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Abstract. We consider a conservation law perturbed by a linear diffusion and non-positive dispersion $u_t + f(u)_x = \varepsilon u_{xx} - \delta(|u_{xx}|^n)_x$. We prove the convergence of the previous solution to the entropy weak solution of the hyperbolic conservation law $u_t + f(u)_x = 0$, in both cases $n = 1$ and $n = 2$.

Keywords. diffusion, dispersion, KdV equation, Burgers equation, hyperbolic conservation laws, entropy measure-valued solutions.

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1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \tag{1.1}$$
$$u(x,0) = u_0(x), \tag{1.2}$$

where ε and δ are small parameters, and g is a non positive function, and we focus on the specific form

$$g(v) = -|v|^n,$$

where $n \geq 1$. Note that if $n = 1$, g is lipschitz, if $1 < n < 2$, g is C^1 and for $n \geq 2$, g is C^2.

When $\delta = 0$ we reduce to the viscous (generalized) Burgers equation and the approximate solutions $u^{\varepsilon,0}$ converge to the entropy solution of the hyperbolic equation (called the *vanishing viscosity method*, see, e.g., Whitham [21] or Kružkov [9])

$$u_t + f(u)_x = 0 \tag{1.3}$$
$$u(x,0) = u_0(x). \tag{1.4}$$

On the other hand, when $\varepsilon = 0$, if we consider the flux function $f(u) = u^2$ and the linear dispersion δu_{xxx} we obtain the Korteweg-de Vries equation. The approximate solutions $u^{0,\delta}$ do not converge in a strong topology (see Lax-Levermore [11]). So, as parameters ε and δ vanish, we are concerned with singular limits and to ensure convergence it is necessary to be in the dominant dissipation regime.

The pioneer study of these singular limits was given by Schonbek [16] about the (generalized) Korteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon\, u_{xx} - \delta\, u_{xxx}.$$

In the case of a convex flux function $f(u)$, she proved the convergence of the solutions of this perturbed equation to the entropy solution of (1.3), when both ε and δ tend to zero, *at least* under the condition $\delta = O(\varepsilon^3)$ (depending on the behavior of the flux f). Also, according to Perthame-Ryzhic [15], the sharp condition should be $\delta = o(\varepsilon)$. LeFloch-Natalini [13] proved the convergence in the case of a nonlinear viscosity function β and linear capillarity

$$u_t + f(u)_x = \varepsilon\, \beta(u_x)_x - \delta\, u_{xxx}.$$

Then, Correia-LeFloch [5] improved the estimates in Schonbek [16] and LeFloch-Natalini [13] and for the first time treated the multidimensional equation, but still in the case of a nonlinear viscosity function and linear capillarity. In fact, the dominant dissipation regime is also assured by the nonlinear viscosity. In our case, we consider the reverse situation.

In general for $\varepsilon = 0$, like for the Korteweg-de Vries equation, the divergent behaviour is expected, as we are considering "pure-dispersive equations". But, Brenier-Levy [3] considered the fully nonlinear equation

$$u_t + f(u)_x = -\delta(u_{xx}^2)_x$$

as a nonlinear generalization of the Korteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions that differs completely from the linear case. In particular, Brenier and Levy [3] conjectured that for strictly convex flux functions f and for the following perturbed problem

$$u_t + f(u)_x = -\delta g(u_{xx})_x - \varepsilon\, u_{xxxx},$$

we have convergence when ε and δ tend to zero under the condition $\varepsilon = o(\delta)$.

The paper is organized as follows. In Section 2, we present the main results of convergence. Section 3 deals with the uniform estimates needed for convergence. Finally, Section 4 is devoted to proving the convergence to the entropy solution of the hyperbolic equation, when both ε and δ go to zero.

2 Main Results

Two main convergence results are presented. The first one is concerned with $g(v) = -|v|$ (i.e. $n = 1$) while the second one is devoted to the case $g(v) = -v^2$ (i.e. $n = 2$).

2.1 Case f convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$

In this case, we prove the following result.

Theorem 2.1 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^2)$, and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \ \textit{where } 0 \leq \beta < 3.$$

Then, setting $u = u_{\varepsilon,\delta}$ the solution of (1.1) − (1.2), the family solutions $(u_{\varepsilon,\delta})$ converges to the entropy solution of (3) − (4).

In the case of the linear dispersion, i.e. $g(u_{xx}) = u_{xx}$ treated in [16], Schonbek gets for a general flux satisfying $f''(u) \leq C$, a convergence with rate $\delta = O(\varepsilon^3)$. Also, when $f''(u) \leq C(1 + |u|)$, the author obtain the convergence with the rate $\delta = O(\varepsilon^4)$. The case of $g(u_{xx}) = -|u_{xx}|$ seems giving a weakly dispersive effects than a classical linear dispersion.

2.2 Case f convex, $\delta > 0$ and $g(u_{xx}) = -u_{xx}^2$

Here function g is regular, and we obtain:

Theorem 2.2 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$, and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2.$$

Then, setting $u = u_{\varepsilon,\delta}$ the solution of (1.1) − (1.2), the family solutions $(u_{\varepsilon,\delta})$ converges to the entropy solution of (3) − (4).

The dispersion here is strongly nonlinear but regular, which provides the well-posedness [1, 2]. We can see that comparing to the results in [16], the rate are quite similar when the flux is convex and satisfies $f''(u) \leq C$ ($\delta = o(\varepsilon^{5/2})$ whereas $\delta = O(\varepsilon^3)$ in [16]).

3 A priori estimates

Assume that η is a regular function and φ a function defined by $\varphi' = \eta' f'$, and let us multiply (1.1) by $\eta'(u)$. We obtain

$$\begin{aligned} \eta(u)_t + \varphi(u)_x &= \varepsilon \left(\eta'(u)\, u_x\right)_x - \varepsilon\, \eta''(u)\, u_x^2 \\ &\quad + \delta \left(\eta'(u)\, g(u_{xx})\right)_x - \delta\, \eta''(u)\, u_x\, g(u_{xx})\,. \end{aligned} \tag{3.1}$$

Integrating over $(0,t) \times \mathbb{R}$ with $\eta(u) = |u|^{\alpha+1}$, the conservative terms vanish and we obtain the following lemma.

Lemma 3.1 *Let $\alpha \geq 1$ and $g : \mathbb{R} \to \mathbb{R}$ be any dispersion function. Each solution of (1.1) satisfies for $t \in [0,T]$*

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^{\alpha+1}\, dx &+ (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1}\, u_x^2\, dxds \\ &+ (\alpha+1)\,\alpha\,\delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1}\, u_x\, g(u_{xx})\, dxds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1}\, dx\,. \end{aligned} \tag{3.2}$$

Usually, taking $\alpha = 1$ in (3.2), we deduce the a priori L^2 first energy estimates.

Let us introduce the functions $\mathcal{G}$, and G defined by $\mathcal{G}'' = G' = g$, i.e,

$$G(u) = -\frac{1}{n+1}|u|^n u, \quad \text{and} \quad \mathcal{G} = -\frac{1}{(n+1)(n+2)}|u|^{n+2}.$$

Using the multiplier $(q+2)(|u_x|^q u_x)_x$ to (1.1), we have

$$\begin{aligned} \left((q+2)\, u_t\, |u_x|^q\, u_x\right)_x - (|u_x|^{q+2})_t &= -(q+2)(q+1)|u_x|^q\, u_{xx}\, f'(u)\, u_x \\ &\quad + \varepsilon\,(q+2)(q+1)\,|u_x|^q\, u_{xx}^2 \\ &\quad + \delta\,(q+2)(q+1)\,|u_x|^q\, u_{xx}\, g'(u_{xx})\, u_{xxx} \\ &= -(q+1)(|u_x|^{q+2})_x\, f'(u) \\ &\quad + \varepsilon\,(q+2)(q+1)\,|u_x|^q\, u_{xx}^2 \\ &\quad + \delta\,(q+2)(q+1)\,n\,|u_x|^q\, G(u_{xx})_x, \end{aligned}$$

and we get the estimate

$$\begin{aligned}\big((q+2)\,u_t\,|u_x|^q\,u_x\big)_x-(|u_x|^{q+2})_t &= -\big((q+1)|u_x|^{q+2}f'(u)\big)_x \\ &\quad +(q+1)|u_x|^{q+2}\,u_x\,f''(u) \\ &\quad +\varepsilon\,(q+2)(q+1)\,|u_x|^q\,u_{xx}^2 \\ &\quad +\big(\delta\,(q+2)(q+1)\,n\,|u_x|^q\,G(u_{xx})\big)_x \\ &\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,|u_x|^{q-2}\,u_x\,\mathcal{G}(u_{xx}).\end{aligned} \tag{3.3}$$

Similarly, using the multiplier $(q+2)(u_x^{q+1})_x$ to (1.1), we can write

$$\begin{aligned}\big((q+2)\,u_t\,u_x^{q+1}\big)_x-(u_x^{q+2})_t &= -(q+2)(q+1)\,u_x^{q+1}\,f'(u)\,u_{xx} \\ &\quad +\varepsilon\,(q+2)(q+1)\,u_x^q\,u_{xx}^2 \\ &\quad +\big(\delta\,(q+2)(q+1)\,n\,u_x^q\,G(u_{xx})\big)_x \\ &\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,u_x^{q-1}\,\mathcal{G}(u_{xx}),\end{aligned}$$

thus

$$\begin{aligned}\big((q+2)\,u_t\,u_x^{q+1}\big)_x-(u_x^{q+2})_t &= -\big((q+1)\,u_x^{q+2}f'(u)\big)_x \\ &\quad +(q+1)\,u_x^{q+3}\,f''(u) \\ &\quad +\varepsilon\,(q+2)(q+1)\,u_x^q\,u_{xx}^2 \\ &\quad +\big(\delta\,(q+2)(q+1)\,n\,u_x^q\,G(u_{xx})\big)_x \\ &\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,u_x^{q-1}\,\mathcal{G}(u_{xx}).\end{aligned} \tag{3.4}$$

Integrating (3.3) and (3.4) over $\mathbb{R}\times[0,t]$ provides

$$\begin{aligned}&\int_{\mathbb{R}}|u_x(t)|^{q+2}\,dx+\varepsilon\,(q+2)(q+1)\int_0^t\int_{\mathbb{R}}|u_x|^q\,u_{xx}^2\,dxds \\ &\qquad=\int_{\mathbb{R}}|u_0'|^{q+2}\,dx-(q+1)\int_0^t\int_{\mathbb{R}}u_x\,|u_x|^{q+2}\,f''(u)\,dxds \\ &\qquad\quad+\delta\,(q+2)(q+1)\,q\,(n+2)\,n\int_0^t\int_{\mathbb{R}}u_x\,|u_x|^{q-2}\,\mathcal{G}(u_{xx})\,dxds\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\int_{\mathbb{R}}u_x(t)^{q+2}\,dx+\varepsilon\,(q+2)(q+1)\int_0^t\int_{\mathbb{R}}u_x^q\,u_{xx}^2\,dxds \\ &\qquad=\int_{\mathbb{R}}(u_0')^{q+2}\,dx-(q+1)\int_0^t\int_{\mathbb{R}}u_x^{q+3}\,f''(u)\,dxds \\ &\qquad\quad+\delta\,(q+2)(q+1)\,q\,(n+2)\,n\int_0^t\int_{\mathbb{R}}u_x^{q-1}\,\mathcal{G}(u_{xx})\,dxds\,.\end{aligned} \tag{3.6}$$

We define now the sets, for $t\in[0,T]$

$$\mathcal{U}_t^+=\{x\in\mathbb{R};\,u_x(x,t)>0\},$$

and

$$\mathcal{U}_t^-=\{x\in\mathbb{R};\,u_x(x,t)<0\}.$$

Adding (3.6) to (3.5) for q odd, we obtain:

Lemma 3.2 *Let q be a odd number. Then, each solution of* (1.1) *satisfies for* $t \in [0,T]$

$$\begin{aligned}\int_{\mathcal{U}_t^+} |u_x(t)|^{q+2}\,dx + \varepsilon\,(q+2)(q+1)\int_0^t\int_{\mathcal{U}_s^+} |u_x|^q\,u_{xx}^2\,dxds \\ +\delta\,(q+2)(q+1)\,q\,(n+2)\,n\int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q-1}\,|\mathcal{G}(u_{xx})|\,dxds \\ +(q+1)\int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q+3}\,f''(u)\,dxds \\ = \int_{\mathcal{U}_0^+} |u_0'|^{q+2}\,dx\,,\end{aligned} \tag{3.7}$$

where the last left hand-side term can be replaced by

$$-(q+2)(q+1)\int_0^t\int_{\mathcal{U}_s{}^+} |u_x|^{q+1}\,f'(u)\,u_{xx}\,dxds\,. \tag{3.8}$$

Now, the combination of Lemmas 3.1 and 3.2 gives the following estimate

Proposition 3.3 *Let $\varepsilon,\delta>0$, and $f:\mathbb{R}\to\mathbb{R}$ be a convex flux function. The solution $u=u_{\varepsilon,\delta}$ of* (1.1) − (1.2) *satisfies the uniform estimate*

$$\int_{\mathbb{R}} |u(t)|^{\alpha+1}\,dx + \varepsilon\int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,u_x^2\,dxds + \delta\int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,|u_x|\,|u_{xx}|^n\,dxds \le C_0, \tag{3.9}$$

for all $\frac{5+n}{2n+1} \le \alpha < \frac{4+n}{n}$.

Proof. When $g(u) = -|u|^n$, equality (3.2) writes

$$\begin{aligned}\int_{\mathbb{R}} |u(t)|^{\alpha+1}\,dx + \alpha(\alpha+1)\,\varepsilon\int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,u_x^2\,dxds = \\ \|u_0\|_{\alpha+1}^{\alpha+1} + \alpha(\alpha+1)\,\delta\int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,u_x\,|u_{xx}|^n\,dxds.\end{aligned} \tag{3.10}$$

Also, when f is convex, we can rewrite (3.7) for $q\ge 1$ odd as

$$\begin{aligned}\int_{\mathcal{U}_t^+} |u_x(t)|^{q+2}\,dx + \varepsilon\int_0^t\int_{\mathcal{U}_s^+} |u_x|^q\,u_{xx}^2\,dxds \\ +\delta\int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q-1}\,|u_{xx}|^{n+2}\,dxds + \int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q+3}\,f''(u)\,dxds \le C_0\end{aligned} \tag{3.11}$$

However, using the Young inequality, we get

$$\begin{aligned}\delta\int_0^t\int_{\mathcal{U}_s^+} |u|^{\alpha-1}\,u_x\,|u_{xx}|^n\,dxds \\ = \int_0^t\int_{\mathcal{U}_s^+} \left(\frac{1}{c\,t^{\frac{\alpha-1}{\alpha+1}}}|u|^{\alpha-1}\right)\left(c\,t^{\frac{\alpha-1}{\alpha+1}}\,|u_x|\right)\,(\delta|u_{xx}|^n)\,dxds \\ \le \left(\frac{1}{t\,c^{\frac{\alpha+1}{\alpha-1}}}\right)\left(\frac{\alpha-1}{\alpha+1}\right)\int_0^t\int_{\mathcal{U}_s^+} |u|^{\alpha+1}\,dxds + \frac{c^k\,t^{k\frac{\alpha-1}{\alpha+1}}}{k}\int_0^t\int_{\mathcal{U}_s^+} |u_x|^k\,dxds \\ +\frac{n}{n+2}\delta^{1+\frac{2}{n}}\int_0^t\int_{\mathcal{U}_s^+} |u_{xx}|^{n+2}\,dxds,\end{aligned} \tag{3.12}$$

where c and k are two constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1),$$

and

$$\frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k = \frac{(n+2)(\alpha+1)}{(4+n) - n\,\alpha},$$

and if $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$, we get $k \geq 3$.
Now, q is chosen odd such that $2+q \geq k$ to obtain

$$|u_x|^k \leq |u_x|^3 + |u_x|^{q+2}.$$

Using (3.11) with $q = 1$ and $q \geq k-2$ odd, we obtain

$$\int_0^t \int_{\mathcal{U}_s^+} |u_x|^k \, dxds + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} \, dxds \leq C_0. \tag{3.13}$$

Now, integrating (3.10) over $[0,t]$, we get

$$\int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} \, dxds \leq \int_0^t \int_{\mathbb{R}} |u|^{\alpha+1} \, dxds$$

$$\leq t\,C_0 + \alpha(\alpha+1)\,t\,\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n \, dxds. \tag{3.14}$$

Thus, injecting (3.13) and (3.14) in (3.12), it comes

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n \, dxds \leq C_0 + \frac{1}{4}\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n \, dxds \tag{3.15}$$

and we obtain

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n \, dxds \leq C_0. \tag{3.16}$$

Finally, injecting (3.16) in (3.10) we obtain (3.9). □

3.1 Case f convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$

We are concerned here with the equation

$$u_t + f(u)_x = \varepsilon\, u_{xx} - \delta\, |u_{xx}|_x. \tag{3.17}$$

Proposition 3.4 *Let $\varepsilon > 0$, and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function, such that*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

Then, the solution $u = u_{\varepsilon,\delta}$ of (3.17) satisfies the estimate

$$\int_{\mathbb{R}} u_x(t)^2 \, dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dxds \leq C + \frac{C}{\delta}, \tag{3.18}$$

where $C > 0$ is a constant independent of ε and δ.
In addition, if $\delta = O(\varepsilon^2)$, the estimate (3.9) with $\alpha = 1$ is

$$\int_{\mathbb{R}} u(t)^2 \, dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 \, dxds + \delta \int_0^t \int_{\mathbb{R}} |u_x|\, |u_{xx}| \, dxds \leq C. \tag{3.19}$$

Proof. On the one hand, (3.5) is rewritten with $q = 0$ as

$$\int_{\mathbb{R}} u_x(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \tag{3.20}$$
$$= \int_{\mathbb{R}} (u_0')^2\,dx + 2\int_0^t \int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds.$$

Since f satisfies $f''(u) \le C(1+|u|^\beta)$, with $0 \le \beta < 3$. Then

$$|f'(u) - f'(0)| \le C(|u| + |u|^{\beta+1}),$$

where C is a generic constant. Thus,

$$\left|\int_0^t \int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds\right| \le C\int_0^t \int_{\mathbb{R}} |u||u_x||u_{xx}|\,dxds$$
$$+C\int_0^t \int_{\mathbb{R}} |u|^{\beta+1}|u_x||u_{xx}|\,dxds. \tag{3.21}$$

Applying (3.9) with $n = 1$, $\alpha = 2$ and $\alpha = \beta + 2 < 5$, we get

$$\left|\int_0^t \int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds\right| \le \frac{C}{\delta}. \tag{3.22}$$

Finally, injecting (3.22) in (3.20) provides (3.18).

On the other hand, estimate (3.2), with $\alpha = 1$ is written as

$$\int_{\mathbb{R}} u(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds = \|u_0\|_2^2 + 2\,\delta \int_0^t \int_{\mathbb{R}} u_x\,|u_{xx}|\,dxds. \tag{3.23}$$

Then

$$\int_{\mathbb{R}} u(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds \le C_0 + 2\,\delta \int_0^t \int_{\mathbb{R}} |u_x|\,|u_{xx}|\,dxds. \tag{3.24}$$

Now, if $\delta \le K\varepsilon^2$

$$\begin{aligned}
\delta \int_0^t \int_{\mathbb{R}} |u_x|\,|u_{xx}|\,dxds &\le \frac{\sqrt{\delta}}{\sqrt{K}\varepsilon} \int_0^t \int_{\mathbb{R}} \left(\sqrt{\varepsilon}|u_x|\right)\,\left(\sqrt{K\,\delta\,\varepsilon}|u_{xx}|\right)\,dxds. \\
&\le \frac{\sqrt{\delta}}{2\sqrt{K}\,\varepsilon}\left(\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds + K\,\delta\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds\right) \\
&\le \frac{1}{2}\left(\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds + K\,\delta\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds\right).
\end{aligned} \tag{3.25}$$

Using (3.18) and injecting (3.25) in (3.24) give (3.19). □

Proposition 3.5 *Let $\varepsilon > 0$,$\delta = o(\varepsilon^2)$, and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \le C(1+|u|^\beta), \text{ where } 0 \le \beta < 3.$$

Then, the solution $u = u_{\varepsilon,\delta}$ satisfies

(a) $\{\varepsilon\,u_x^2\}$ is bounded in $L^1((0,t)\times\mathbb{R})$.
(b) $\{\varepsilon\,u_x\} \to 0$ as $\varepsilon \to 0$, in $L^2((0,t)\times\mathbb{R})$.
(c) $\{\delta\,u_x\,|u_{xx}|\} \to 0$ as $\varepsilon \to 0$, in $L^1((0,t)\times\mathbb{R})$.
(d) $\{\delta\,|u_{xx}|\} \to 0$ as $\varepsilon \to 0$, in $L^2((0,t)\times\mathbb{R})$.

Proof. The statements $(a), (b)$ are obtained thanks to (3.19).

Now, in the same manner we obtained (3.25), we have

$$\delta \int_0^t \int_{\mathbb{R}} |u_x|\,|u_{xx}|\,dxds \leq \frac{\sqrt{\delta}}{2\varepsilon}\left(\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds + \delta\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds\right),$$

and using (3.18) and (3.19) it gets

$$\delta \int_0^t \int_{\mathbb{R}} |u_x|\,|u_{xx}|\,dxds \leq C\frac{\sqrt{\delta}}{\varepsilon}, \tag{3.26}$$

which gives (c) as soon as $\delta = o(\varepsilon^2)$.

Finally, (d) is obtained thanks to (3.18) since,

$$\delta^2 \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \leq \frac{\delta}{\varepsilon}\left(\delta\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds\right) \leq C_0\,\frac{\delta}{\varepsilon}. \tag{3.27}$$

□

3.2 Case f convex, $\delta > 0$ and $g(u_{xx}) = -u_{xx}^2$

We are concerned here with the equation

$$u_t + f(u)_x = \varepsilon\,u_{xx} - \delta\,(u_{xx}^2)_x. \tag{3.28}$$

It is noteworthy that the initial value problem associated with (3.28) is well-posed [1, 2].

Proposition 3.6 *Let $\varepsilon > 0$, $\delta = O(\varepsilon)$ and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function. Then, the solution $u = u_{\varepsilon,\delta}$ of (3.28) satisfies the estimate (3.9) with $\alpha = 1$, i.e.,*

$$\int_{\mathbb{R}} u(t)^2\,dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds + \delta \int_0^t \int_{\mathbb{R}} |u_x|\,u_{xx}^2\,dxds \leq C_0. \tag{3.29}$$

If in addition

$$f''(u) \leq C(1 + |u|^\beta), \ \textit{where } 0 \leq \beta < 1/2,$$

then the solution $u = u_{\varepsilon,\delta}$ checks

$$\int_{\mathbb{R}} u_x(t)^2\,dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \leq C_0 + C_0\,\delta^{-1/2}\varepsilon^{-1/4}. \tag{3.30}$$

Proof. On the one hand, (3.2) with $n = 2$, $\alpha = 1$ is written

$$\int_{\mathbb{R}} u(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds = \|u_0\|_2^2 + 2\,\delta \int_0^t \int_{\mathbb{R}} u_x\,u_{xx}^2\,dxds, \tag{3.31}$$

and, from (3.7) with $q = 1$, we get

$$\varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x\,u_{xx}^2\,dxds \leq C_0. \tag{3.32}$$

Thus, if $\delta \leq (k\,\varepsilon)$, (3.32) and (3.31) give (3.29).

On the other hand, assuming $f''(u) \leq C(1 + |u|^\beta)$, with $0 \leq \beta < 1/2$, we have

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}).$$

Then, to estimate the last term in (3.20), we procced as in the case $n = 1$:

$$\begin{aligned}
|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds| &\leq C\int_0^t\int_{\mathbb{R}} |u||u_x||u_{xx}|\,dxds + C\int_0^t\int_{\mathbb{R}} |u|^{\beta+1}|u_x||u_{xx}|\,dxds \\
&\leq C\delta^{-1/2}\varepsilon^{-1/4}\int_0^t\int_{\mathbb{R}} \left(\delta^{1/2}|u|^{1/2}|u_x|^{1/2}|u_{xx}|\right)\left(|u|^{1/2}\right)\left(\varepsilon^{1/4}|u_x|^{1/2}\right)\,dxds \\
&\quad + C\delta^{-1/2}\varepsilon^{-1/4}\int_0^t\int_{\mathbb{R}} \left(\delta^{1/2}|u|^{\beta+1/2}|u_x|^{1/2}|u_{xx}|\right)\left(|u|^{1/2}\right)\left(\varepsilon^{1/4}|u_x|^{1/2}\right)\,dxds.
\end{aligned}$$

Using the Young inequality, it comes

$$\begin{aligned}
|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds| &\leq C\delta^{-1/2}\varepsilon^{-1/4}\left(\frac{\delta}{2}\int_0^t\int_{\mathbb{R}} |u|^{2\beta+1}|u_x|u_{xx}^2\,dxds\right. \\
&\quad \left. +\frac{\delta}{2}\int_0^t\int_{\mathbb{R}} |u||u_x|u_{xx}^2\,dxds + \frac{1}{2}\int_0^t\int_{\mathbb{R}} u^2\,dxds + \frac{\varepsilon}{2}\int_0^t\int_{\mathbb{R}} u_x^2\,dxds\right). \quad (3.33)
\end{aligned}$$

Inequality (3.9) with $n = 2$, $\alpha = 2$ and $\alpha = 2\beta + 2 < 3$ is written as

$$\delta\int_0^t\int_{\mathbb{R}} |u||u_x|u_{xx}^2\,dxds + \delta\int_0^t\int_{\mathbb{R}} |u|^{2\beta+1}|u_x|u_{xx}^2\,dxds \leq C_0. \quad (3.34)$$

Now, injecting (3.34) and (3.29) in (3.33), we obtain

$$|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds| \leq C\delta^{-1/2}\varepsilon^{-1/4}. \quad (3.35)$$

Finally, injecting (3.35) in (3.20) we get the required estimate (3.30). □

Proposition 3.7 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$, and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2.$$

Then, the solution $u = u_{\varepsilon,\delta}$ of (3.28) satisfies
(a) $\{\varepsilon\,u_x^2\}$ is bounded in $L^1((0,t)\times\mathbb{R})$.
(b) $\{\varepsilon\,u_x\} \to 0$ as $\varepsilon \to 0$, in $L^2((0,t)\times\mathbb{R})$.
(c) $\{\delta\,u_x^-\,u_{xx}^2\}$, as $u_x^- = \max(0,-u_x)$, is bounded in $L^1((0,t)\times\mathbb{R})$.
(d) $\{\delta\,u_x^+\,u_{xx}^2\} \to 0$, as $u_x^+ = \max(0,u_x)$ when $\varepsilon \to 0$ in $L^1((0,t)\times\mathbb{R})$.
(e) $\{\delta\,u_{xx}^2\} \to 0$ as $\varepsilon \to 0$, in $L^1((0,t)\times\mathbb{R})$.

Proof. The statements $(a), (b)$ and (c) are obtained from (3.29).
Now, (d) is obtained from (3.7) with $q = 1$ since

$$\delta\int_0^t\int_{\mathcal{U}_s{}^+} u_x\,u_{xx}^2\,dxds \leq \frac{\delta}{\varepsilon}\left(\varepsilon\int_0^t\int_{\mathcal{U}_s{}^+} u_x\,u_{xx}^2\,dxds\right) \leq C\,\frac{\delta}{\varepsilon}. \quad (3.36)$$

Finally, (3.30) provides (e) since

$$\delta\int_0^t\int_{\mathbb{R}} u_{xx}^2\,dxds \leq \delta^{\frac{1}{2}}\,\varepsilon^{-5/4}(\delta^{\frac{1}{2}}\,\varepsilon^{5/4}\int_0^t\int_{\mathbb{R}} u_{xx}^2\,dxds) \leq C\,\sqrt{\delta\varepsilon^{-5/2}}. \quad (3.37)$$

□

4 Convergence Proof

We now define the measure-valued solutions to the first order Cauchy problem (1.3)-(1.4) as DiPerna [7].

Definition 4.1 *Assume that $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R})$ satisfies the growth condition*

$$|f(u)| \leq \mathcal{O}(|u|^m) \text{ as } |u| \to \infty, \qquad \text{for some } m \in [0, q). \tag{4.1}$$

A Young measure ν is called an entropy measure-valued (e.m.-v.) solution to (1.3)-(1.4) *if*

$$\langle \nu, |u-k| \rangle_t + \langle \nu, \operatorname{sgn}(u-k)(f(u)-f(k)) \rangle_x \leq 0, \qquad \text{for all } k \in \mathbb{R}, \tag{4.2}$$

in the sense of distributions on $(0,T) \times \mathbb{R}$, and

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle \, dxds = 0, \qquad \text{for all compact set } K \subseteq \mathbb{R}. \tag{4.3}$$

A representation theorem of Young's measures associated with a sequence of uniformly bounded functions of L^q is used to link the structure of measure and the strong convergence [16].

Lemma 4.2 *Let $\{u_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $L^\infty((0,T); L^q(\mathbb{R}))$. Then there exists a subsequence denoted by $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ and a weakly-$\star$ measurable mapping $\nu : \mathbb{R} \times (0,T) \to Prob(\mathbb{R})$ such that, for all functions $h \in \mathcal{C}(\mathbb{R})$ satisfying* (4.1)*, $\langle \nu_{(x,t)}, h \rangle$ belongs to $L^\infty((0,T); L^{q/m}_{loc}(\mathbb{R}))$ and the following limit representation holds:*

$$\iint_{\mathbb{R} \times (0,T)} \langle \nu_{(x,t)}, h \rangle \, \phi(x,t) \, dxdt = \lim_{n \to \infty} \iint_{\mathbb{R} \times (0,T)} h(\tilde{u}_n(x,t)) \, \phi(x,t) \, dxdt, \tag{4.4}$$

for all $\phi \in L^1(\mathbb{R} \times (0,T)) \cap L^\infty(\mathbb{R} \times (0,T))$.

Conversely, given ν, there exists a sequence $\{u_n\}$ satisfying the same conditions as above and such that (4.4) *holds for any h satisfying* (4.1)*.*

Proof of the main results. We begin proving (4.2) by using Proposition 3.3, resp. Proposition-3.5, for $n = 1$, resp. $n = 2$, and we apply the Young measure representation theorem in the suitable L^q space (4.4) to show that ν satisfies (4.2). Also, we use a standard regularization of $\operatorname{sgn}(u-k)(f(u)-f(k))$ and $|u-k|$ ($k \in \mathbb{R}$), since it is sufficient to show that there exists a bounded measure $\mu \leq 0$ such that

$$\eta(u)_t + q(u)_x \longrightarrow \mu \quad \text{ in } \quad \mathcal{D}'(\mathbb{R} \times (0,T)) \tag{4.5}$$

for an arbitrary convex function η (we assume that η' and η'' are bounded on $\mathbb{R}$).

Now, to prove (4.5), we rewrite the formulae (3.1) in the form

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4, \tag{4.6}$$

where,

$$\begin{aligned}
\mu_1 : &= \varepsilon \left(\eta'(u) \, u_x \right)_x; \\
\mu_2 : &= -\varepsilon \, \eta''(u) \, u_x^2; \\
\mu_3 : &= \delta \left(\eta'(u) \, g(u_{xx}) \right)_x; \\
\mu_4 : &= -\delta \, \eta''(u) \, u_x \, g(u_{xx}).
\end{aligned}$$

We distinguish the case $n = 1$ from $n = 2$.

Case n = 1: $g(u_{xx}) = -|u_{xx}|$, f convex and $\delta = o(\varepsilon^2)$. We have

$$| < \mu_1, \theta > | \quad \leq \quad \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x \, \eta'(u) \, u_x| \, dxds \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x \, u_x| \, dxds \leq C||\theta_x||_{L^2}||\varepsilon \, u_x||_{L^2}, \quad (4.7)$$

$$| < \mu_2, \theta > | \quad \leq \quad \varepsilon \int_0^T \int_{\mathbb{R}} |\theta \, \eta''(u) \, u_x^2| \, dxds \leq C||\theta||_{L^\infty}||\varepsilon \, u_x^2||_{L^1}. \quad (4.8)$$

Since η is a convex function, we notice for a non negative function θ

$$< \mu_2, \theta > \quad = \quad -\varepsilon \int_0^T \int_{\mathbb{R}} \theta \, \eta''(u) \, u_x^2 \, dxds \leq 0. \quad (4.9)$$

In the same way, we have

$$\begin{aligned} | < \mu_3, \theta > | \quad &\leq \quad \delta \int_0^T \int_{\mathbb{R}} |\theta_x \, \eta'(u) \, |u_{xx}|| \, dxds \leq C \, \delta \int_0^T \int_{\mathbb{R}} |\theta_x \, |u_{xx}|| \, dxds \\ &\leq \quad C||\theta_x||_{L^2}||\delta \, |u_{xx}|||_{L^2}, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} | < \mu_4, \theta > | \quad &\leq \quad \delta \int_0^T \int_{\mathbb{R}} |\theta \, \eta''(u) \, u_x |u_{xx}|| \, dxds \leq C \, \delta \int_0^T \int_{\mathbb{R}} |\theta \, u_x |u_{xx}|| \, dxds \\ &\leq \quad C||\theta||_{L^\infty}||\delta \, u_x |u_{xx}|||_{L^1}. \end{aligned} \quad (4.11)$$

Combining (4.7), (4.8), (4.9), (4.10) and (4.11), with (a), (b), (c), (d) in Proposition 3.3, gives (4.5) where μ is non positive bounded measure.

Case n = 2: $g(u_{xx}) = -u_{xx}^2$, f convex and $\delta = o(\varepsilon^{5/2})$. Estimates (4.7), (4.8), (4.9) remain true. Concerning μ_3, we have

$$\begin{aligned} | < \mu_3, \theta > | \quad &\leq \quad \delta \int_0^T \int_{\mathbb{R}} |\theta_x \, \eta'(u) \, u_{xx}^2| \, dxds \leq C \, \delta \int_0^T \int_{\mathbb{R}} |\theta_x \, u_{xx}^2| \, dxds \\ &\leq \quad C||\theta_x||_{L^\infty}||\delta \, u_{xx}^2||_{L^1}. \end{aligned} \quad (4.12)$$

Now, μ_4 is split as

$$\mu_4 = \mu_{41} + \mu_{42},$$

with

$$\begin{aligned} \mu_{41} \quad &:= \quad \delta \ \eta''(u) \, u_x^+ \, u_{xx}^2; \\ \mu_{42} \quad &:= \quad -\delta \ \eta''(u) \, u_x^- \, u_{xx}^2, \end{aligned}$$

where $u_x^+ = \max(0, u_x)$ and $u_x^- = \max(0, -u_x)$. Then we have

$$\begin{aligned} | < \mu_{41}, \theta > | \quad &\leq \quad \delta \int_0^T \int_{\mathbb{R}} |\theta \, \eta''(u) \, u_x^+ u_{xx}^2| \, dxds \leq C \, \delta \int_0^T \int_{\mathbb{R}} |\theta \, u_x^+ \, u_{xx}^2| \, dxds \\ &\leq \quad C||\theta||_{L^\infty}||\delta \, u_x^+ \, u_{xx}^2||_{L^1}, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} | < \mu_{42}, \theta > | \quad &\leq \quad \delta \int_0^T \int_{\mathbb{R}} |\theta \, \eta''(u) \, u_x^- u_{xx}^2| \, dxds \leq C \, \delta \int_0^T \int_{\mathbb{R}} |\theta \, u_x^- \, u_{xx}^2| \, dxds \\ &\leq \quad C||\theta||_{L^\infty}||\delta \, u_x^- \, u_{xx}^2||_{L^1}. \end{aligned} \quad (4.14)$$

Again, since $\eta'' \geq 0$, it gets for a non negative function θ

$$< \mu_{42}, \theta > \quad = \quad -\delta \int_0^T \int_{\mathbb{R}} \theta\, \eta''(u)\, u_x^- u_{xx}^2 \, dxds \leq 0. \tag{4.15}$$

Finally, from inequalities (4.7), (4.8), (4.9), (4.12), (4.13), (4.14) and (4.15), combined with (a), (b), (c) and (d) in Proposition 3.5, we obtain (4.5) where μ is non positive bounded measure.

Now we will prove (4.3). We follow arguments of DiPerna [7] and Szepessy [18]: we have to check that, for each compact set K of $\mathbb{R}$,

$$\lim_{t\to 0+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle \, dxds = \lim_{t\to 0+} \lim_{\varepsilon \to 0+} \frac{1}{t} \int_0^t \int_K \left| u^{\varepsilon,\delta}(x,s) - u_0(x) \right| \, dxds = 0.$$

By Jensen's inequality

$$\frac{1}{t} \int_0^t \int_K \left| u^{\varepsilon,\delta}(x,s) - u_0(x) \right| \, dxds \leq m(K)^{1/2} \left(\frac{1}{t} \int_0^t \int_K \left(u^{\varepsilon,\delta}(x,s) - u_0(x) \right)^2 \, dxds \right)^{1/2},$$

where $m(K)$ denotes the Lebesgue measure of K. Then we will establish that

$$\lim_{t\to 0+} \lim_{\varepsilon \to 0+} \frac{1}{t} \int_0^t \int_K \left(u^{\varepsilon,\delta}(x,s) - u_0(x) \right)^2 \, dxds = 0.$$

Let $K_i \subset K_{i+1}$ ($i = 0, 1, \dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i\geq 0} K_i = \mathbb{R}$. Using the identity $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$, we get for all $i = 0, 1, \dots$

$$\begin{aligned}
&\frac{1}{t} \int_0^t \int_K \left(u^{\varepsilon,\delta}(\cdot,s) - u_0 \right)^2 \, dxds \\
&\leq \frac{1}{t} \int_0^t \left(\int_{K_i} |u^{\varepsilon,\delta}(\cdot,s)|^2 \, dx - \int_{K_i} u_0^2 \, dx - 2 \int_{K_i} u_0 \left(u^{\varepsilon,\delta}(\cdot,s) - u_0 \right) \, dx \right) ds \\
&\leq \int_{\mathbb{R}\setminus K_i} u_0^2 \, dx + \frac{1}{t} \int_0^t \left(\int_{\mathbb{R}} |u^{\varepsilon,\delta}(\cdot,s)|^2 \, dx - \int_{\mathbb{R}} u_0^2 \, dx \right) \\
&\qquad\qquad + \frac{2}{t} \int_0^t \left| \int_{K_i} u_0 \left(u^{\varepsilon,\delta}(\cdot,s) - u_0 \right) \, dx \right| ds.
\end{aligned} \tag{4.16}$$

For the first term of the right hand side, we clearly have

$$\lim_{i\to\infty} \int_{\mathbb{R}\setminus K_i} u_0^2 \, dx = 0.$$

Now, injecting (3.26) in (3.23), in the case $n = 1$, and respectively (3.36) in (3.31), in the case $n = 2$, we obtain

$$\int_{\mathbb{R}} |u^{\varepsilon,\delta}(\cdot,s)|^2 \, dx - \int_{\mathbb{R}} u_0^2 \, dx \leq C \frac{\sqrt{\delta}}{\varepsilon},$$

and respectively,

$$\int_{\mathbb{R}} |u^{\varepsilon,\delta}(\cdot,s)|^2 \, dx - \int_{\mathbb{R}} u_0^2 \, dx \leq C \frac{\delta}{\varepsilon}.$$

In both cases, the right hand side of these inequalities tends to zero when $\varepsilon \to 0$.

To estimate the last term in the inequality (4.16), we choose $\{\theta_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \theta_n = u_0 \quad \text{ in } \quad L^2(\mathbb{R}).$$

Then, the Cauchy-Schwarz inequality gives

$$\begin{aligned}\Big|\int_{K_i} \quad & u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\Big| \le \int_{K_i}|u_0-\theta_n|\left|u^{\varepsilon,\delta}(\cdot,s)-u_0\right|\,dx\\ &+\left|\int_{K_i}\theta_n\left(u_0^{\varepsilon,\delta}-u_0\right)+\int_{K_i}\theta_n\left(u^{\varepsilon,\delta}(\cdot,s)-u_0^{\varepsilon,\delta}\right)\,dx\right|\\ &\le \|u_0-\theta_n\|_{L^2(\mathbb{R})}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathbb{R})}+\|u_0\|_{L^2(\mathbb{R})}\right)\\ &+\|\theta_n\|_{L^2(\mathbb{R})}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathbb{R})}+\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|.\end{aligned}$$

In view of (3.19) for $n=1$ and respectively (3.29) for $n=2$, we have

$$\|u_0-\theta_n\|_{L^2(\mathbb{R})}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathbb{R})}+\|u_0\|_{L^2(\mathbb{R})}\right)\le\left(\|u_0\|_{L^2(\mathbb{R})}+C\right)\|u_0-\theta_n\|_{L^2(\mathbb{R})},$$

which tends to zero when $n\to\infty$, since $\lim_{\varepsilon\to 0+}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathbb{R})}=0$. Finally, it remains to prove that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|ds=0.$$

We have, by (1.1),

$$\begin{aligned}\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right| &= \left|\int_0^s\int_{K_i}\theta_n\big(-f(u^{\varepsilon,\delta})_x+\varepsilon\,u_{xx}^{\varepsilon,\delta}+\delta\,g\big(u_{xx}^{\varepsilon,\delta}\big)_x\big)\,dxd\tau\right|\\ &\le \int_0^s\int_{K_i}\left|(\theta_n)_x\,f\big(u^{\varepsilon,\delta}\big)\right|\,dxd\tau+\varepsilon\int_0^s\int_{K_i}\left|(\theta_n)_x\,u_x^{\varepsilon,\delta}\right|\,dxd\tau\\ &\quad+\delta\int_0^s\int_{K_i}\left|(\theta_n)_x\,g\big(u_{xx}^{\varepsilon,\delta}\big)\right|\,dxd\tau\\ &:= I_1+I_2+I_3.\end{aligned}$$

To compute each quantity I_1, I_2 and I_3, we distinguish the cases $n=1$ from $n=2$.

Case n = 1: $g(u_{xx})=-|u_{xx}|$, f convex and $\delta=o(\varepsilon^2)$. Since f is such that

$$0\le f''(u)\le C(1+|u|^\beta),$$

where $\beta<3$. Thus,

$$|f(u)|\le C(1+|u|^m),$$

where $m<5$. Then, Proposition 3.1 implies

$$\int_0^s\int_{K_i}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\le\int_0^s\int_{\mathbb{R}}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\le C\,s,\tag{4.17}$$

and

$$\begin{aligned}I_1 &= \int_0^s\int_{K_i}|(\theta_n)_x|\left|f\big(u^{\varepsilon,\delta}\big)\right|\,dx\,d\tau\\ &\le C\int_0^s\int_{K_i}|(\theta_n)_x|\,dx\,d\tau+C\int_0^s\int_{K_i}|(\theta_n)_x|\,|u^{\varepsilon,\delta}|^m\,dx\,d\tau\\ &\le C\int_0^s\int_{K_i}|(\theta_n)_x|\,dx\,d\tau+C\|(\theta_n)_x\|_{L^\infty(\mathbb{R})}\int_0^s\int_{K_i}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\\ &\le C\,s\,\|(\theta_n)_x\|_{L^1(\mathbb{R})}+C\,s\,\|(\theta_n)_x\|_{L^\infty(\mathbb{R})}.\end{aligned}\tag{4.18}$$

Thanks to (3.19), we can write

$$\begin{aligned} I_2 \quad &= \varepsilon \int_0^s \int_{K_i} |(\theta_n)_x| \left|u_x^{\varepsilon,\delta}\right| \, dx \, d\tau \\ &\leq \left(\varepsilon \int_0^s \int_{K_i} |(\theta_n)_x|^2 \, dx d\tau\right)^{\frac{1}{2}} \left(\varepsilon \int_0^s \int_{K_i} \left|u_x^{\varepsilon,\delta}\right|^2 \, dx d\tau\right)^{\frac{1}{2}} \\ &\leq C \, \varepsilon^{\frac{1}{2}} \, s^{\frac{1}{2}} \, \| \, (\theta_n)_x \, \|_{L^2(\mathbb{R})}. \end{aligned} \tag{4.19}$$

Finally for I_3, using (3.27), we get

$$\begin{aligned} I_3 \quad &= \delta \int_0^s \int_{K_i} \left|(\theta_n)_x \, g\big(u_{xx}^{\varepsilon,\delta}\big)\right| \, dx d\tau = \int_0^s \int_{K_i} \delta \, |(\theta_n)_x| \left|u_{xx}^{\varepsilon,\delta}\right| \, dx d\tau \\ &\leq \left(\int_0^s \int_{K_i} |(\theta_n)_x|^2 \, dx d\tau\right)^{\frac{1}{2}} \left(\delta^2 \int_0^s \int_{K_i} \left|u_{xx}^{\varepsilon,\delta}\right|^2 \, dx d\tau\right)^{\frac{1}{2}} \\ &\leq \, s^{\frac{1}{2}} \, \| \, (\theta_n)_x \, \|_{L^2(\mathbb{R})} \left(\delta^2 \int_0^s \int_{K_i} \left|u_{xx}^{\varepsilon,\delta}\right|^2 \, dx d\tau\right)^{\frac{1}{2}} \\ &\leq s^{\frac{1}{2}} \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}} \, \| \, (\theta_n)_x \, \|_{L^2(\mathbb{R})} \end{aligned} \tag{4.20}$$

Now, from (4.18), (4.19) and (4.20), we deduce

$$\begin{aligned} &\lim_{t\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n \, u_s^{\varepsilon,\delta} \, dx d\tau \right| ds \\ &\quad \leq \lim_{t\to 0+} \lim_{\varepsilon\to 0+} \frac{C}{t} \Big(t^2 \left(\| \, (\theta_n)_x \, \|_{L^1(\mathbb{R})} + \| \, (\theta_n)_x \, \|_{L^\infty(\mathbb{R})}\right) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \, \| \, (\theta_n)_x \, \|_{L^2(\mathbb{R})} \\ &\qquad + t^{\frac{3}{2}} \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}} \, \| \, (\theta_n)_x \, \|_{L^2(\mathbb{R})}. \Big) \\ &\quad \leq \lim_{t\to 0+} \lim_{\varepsilon\to 0+} C \left(t + \, \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} \, + t^{\frac{1}{2}} \, \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}}\right), \end{aligned}$$

and since $\delta = o(\varepsilon^2)$, we obtain the desired conclusion, and Theorem 2.1 is proved.

Case n = 2: $g(u_{xx}) = -u_{xx}^2$, f convex and $\delta = o(\varepsilon^{5/2})$. Here, f is such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

where $\beta < 1/2$, thus,

$$|f(u)| \leq C(1 + |u|^m),$$

where $m < 5/2$.

Estimates I_1 and I_2 are obtained in the same manner as $n = 1$ using (3.29) instead of (3.18). From (3.37) we obtain

$$\begin{aligned} I_3 \quad &= \delta \int_0^s \int_{K_i} \left|(\theta_n)_x \, g\big(u_{xx}^{\varepsilon,\delta}\big)\right| \, dx d\tau = \int_0^s \int_{K_i} \delta \, |(\theta_n)_x| \left|u_{xx}^{\varepsilon,\delta}\right|^2 \, dx d\tau \\ &\leq \| \, (\theta_n)_x \, \|_{L^\infty(\mathbb{R})} \left(\delta \int_0^s \int_{K_i} \left|u_{xx}^{\varepsilon,\delta}\right|^2 \, dx d\tau\right) \\ &\leq \sqrt{\delta \varepsilon^{-5/2}} \, \| \, (\theta_n)_x \, \|_{L^\infty(\mathbb{R})} \end{aligned} \tag{4.21}$$

Finally, using (4.18), (4.19) and (4.21), we obtain

$$\begin{aligned}
&\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,u_s^{\varepsilon,\delta}\,dxd\tau\right|ds\\
&\quad\le \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{C}{t}\Big(t^2\left(\|\left(\theta_n\right)_x\|_{L^1(\mathbb{R})}+\|\left(\theta_n\right)_x\|_{L^\infty(\mathbb{R})}\right)+\varepsilon^{\frac{1}{2}}t^{\frac{3}{2}}\,\|\left(\theta_n\right)_x\|_{L^2(\mathbb{R})}\\
&\qquad +t\sqrt{\delta\varepsilon^{-5/2}}\,\|\left(\theta_n\right)_x\|_{L^\infty(\mathbb{R})}\cdot\Big)\\
&\quad\le \lim_{t\to 0+}\lim_{\varepsilon\to 0+}C\left(t+\varepsilon^{\frac{1}{2}}t^{\frac{1}{2}}+\sqrt{\delta\varepsilon^{-5/2}}\right),
\end{aligned}$$

and since $\delta = o(\varepsilon^{5/2})$, we obtain the desired conclusion, and Theorem 2.2 is proved.

□

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