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► **To cite this version:**

Nabil Bedjaoui, Joaquim M.C. Correia, Youcef Mammeri. On a limit of perturbed conservation laws with diffusion and non-positive dispersion. Communications in Mathematical Sciences, 2016, 10.4310/CMS.2016.v14.n6.a2 . hal-01191885

**HAL Id: hal-01191885**

**<https://hal.science/hal-01191885v1>**

Submitted on 2 Sep 2015

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# On a limit of perturbed conservation laws with diffusion and non-positive dispersion

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**Abstract.** We consider a conservation law perturbed by a linear diffusion and non-positive dispersion  $u_t + f(u)_x = \varepsilon u_{xx} - \delta(|u_{xx}|^n)_x$ . We prove the convergence of the previous solution to the entropy weak solution of the hyperbolic conservation law  $u_t + f(u)_x = 0$ , in both cases  $n = 1$  and  $n = 2$ .

**Keywords.** diffusion, dispersion, KdV equation, Burgers equation, hyperbolic conservation laws, entropy measure-valued solutions.

**MS Codes.** 35G25,35L65,35Q53,76B15.

## 1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad (1.2)$$

where  $\varepsilon$  and  $\delta$  are small parameters, and  $g$  is a non positive function, and we focus on the specific form

$$g(v) = -|v|^n,$$

where  $n \geq 1$ . Note that if  $n = 1$ ,  $g$  is lipschitz, if  $1 < n < 2$ ,  $g$  is  $C^1$  and for  $n \geq 2$ ,  $g$  is  $C^2$ .

When  $\delta = 0$  we reduce to the viscous (generalized) Burgers equation and the approximate solutions  $u^{\varepsilon,0}$  converge to the entropy solution of the hyperbolic equation (called the *vanishing viscosity method*, see, e.g., Whitham [21] or Kruřkov [9])

$$u_t + f(u)_x = 0 \quad (1.3)$$

$$u(x, 0) = u_0(x). \quad (1.4)$$

On the other hand, when  $\varepsilon = 0$ , if we consider the flux function  $f(u) = u^2$  and the linear dispersion  $\delta u_{xxx}$  we obtain the Korteweg-de Vries equation. The approximate solutions  $u^{0,\delta}$  do not converge in a strong topology (see Lax-Levermore [11]). So, as parameters  $\varepsilon$  and  $\delta$  vanish, we are concerned with singular limits and to ensure convergence it is necessary to be in the dominant dissipation regime.

The pioneer study of these singular limits was given by Schonbek [16] about the (generalized) Korteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta u_{xxx}.$$

In the case of a convex flux function  $f(u)$ , she proved the convergence of the solutions of this perturbed equation to the entropy solution of (1.3), when both  $\varepsilon$  and  $\delta$  tend to zero, *at least* under the condition  $\delta = O(\varepsilon^3)$  (depending on the behavior of the flux  $f$ ). Also, according to Perthame-Ryzhik [15], the sharp condition should be  $\delta = o(\varepsilon)$ . LeFloch-Natalini [13] proved the convergence in the case of a nonlinear viscosity function  $\beta$  and linear capillarity

$$u_t + f(u)_x = \varepsilon \beta(u_x)_x - \delta u_{xxx}.$$

Then, Correia-LeFloch [5] improved the estimates in Schonbek [16] and LeFloch-Natalini [13] and for the first time treated the multidimensional equation, but still in the case of a nonlinear viscosity function and linear capillarity. In fact, the dominant dissipation regime is also assured by the nonlinear viscosity. In our case, we consider the reverse situation.

In general for  $\varepsilon = 0$ , like for the Korteweg-de Vries equation, the divergent behaviour is expected, as we are considering "pure-dispersive equations". But, Brenier-Levy [3] considered the fully nonlinear equation

$$u_t + f(u)_x = -\delta(u_{xx}^2)_x$$

as a nonlinear generalization of the Korteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions that differs completely from the linear case. In particular, Brenier and Levy [3] conjectured that for strictly convex flux functions  $f$  and for the following perturbed problem

$$u_t + f(u)_x = -\delta g(u_{xx})_x - \varepsilon u_{xxx},$$

we have convergence when  $\varepsilon$  and  $\delta$  tend to zero under the condition  $\varepsilon = o(\delta)$ .

The paper is organized as follows. In Section 2, we present the main results of convergence. Section 3 deals with the uniform estimates needed for convergence. Finally, Section 4 is devoted to proving the convergence to the entropy solution of the hyperbolic equation, when both  $\varepsilon$  and  $\delta$  go to zero.

## 2 Main Results

Two main convergence results are presented. The first one is concerned with  $g(v) = -|v|$  (i.e.  $n = 1$ ) while the second one is devoted to the case  $g(v) = -v^2$  (i.e.  $n = 2$ ).

### 2.1 Case $f$ convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$

In this case, we prove the following result.

**Theorem 2.1** *Let  $\varepsilon > 0$ ,  $\delta = o(\varepsilon^2)$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

*Then, setting  $u = u_{\varepsilon, \delta}$  the solution of (1.1) – (1.2), the family solutions  $(u_{\varepsilon, \delta})$  converges to the entropy solution of (3) – (4).*

In the case of the linear dispersion, i.e.  $g(u_{xx}) = u_{xx}$  treated in [16], Schonbek gets for a general flux satisfying  $f''(u) \leq C$ , a convergence with rate  $\delta = O(\varepsilon^3)$ . Also, when  $f''(u) \leq C(1 + |u|)$ , the author obtain the convergence with the rate  $\delta = O(\varepsilon^4)$ . The case of  $g(u_{xx}) = -|u_{xx}|$  seems giving a weakly dispersive effects than a classical linear dispersion.

## 2.2 Case $f$ convex, $\delta > 0$ and $g(u_{xx}) = -u_{xx}^2$

Here function  $g$  is regular, and we obtain:

**Theorem 2.2** *Let  $\varepsilon > 0$ ,  $\delta = o(\varepsilon^{5/2})$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2.$$

*Then, setting  $u = u_{\varepsilon, \delta}$  the solution of (1.1) – (1.2), the family solutions  $(u_{\varepsilon, \delta})$  converges to the entropy solution of (3) – (4).*

The dispersion here is strongly nonlinear but regular, which provides the well-posedness [1, 2]. We can see that comparing to the results in [16], the rate are quite similar when the flux is convex and satisfies  $f''(u) \leq C$  ( $\delta = o(\varepsilon^{5/2})$ ) whereas  $\delta = O(\varepsilon^3)$  in [16].

## 3 A priori estimates

Assume that  $\eta$  is a regular function and  $\varphi$  a function defined by  $\varphi' = \eta' f'$ , and let us multiply (1.1) by  $\eta'(u)$ . We obtain

$$\begin{aligned} \eta(u)_t + \varphi(u)_x &= \varepsilon (\eta'(u) u_x)_x - \varepsilon \eta''(u) u_x^2 \\ &\quad + \delta (\eta'(u) g(u_{xx}))_x - \delta \eta''(u) u_x g(u_{xx}). \end{aligned} \quad (3.1)$$

Integrating over  $(0, t) \times \mathbb{R}$  with  $\eta(u) = |u|^{\alpha+1}$ , the conservative terms vanish and we obtain the following lemma.

**Lemma 3.1** *Let  $\alpha \geq 1$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be any dispersion function. Each solution of (1.1) satisfies for  $t \in [0, T]$*

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^{\alpha+1} dx &+ (\alpha + 1) \alpha \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds \\ &+ (\alpha + 1) \alpha \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x g(u_{xx}) dx ds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1} dx. \end{aligned} \quad (3.2)$$

Usually, taking  $\alpha = 1$  in (3.2), we deduce the a priori  $L^2$  first energy estimates.

Let us introduce the functions  $\mathcal{G}$ , and  $G$  defined by  $\mathcal{G}'' = G' = g$ , i.e,

$$G(u) = -\frac{1}{n+1} |u|^n u, \quad \text{and} \quad \mathcal{G} = -\frac{1}{(n+1)(n+2)} |u|^{n+2}.$$

Using the multiplier  $(q+2)(|u_x|^q u_x)_x$  to (1.1), we have

$$\begin{aligned} ((q+2) u_t |u_x|^q u_x)_x - (|u_x|^{q+2})_t &= -(q+2)(q+1) |u_x|^q u_{xx} f'(u) u_x \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + \delta (q+2)(q+1) |u_x|^q u_{xx} g'(u_{xx}) u_{xxx} \\ &= -(q+1) (|u_x|^{q+2})_x f'(u) \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + \delta (q+2)(q+1) n |u_x|^q G(u_{xx})_x, \end{aligned}$$

and we get the estimate

$$\begin{aligned}
((q+2)u_t|u_x|^q u_x)_x - (|u_x|^{q+2})_t &= -((q+1)|u_x|^{q+2}f'(u))_x \\
&+ (q+1)|u_x|^{q+2}u_x f''(u) \\
&+ \varepsilon(q+2)(q+1)|u_x|^q u_{xx}^2 \\
&+ (\delta(q+2)(q+1)n|u_x|^q G(u_{xx}))_x \\
&- \delta(q+2)(q+1)q(n+2)n|u_x|^{q-2}u_x \mathcal{G}(u_{xx}).
\end{aligned} \tag{3.3}$$

Similarly, using the multiplier  $(q+2)(u_x^{q+1})_x$  to (1.1), we can write

$$\begin{aligned}
((q+2)u_t u_x^{q+1})_x - (u_x^{q+2})_t &= -(q+2)(q+1)u_x^{q+1}f'(u)u_{xx} \\
&+ \varepsilon(q+2)(q+1)u_x^q u_{xx}^2 \\
&+ (\delta(q+2)(q+1)n u_x^q G(u_{xx}))_x \\
&- \delta(q+2)(q+1)q(n+2)n u_x^{q-1} \mathcal{G}(u_{xx}),
\end{aligned}$$

thus

$$\begin{aligned}
((q+2)u_t u_x^{q+1})_x - (u_x^{q+2})_t &= -((q+1)u_x^{q+2}f'(u))_x \\
&+ (q+1)u_x^{q+3}f''(u) \\
&+ \varepsilon(q+2)(q+1)u_x^q u_{xx}^2 \\
&+ (\delta(q+2)(q+1)n u_x^q G(u_{xx}))_x \\
&- \delta(q+2)(q+1)q(n+2)n u_x^{q-1} \mathcal{G}(u_{xx}).
\end{aligned} \tag{3.4}$$

Integrating (3.3) and (3.4) over  $\mathbb{R} \times [0, t]$  provides

$$\begin{aligned}
&\int_{\mathbb{R}} |u_x(t)|^{q+2} dx + \varepsilon(q+2)(q+1) \int_0^t \int_{\mathbb{R}} |u_x|^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} |u'_0|^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q+2} f''(u) dx ds \\
&\quad + \delta(q+2)(q+1)q(n+2)n \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q-2} \mathcal{G}(u_{xx}) dx ds,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\int_{\mathbb{R}} u_x(t)^{q+2} dx + \varepsilon(q+2)(q+1) \int_0^t \int_{\mathbb{R}} u_x^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} (u'_0)^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x^{q+3} f''(u) dx ds \\
&\quad + \delta(q+2)(q+1)q(n+2)n \int_0^t \int_{\mathbb{R}} u_x^{q-1} \mathcal{G}(u_{xx}) dx ds.
\end{aligned} \tag{3.6}$$

We define now the sets, for  $t \in [0, T]$

$$\mathcal{U}_t^+ = \{x \in \mathbb{R}; u_x(x, t) > 0\},$$

and

$$\mathcal{U}_t^- = \{x \in \mathbb{R}; u_x(x, t) < 0\}.$$

Adding (3.6) to (3.5) for  $q$  odd, we obtain:

**Lemma 3.2** *Let  $q$  be an odd number. Then, each solution of (1.1) satisfies for  $t \in [0, T]$*

$$\begin{aligned}
& \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\
& \quad + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |\mathcal{G}(u_{xx})| dx ds \\
& \quad + (q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \\
& = \int_{\mathcal{U}_0^+} |u'_0|^{q+2} dx,
\end{aligned} \tag{3.7}$$

where the last left hand-side term can be replaced by

$$-(q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+1} f'(u) u_{xx} dx ds. \tag{3.8}$$

Now, the combination of Lemmas 3.1 and 3.2 gives the following estimate

**Proposition 3.3** *Let  $\varepsilon, \delta > 0$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function. The solution  $u = u_{\varepsilon, \delta}$  of (1.1) – (1.2) satisfies the uniform estimate*

$$\int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} |u_x| |u_{xx}|^n dx ds \leq C_0, \tag{3.9}$$

for all  $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$ .

**Proof.** When  $g(u) = -|u|^n$ , equality (3.2) writes

$$\begin{aligned}
& \int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \alpha(\alpha+1) \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds = \\
& \quad \|u_0\|_{\alpha+1}^{\alpha+1} + \alpha(\alpha+1) \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} |u_x| |u_{xx}|^n dx ds.
\end{aligned} \tag{3.10}$$

Also, when  $f$  is convex, we can rewrite (3.7) for  $q \geq 1$  odd as

$$\begin{aligned}
& \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\
& \quad + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |u_{xx}|^{n+2} dx ds + \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \leq C_0
\end{aligned} \tag{3.11}$$

However, using the Young inequality, we get

$$\begin{aligned}
& \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} |u_x| |u_{xx}|^n dx ds \\
& = \int_0^t \int_{\mathcal{U}_s^+} \left( \frac{1}{c t^{\frac{\alpha-1}{\alpha+1}}} |u|^{\alpha-1} \right) \left( c t^{\frac{\alpha-1}{\alpha+1}} |u_x| \right) (\delta |u_{xx}|^n) dx ds \\
& \leq \left( \frac{1}{t c^{\frac{\alpha+1}{\alpha-1}}} \right) \left( \frac{\alpha-1}{\alpha+1} \right) \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds + \frac{c^k t^k \frac{\alpha-1}{\alpha+1}}{k} \int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds \\
& \quad + \frac{n}{n+2} \delta^{1+\frac{2}{n}} \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds,
\end{aligned} \tag{3.12}$$

where  $c$  and  $k$  are two constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1),$$

and

$$\frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k = \frac{(n+2)(\alpha+1)}{(4+n) - n\alpha},$$

and if  $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$ , we get  $k \geq 3$ .

Now,  $q$  is chosen odd such that  $2+q \geq k$  to obtain

$$|u_x|^k \leq |u_x|^3 + |u_x|^{q+2}.$$

Using (3.11) with  $q = 1$  and  $q \geq k - 2$  odd, we obtain

$$\int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds \leq C_0. \quad (3.13)$$

Now, integrating (3.10) over  $[0, t]$ , we get

$$\begin{aligned} \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds &\leq \int_0^t \int_{\mathbb{R}} |u|^{\alpha+1} dx ds \\ &\leq t C_0 + \alpha(\alpha+1) t \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds. \end{aligned} \quad (3.14)$$

Thus, injecting (3.13) and (3.14) in (3.12), it comes

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0 + \frac{1}{4} \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \quad (3.15)$$

and we obtain

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0. \quad (3.16)$$

Finally, injecting (3.16) in (3.10) we obtain (3.9).  $\square$

### 3.1 Case $f$ convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$

We are concerned here with the equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta |u_{xx}|_x. \quad (3.17)$$

**Proposition 3.4** *Let  $\varepsilon > 0$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function, such that*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

*Then, the solution  $u = u_{\varepsilon, \delta}$  of (3.17) satisfies the estimate*

$$\int_{\mathbb{R}} u_x(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C + \frac{C}{\delta}, \quad (3.18)$$

*where  $C > 0$  is a constant independent of  $\varepsilon$  and  $\delta$ .*

*In addition, if  $\delta = O(\varepsilon^2)$ , the estimate (3.9) with  $\alpha = 1$  is*

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds \leq C. \quad (3.19)$$

**Proof.** On the one hand, (3.5) is rewritten with  $q = 0$  as

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds & \quad (3.20) \\ & = \int_{\mathbb{R}} (u'_0)^2 dx + 2 \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds. \end{aligned}$$

Since  $f$  satisfies  $f''(u) \leq C(1 + |u|^\beta)$ , with  $0 \leq \beta < 3$ . Then

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}),$$

where  $C$  is a generic constant. Thus,

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| & \leq C \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}| dx ds \\ & \quad + C \int_0^t \int_{\mathbb{R}} |u|^{\beta+1} |u_x| |u_{xx}| dx ds. \end{aligned} \quad (3.21)$$

Applying (3.9) with  $n = 1$ ,  $\alpha = 2$  and  $\alpha = \beta + 2 < 5$ , we get

$$\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| \leq \frac{C}{\delta}. \quad (3.22)$$

Finally, injecting (3.22) in (3.20) provides (3.18).

On the other hand, estimate (3.2), with  $\alpha = 1$  is written as

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds = \|u_0\|_2^2 + 2\delta \int_0^t \int_{\mathbb{R}} u_x |u_{xx}| dx ds. \quad (3.23)$$

Then

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \leq C_0 + 2\delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds. \quad (3.24)$$

Now, if  $\delta \leq K\varepsilon^2$

$$\begin{aligned} \delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds & \leq \frac{\sqrt{\delta}}{\sqrt{K}\varepsilon} \int_0^t \int_{\mathbb{R}} (\sqrt{\varepsilon}|u_x|) (\sqrt{K\delta\varepsilon}|u_{xx}|) dx ds \\ & \leq \frac{\sqrt{\delta}}{2\sqrt{K}\varepsilon} \left( \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + K\delta\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right) \\ & \leq \frac{1}{2} \left( \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + K\delta\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right). \end{aligned} \quad (3.25)$$

Using (3.18) and injecting (3.25) in (3.24) give (3.19).  $\square$

**Proposition 3.5** *Let  $\varepsilon > 0, \delta = o(\varepsilon^2)$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 3.$$

*Then, the solution  $u = u_{\varepsilon, \delta}$  satisfies*

- (a)  $\{\varepsilon u_x^2\}$  is bounded in  $L^1((0, t) \times \mathbb{R})$ .
- (b)  $\{\varepsilon u_x\} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , in  $L^2((0, t) \times \mathbb{R})$ .
- (c)  $\{\delta u_x |u_{xx}|\} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , in  $L^1((0, t) \times \mathbb{R})$ .
- (d)  $\{\delta |u_{xx}|\} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , in  $L^2((0, t) \times \mathbb{R})$ .



**Proof.** The statements (a), (b) are obtained thanks to (3.19).

Now, in the same manner we obtained (3.25), we have

$$\delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds \leq \frac{\sqrt{\delta}}{2\varepsilon} \left( \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right),$$

and using (3.18) and (3.19) it gets

$$\delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}| dx ds \leq C \frac{\sqrt{\delta}}{\varepsilon}, \quad (3.26)$$

which gives (c) as soon as  $\delta = o(\varepsilon^2)$ .

Finally, (d) is obtained thanks to (3.18) since,

$$\delta^2 \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq \frac{\delta}{\varepsilon} \left( \delta \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right) \leq C_0 \frac{\delta}{\varepsilon}. \quad (3.27)$$

□

### 3.2 Case $f$ convex, $\delta > 0$ and $g(u_{xx}) = -u_{xx}^2$

We are concerned here with the equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta (u_{xx}^2)_x. \quad (3.28)$$

It is noteworthy that the initial value problem associated with (3.28) is well-posed [1, 2].

**Proposition 3.6** *Let  $\varepsilon > 0$ ,  $\delta = O(\varepsilon)$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function. Then, the solution  $u = u_{\varepsilon, \delta}$  of (3.28) satisfies the estimate (3.9) with  $\alpha = 1$ , i.e.,*

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u_x| u_{xx}^2 dx ds \leq C_0. \quad (3.29)$$

If in addition

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2,$$

then the solution  $u = u_{\varepsilon, \delta}$  checks

$$\int_{\mathbb{R}} u_x(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C_0 + C_0 \delta^{-1/2} \varepsilon^{-1/4}. \quad (3.30)$$

**Proof.** On the one hand, (3.2) with  $n = 2$ ,  $\alpha = 1$  is written

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds = \|u_0\|_2^2 + 2\delta \int_0^t \int_{\mathbb{R}} u_x u_{xx}^2 dx ds, \quad (3.31)$$

and, from (3.7) with  $q = 1$ , we get

$$\varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds \leq C_0. \quad (3.32)$$

Thus, if  $\delta \leq (k\varepsilon)$ , (3.32) and (3.31) give (3.29).

On the other hand, assuming  $f''(u) \leq C(1 + |u|^\beta)$ , with  $0 \leq \beta < 1/2$ , we have

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}).$$

Then, to estimate the last term in (3.20), we proceed as in the case  $n = 1$ :

$$\begin{aligned}
\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}| dx ds + C \int_0^t \int_{\mathbb{R}} |u|^{\beta+1} |u_x| |u_{xx}| dx ds \\
&\leq C \delta^{-1/2} \varepsilon^{-1/4} \int_0^t \int_{\mathbb{R}} \left( \delta^{1/2} |u|^{1/2} |u_x|^{1/2} |u_{xx}| \right) \left( |u|^{1/2} \right) \left( \varepsilon^{1/4} |u_x|^{1/2} \right) dx ds \\
&\quad + C \delta^{-1/2} \varepsilon^{-1/4} \int_0^t \int_{\mathbb{R}} \left( \delta^{1/2} |u|^{\beta+1/2} |u_x|^{1/2} |u_{xx}| \right) \left( |u|^{1/2} \right) \left( \varepsilon^{1/4} |u_x|^{1/2} \right) dx ds.
\end{aligned}$$

Using the Young inequality, it comes

$$\begin{aligned}
\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \delta^{-1/2} \varepsilon^{-1/4} \left( \frac{\delta}{2} \int_0^t \int_{\mathbb{R}} |u|^{2\beta+1} |u_x| u_{xx}^2 dx ds \right. \\
&\quad \left. + \frac{\delta}{2} \int_0^t \int_{\mathbb{R}} |u| |u_x| u_{xx}^2 dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^2 dx ds + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right). \quad (3.33)
\end{aligned}$$

Inequality (3.9) with  $n = 2$ ,  $\alpha = 2$  and  $\alpha = 2\beta + 2 < 3$  is written as

$$\delta \int_0^t \int_{\mathbb{R}} |u| |u_x| u_{xx}^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{2\beta+1} |u_x| u_{xx}^2 dx ds \leq C_0. \quad (3.34)$$

Now, injecting (3.34) and (3.29) in (3.33), we obtain

$$\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| \leq C \delta^{-1/2} \varepsilon^{-1/4}. \quad (3.35)$$

Finally, injecting (3.35) in (3.20) we get the required estimate (3.30).  $\square$

**Proposition 3.7** *Let  $\varepsilon > 0$ ,  $\delta = o(\varepsilon^{5/2})$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function satisfying*

$$f''(u) \leq C(1 + |u|^\beta), \text{ where } 0 \leq \beta < 1/2.$$

*Then, the solution  $u = u_{\varepsilon, \delta}$  of (3.28) satisfies*

- (a)  $\{\varepsilon u_x^2\}$  is bounded in  $L^1((0, t) \times \mathbb{R})$ .
- (b)  $\{\varepsilon u_x\} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , in  $L^2((0, t) \times \mathbb{R})$ .
- (c)  $\{\delta u_x^- u_{xx}^2\}$ , as  $u_x^- = \max(0, -u_x)$ , is bounded in  $L^1((0, t) \times \mathbb{R})$ .
- (d)  $\{\delta u_x^+ u_{xx}^2\} \rightarrow 0$ , as  $u_x^+ = \max(0, u_x)$  when  $\varepsilon \rightarrow 0$  in  $L^1((0, t) \times \mathbb{R})$ .
- (e)  $\{\delta u_{xx}^2\} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , in  $L^1((0, t) \times \mathbb{R})$ .

**Proof.** The statements (a), (b) and (c) are obtained from (3.29).

Now, (d) is obtained from (3.7) with  $q = 1$  since

$$\delta \int_0^t \int_{\mathcal{U}_{\varepsilon^+}} u_x u_{xx}^2 dx ds \leq \frac{\delta}{\varepsilon} \left( \varepsilon \int_0^t \int_{\mathcal{U}_{\varepsilon^+}} u_x u_{xx}^2 dx ds \right) \leq C \frac{\delta}{\varepsilon}. \quad (3.36)$$

Finally, (3.30) provides (e) since

$$\delta \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq \delta^{\frac{1}{2}} \varepsilon^{-5/4} (\delta^{\frac{1}{2}} \varepsilon^{5/4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds) \leq C \sqrt{\delta \varepsilon^{-5/2}}. \quad (3.37)$$

$\square$

## 4 Convergence Proof

We now define the measure-valued solutions to the first order Cauchy problem (1.3)-(1.4) as DiPerna [7].

**Definition 4.1** *Assume that  $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$  and  $f \in \mathcal{C}(\mathbb{R})$  satisfies the growth condition*

$$|f(u)| \leq \mathcal{O}(|u|^m) \text{ as } |u| \rightarrow \infty, \quad \text{for some } m \in [0, q]. \quad (4.1)$$

*A Young measure  $\nu$  is called an entropy measure-valued (e.m.-v.) solution to (1.3)-(1.4) if*

$$\langle \nu, |u - k| \rangle_t + \langle \nu, \text{sgn}(u - k)(f(u) - f(k)) \rangle_x \leq 0, \quad \text{for all } k \in \mathbb{R}, \quad (4.2)$$

*in the sense of distributions on  $(0, T) \times \mathbb{R}$ , and*

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds = 0, \quad \text{for all compact set } K \subseteq \mathbb{R}. \quad (4.3)$$

A representation theorem of Young's measures associated with a sequence of uniformly bounded functions of  $L^q$  is used to link the structure of measure and the strong convergence [16].

**Lemma 4.2** *Let  $\{u_n\}_{n \in \mathbb{N}}$  be a bounded sequence in  $L^\infty((0, T); L^q(\mathbb{R}))$ . Then there exists a subsequence denoted by  $\{\tilde{u}_n\}_{n \in \mathbb{N}}$  and a weakly- $\star$  measurable mapping  $\nu : \mathbb{R} \times (0, T) \rightarrow \text{Prob}(\mathbb{R})$  such that, for all functions  $h \in \mathcal{C}(\mathbb{R})$  satisfying (4.1),  $\langle \nu_{(x,t)}, h \rangle$  belongs to  $L^\infty((0, T); L_{loc}^{q/m}(\mathbb{R}))$  and the following limit representation holds:*

$$\iint_{\mathbb{R} \times (0, T)} \langle \nu_{(x,t)}, h \rangle \phi(x, t) dx dt = \lim_{n \rightarrow \infty} \iint_{\mathbb{R} \times (0, T)} h(\tilde{u}_n(x, t)) \phi(x, t) dx dt, \quad (4.4)$$

for all  $\phi \in L^1(\mathbb{R} \times (0, T)) \cap L^\infty(\mathbb{R} \times (0, T))$ .

Conversely, given  $\nu$ , there exists a sequence  $\{u_n\}$  satisfying the same conditions as above and such that (4.4) holds for any  $h$  satisfying (4.1).

**Proof of the main results.** We begin proving (4.2) by using Proposition 3.3, resp. Proposition 3.5, for  $n = 1$ , resp.  $n = 2$ , and we apply the Young measure representation theorem in the suitable  $L^q$  space (4.4) to show that  $\nu$  satisfies (4.2). Also, we use a standard regularization of  $\text{sgn}(u - k)(f(u) - f(k))$  and  $|u - k|$  ( $k \in \mathbb{R}$ ), since it is sufficient to show that there exists a bounded measure  $\mu \leq 0$  such that

$$\eta(u)_t + q(u)_x \longrightarrow \mu \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)) \quad (4.5)$$

for an arbitrary convex function  $\eta$  (we assume that  $\eta'$  and  $\eta''$  are bounded on  $\mathbb{R}$ ).

Now, to prove (4.5), we rewrite the formulae (3.1) in the form

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4, \quad (4.6)$$

where,

$$\begin{aligned} \mu_1 &: = \varepsilon (\eta'(u) u_x)_x; \\ \mu_2 &: = -\varepsilon \eta''(u) u_x^2; \\ \mu_3 &: = \delta (\eta'(u) g(u_{xx}))_x; \\ \mu_4 &: = -\delta \eta''(u) u_x g(u_{xx}). \end{aligned}$$

We distinguish the case  $n = 1$  from  $n = 2$ .

**Case n = 1:**  $g(u_{xx}) = -|u_{xx}|$ ,  $f$  convex and  $\delta = o(\varepsilon^2)$ . We have

$$|\langle \mu_1, \theta \rangle| \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) u_x| dx ds \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x u_x| dx ds \leq C \|\theta_x\|_{L^2} \|\varepsilon u_x\|_{L^2}, \quad (4.7)$$

$$|\langle \mu_2, \theta \rangle| \leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^2| dx ds \leq C \|\theta\|_{L^\infty} \|\varepsilon u_x^2\|_{L^1}. \quad (4.8)$$

Since  $\eta$  is a convex function, we notice for a non negative function  $\theta$

$$\langle \mu_2, \theta \rangle = -\varepsilon \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^2 dx ds \leq 0. \quad (4.9)$$

In the same way, we have

$$\begin{aligned} |\langle \mu_3, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) |u_{xx}|| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta_x |u_{xx}|| dx ds \\ &\leq C \|\theta_x\|_{L^2} \|\delta |u_{xx}|\|_{L^2}, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} |\langle \mu_4, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x |u_{xx}|| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x |u_{xx}|| dx ds \\ &\leq C \|\theta\|_{L^\infty} \|\delta u_x |u_{xx}|\|_{L^1}. \end{aligned} \quad (4.11)$$

Combining (4.7), (4.8), (4.9), (4.10) and (4.11), with (a), (b), (c), (d) in Proposition 3.3, gives (4.5) where  $\mu$  is non positive bounded measure.

**Case n = 2:**  $g(u_{xx}) = -u_{xx}^2$ ,  $f$  convex and  $\delta = o(\varepsilon^{5/2})$ . Estimates (4.7), (4.8), (4.9) remain true. Concerning  $\mu_3$ , we have

$$\begin{aligned} |\langle \mu_3, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) u_{xx}^2| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta_x u_{xx}^2| dx ds \\ &\leq C \|\theta_x\|_{L^\infty} \|\delta u_{xx}^2\|_{L^1}. \end{aligned} \quad (4.12)$$

Now,  $\mu_4$  is split as

$$\mu_4 = \mu_{41} + \mu_{42},$$

with

$$\begin{aligned} \mu_{41} &:= \delta \eta''(u) u_x^+ u_{xx}^2; \\ \mu_{42} &:= -\delta \eta''(u) u_x^- u_{xx}^2, \end{aligned}$$

where  $u_x^+ = \max(0, u_x)$  and  $u_x^- = \max(0, -u_x)$ . Then we have

$$\begin{aligned} |\langle \mu_{41}, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^+ u_{xx}^2| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x^+ u_{xx}^2| dx ds \\ &\leq C \|\theta\|_{L^\infty} \|\delta u_x^+ u_{xx}^2\|_{L^1}, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} |\langle \mu_{42}, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^- u_{xx}^2| dx ds \leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x^- u_{xx}^2| dx ds \\ &\leq C \|\theta\|_{L^\infty} \|\delta u_x^- u_{xx}^2\|_{L^1}. \end{aligned} \quad (4.14)$$

Again, since  $\eta'' \geq 0$ , it gets for a non negative function  $\theta$

$$\langle \mu_{42}, \theta \rangle = -\delta \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^- u_{xx}^2 dx ds \leq 0. \quad (4.15)$$

Finally, from inequalities (4.7), (4.8), (4.9), (4.12), (4.13), (4.14) and (4.15), combined with (a), (b), (c) and (d) in Proposition 3.5, we obtain (4.5) where  $\mu$  is non positive bounded measure.

Now we will prove (4.3). We follow arguments of DiPerna [7] and Szepessy [18]: we have to check that, for each compact set  $K$  of  $\mathbb{R}$ ,

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds = \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds = 0.$$

By Jensen's inequality

$$\frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds \leq m(K)^{1/2} \left( \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds \right)^{1/2},$$

where  $m(K)$  denotes the Lebesgue measure of  $K$ . Then we will establish that

$$\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds = 0.$$

Let  $K_i \subset K_{i+1}$  ( $i = 0, 1, \dots$ ) be an increasing sequence of compact sets such that  $K_0 = K$  and  $\cup_{i \geq 0} K_i = \mathbb{R}$ . Using the identity  $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$ , we get for all  $i = 0, 1, \dots$

$$\begin{aligned} & \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(\cdot, s) - u_0)^2 dx ds \\ & \leq \frac{1}{t} \int_0^t \left( \int_{K_i} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{K_i} u_0^2 dx - 2 \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right) ds \\ & \leq \int_{\mathbb{R} \setminus K_i} u_0^2 dx + \frac{1}{t} \int_0^t \left( \int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \right) \\ & \quad + \frac{2}{t} \int_0^t \left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| ds. \end{aligned} \quad (4.16)$$

For the first term of the right hand side, we clearly have

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R} \setminus K_i} u_0^2 dx = 0.$$

Now, injecting (3.26) in (3.23), in the case  $n = 1$ , and respectively (3.36) in (3.31), in the case  $n = 2$ , we obtain

$$\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \leq C \frac{\sqrt{\delta}}{\varepsilon},$$

and respectively,

$$\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \leq C \frac{\delta}{\varepsilon}.$$

In both cases, the right hand side of these inequalities tends to zero when  $\varepsilon \rightarrow 0$ .

To estimate the last term in the inequality (4.16), we choose  $\{\theta_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$  such that

$$\lim_{n \rightarrow \infty} \theta_n = u_0 \quad \text{in } L^2(\mathbb{R}).$$

Then, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| &\leq \int_{K_i} |u_0 - \theta_n| |u^{\varepsilon, \delta}(\cdot, s) - u_0| dx \\
&\quad + \left| \int_{K_i} \theta_n (u_0^{\varepsilon, \delta} - u_0) + \int_{K_i} \theta_n (u^{\varepsilon, \delta}(\cdot, s) - u_0^{\varepsilon, \delta}) dx \right| \\
&\leq \|u_0 - \theta_n\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \\
&\quad + \|\theta_n\|_{L^2(\mathbb{R})} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} + \left| \int_0^s \int_{K_i} \theta_n \partial_s u^{\varepsilon, \delta} dx d\tau \right|.
\end{aligned}$$

In view of (3.19) for  $n = 1$  and respectively (3.29) for  $n = 2$ , we have

$$\|u_0 - \theta_n\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \leq (\|u_0\|_{L^2(\mathbb{R})} + C) \|u_0 - \theta_n\|_{L^2(\mathbb{R})},$$

which tends to zero when  $n \rightarrow \infty$ , since  $\lim_{\varepsilon \rightarrow 0^+} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} = 0$ . Finally, it remains to prove that

$$\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n \partial_s u^{\varepsilon, \delta} dx d\tau \right| ds = 0.$$

We have, by (1.1),

$$\begin{aligned}
\left| \int_0^s \int_{K_i} \theta_n \partial_s u^{\varepsilon, \delta} dx d\tau \right| &= \left| \int_0^s \int_{K_i} \theta_n (-f(u^{\varepsilon, \delta})_x + \varepsilon u_{xx}^{\varepsilon, \delta} + \delta g(u_{xx}^{\varepsilon, \delta})_x) dx d\tau \right| \\
&\leq \int_0^s \int_{K_i} |(\theta_n)_x f(u^{\varepsilon, \delta})| dx d\tau + \varepsilon \int_0^s \int_{K_i} |(\theta_n)_x u_x^{\varepsilon, \delta}| dx d\tau \\
&\quad + \delta \int_0^s \int_{K_i} |(\theta_n)_x g(u_{xx}^{\varepsilon, \delta})| dx d\tau \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

To compute each quantity  $I_1$ ,  $I_2$  and  $I_3$ , we distinguish the cases  $n = 1$  from  $n = 2$ .

**Case  $n = 1$ :**  $g(u_{xx}) = -|u_{xx}|$ ,  $f$  convex and  $\delta = o(\varepsilon^2)$ . Since  $f$  is such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

where  $\beta < 3$ . Thus,

$$|f(u)| \leq C(1 + |u|^m),$$

where  $m < 5$ . Then, Proposition 3.1 implies

$$\int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \leq \int_0^s \int_{\mathbb{R}} |u^{\varepsilon, \delta}|^m dx d\tau \leq C s, \quad (4.17)$$

and

$$\begin{aligned}
I_1 &= \int_0^s \int_{K_i} |(\theta_n)_x| |f(u^{\varepsilon, \delta})| dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_n)_x| dx d\tau + C \int_0^s \int_{K_i} |(\theta_n)_x| |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_n)_x| dx d\tau + C \|(\theta_n)_x\|_{L^\infty(\mathbb{R})} \int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C s \|(\theta_n)_x\|_{L^1(\mathbb{R})} + C s \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}.
\end{aligned} \quad (4.18)$$

Thanks to (3.19), we can write

$$\begin{aligned}
I_2 &= \varepsilon \int_0^s \int_{K_i} |(\theta_n)_x| |u_x^{\varepsilon, \delta}| \, dx \, d\tau \\
&\leq \left( \varepsilon \int_0^s \int_{K_i} |(\theta_n)_x|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \varepsilon \int_0^s \int_{K_i} |u_x^{\varepsilon, \delta}|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
&\leq C \varepsilon^{\frac{1}{2}} s^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{4.19}$$

Finally for  $I_3$ , using (3.27), we get

$$\begin{aligned}
I_3 &= \delta \int_0^s \int_{K_i} |(\theta_n)_x g(u_{xx}^{\varepsilon, \delta})| \, dx \, d\tau = \int_0^s \int_{K_i} \delta |(\theta_n)_x| |u_{xx}^{\varepsilon, \delta}| \, dx \, d\tau \\
&\leq \left( \int_0^s \int_{K_i} |(\theta_n)_x|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \delta^2 \int_0^s \int_{K_i} |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
&\leq s^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \left( \delta^2 \int_0^s \int_{K_i} |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
&\leq s^{\frac{1}{2}} \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})}
\end{aligned} \tag{4.20}$$

Now, from (4.18), (4.19) and (4.20), we deduce

$$\begin{aligned}
&\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n u_s^{\varepsilon, \delta} \, dx \, d\tau \right| \, ds \\
&\leq \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{C}{t} \left( t^2 (\|(\theta_n)_x\|_{L^1(\mathbb{R})} + \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \right. \\
&\quad \left. + t^{\frac{3}{2}} \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \right) \\
&\leq \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} C \left( t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + t^{\frac{1}{2}} \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \right),
\end{aligned}$$

and since  $\delta = o(\varepsilon^2)$ , we obtain the desired conclusion, and Theorem 2.1 is proved.

**Case  $n = 2$ :**  $g(u_{xx}) = -u_{xx}^2$ ,  $f$  convex and  $\delta = o(\varepsilon^{5/2})$ . Here,  $f$  is such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

where  $\beta < 1/2$ , thus,

$$|f(u)| \leq C(1 + |u|^m),$$

where  $m < 5/2$ .

Estimates  $I_1$  and  $I_2$  are obtained in the same manner as  $n = 1$  using (3.29) instead of (3.18). From (3.37) we obtain

$$\begin{aligned}
I_3 &= \delta \int_0^s \int_{K_i} |(\theta_n)_x g(u_{xx}^{\varepsilon, \delta})| \, dx \, d\tau = \int_0^s \int_{K_i} \delta |(\theta_n)_x| |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \\
&\leq \|(\theta_n)_x\|_{L^\infty(\mathbb{R})} \left( \delta \int_0^s \int_{K_i} |u_{xx}^{\varepsilon, \delta}|^2 \, dx \, d\tau \right) \\
&\leq \sqrt{\delta \varepsilon^{-5/2}} \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}
\end{aligned} \tag{4.21}$$

Finally, using (4.18), (4.19) and (4.21), we obtain

$$\begin{aligned}
& \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_n u_s^{\varepsilon, \delta} dx d\tau \right| ds \\
& \leq \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{C}{t} \left( t^2 (\|(\theta_n)_x\|_{L^1(\mathbb{R})} + \|(\theta_n)_x\|_{L^\infty(\mathbb{R})}) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \|(\theta_n)_x\|_{L^2(\mathbb{R})} \right. \\
& \quad \left. + t \sqrt{\delta \varepsilon^{-5/2}} \|(\theta_n)_x\|_{L^\infty(\mathbb{R})} \right) \\
& \leq \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} C \left( t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\delta \varepsilon^{-5/2}} \right),
\end{aligned}$$

and since  $\delta = o(\varepsilon^{5/2})$ , we obtain the desired conclusion, and Theorem 2.2 is proved.  $\square$

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