in \overline{\Omega} = [0, 1] \times \mathbb{R}$ the
23 phase space variable, where x is the particle position and v the particle velocity.
24 This work is concerned with the linear stability of an equilibrium for the two species
25 Vlasov-Poisson system in the presence of spatial boundaries

26 (1)
$$\begin{cases} \partial_t f_i + v \partial_x f_i + E \partial_v f_i = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_t f_e + v \partial_x f_e - \frac{1}{\mu} E \partial_v f_e = 0 & \text{in } (0, +\infty) \times \Omega, \end{cases}$$

27
28 (2)
$$-\varepsilon^2 \partial_{xx} \phi = n_i - n_e \quad \text{in } (0, +\infty) \times (0, 1),$$

29 where $f_i : [0, +\infty) \times \overline{\Omega} \rightarrow \mathbb{R}^+$, $f_e : [0, +\infty) \times \overline{\Omega} \rightarrow \mathbb{R}^+$ are the ions and electrons
30 distribution functions in the phase-space and $\phi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ is the electric
31 potential. Here the physical parameters μ and ε stand respectively for the mass ratio
32 between electrons and ions, and a normalized Debye length that will be (for simplicity)
33 in the sequel taken equal to 1. We also denote

34
$$E = -\partial_x \phi, \quad n_i = \int_{\mathbb{R}} f_i dv, \quad n_e = \int_{\mathbb{R}} f_e dv,$$

35 the electric field, the ion density and the electron density. The boundary conditions
36 are given for all $t \in (0, +\infty)$ by

37 (3)
$$\begin{cases} f_i(t, 0, v > 0) = f_i^{in}(v), & f_i(t, 1, v < 0) = 0, \\ f_e(t, 0, v > 0) = f_e^{in}(v), & f_e(t, 1, v < 0) = \alpha f_e(t, 1, -v), \end{cases}$$

*UPMC-Paris06, CNRS UMR 7598, Laboratoire Jacques Louis Lions., 4 pl. Jussieu, F75252 Paris
Cedex 05, France (badsi@ljl.math.upmc.fr).

38 (4)
$$\phi(t, 0) = 0, \quad E(t, 1) = E^*(t, f_i, f_e).$$

39 where f_i^{in} and f_e^{in} denote two given incoming particles velocity distributions that are
 40 time independent. The scalar parameter α belongs to the interval $[0, 1)$ and represents
 41 the rate of re-emitted electrons in the domain $(0, 1)$. The scalar $E^*(t, f_i, f_e)$ depends
 42 on the unknown (f_i, f_e, ϕ) via the formula

43 (5)
$$E^*(t, f_i, f_e) = \left(E_w^0 - \int_0^t \int_{\mathbb{R}} (f_i(\tau, 1, v) - f_e(\tau, 1, v)) v dv d\tau \right).$$

44 Up to our knowledge, the theory of existence and uniqueness for such a initial bound-
 45 ary value problem (1)-(4) has not been treated in full details. In the one dimensional
 46 case, there is the result of Bostan [5] which establishes the existence and uniqueness
 47 of the mild solution to a Vlasov-Poisson system in which the boundary conditions do
 48 not depend on the solution itself. Still in the one dimensional case, the work of Guo
 49 [10] studies the dynamic of a plane diode. Also, the result of BenAbdallah [3] shows
 50 the existence of weak solutions for the Vlasov-Poisson system in dimension greater
 51 than or equal to one, but once again the boundaries are not coupled to the solution
 52 itself. The existence and uniqueness of weak solutions in the half-space with specular
 53 reflection condition is obtained in [11]. The case of partially absorbing boundary condi-
 54 tion is treated in [9]. The existence of a stationary solution to the system (1)-(4)
 55 was proven in [1], the stationary solution corresponds to the Debye sheath (see [17]
 56 for further physical details). This work can be considered as a continuation of the
 57 work [1], and a first step in the study of the wellposedness of the non-linear system
 58 (1)-(4).

59 **1.2. Physical interpretation of the model.** The bi-kinetic model (1)-(4)
 60 models the dynamical transition between the core of a plasma and a wall (see for
 61 instance [13]). The region of plasma we consider is modeled by the line segment $[0, 1]$
 62 where $x = 0$ is assumed to be somewhere in the bulk plasma and thus a source of
 63 particles. The sources here are modeled by the injection of particles that are math-
 64 ematically encoded in the given distributions f_i^{in} and f_e^{in} . The wall at $x = 1$ is
 65 supposed to be metallic and partially absorbing: it absorbs completely the ions and
 66 re-emits a fraction α of the electrons. The parameter α can be seen as a constitutive
 67 parameter of the wall. The accumulation of charged particles at the wall induces an
 68 electric-field that is given by (5) (the number E_w^0 denotes the initial electric field at
 69 the wall). The boundary condition of the electric-field at the wall can be formally
 70 re-written as $\partial_t E(t, 1) + j(t, 1) = 0$ where $j(t, 1) := \int_{\mathbb{R}} (f_i(t, 1, v) - f_e(t, 1, v)) v dv$ is the
 71 current density at the wall. At a formal level, one easily verifies that this boundary
 72 condition ensures the compatibility of the solutions to (1)-(4) with the Vlasov-Ampère
 73 system made of equations (1), (3) with the Maxwell-Ampère equation

74 (6)
$$\varepsilon^2 \partial_t E = -j \text{ in } (0, +\infty) \times [0, 1], \quad j := \int_{\mathbb{R}} (f_i - f_e) v dv.$$

75 provided the initial data satisfy the Poisson equation

76
$$\partial_x E(0, \cdot) = \int_{\mathbb{R}} f_i(0, \cdot, \cdot) - f_e(0, \cdot, \cdot) dv.$$

77 Because of this equivalence, we shall rather consider the Vlasov-Ampère system (1),
 78 (3) and (6).

79 **1.3. Statement of the main result.** The mathematical and physical aspect
 80 we investigate in this work is the linear stability of the Debye sheath for the Vlasov-
 81 Ampère model (1),(3) and (6). The rigorous mathematical construction of such an
 82 equilibrium was obtained for the first time for the model (1), (3) together with (6) in
 83 [1]. The main result of this work can be roughly summarized as follows: if the initial
 84 data that is a small perturbation of the sheath equilibrium, then the solution of the
 85 system (1),(3) and (6) remains close to the sheath equilibrium for all times provided
 86 the ions are frozen. To make things more precise at this stage, we need supplementary
 87 materials. Let us denote by $(f_i^\infty, f_e^\infty, \phi^\infty)$ the sheath equilibrium to (1),(3) and (6).
 88 Let us write the solution of (1),(3) and (6) as the sum of the sheath equilibrium
 89 plus an interior perturbation that affects only the electrons and the electrostatic field,
 90 namely: $(f_i, f_e, \phi) = (f_i^\infty, f_e^\infty + \tilde{f}_e, \phi^\infty + \tilde{\phi})$. The formal linearization yields the
 91 linearized Vlasov-Ampère system (after dropping the \sim)

$$92 \quad (\text{LVA}): \begin{cases} \partial_t f_e + D f_e = E \partial_v f_e^\infty, & \text{in } (0, +\infty) \times \Omega \\ \partial_t E = \int_{\mathbb{R}} f_e v dv, & \text{in } (0, +\infty) \times [0, 1] \\ f_e(t, 0, v > 0) = 0, \quad f_e(t, 1, v < 0) = \alpha f_e(t, 1, -v) & \text{in } (0, +\infty) \end{cases}$$

93 where D denotes the first order linear differential operator defined formally by

$$94 \quad (7) \quad D := v \partial_x - E^\infty \partial_v \text{ with } E^\infty = -(\phi^\infty)' \text{ being the equilibrium electric field.}$$

95 Because the equilibrium density f_e^∞ is discontinuous across the curve of equation
 96 $v = v_e(x)$ where the function v_e is defined for all $x \in [0, 1]$ by

$$97 \quad (8) \quad v_e(x) := -\sqrt{2(\phi^\infty(x) - \phi_w)},$$

98 its velocity derivative takes the form

$$99 \quad (9) \quad \partial_v f_e^\infty = [f_e^\infty] \delta^{v_e} - v f_e^\infty,$$

100 where δ^{v_e} is a Dirac distribution supported on the curve of equation $v = v_e(x)$,
 101 namely:

$$102 \quad (10) \quad \langle \delta^{v_e}, \varphi \rangle = \int_0^1 \varphi(x, v_e(x)) dx \quad \forall \varphi \in \mathcal{D}(\Omega).$$

103 and where

$$104 \quad (11) \quad [f_e^\infty] := \lim_{v \rightarrow v_e(x)^+} f_e^\infty(x, v) - \lim_{v \rightarrow v_e(x)^-} f_e^\infty(x, v)$$

105 denotes the constant jump of f_e^∞ across the characteristic curve $v = v_e(x)$. As
 106 a consequence, it is natural to look for solutions to (LVA) that decompose into a
 107 singular part plus a regular one, as

$$108 \quad (12) \quad f_e = \eta_e(t, x) \delta^{v_e} + g_e(t, x, v)$$

109 with η_e and g_e two functions. The main result can be in rough terms stated as
 110 follows: *For any initial data (f_e^0, E^0) with f_e^0 of the form $f_e^0(x, v) = \eta^0(x) \delta^{v_e} + g_e^0(x, v)$*
 111 *where η_0 and g_e^0 are two functions, there exists a couple of functions (η_e, g_e) and an*
 112 *electric field E such that if we define $f_e(t, x, v) = \eta_e(t, x) \delta^{v_e} + g_e(t, x, v)$, then the*

113 couple (f_e, E) is solution to the linearized Vlasov-Ampère system with initial condition
 114 $f_e(t=0, x, v) = f_e^0(x, v)$ and $E(t=0, x) = E^0(x)$. Moreover the non negative energy
 115 functional defined for $t \geq 0$ by

$$116 \quad (13) \quad \mathcal{E}(t) = \frac{1}{2} \left(\int_0^1 \frac{\eta_e^2(t, x) |v_e(x)| dx}{[f_e^\infty]} + \int_\Omega \frac{g_e(t, x, v)^2}{f_e^\infty(x, v)} dx dv + \int_0^1 E(t, x)^2 dx \right)$$

117 is non increasing.

118 **1.4. The mathematical approach and its difficulty.** The main difficulty
 119 in the analysis lies in the fact that for $\alpha \in [0, 1)$ the electron sheath equilibrium
 120 is a discontinuous function of the particle energy. This is due to the absorption of
 121 particles with positive velocities at the wall ($x = 1$), which creates a discontinuity
 122 that propagates into the domain along the characteristic curve $v = v_e(x)$. Thus the
 123 linearization of the Vlasov-Ampère system (1),(3),(6) around the sheath equilibrium
 124 $(f_i^\infty, f_e^\infty, E^\infty)$ with no perturbation on the ions, yields a linear system whose solution
 125 still denoted (f_e, E) is singular. Assuming a decomposition of the form

$$126 \quad f_e(t, x, v) = \eta_e(t, x) \delta^{v_e} + g_e(t, x, v)$$

127 where η_e and g_e are two functions and δ^{v_e} is a Dirac mass supported by the char-
 128 acteristic curve of equation $v = v_e(x)$ yields another linear system on (η_e, g_e, E) .
 129 Making a suitable change of variable leads to the system (VAL). The system (VAL)
 130 contains a degenerate transport equation because the given velocity field v_e vanishes
 131 at $x = 1$ and its derivate $\partial_x v_e$ is not essentially bounded as the Diperna-Lions theory
 132 of transport equation [8] requires. This difficulty is overcome using the fact that the
 133 velocity field is only weakly degenerated, it vanishes at the border like a squareroot.
 134 This allows us to prove a Hardy-Poincaré type inequality. Ultimately by applying the
 135 Hille-Yosida theorem, we show that the linearized system (VAL) is well-posed and
 136 that the energy of the system is non increasing.

137 **1.5. Previous works.** Stability issues for Vlasov-Poisson systems in bounded
 138 geometry are of a tremendous importance for practical applications, be it in the
 139 modeling of laboratory plasmas, or in the design of numerical methods. Unfortunately,
 140 and despite its worthy interest, it seems that it has not been studied in full details.
 141 Stability analysis for such a Vlasov-Poisson system has already been performed in
 142 the absence of spatial boundaries, that is, either in all space or in a periodical setting
 143 [16, 14, 12]. However, it seems that in the presence of spatial boundaries, the question
 144 of stability has not been extensively adressed. Up to our knowledge, it is only very
 145 recently, with the work of Nguyen and Strauss in [15] that the question was raised.
 146 The authors considered the Vlasov-Maxwell system in a cylindrical geometry and
 147 assumed an equilibrium that is a C^1 function of the particle energy and momentum.
 148 Additionally, they assumed as in the previous works [16, 12] that the equilibrium, on
 149 its support, is monotone in the particle energy, which seem to be a key ingredient to
 150 prove stability results. Only the recent work of Ben-Artzi in [4] deals with equilibria
 151 that are not monotone in the particle energy. The author gives sufficient conditions
 152 for an instability, but on the other hand the analysis is performed in an unbounded
 153 geometry.

154 **1.6. Organization of the paper.** The plan of this paper is as follows. In
 155 section 2, we derive the system (VAL) and give a precise statement of the main result,
 156 including the functional spaces, and the notions of solutions we consider. In section 3,

157 we state the Hardy-Poincaré type inequality and give some technical lemmas needed
 158 to prove the main result. We eventually prove the main result. In the appendix A,
 159 we briefly discuss the regularity of the solution.

160 2. The stability result.

161 **2.1. Description of the sheath equilibrium.** The equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$
 162 is associated with an electron boundary condition which is a Maxwellian, namely it
 163 takes the form

$$164 \quad (14) \quad f_e^{in}(v) = n_0 \sqrt{\frac{2}{\pi}} e^{-\frac{v^2}{2}} \text{ for } v > 0.$$

165 where $n_0 > 0$ is an electron reference density. The sheath equilibrium is a stationary
 166 and weak solution of the Vlasov-Ampère system (1), (3), (6). It belongs to the space
 167 $(L^1 \cap L^\infty(\Omega))^2 \times C^2[0, 1]$ and enjoys the following properties:

- 168 1. For all $x \in [0, 1]$, $(\phi^\infty)''(x) \leq 0$, $E^\infty(x) := -(\phi^\infty)'(x) \geq 0$, $\phi^\infty(0) = 0$,
 169 $\phi^\infty(1) =: \phi_w$ and $E^\infty(1) =: E_w^\infty > 0$.
- 170 2. $f_i^\infty(x, v) = \begin{cases} f_i^{in}(\sqrt{v^2 + 2\phi^\infty(x)}) & \text{for } (x, v) \text{ s.t } v > \sqrt{-2\phi^\infty(x)} \\ 0 & \text{elsewhere.} \end{cases}$
- 171 3. $f_e^\infty(x, v) = n_0 \sqrt{\frac{2}{\pi}} \begin{cases} e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} & \text{for } (x, v) \text{ s.t } v \geq v_e(x) \\ \alpha e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} & \text{for } (x, v) \text{ s.t } v < v_e(x), \end{cases}$
- 172 4. $\int_{\mathbb{R}} f_i^\infty(x, v) v dv = \int_{\mathbb{R}} f_e^\infty(x, v) v dv$ for all $x \in [0, 1]$.

173 Such an equilibrium is proven to exist in [1] under the necessary and sufficient condi-
 174 tion that the following kinetic Bohm criterion

$$175 \quad \frac{\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv}{\int_{\mathbb{R}^+} f_i^{in}(v) dv} < \frac{\left(\sqrt{2\pi} + (1 - \alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv \right)}{\left(\sqrt{2\pi} - (1 - \alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right)}.$$

176 holds true for $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$. The physical meaning of this inequality is that
 177 there cannot be too many ions particles entering the domain with low velocities.
 178 It is instructive to have a representation of the ions and electrons characteristics
 179 in the phase space (see Figure 1). We see that for the electrons, the equilibrium
 180 is a truncated Maxwellian distribution. Especially, it is discontinuous across the
 181 characteristic curve $S := \{(x, v_e(x)) / x \in [0, 1]\}$ where the function v_e is defined in
 182 (8). To be more precise we have the following:

183 **LEMMA 1.** a) $f_e^\infty \in C^2(\Omega \setminus S)$.
 184 b)

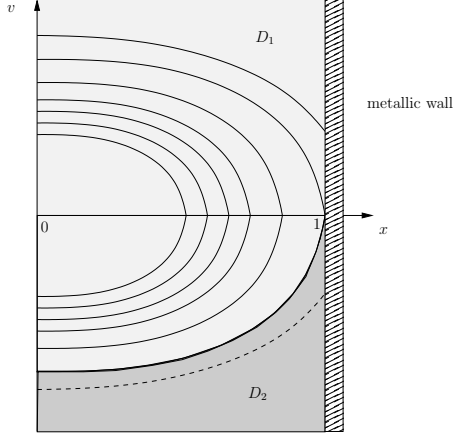
$$184 \quad \partial_v f_e^\infty = [f_e^\infty] \delta^{v_e} - v f_e^\infty.$$

185 where $[f_e^\infty] = n_0 \sqrt{\frac{2}{\pi}} (1 - \alpha) e^{\phi_w} > 0$ is the jump f_e^∞ across the characteristic
 186 curve S defined in (11) and δ^{v_e} is the Dirac mass supported by the function
 187 v_e defined in (8).

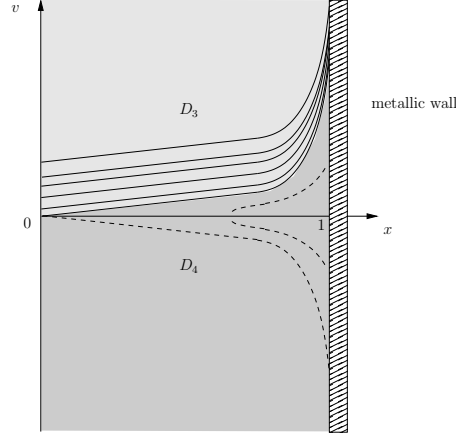
188 *Proof.* a) It is straightforward from its definition that f_e^∞ belongs to $C^2(\Omega \setminus S)$.

Fig. 1: Ions and electrons phase space

(a) Schematic characteristic electron trajectories associated with the potential ϕ^∞ . The electron equilibrium f_e^∞ is discontinuous across the curve $v = v_e(x)$ (bold curve).



(b) Schematic characteristic ions trajectories associated with the potential ϕ^∞ . The ion equilibrium f_i^∞ vanishes outside D_3 (lighter gray).



189 b) It follows from an integration by parts. Indeed, for all $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}
190 \quad & \langle \partial_v f_e^\infty, \varphi \rangle = - \langle f_e^\infty, \partial_v \varphi \rangle \\
191 \quad & = - \int_0^1 \int_{\mathbb{R}} f_e^\infty(x, v) \partial_v \varphi(x, v) dv dx \\
192 \quad & = - \int_0^1 \int_{v \geq v_e(x)} f_e^\infty(x, v) \partial_v \varphi(x, v) dv dx \\
193 \quad & \quad - \int_0^1 \int_{v < v_e(x)} f_e^\infty(x, v) \partial_v \varphi(x, v) dv dx \\
194 \quad & = - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \left[e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x, v) \right]_{v_e(x)}^{+\infty} dx \\
195 \quad & \quad - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \int_{v \geq v_e(x)} v e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x, v) dv dx \\
196 \quad & \quad - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \left[\alpha e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x, v) \right]_{-\infty}^{v_e(x)} dx \\
197 \quad & \quad - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \int_{v < v_e(x)} \alpha v e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x, v) dv dx.
\end{aligned}$$

198

200 It is then easy to conclude. □

201 LEMMA 2. The function $x \in [0, 1] \mapsto v_e(x) := -\sqrt{2(\phi^\infty(x) - \phi_w)}$ has the follow-
 202 ing properties:

203 a) $v_e \in C^0([0, 1]) \cap C^2([0, 1))$

204 b) For all $x \in [0, 1)$, $v_e(x) < 0$, $v_e(1) = 0$, $v_e(x) \underset{x \rightarrow 1^-}{\sim} -\nu \sqrt{1-x}$ where $\nu =$

- 205 $\sqrt{2E_w^\infty}$.
 206 c) For all $x \in (0, 1)$, $v_e'(x)v_e(x) + E^\infty(x) = 0$.
 207 d) $v_e \in W^{1,1}(0, 1)$ and $\frac{1}{v_e} \in L^1(0, 1)$.

208 *Proof.* We skip the proof because it is essentially a consequence of the regu-
 209 larity of ϕ^∞ . \square

210 This lemma is important because it makes precise the regularity of v_e which must be
 211 known insofar as it will appear as the velocity field of a one dimensional transport
 212 equation of the form $(\partial_t + v_e \partial_x)u = s$. The fact that $\frac{1}{v_e} \in L^1(0, 1)$ implies that the
 213 characteristics, namely the solutions of the ode

$$214 \quad \dot{x}(t) = v_e(x(t))$$

215 have a finite incoming time into the domain $[0, 1]$. The time for a characteristic
 216 starting from a point $x \in [0, 1]$ to reach $x = 1$ is $\int_x^1 \frac{-1}{v_e(s)} ds$. We give a sketch of the
 characteristics in the plan (x, t) .

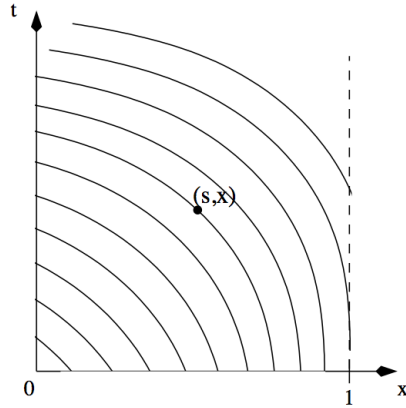


Fig. 2: Characteristics curves in the plan (x, t) . The backward in time characteristics cross $x = 1$ in finite time.

217

218 **2.2. Derivation of the linearized system (VAL).** We now derive the linear
 219 system (VAL). Let us give the definition of solution we consider for the linearized
 220 Vlasov-Ampère system.

221 **DEFINITION 3.** Assume f_e^0 a measure on $\overline{\Omega}$ and $E^0 \in L^2(0, 1)$. Let f_e be a mea-
 222 sure on $[0, +\infty) \times \overline{\Omega}$ and $E \in W_{loc}^{1,\infty}([0, +\infty); L^2(0, 1))$. We say that (f_e, E) is a weak
 223 solution of (LVA) system iff:

- 224 a) For all $\varphi \in C_c^1([0, +\infty) \times \overline{\Omega})$ such that $\varphi(t, 0, v \leq 0) = 0$ and $\varphi(t, 1, v \geq 0) =$
 225 $-\alpha\varphi(t, 1, -v)$

$$226 \quad - \langle f_e, \partial_t \varphi + D\varphi \rangle_{[0, +\infty) \times \overline{\Omega}}$$

$$227 \quad = \langle f_e^0, \varphi(0, \cdot) \rangle_{\overline{\Omega}}$$

$$228 \quad + [f_e^\infty] \int_0^{+\infty} \int_0^1 E(t, x) \varphi(t, x, v_e(x)) dx dt$$

$$229 \quad - \int_0^{+\infty} \int_0^1 \int_{\mathbb{R}} E(t, x) v f_e^\infty(x, v) \varphi(t, x, v) dx dv dt.$$

230

231 *b) The current density $(t, x) \in (0, +\infty) \times (0, 1) \mapsto j_e(t, x) := \langle f_e(t, x, \cdot), v \rangle_{\mathbb{R}}$ be-*
 232 *longs to $L_{loc}^\infty([0, +\infty); L^2(0, 1))$, $\partial_t E(t, x) = j_e(t, x)$ for a.e $(t, x) \in (0, +\infty) \times$*
 233 *$(0, 1)$ and $E(t = 0, x) = E^0(x)$ for a.e $x \in (0, 1)$.*

234 One readily checks that f_e satisfies

$$235 \quad (15) \quad \partial_t f_e + Df_e = [f_e^\infty]E\delta^{v_e} - Evf_e^\infty \text{ in } \mathcal{D}'((0, +\infty) \times \Omega).$$

236 Since the right hand side of (15) is the sum of Dirac mass supported by v_e plus a
 237 function, a natural idea is to look for $f_e \in \mathcal{D}'((0, +\infty) \times \Omega)$ under the following form

$$238 \quad f_e = \eta_e(t, x)\delta^{v_e} + g_e(t, x, v)$$

239 where $\eta_e : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ and $g_e : [0, +\infty) \times \bar{\Omega}$ are a priori two regular functions.
 240 Note that for all $\varphi \in \mathcal{D}((0, +\infty) \times \Omega)$

$$241 \quad \langle f_e, \varphi \rangle = \int_0^{+\infty} \int_0^1 \eta_e(t, x)\varphi(t, x, v_e(x))dxdt + \int_0^{+\infty} \int_\Omega g_e(t, x, v)\varphi(t, x, v)dxdvdt$$

242

243 **PROPOSITION 4.** *Let (f_e, E) be a weak solution of (LVA) with f_e of the form (12)*
 244 *where $\eta_e \in L_{loc}^1([0, +\infty) \times (0, 1))$ and $g_e \in L_{loc}^1([0, +\infty) \times \Omega)$. Then f_e is solution of*
 245 *the Vlasov equation (15) if and only if η_e and g_e satisfy*

$$246 \quad (16) \quad \partial_t \eta_e + \partial_x(v_e \eta_e) = [f_e^\infty]E \text{ in } \mathcal{D}'((0, +\infty) \times (0, 1)),$$

247

$$248 \quad (17) \quad \partial_t g_e + Dg_e = -Evf_e^\infty \text{ in } \mathcal{D}'((0, +\infty) \times \Omega).$$

249

250 *Proof.* We omit the proof because it follows from standard calculations. \square

251 Let us now introduce the change of unknown

$$252 \quad (18) \quad w_e(t, x) := v_e(x)\eta_e(t, x), \quad h_e(t, x, v) := \begin{cases} \frac{g_e(t, x, v)}{\sqrt{f_e^\infty(x, v)}} & \text{if } f_e^\infty(x, v) \neq 0 \\ g_e(t, x, v) & \text{if } f_e^\infty(x, v) = 0. \end{cases}$$

253 We recall that $Df_e^\infty = 0$ in $\mathcal{D}'(\Omega)$ and we note that by definition f_e^∞ never vanishes
 254 when $\alpha \in (0, 1)$. One also checks by a straightforward calculation that the couple
 255 (η_e, g_e) is a solution of (16)-(17) if and only if (w_e, h_e) is a solution to

$$256 \quad (19) \quad \partial_t w_e + v_e \partial_x w_e = [f_e^\infty]v_e E \text{ in } \mathcal{D}'((0, +\infty) \times (0, 1)),$$

257

$$258 \quad (20) \quad \partial_t h_e + Dh_e = -Ev\sqrt{f_e^\infty} \text{ in } \mathcal{D}'((0, +\infty) \times \Omega).$$

259 The transport equation (19) has a negative on $[0, 1)$ and vanishing at $x = 1$ velocity
 260 field v_e defined by (8). It is a priori not clear whether a boundary condition is needed
 261 for this equation. Having a closer look at the characteristics that are drawn in figure
 262 ??, we see that a boundary condition at $x = 1$ is necessary if one wants to solve the
 263 equation on the domain $(0, +\infty) \times (0, 1)$. Physically speaking, the number $w_e(t, x)$
 264 is the current of electrons at the position $x \in [0, 1]$ carried by the singular part of f_e
 265 notably

$$266 \quad w_e(t, x) = \eta_e(t, x)\langle \delta_{v=v_e(x)}, v \rangle$$

267 where $\delta_{v=v_e(x)}$ denotes the classical Dirac mass supported at $v = v_e(x)$. For $\eta_e \underset{x \rightarrow 1^-}{=} o(\frac{1}{v_e})$
 268 this yields $w_e(t, 1) = 0$. We thus impose the homogeneous Dirichlet boundary
 269 condition $w_e(t, 1) = 0$. The boundary conditions for the transport equation (20) are
 270 easily derived from the original boundary condition on f_e , they write for all $t > 0$:

$$271 \quad (21) \quad h_e(t, 0, v > 0) = 0, \quad h_e(t, 1, v < 0) = \sqrt{\alpha} h_e(t, 1, -v).$$

272 The initial boundary value problem then writes: given $w_e^0 : [0, 1] \rightarrow \mathbb{R}$, $h_e^0 : \overline{\Omega} \rightarrow \mathbb{R}$
 273 and $E^0 \in [0, 1] \rightarrow \mathbb{R}$, find $w_e : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$, $h_e : [0, +\infty) \times \overline{\Omega} \rightarrow \mathbb{R}$ and
 274 $E : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ solution to

$$275 \quad (\text{VAL}) : \begin{cases} \partial_t w_e + v_e \partial_x w_e = [f_e^\infty] v_e E \text{ in } (0, +\infty) \times (0, 1), \\ \partial_t h_e + D h_e = -E v \sqrt{f_e^\infty} \text{ in } (0, +\infty) \times \Omega, \\ \partial_t E = w_e + \int_{\mathbb{R}} v \sqrt{f_e^\infty} h_e dv \text{ in } (0, +\infty) \times [0, 1], \\ h_e(t > 0, 0, v > 0) = 0, h_e(t > 0, 1, v < 0) = \sqrt{\alpha} h_e(t, 1, -v), \\ w_e(t > 0, 1) = 0, \end{cases}$$

276 satisfying $w_e(t = 0, \cdot) = w_e^0$, $h_e(t = 0, \cdot, \cdot) = h_e^0$, $E(t = 0, \cdot) = E^0$.

277 **2.3. The main result.** We are now in position to state precisely our main result.
 278 First let us begin with defining what linear stability stands for in this work.

279 **DEFINITION 5.** Let G and H be two Hilbert spaces equipped respectively with the
 280 norm $\|\cdot\|_H$, $\|\cdot\|_G$ and with the continuous embedding $G \hookrightarrow H$. We say that the
 281 equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is linearly stable by interior electron perturbation of the
 282 form (12) iff:

- 283 a) For all $(w_e^0, h_e^0, E^0) \in G$ the system (VAL) admits a unique strong solution
 284 $(w_e, h_e, E) \in C^0([0, +\infty); G) \cap C^1([0, +\infty); H)$.
 285 b) For all $\epsilon > 0$, there is $\eta > 0$ such that $\|(w_e^0, h_e^0, E^0)\|_H < \eta \Rightarrow \|(w_e, h_e, E)\|_H < \epsilon$
 286 $\forall t \geq 0$.

287 The Hilbert spaces to be considered are the following

$$288 \quad G := H_{\frac{1}{v_e}, 0}^1(0, 1) \times W_{0, \alpha}^2(\Omega) \times L^2(0, 1), \quad H := L_{v_e, -\frac{1}{2}}^2(0, 1) \times L^2(\Omega) \times L^2(0, 1)$$

289 where the spaces $L_{v_e, -\frac{1}{2}}^2(0, 1)$, $H_{\frac{1}{v_e}, 0}^1(0, 1)$ and $W_{0, \alpha}^2(\Omega)$ are defined in section 3.

290 **THEOREM 6.** The equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is linearly stable in the sense of Def-
 291 inition 5. More precisely,

- 292 a) For all $(w_e^0, h_e^0, E^0) \in G$ there exists a unique strong solution $(w_e, h_e, E) \in$
 293 $C^0([0, +\infty); G) \cap C^1([0, +\infty); H)$ to (VAL).
 294 b) The energy defined in (13) is well-defined and for all times $t \geq 0$, re-writes

$$295 \quad \mathcal{E}(t) = \frac{1}{2} \left(\int_0^1 \frac{w_e(t, x)^2}{[f_e^\infty] |v_e(x)|} dx + \int_{\Omega} h_e(t, x, v)^2 dx dv + \int_0^1 E(t, x)^2 dx \right)$$

296 and is non increasing.

297 One of the key ingredient in the proof of the Theorem 6 is the following energy identity.

298

299 **PROPOSITION 7 (Energy dissipation).** Strong solutions to (VAL) satisfy

$$300 \quad \frac{d}{dt} \mathcal{E}(t) = -\frac{1}{2} \left(\frac{1}{[f_e^\infty]} w_e^2(t, 0) + (1 - \alpha) \int_{\mathbb{R}^+} v h_e^2(t, 1, v) dv - \int_{\mathbb{R}^-} v h_e^2(t, 0, v) dv \right) \leq 0.$$

301 *In particular, for all $0 \leq t \leq t'$, $0 \leq \mathcal{E}(t') \leq \mathcal{E}(t)$.*

302 *Proof.* Let $(w_e, h_e, E) \in C^0([0, +\infty); G) \cap C^1([0, +\infty); H)$ a solution to (VAL).
 303 We set $\mathcal{E}_{w_e}(t) := \frac{1}{[f_e^\infty]} \int_0^1 \frac{w_e(t, x)^2}{|v_e(x)|} dx$, $\mathcal{E}_{h_e}(t) := \int_\Omega h_e(t, x, v)^2 dx dv$ and $\mathcal{E}_{pot}(t) :=$
 304 $\int_0^1 E(t, x)^2 dx$. We compute each terms separately. Because of the regularity of
 305 (w_e, h_e, E) we can differentiate under the integral sign.

$$\begin{aligned} 306 \quad \frac{d}{dt} \mathcal{E}_{w_e}(t) &= \frac{2}{[f_e^\infty]} \int_0^1 \frac{\partial_t w_e(t, x) w_e(t, x)}{|v_e(x)|} dx \\ 307 \quad &= \frac{2}{[f_e^\infty]} \int_0^1 ([f_e^\infty] E(t, x) v_e(x) - v_e(x) \partial_x w_e(t, x)) \frac{w_e(t, x)}{|v_e(x)|} dx \end{aligned}$$

308

310 Once again because $w_e(t, \cdot) \in H^1_{v_e^{\frac{1}{2}}, 0}(0, 1) \subset L^2_{v_e^{-\frac{1}{2}}}(0, 1)$ the second integral is conver-
 311 gent

$$312 \quad \left| \int_0^1 (v_e(x) \partial_x w_e(t, x)) \frac{w_e(t, x)}{|v_e(x)|} dx \right| < +\infty,$$

313 and we deduce

$$314 \quad \frac{d}{dt} \mathcal{E}_{w_e}(t) = -2 \int_0^1 E(t, x) w_e(x) dx - \frac{1}{[f_e^\infty]} w_e^2(t, 0).$$

316 We now compute the energy part associated with h_e . Using the boundary conditions
 317 and an integration by parts we get

$$\begin{aligned} 318 \quad \frac{d}{dt} \mathcal{E}_{h_e}(t) &= 2 \int_\Omega \partial_t h_e(t, x, v) h_e(t, x, v) dx dv \\ 319 \quad &= 2 \int_\Omega \left(-E(t, x) v \sqrt{f_e^\infty(x, v)} - D f_e^\infty(x, v) \right) h_e(t, x, v) dx dv \\ 320 \quad &= -2 \int_\Omega E(t, x) v \sqrt{f_e^\infty(x, v)} h_e(t, x, v) dx dv \\ 321 \quad &\quad - (1 - \alpha) \int_{\mathbb{R}^+} v h_e^2(t, 1, v) dv + \int_{\mathbb{R}^-} v h_e^2(t, 0, v) dv. \end{aligned}$$

323 Note that since $h_e(t, \cdot, \cdot) \in W^2_{0, \alpha}(\Omega)$ the boundary terms make sense. We lastly turn
 324 to the electric part of the energy.

$$\begin{aligned} 325 \quad \frac{d}{dt} \mathcal{E}_{pot}(t) &= 2 \int_0^1 \partial_t E(t, x) E(t, x) dx \\ 326 \quad &= 2 \int_0^1 \left(w_e(t, x) + \int_{\mathbb{R}} v \sqrt{f_e^\infty(x, v)} h_e(t, x, v) dv \right) E(t, x) dx. \end{aligned}$$

328 Gathering all terms together enables to get the desired identity. In particular, we
 329 deduce that $t \in [0, +\infty) \mapsto \mathcal{E}(t)$ is non increasing. Hence $\mathcal{E}(t) \leq \mathcal{E}(t')$ for all $0 \leq t \leq$
 330 t' . \square

331 **3. Functional spaces, technical lemmas and proof of the main result.**

332 In this section, we define the functional framework that is part of the main result 6.

333 We eventually prove the main result by showing that the Hill-Yosida theorem applies.

334 **3.1. Functional spaces.** We define the following spaces

$$335 \quad H_{v_e}^1(0, 1) := \{u \in L^2(0, 1) \text{ s.t. } \sqrt{|v_e|}u' \in L^2(0, 1)\}$$

336 where v_e is the function defined by (8) and note that it is such that $\frac{1}{v_e} \in L^1(0, 1)$. It
337 is a Hilbert space endowed with the inner product

$$338 \quad (u, v)_{H_{v_e}^1(0,1)} := \int_0^1 u(x)v(x) + |v_e(x)|u'(x)v'(x)dx \quad \forall (u, v) \in H_{v_e}^1(0, 1)^2.$$

339 Moreover, we can prove the imbedding $H_{v_e}^1(0, 1) \hookrightarrow C^0[0, 1]$ so that we can define the
340 space

$$341 \quad H_{v_e, 0}^1(0, 1) := \{u \in H_{v_e}^1(0, 1) \text{ s.t. } u(1) = 0\}.$$

342 We also defined the weighted Lebesgue space

$$343 \quad \mathcal{L}_{v_e}^2(0, 1) := \{u : (0, 1) \rightarrow \mathbb{R} \text{ measurable s.t. } \int_0^1 \frac{u^2(x)}{|v_e(x)|}dx < +\infty\}$$

344 and the quotient space

$$345 \quad L_{v_e}^2(0, 1) := \mathcal{L}_{v_e}^2(0, 1)/\mathcal{R}$$

346 where \mathcal{R} denotes the usual equivalence relation of almost everywhere equality for the
347 Lebesgue measure. The space $L_{v_e}^2(0, 1)$ is an Hilbert space endowed with the inner
348 product

$$349 \quad (u, v)_{L_{v_e}^2(0,1)} := \int_0^1 \frac{u(x)}{\sqrt{|v_e(x)|}} \frac{v(x)}{\sqrt{|v_e(x)|}} dx \quad \forall (u, v) \in L_{v_e}^2(0, 1)^2.$$

350 An important tool in this work is the following inequality.

351 **LEMMA 8** (Hardy-Poincaré type inequality). *For all $\varphi \in H_{v_e, 0}^1(0, 1)$,*

$$352 \quad \int_0^1 \frac{\varphi(x)^2}{|v_e(x)|} dx \leq \left\| \frac{1}{v_e} \right\|_{L^1(0,1)}^2 \int_0^1 |v_e(x)|\varphi'(x)^2 dx.$$

353 *Consequently, $H_{v_e, 0}^1(0, 1) \hookrightarrow L_{v_e}^2(0, 1)$.*

354 *Proof.* We prove the inequality for $\varphi \in C_c^\infty(0, 1)$. Let $\delta > 0$, one has for all
355 $x \in [0, 1 - \delta]$

$$356 \quad \varphi(x) - \varphi(1 - \delta) = - \int_x^{1-\delta} \frac{\varphi'(s)\sqrt{|v_e(s)|}}{\sqrt{|v_e(s)|}} ds.$$

357 Using the Cauchy-Schwarz inequality yields

$$358 \quad |\varphi(x) - \varphi(1 - \delta)| \leq \|\sqrt{|v_e|}\varphi'\|_{L^2(x, 1-\delta)} \left\| \frac{1}{\sqrt{|v_e|}} \right\|_{L^2(x, 1-\delta)}$$

$$359 \quad \leq \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx \right)^{\frac{1}{2}} \left\| \frac{1}{v_e} \right\|_{L^1(0,1)}^{\frac{1}{2}}.$$

360

361 Taking the limit as $\delta \rightarrow 0^+$ yields for all $x \in [0, 1]$.

$$362 \quad |\varphi(x)| \leq \left(\int_0^1 |v_e(x)| \varphi'(x)^2 dx \right)^{\frac{1}{2}} \left\| \frac{1}{v_e} \right\|_{L^1(0,1)}^{\frac{1}{2}}.$$

363 Therefore for all $x \in [0, 1)$ we have

$$364 \quad \frac{\varphi(x)^2}{|v_e(x)|} \leq \frac{1}{|v_e(x)|} \left(\int_0^1 |v_e(x)| \varphi'(x)^2 dx \right) \left\| \frac{1}{v_e} \right\|_{L^1(0,1)}.$$

365 One has therefore for $\delta > 0$

$$366 \quad \int_0^{1-\delta} \frac{\varphi(x)^2}{|v_e(x)|} dx \leq \int_0^{1-\delta} \frac{1}{|v_e(x)|} dx \left(\int_0^1 |v_e(x)| \varphi'(x)^2 dx \right) \left\| \frac{1}{v_e} \right\|_{L^1(0,1)}$$

$$367 \quad \leq \left\| \frac{1}{v_e} \right\|_{L^1(0,1)}^2 \int_0^1 |v_e(x)| \varphi'(x)^2 dx.$$

369 Taking the limit at $\delta \rightarrow 0^+$ yields the desired inequality. The result extends to
 370 functions of the space $H_{v_e, 0}^1(0, 1)$ by using the density of $C_c^\infty(0, 1)$ in $H_{v_e, 0}^1(0, 1)$ (see
 371 [7] Lemma 2.6 for the proof of density). \square

372 Thanks to Lemma 8 we can define on $H_{v_e, 0}^1(0, 1)$ the following norm

$$373 \quad \|u\|_{H_{v_e, 0}^1(0,1)} := \left(\int_0^1 |v_e(x)| u'(x)^2 dx \right)^{\frac{1}{2}} \quad \forall u \in H_{v_e, 0}^1(0, 1).$$

374 It is an equivalent norm to the $H_{v_e, 0}^1(0, 1)$ -norm. We will also need the following density
 375 result.

376 LEMMA 9. $H_{v_e, 0}^1(0, 1)$ is dense in $L_{v_e, 0}^2(0, 1)$.

377 *Proof.* Let $u \in L_{v_e, 0}^2(0, 1)$. Let us build a sequence $(u_n)_{n \in \mathbb{N}} \subset H_{v_e, 0}^1(0, 1)$ such
 378 that $\|u_n - u\|_{L_{v_e, 0}^2(0,1)} \xrightarrow{n \rightarrow +\infty} 0$. We firstly remark that since $\frac{u}{\sqrt{-v_e}} \in L^2(0, 1)$ and
 379 because $H_0^1(0, 1)$ is dense in $L^2(0, 1)$ there is a sequence $(\tilde{u}_n)_{n \in \mathbb{N}} \subset H_0^1(0, 1)$ such that
 380 $\int_0^1 \left| \tilde{u}_n - \frac{u}{\sqrt{-v_e}} \right|^2(x) dx \xrightarrow{n \rightarrow +\infty} 0$. Let us then define for all $n \in \mathbb{N}$ $u_n := \tilde{u}_n \sqrt{-v_e}$. To
 381 conclude the proof, it suffices to check that for all $n \in \mathbb{N}$, $u_n \in H_{v_e, 0}^1(0, 1)$. Because
 382 $v_e \in C^0[0, 1]$ one readily verifies that $u_n \in C^0[0, 1] \cap L^2(0, 1)$. Let us now compute the
 383 derivative. One has in $\mathcal{D}'(0, 1)$

$$384 \quad u_n' = \tilde{u}_n' \sqrt{-v_e} - \tilde{u}_n \frac{v_e'}{2\sqrt{-v_e}} \Rightarrow \sqrt{-v_e} u_n' = -\tilde{u}_n v_e' - \frac{\tilde{u}_n v_e'}{2}.$$

385 One has $\tilde{u}_n v_e' \in L^2(0, 1)$, it therefore suffices to prove that $\tilde{u}_n v_e' \in L^2(0, 1)$. Using
 386 Lemma 2 d) we have for all $x \in (0, 1)$

$$387 \quad \tilde{u}_n(x) v_e'(x) = -\tilde{u}_n(x) \frac{E^\infty(x)}{v_e(x)}.$$

388 Since $E^\infty \in C^0[0, 1]$, it suffices to show that $\frac{\tilde{u}_n}{v_e} \in L^2(0, 1)$. Still using Lemma 2 b),
 389 we have $\frac{\tilde{u}_n^2(x)}{v_e^2(x)} \underset{x \rightarrow 1^-}{\sim} \nu^2 \frac{\tilde{u}_n(x)^2}{(1-x)}$. But $\tilde{u}_n \in H_0^1(0, 1)$ and a classical Hardy inequality
 390 (see [6, p.147] for instance) enables us to conclude that $\frac{\tilde{u}_n}{v_e} \in L^2(0, 1)$. \square

391 We now define

$$392 \quad W^2(\Omega) := \{h \in L^2(\Omega) \text{ s.t } Dh \in L^2(\Omega)\}.$$

393 Following [2], for a function $h \in W^2(\Omega)$ we can define its restriction to $\Sigma_- := \{0\} \times$
 394 $(0, +\infty) \cup \{1\} \times (-\infty, 0)$ and $\Sigma_+ := \{0\} \times (-\infty, 0) \cup \{1\} \times (0, +\infty)$. Moreover, $h|_{\Sigma_-}$
 395 and $h|_{\Sigma_+}$ belongs respectively to $L_{loc}^2(\Sigma_-)$ and $L_{loc}^2(\Sigma_+)$. Thus, we can also define

$$396 \quad W_{0,\alpha}^2(\Omega) := \{h \in W^2(\Omega) \text{ s.t } h(0, v > 0) = 0 \text{ and } h(1, v < 0) = \sqrt{\alpha}h(1, -v)\}.$$

397

398 LEMMA 10. $W_{0,\alpha}^2(\Omega)$ is dense in $L^2(\Omega)$ for the L^2 -norm.

399 *Proof.* It suffices to remark that $C_c^1(\Omega) \subset W_{0,\alpha}^2(\Omega)$. Then we deduce that $\overline{C_c^1(\Omega)} \subset$
 400 $\overline{W_{0,\alpha}^2(\Omega)} \subset L^2(\Omega)$ where \overline{X} denotes the closure of the set X in $L^2(\Omega)$ for the L^2 -norm.
 401 But, $C_c^1(\Omega)$ is dense in $L^2(\Omega)$ so $\overline{C_c^1(\Omega)} = L^2(\Omega)$ and then $\overline{W_{0,\alpha}^2(\Omega)} = L^2(\Omega)$. \square

402 To finish with this section, we state a Lemma due to Bardos [2] and justify a Green
 403 formula.

404 LEMMA 11. $C_c^\infty(\overline{\Omega})$ is dense in $W_{0,\alpha}^2(\Omega)$ for the norm defined by

$$405 \quad \|h\|_{W_{0,\alpha}^2}^2 := \|h\|_{L^2(\Omega)}^2 + \|Dh\|_{L^2(\Omega)}^2 \quad \forall h \in W_{0,\alpha}^2.$$

406

407 *Proof.* See [2] \square

408 LEMMA 12. (Traces integrability) Let $h \in W_{0,\alpha}^2(\Omega)$ then $h(1, \cdot) \in L^2(\mathbb{R}^+, |v|dv)$
 409 and $h(0, \cdot) \in L^2(\mathbb{R}^-, |v|dv)$ and the following Green Formula holds :

$$410 \quad \int_{\Omega} Dh(x, v)h(x, v)dx dv = (1 - \alpha) \int_{\mathbb{R}^+} \frac{v}{2} h^2(1, v)dv - \int_{\mathbb{R}^-} \frac{v}{2} h^2(0, v)dv.$$

411

412 *Proof.* We argue by density. Let $h \in W_{0,\alpha}^2(\Omega)$ then in virtue of Lemma 11 there
 413 is $(h_n)_{n \in \mathbb{N}} \subset C_c^\infty(\overline{\Omega})$ such that $\|h_n - h\|_{W_{0,\alpha}^2} \xrightarrow{n \rightarrow +\infty} 0$. Using the boundary conditions
 414 and a Green Formula (that is valid for regular functions) we have for all $n \in \mathbb{N}$,

$$415 \quad \int_{\Omega} Dh_n(x, v)h_n(x, v)dx dv = (1 - \alpha) \int_{\mathbb{R}^+} \frac{v}{2} h_n^2(1, v)dv - \int_{\mathbb{R}^-} \frac{v}{2} h_n^2(0, v)dv.$$

416 By standard arguments, it is easy to see that

$$417 \quad \int_0^1 \int_{\mathbb{R}} Dh_n(x, v)h_n(x, v)dx dv \xrightarrow{n \rightarrow +\infty} \int_0^1 \int_{\mathbb{R}} Dh(x, v)h(x, v)dx dv.$$

418 Then the sequences $\left(\int_{\mathbb{R}^+} \frac{v}{2} h_n^2(1, v)dv\right)_{n \in \mathbb{N}}$ and $\left(\int_{\mathbb{R}^-} \frac{v}{2} h_n^2(0, v)dv\right)_{n \in \mathbb{N}}$ are bounded
 419 and converge (up to an extraction). Lastly, we can show that the trace operators

$$420 \quad \gamma_0 : h \in C_c^\infty(\overline{\Omega}) \cap W_{0,\alpha}^2(\Omega) \mapsto h(0, \cdot) \in L^2(\mathbb{R}^-, |v|dv),$$

421

422

$$\gamma_1 : h \in C_c^\infty(\overline{\Omega}) \cap W_{0,\alpha}^2(\Omega) \mapsto h(1, \cdot) \in L^2(\mathbb{R}^+, |v|dv)$$

423

424

extend both continuously to $W_{0,\alpha}^2(\Omega)$. This finally enables us to pass to the limit at both side of the previous equality so that the formula holds. \square

425

426

We lastly mention that by a similar density argument we can also prove the following Green-formula.

427

428

LEMMA 13. For all $h \in W_{0,\alpha}^2(\Omega)$ and for all $\psi \in W^2(\Omega)$ such that $\psi(0, v < 0) = 0$ and $\psi(1, v > 0) = \sqrt{\alpha}\psi(1, -v)$ a.e we have,

429

$$\int_{\Omega} Dh(x, v)\psi(x, v)dx dv = - \int_{\Omega} h(x, v)D\psi(x, v)dx dv.$$

430

431

432

3.2. Proof of the main result. We prove the main result by checking that the Hille-Yosida's Theorem applies (see [6, p.105] for a precise statement). The Hilbert space $H = L^2_{v_e^{-\frac{1}{2}}}(0, 1) \times L^2(\Omega) \times L^2(0, 1)$ is equipped with the inner product

433

$$(U_1, U_2)_H := \frac{1}{[f_e^\infty]} (w_1, w_2)_{L^2_{v_e^{-\frac{1}{2}}}(0,1)} + (h_1, h_2)_{L^2(\Omega)} + (E_1, E_2)_{L^2(0,1)},$$

434

435

for all $U_1 := (w_1, h_1, E_1)$, $U_2 := (w_2, h_2, E_2)$ in H . We introduce the unbounded operator $A : D(A) \subset H \rightarrow H$ defined by :

436

$$\left\{ \begin{array}{l} AU := \left(\begin{array}{l} v_e \partial_x w - [f_e^\infty] v_e E \\ Dh + Ev \sqrt{f_e^\infty} \\ - \left(w + \int_{\mathbb{R}} v \sqrt{f_e^\infty} h dv \right) \end{array} \right), \\ \forall U := \left(\begin{array}{l} w \\ h \\ E \end{array} \right) \in D(A) = H^1_{v_e^{\frac{1}{2}}, 0}(0, 1) \times W_{0,\alpha}^2(\Omega) \times L^2(0, 1), \end{array} \right.$$

437

438

We are going to check the assumptions of the Hille-Yosida's Theorem. For precise definitions we refer the reader to the appendix B.

439

440

LEMMA 14. The unbounded operator $A : D(A) \subset H \rightarrow H$ has the following properties:

441

442

- a) It is dissipative, in the sense that $(AU, U)_H \geq 0$.
- b) $D(A)$ is dense in H .

443 *Proof.* a) We prove that A is dissipative. Let $U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A)$. We compute

$$\begin{aligned}
 444 \quad (AU, U)_H &= \frac{1}{[f_e^\infty]} \underbrace{\left(v_e \partial_x w - \frac{[f_e^\infty]}{\mu} v_e E, w \right)}_{:=I_1} \Big|_{L^2_{v_e^{-\frac{1}{2}}}(0,1)} \\
 445 \quad &+ \underbrace{\left(Dh + Ev \sqrt{f_e^\infty}, h \right)}_{:=I_2} \Big|_{L^2(\Omega)} \\
 446 \quad &- \underbrace{\left(w + \int_{\mathbb{R}} hv \sqrt{f_e^\infty} dv, E \right)}_{:=I_3} \Big|_{L^2(0,1)}.
 \end{aligned}$$

447
448 We can compute I_1 and I_2 by an integration by parts so that we obtain :

$$449 \quad I_1 = \int_0^1 (-\partial_x w(x) + [f_e^\infty] E(x)) w(x) dx = \frac{w^2(0)}{2} + [f_e^\infty] \int_0^1 E(x) w(x) dx.$$

$$\begin{aligned}
 451 \quad I_2 &= \int_0^1 \int_{\mathbb{R}} \left(Dh(x, v) + E(x) v \sqrt{f_e^\infty(x, v)} \right) h(x, v) dx dv \\
 452 \quad &= (1 - \alpha) \int_{\mathbb{R}^+} \frac{vh^2(1, v)}{2} dv - \int_{\mathbb{R}^-} \frac{vh^2(0, v)}{2} dv + \int_{\Omega} E(x) v \sqrt{f_e^\infty(x, v)} h(x, v) dx dv.
 \end{aligned}$$

$$453 \quad I_3 = \int_0^1 E(x) w(x) dx + \int_{\Omega} E(x) v \sqrt{f_e^\infty(x, v)} h(x, v) dx dv.$$

454
455 Collecting all terms together, we finally deduce

$$456 \quad (AU, U)_H = \frac{1}{[f_e^\infty]} \frac{w^2(0)}{2} + (1 - \alpha) \int_{\mathbb{R}^+} \frac{vh^2(1, v)}{2} dv - \int_{\mathbb{R}^-} \frac{vh^2(0, v)}{2} dv \geq 0.$$

457
458 b) The fact that $D(A)$ is dense in H is a consequence of Lemmas 9 and 10. \square

459 LEMMA 15. *The unbounded operator $I + A : D(A) \subset H \rightarrow H$ is such that $R(I +$*
460 *$A) = H$.*

461 *Proof.* We are going to apply the Proposition 17. Of course, $D(A)$ is dense in H
462 as we have already proven. Since H is a Hilbert space, by the Riesz representation
463 theorem, we can identify H with its dual, namely $H' \cong H$ so that the adjoint operator
464 of $I + A$ is the unbounded operator $(I + A)^* : D(A^*) \subset H \rightarrow D(A)'$ defined by:

$$\begin{cases}
 465 \quad (I + A)^* U := U + A^* U \text{ with } A^* U := \begin{pmatrix} -v_e \partial_x w + [f_e^\infty] v_e E \\ -Dh - Ev \sqrt{f_e^\infty} \\ \left(w + \int_{\mathbb{R}} v \sqrt{f_e^\infty} h dv \right) \end{pmatrix}, \\
 \forall U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A^*) = \{u \in H^1_{v_e^{-\frac{1}{2}}}(0, 1) \text{ s.t } u(0) = 0\} \times \\
 \{h \in W^2(\Omega) \text{ s.t } h(0, v < 0) = 0 \text{ and } h(1, v > 0) = \sqrt{\alpha} h(1, -v)\} \times L^2(0, 1)
 \end{cases}$$

466 Of course, $D(A^*)$ is also dense in H . This enables to prove that $I + A$ is closed (see
 467 [6, Proposition II.16, p.28]). Lastly, straightforward integrations by parts allow us to
 468 prove that A^* is also dissipative which in the end turns out to be sufficient to prove
 469 that

$$470 \quad ((I + A)^*U, U)_H = \|U\|_H^2 + (A^*U, U)_H \geq \|U\|_H^2.$$

471 Therefore a Cauchy-Schwarz inequality yields $\|(I + A)^*U\|_H \geq \|U\|_H$. Therefore the
 472 Proposition 17 applies, it concludes the proof. \square

473 The proof of the main result is now straightforward. Combining Lemmas 14 and 15
 474 we can apply the Hille-Yosida theorem. For all $(w_e^0, h_e^0, E^0) \in D(A)$ there is a unique

$$475 \quad \begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} \in C^0([0, +\infty); D(A)) \cap C^1([0, +\infty), H) \text{ such that}$$

$$476 \quad \frac{d}{dt} \begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} + A \begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

477 with

$$478 \quad \begin{pmatrix} w(0, \cdot) \\ h(0, \cdot, \cdot) \\ E(0, \cdot) \end{pmatrix} = \begin{pmatrix} w_e^0 \\ h_e^0 \\ E^0 \end{pmatrix}.$$

479 The boundary conditions are included in the space $D(A)$ so that they are satisfied by
 480 the solution. The solution also satisfies for all times $0 \leq t \leq t'$ the inequality

$$481 \quad \left\| \begin{pmatrix} w_e(t', \cdot, \cdot) \\ h_e(t', \cdot, \cdot) \\ E(t', \cdot) \end{pmatrix} \right\|_H^2 \leq \left\| \begin{pmatrix} w_e(t, \cdot, \cdot) \\ h_e(t, \cdot, \cdot) \\ E(t, \cdot, \cdot) \end{pmatrix} \right\|_H^2$$

482 which re-writes in terms of the energy $\mathcal{E}(t') \leq \mathcal{E}(t)$. It also implies the stability with
 483 respect to perturbation. Indeed, for any $\epsilon > 0$, it suffices to choose (w_e^0, h_e^0, E^0) such
 484 that $\mathcal{E}(0) < \epsilon$ to get that $\mathcal{E}(t) < \epsilon$ for all times $t \geq 0$.

485 **Appendix A. On the regularity of η_e .** We want to explain why we have
 486 worked on the flux variable w_e rather than on the density number η_e . First notice
 487 that one has the following imbedding

$$488 \quad H_{v_e^{\frac{1}{2}}, 0}^1(0, 1) \hookrightarrow C^0[0, 1]$$

489 which implies that $w_e(t, \cdot) \in C^0[0, 1]$ for all $t \geq 0$. The boundary condition on w_e
 490 therefore makes sense. As far as the electron density η_e is concerned, we now observe
 491 that because $\frac{1}{v_e} \in L^1(0, 1)$ one has

$$492 \quad \eta_e \in C^0([0, +\infty); C^0[0, 1) \cap L^1(0, 1)) \cap C^1([0, +\infty); L^1(0, 1)),$$

493 and $\eta_e \underset{x \rightarrow 1^-}{=} o(\frac{1}{v_e})$. However, it is not clear whether η_e can be extended by continuity
 494 at $x = 1$. In fact, we cannot expect any more integrability on the spatial derivative of
 495 η_e and thus the notion of boundary condition at $x = 1$ is not obvious. To illustrate
 496 this lack of regularity, a simple calculation shows that

$$497 \quad v_e \partial_x \eta_e = \partial_x w_e - \frac{v_e'}{v_e} w_e \text{ in } \mathcal{D}'(0, 1).$$

498 The first term at the right hand side of the equality belongs to $L^1(0, 1)$ while the
 499 second term does not :

500
$$\forall t \geq 0 \forall x \in (0, 1) \quad \left| \frac{v'_e(x)}{v_e(x)} w_e(t, x) \right| \geq \min_{x \in [0, 1]} |w_e(t, x)| \left| \frac{v'_e(x)}{v_e(x)} \right|,$$

501 and $\int_0^1 \left| \frac{v'_e(x)}{v_e(x)} \right| dx = +\infty$. This lack of integrability of $\partial_x \eta_e$ is inherent in the nature
 502 of the functional space $H^1_{\frac{1}{2}, 0}(0, 1)$. For instance, we can show that a function of the
 503 form $w : x \in [0, 1] \mapsto (1-x)^s$ belongs to $H^1_{\frac{1}{2}, 0}(0, 1)$ if and only if $s > \frac{1}{4}$. The quotient
 504 $\eta := \frac{w}{v_e}$ has the following behavior $\eta(x) \underset{x \rightarrow 1^-}{\sim} \frac{-(1-x)^{s-\frac{1}{2}}}{\nu}$ with $s - \frac{1}{2} > -\frac{1}{4}$. Then for
 505 $\frac{1}{4} < s < \frac{1}{2}$ we have $\lim_{x \rightarrow 1^-} \eta(x) = -\infty$ and $\eta \in L^1(0, 1)$ but $\partial_x \eta \notin L^1(0, 1)$.

506 **Appendix B. Reminder on linear evolution equation in infinite dimen-**
 507 **sion.**

508 DEFINITION 16. Let H a Hilbert space and $A : D(A) \subset H \rightarrow H$ an unbounded
 509 linear operator. We say that A is dissipative if

510
$$(Av, v)_H \geq 0 \quad \forall v \in D(A),$$

511 A is maximal dissipative if moreover $R(I + A) = H$.

512 We recall a result that characterizes surjective operators.

513 PROPOSITION 17 ([6] Theorem II.19 page 30). Let $A : D(A) \subset H \rightarrow H$ an
 514 unbounded linear operator, closed with $D(A)$ dense in H . Then

515
$$R(A) = H \Leftrightarrow \exists C \geq 0 \text{ such that } \|v\|_{H'} \leq C \|A^*v\|_{H'} \quad \forall v \in D(A^*),$$

516 where A^* is the adjoint-operator of A and H' is the dual space of H .

517 THEOREM 18 (Hille-Yosida). Let A be a maximal monotone operator in a Hilbert
 518 space H . Then for all $u_0 \in D(A)$ there is a unique $u \in C^1([0, +\infty); H) \cap C^0([0, +\infty); D(A))$ ■
 519 solution of the problem :

520
$$\begin{cases} \frac{du}{dt} + Au = 0 \text{ on } [0, +\infty) \\ u(0) = u_0. \end{cases}$$

521 Moreover, one has

522
$$\|u(t)\|_H \leq \|u_0\| \text{ and } \left\| \frac{du}{dt}(t) \right\|_H \leq \|Au_0\|_H \text{ for all } t \geq 0.$$

523 REFERENCES

524 [1] M. BADSI, M. CAMPOS PINTO, AND B. DÉSPRES, *A minimization formulation of a bikinetic*
 525 *sheath*, hal-01075646v2, (2014).
 526 [2] BARDOS, *Problèmes aux limites pour les équations aux dérivées partielles du premier ordre*
 527 *à coefficients réels; théorèmes d'approximations; application à l'équation de transport*,
 528 *Annales scientifiques de l'E.N.S.*, (1970), pp. 185–233.
 529 [3] N. BEN ABDALLAH, *Weak solutions of the initial-boundary value problem for the vlasov-poisson*
 530 *system*, M2AS, 17 (1994), pp. 451–476.

- 531 [4] J. BEN-ARTZI, *Instability of nonsymmetric nonmonotone equilibria of the vlasov-maxwell system*, J. Math. Phys, 52 (2011).
532
- 533 [5] M. BOSTAN, *Existence and uniqueness of the mild solution for the 1d vlasov-poisson initial-boundary value problem*, SIAM J. Math.Anal., 37 (2005), pp. 156–188.
534
- 535 [6] H. BREZIS, *Analyse fonctionnelle : Théorie et applications*, Dunod, (1983).
536
- 537 [7] M. CAMPITI, G. METAFUNE, AND D. PALLARA, *Degenerate self-adjoint evolution equation on the unit interval*, Semigroup Forum, 57 (1998), pp. 1–36.
538
- 539 [8] R. DIPERNA AND P. LIONS, *Ordinary differential equations, transport theory and sobolev spaces*, Inventiones mathematicae, 98 (1989).
540
- 541 [9] Y. GUO, *Regularity for the vlasov equations in a half space*, Univ. Math. J, (1994).
542
- 543 [10] Y. GUO, *The dynamics of a plane diode*, SIAM J. Math.Anal., 35 (2004), pp. 1617–1635.
544
- 545 [11] J.-H. HWANG AND J. SCHAEFFER, *Uniqueness for weak solutions of a one-dimensional boundary value problem for the vlasov-poisson system*, Journal of differential equations, (2008).
546
- 547 [12] M. LEMOU, F. MÉHATS, AND R. P., *Orbital stability of spherical galactic models*, Inventiones mathematicae, 187 (2012), pp. 145–194.
548
- 549 [13] G. MANFREDI AND S. DEVAUX, *Plasma-wall transition in weakly collisional plasmas*, AIP conference proceedings, (2008).
550
- 551 [14] C. MOUHOT AND C. VILLANI, *On landau damping*, Acta mathematica, (2011).
552
- 553 [15] T. NGUYEN AND W. STRAUSS, *Stability analysis of collisionless plasmas with specularly reflecting boundary*, SIAM J. Math.Anal., 45 (2013), pp. 777–808.
554
- 555 [16] G. REIN, *Non-linear stability for the vlasov-poisson system. the energy-casimir method*, Mathematical methods in the applied sciences, 17 (1994), pp. 1129–1140.
- 556 [17] P. STANGEBY, *The plasma boundary of magnetic fusion devices*, Institute of Physics Publishing, (2000).