Convergence of the MAC scheme for the incompressible Navier-Stokes equations

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CONVERGENCE OF THE MAC SCHEME FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We prove in this paper the convergence of the Marker and cell (MAC) scheme for the discretization of the steady-state and unsteady-state incompressible Navier-Stokes equations in primitive variables on non-uniform Cartesian grids, without any regularity assumption on the solution. *A priori* estimates on solutions to the scheme are proven ; they yield the existence of discrete solutions and the compactness of sequences of solutions obtained with family of meshes the space step of which tends to zero. We then establish that the limit is a weak solution to the continuous problem.

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1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^d$ with $d = 2$ or $d = 3$. We consider the steady-state incompressible Navier-Stokes equations, which read:

$$\begin{aligned} &\mathrm{div}\bar{\boldsymbol{u}} = 0, && \text{in } \Omega, && \text{(1a)}\\ &-\Delta\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \boldsymbol{\nabla}\bar{p} = \boldsymbol{f}, && \text{in } \Omega, && \text{(1b)}\\ &\bar{\boldsymbol{u}} = 0, && \text{on } \partial\Omega. && \text{(1c)}\end{aligned}$$

where $\bar{\boldsymbol{u}}$ stands for the (vector-valued) velocity of the flow, $\bar{p}$ for the pressure and $\boldsymbol{f}$ is a given field of $L^2(\Omega)^d$, and where for two given vector fields $\boldsymbol{v} = (v_1, \ldots, v_d)$ and $\boldsymbol{w} = (w_1, \ldots, w_d)$, the quantity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}$ is a vector field whose components are $((\boldsymbol{v}\cdot\nabla)\boldsymbol{w})_i = \sum_{k=1}^d v_k\partial_k w_i$, $i = 1, \ldots, d$. A weak formulation of Problem (1) reads:

$$\begin{aligned}&\text{Find } (\bar{\boldsymbol{u}}, \bar{p}) \in H^1_0(\Omega)^d \times L^2_0(\Omega) \text{ such that, } \forall (\boldsymbol{v}, q) \in H^1_0(\Omega)^d \times L^2_0(\Omega),\\ &\int_\Omega \nabla\bar{\boldsymbol{u}} : \nabla\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega ((\bar{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} - \int_\Omega \bar{p}\,\mathrm{div}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}, && \text{(2a)}\\ &\int_\Omega q\ \mathrm{div}\bar{\boldsymbol{u}}\ \mathrm{d}\boldsymbol{x} = 0, && \text{(2b)}\end{aligned}$$

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where $L^2_0(\Omega)$ stands for the subspace of $L^2(\Omega)$ of zero mean-valued functions.
We shall consider the transient Navier-Stokes equations:

$$\begin{aligned}
&\mathrm{div}\bar{\boldsymbol{u}} = 0 && \text{in } \Omega\times(0,T), && (3a)\\
&\partial_t\bar{\boldsymbol{u}} - \Delta\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \boldsymbol{\nabla}\bar{p} = \boldsymbol{f} && \text{in } \Omega\times(0,T), && (3b)\\
&\bar{\boldsymbol{u}} = 0 && \text{on } \partial\Omega\times(0,T), && (3c)\\
&\bar{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{u}_0. && \text{in } \Omega. && (3d)
\end{aligned}$$

This problem is posed for $(\boldsymbol{x},t)$ in $\Omega\times(0,T)$ where $T\in\mathbb{R}^*_+$ and Ω is an open bounded domain of $\mathbb{R}^d; d=2$ or 3, $\bar{\boldsymbol{u}}$ stands for the (vector-valued) velocity of the flow, $\bar{p}$ for the pressure, $\boldsymbol{f}$ is a given vector field of $L^2(\Omega\times(0,T))^d$ and $\boldsymbol{u}_0\in L^2(\Omega)$. Denoting by $\boldsymbol{E}(\Omega)=\{\boldsymbol{u}\in H^1_0(\Omega)^d\ ;\ \mathrm{div}\ \boldsymbol{u}=0,\ a.e.\ \text{in}\ \Omega\}$ the set of divergence free functions, we consider the following weak formulation of the transient problem (3) (see e.g. [3]).

$$\begin{aligned}
&\text{Find } \boldsymbol{u}\in L^2(0,T;\boldsymbol{E}(\Omega))\cap L^\infty(0,T;L^2(\Omega)^d)\ ;\ \text{such that, } \forall \boldsymbol{v}\in L^2(0,T;\boldsymbol{E}(\Omega))\cap C^\infty_c(\Omega\times[0,T))\\
&-\int_0^T\int_\Omega \bar{\boldsymbol{u}}(\boldsymbol{x},t)\cdot\partial_t\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t - \int_\Omega \boldsymbol{u}_0(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x},0)\ \mathrm{d}\boldsymbol{x} + \int_0^T\int_\Omega \nabla\bar{\boldsymbol{u}}(\boldsymbol{x},t):\nabla\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t \qquad (4)\\
&\qquad + \int_0^T\int_\Omega ((\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}})(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.
\end{aligned}$$

The aim of this paper is to show, under minimal regularity assumptions on the solution, that sequences of approximate solutions obtained by the discretization of problem (1)(resp. (3)) by the Marker-And-cell (MAC) scheme converge to a solution of (2)(resp. (4)) as the mesh size tends to 0.
The Marker-And-Cell (MAC) scheme, introduced in the middle of the sixties [20], is one of the most popular methods [25, 29] for the approximation of the Navier-Stokes equations in the engineering framework, because of its simplicity, its efficiency and its remarkable mathematical properties. The first error analysis seems to be that of [26] in the case of the time-dependent Stokes equations on uniform square grids. The mathematical analysis of the scheme was performed for the steady-state Stokes equations in [24] for uniform rectangular meshes with H^2 regularity assumption on the pressure. Error estimates for the MAC scheme applied to the Stokes equations have been obtained by viewing the MAC scheme as a mixed finite element method [18, 19] or a divergence conforming DG method [21]. Error estimates for rectangular meshes were also obtained for the related covolume method, see [6] and references therein. Using the tools that were developed for the finite volume theory [10, 11], an order 1 error estimate for non-uniform meshes was obtained in [1], with order 2 convergence for uniform meshes, under the usual regularity assumptions (H^2 for the velocities, H^1 for the pressure). It was recently shown in [22] that under higher regularity assumptions (C^4 for the velocities and C^3 for the pressure) and an additional convergence assumption on the pressure, superconvergence is obtained for non uniform meshes. Note also that the convergence of the MAC scheme for the Stokes equations with a right-hand-side in $H^{-1}(\Omega)$ was proven in [2].

Mathematical studies of the MAC scheme for the non linear Navier-Stokes equations are scarcer. A pioneering work was that of [24] for the steady-state Navier-Stokes equations and for uniform rectangular grids. More recently, a variant of the MAC scheme was defined on locally refined grids and the convergence proof was performed for both the steady-state and time dependent cases in two or three space dimensions [4]. For the Stokes equations on uniform grids, this latter scheme coincides with the usual MAC scheme that is classically used in CFD codes. However, for the Navier-Stokes equations, the nonlinear convection term is discretised in a manner which is similar to the finite element framework (see e.g. [28]), which no longer coincides with the usual MAC scheme, even on uniform grids. This discretization entails in a larger stencil, and numerical experiments [5] tend to show that is not as efficient as the classical MAC scheme. Our purpose here is to analyse the classical MAC scheme for the Navier-Stokes equations in primitive variables on a non-uniform rectangular mesh in two or three dimensions, and, as in [4], without regularity assumptions on the solutions.

In section 2 we introduce the MAC space grid and the discrete operators. In particular, the velocity convection operator is approximated so as to be compatible with a discrete continuity equation on the duals cells ; this discretization coincides with the usual discretization on uniform meshes [25], contrary to the scheme of [4].

We introduce the MAC scheme for the steady state Navier-Stokes equations in Section 3. We give a weak formulation of the scheme. Velocity and pressure estimates are thus obtained, which lead to the compactness of sequences of approximate solutions. We then show that any prospective limit is a weak solution of the Navier-Stokes equations.

In Section 4, we turn to the unsteady Navier-Stokes equations. An essential feature of the studied scheme is that the (discrete) kinetic energy remains controlled. We show the compactness of approximate sequences of solutions thanks to a discrete Aubin-Simon argument, and again conclude that any prospective limit of the approximate velocities is a weak solution of the Navier-Stokes equations thanks to a passage to the limit in the scheme. In the case of the unsteady Stokes equations, we are able to obtain some estimates which yield the compactness of sequences of approximate pressures; we are then able to conclude that the approximate pressure converges to a weak solution of the Stokes equations as the mesh size and time steps tend to 0.

2. Space discretization

We assume that the domain Ω is a union of rectangles ($d = 2$) or orthogonal parallelepipeds ($d = 3$), and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$.

Definition 2.1 (MAC grid)**.** A discretization of Ω with MAC grid, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, where:

- the pressure (or primal) grid denoted by $\mathcal{M}$, which consists of a union of possibly non uniform rectangles; a generic cell of this grid is denoted by K, and its mass center $\boldsymbol{x}_K$. A generic face (or edge in the two-dimensional case) of such a cell is denoted by $\sigma \in \mathcal{E}(K)$, and its mass center $\boldsymbol{x}_\sigma$, where $\mathcal{E}(K)$ denotes the set of all faces of K. The set of all faces of the mesh is denoted by $\mathcal{E}$; we have $\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$, where $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$) are the edges of $\mathcal{E}$ that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector $\boldsymbol{e}_i$ of the canonical basis of $\mathbb{R}^d$ is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \ldots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\rm int} \cup \mathcal{E}^{(i)}_{\rm ext}$, where $\mathcal{E}^{(i)}_{\rm int}$ (resp. $\mathcal{E}^{(i)}_{\rm ext}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain.
- For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = \partial K \cap \partial L$ and we write that $\sigma = \overrightarrow{K|L}$ if, furthermore, $\sigma \in \mathcal{E}^{(i)}$ and $\overrightarrow{\boldsymbol{x}_K\boldsymbol{x}_L} \cdot \boldsymbol{e}_i > 0$ for some $i \in [\![1, d]\!]$. A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:
  - if $\sigma = K|L \in \mathcal{E}_{\rm int}$ then $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$, where $D_{K,\sigma}$ (resp. $D_{L,\sigma}$) is the half-part of K (resp. L) adjacent to σ (see Fig. 1 for the two-dimensional case) ;
  - if $\sigma \in \mathcal{E}_{\rm ext}$ is adjacent to the cell K, then $D_\sigma = D_{K,\sigma}$.

  A primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \ldots, d$ are such that $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}_i > 0$. A dual face separating two duals cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$ or $\epsilon = \overrightarrow{\sigma|\sigma'}$ when specifying its orientation: more precisely we write that $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $\overrightarrow{\boldsymbol{x}_\sigma\boldsymbol{x}_{\sigma'}} \cdot \boldsymbol{e}_j > 0$ for some $j \in [\![1, d]\!]$. To any dual face ϵ, we associate a distance d_ϵ as sketched on Figure 1. For a dual face $\epsilon \subset \partial D_\sigma, \sigma \in \mathcal{E}^{(i)}$, $i \in [\![1, d]\!]$, the distance d_ϵ is defined by:

$$d_\epsilon = \begin{cases} d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}) & \text{if } \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}^{(i)}_{\rm int}, \\ d(\boldsymbol{x}_\sigma, \epsilon) & \text{if } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\rm ext} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \tag{5}$$

  where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$, and the set $\widetilde{\mathcal{E}}^{(i)}$ of the faces of the i-th dual mesh (associated to the ith velocity component) is decomposed into the internal and boundary edges: $\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{E}}^{(i)}_{\rm int} \cup \widetilde{\mathcal{E}}^{(i)}_{\rm ext}$.

We define the regularity of the mesh $\mathcal{M}$ by:

$$\eta_{\mathcal{M}} = \max \left\{ \frac{|\sigma|}{|\sigma'|},\ \sigma \in \mathcal{E}^{(i)}, \forall \sigma' \in \mathcal{E}^{(j)}, i, j \in [\![1, d]\!], i \neq j \right\}, \tag{6}$$

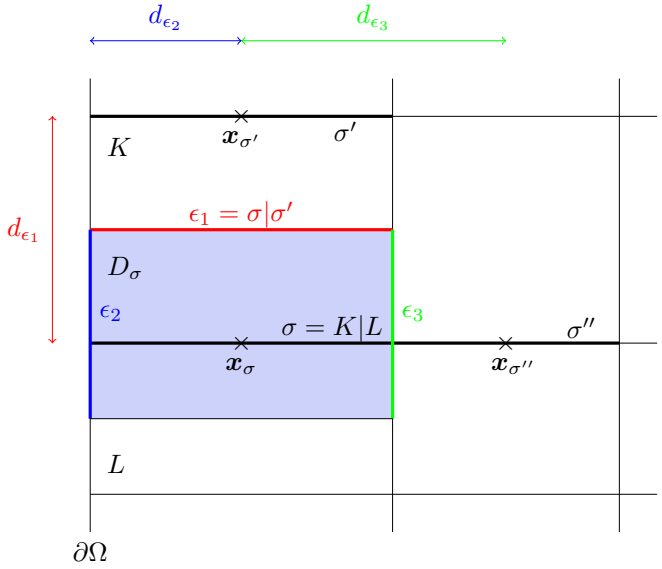


FIGURE 1. Notations for control volumes and dual cells (for the second component of the velocity).

where $|\cdot|$ stands for the $(d-1)$-dimensional measure of a subset of $\mathbb{R}^{d-1}$ (in the sequel, it is also be used to denote or d-dimensional measure of a subset $\mathbb{R}^d$). We also define the size of the mesh by

$$h_{\mathcal{M}} = \max\{\mathrm{diam}(K), K \in \mathcal{M}\}.$$

The discrete velocity unknowns are associated to the velocity cells and are denoted by $(u_\sigma)_{\sigma\in\mathcal{E}^{(i)}}$, $i = 1, \ldots, d$, while the discrete pressure unknowns are associated to the primal cells and are are denoted by $(p_K)_{K\in\mathcal{M}}$.

Definition 2.2 (Discrete spaces)**.** Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid in the sense of Definition 2.1. The discrete pressure space $L_{\mathcal{M}}$ is defined as the set of piecewise constant functions over each of the grid cells K of $\mathcal{M}$, and the discrete $i-th$ velocity space $H^{(i)}_{\mathcal{E}}$ as the set of piecewise constant functions over each of the grid cells D_σ , $\sigma \in \mathcal{E}^{(i)}$. We shall denote by $L_{\mathcal{M},0}$ the functions of $L_{\mathcal{M}}$ with zero mean value. As in the continuous case, the Dirichlet boundary conditions are (partly) incorporated in the definition of the velocity spaces, and, to this purpose, we introduce $H^{(i)}_{\mathcal{E},0} \subset H^{(i)}_{\mathcal{E}}, i = 1, \ldots, d$, defined as follows:

$$H^{(i)}_{\mathcal{E},0} = \Big\{u \in H^{(i)}_{\mathcal{E}},\ u(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}},\ i = 1, \ldots, d\Big\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H^{(i)}_{\mathcal{E},0}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions $\chi_K, K \in \mathcal{M}$ and $\chi_{D_\sigma}, \sigma \in \mathcal{E}$ of the pressure and velocity cells, defined by

$$\chi_K(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in K, \\ 0 \text{ if } \boldsymbol{x} \notin K, \end{cases} \quad \chi_{D_\sigma}(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in D_\sigma, \\ 0 \text{ if } \boldsymbol{x} \notin D_\sigma. \end{cases}$$

We can then write a function $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ as $\boldsymbol{u} = (u_1, \ldots, u_d)$ with $u_i = \sum_{\sigma\in\mathcal{E}^{(i)}} u_\sigma \chi_{D_\sigma}$, $i \in [\![1, d]\!]$ and a function $p \in L_{\mathcal{M}}$ as $p = \sum_{K\in\mathcal{M}} p_K \chi_K$.

Let us now introduce the discrete operators which are used to write the numerical scheme.

Discrete divergence and gradient operators The discrete divergence operator $\mathrm{div}_{\mathcal{M}}$ is defined by:

$$\left.\begin{array}{l|l} \mathrm{div}_{\mathcal{M}}: & \mathbf{H}_{\mathcal{E},0} \longrightarrow L_{\mathcal{M}} \\ & \boldsymbol{u} \longmapsto \mathrm{div}_{\mathcal{M}}\boldsymbol{u} = \displaystyle\sum_{K\in\mathcal{M}} \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| u_{K,\sigma}\ \chi_K, \end{array}\right. \tag{7}$$

$$\text{with } u_{K,\sigma} = u_\sigma \boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{e}_i \text{ for } \sigma\in\mathcal{E}^{(i)}\cap\mathcal{E}(K), i=1,\dots,d. \tag{8}$$

where $\boldsymbol{n}_{K,\sigma}$ denotes the unit normal vector to σ outward K. Note that we have the usual finite volume property of local conservativity of the flux through an interface $\sigma = K|L$ between the cells $K, L\in\mathcal{M}$, *i.e.*

$$u_{K,\sigma} = -u_{L,\sigma}, \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}. \tag{9}$$

We can now define the discrete divergence free velocity space $\boldsymbol{E}_{\mathcal{E}}(\Omega) = \{\boldsymbol{u}\in\mathbf{H}_{\mathcal{E},0}\ ;\ \mathrm{div}_{\mathcal{M}}\ \boldsymbol{u} = 0\}$.

The discrete divergence of $\boldsymbol{u} = (u_1,\dots,u_d)\in\mathbf{H}_{\mathcal{E},0}$ may also be written as

$$\mathrm{div}_{\mathcal{M}}(\boldsymbol{u}) = \sum_{i=1}^{d} (\eth_i u_i)_K \chi_K, \tag{10}$$

where the discrete derivative $(\eth_i u_i)_K$ of u_i on K is defined by

$$(\eth_i u_i)_K = \frac{|\sigma|}{|K|}(u_{\sigma'} - u_\sigma) \text{ with } K = [\overrightarrow{\sigma\sigma'}], \sigma,\sigma'\in\mathcal{E}^{(i)}. \tag{11}$$

The discrete derivatives and divergence are consistent in the following sense:

Lemma 2.3 (Discrete derivative and divergence consistency)**.** *Let $\mathcal{D} = (\mathcal{M},\mathcal{E})$ be a MAC grid, and let $\Pi_{\mathcal{E}}$ be an interpolator from $C_c^\infty(\Omega)^d$ to $\mathbf{H}_{\mathcal{E},0}$ such that, for any $\boldsymbol{\varphi} = (\varphi_1,\cdots,\varphi_d)^t \in (C_c^\infty(\Omega))^d$, there exists $C_{\boldsymbol{\varphi}}\geq 0$ depending only on $\boldsymbol{\varphi}$ such that*

$$\begin{array}{l} \Pi_{\mathcal{E}}\boldsymbol{\varphi} = \left(\Pi_{\mathcal{E}}^{(1)}\varphi_1,\cdots,\Pi_{\mathcal{E}}^{(d)}\varphi_d\right) \in H_{\mathcal{E},0}^{(1)}\times\cdots\times H_{\mathcal{E},0}^{(d)}, \text{ where} \\ |\Pi_{\mathcal{E}}^{(i)}\varphi_i(\boldsymbol{x}) - \varphi_i(\boldsymbol{x}_\sigma)| \leq C_{\boldsymbol{\varphi}} h_{\mathcal{M}}^2\ \forall \boldsymbol{x}\in D_\sigma,\ \forall\sigma\in\mathcal{E}^{(i)},\ \forall i = 1,\cdots,d, \end{array} \tag{12}$$

Then there exists $C_{\boldsymbol{\varphi},\eta}\geq 0$, where η is the regularity of the mesh defined by (6)*, such that; $|\eth_i\Pi_{\mathcal{E}}^{(i)}\varphi_i(\boldsymbol{x}) - \partial_i\varphi_i(\boldsymbol{x})| \leq C_{\boldsymbol{\varphi},\eta} h_{\mathcal{M}}$ for a.e. $\boldsymbol{x}\in\Omega$.*

As a consequence, if $(\mathcal{D}_n)_{n\in\mathbb{N}} = (\mathcal{M}_n,\mathcal{E}_n)_{n\in\mathbb{N}}$ is a sequence of MAC grids such that $\eta_n\leq\eta$ for all n and $h_{\mathcal{M}_n}\to 0$ as $n\to+\infty$, then $\mathrm{div}_{\mathcal{M}_n}(\Pi_{\mathcal{E}_n}\boldsymbol{\varphi})\to\mathrm{div}\boldsymbol{\varphi}$ uniformly as $n\to+\infty$.

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence, and reads:

$$\left.\begin{array}{l|l} \boldsymbol{\nabla}_{\mathcal{E}}: & L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ & p \longmapsto \boldsymbol{\nabla}_{\mathcal{E}} p \\ & \boldsymbol{\nabla}_{\mathcal{E}} p(\boldsymbol{x}) = (\eth_1 p(\boldsymbol{x}),\dots,\eth_d p(\boldsymbol{x}))^t, \end{array}\right. \tag{13}$$

where $\eth_i p\in H_{\mathcal{E},0}^{(i)}$ is the discrete derivative of p in the i-th direction, defined by:

$$\eth_i p(\boldsymbol{x}) = \frac{|\sigma|}{|D_\sigma|}\ (p_L - p_K) \quad \forall \boldsymbol{x}\in D_\sigma, \text{ for } \sigma = \overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}^{(i)},\ i=1,\dots,d. \tag{14}$$

Note that in fact, the discrete gradient of a function of $L_{\mathcal{M}}$ should only be defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) for the sake of simplicity. Again, the definition of the discrete derivatives of the pressure on the MAC grid is evidently consistent in the following sense:

Lemma 2.4 (Discrete gradient consistency)**.** *Let $\Pi_{\mathcal{M}}$ be an interpolator from $C_c^\infty(\Omega)$ to $L_{\mathcal{M}}$ such that, for any $\psi \in C_c^\infty(\Omega)$, there exists $C_\psi \geq 0$ depending only on ψ such that*

$$|\Pi_{\mathcal{M}}\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_K)| \leq C_\psi h_{\mathcal{M}}^2, \ \forall \boldsymbol{x} \in K, \ \forall K \in \mathcal{M}. \tag{15}$$

then there exists $C_{\psi,\eta} \geq 0$ depending only on ψ and η such that

$$|\eth_i \Pi_{\mathcal{M}}\psi(\boldsymbol{x}) - \partial_i\psi(\boldsymbol{x})| \leq C_{\psi,\eta} h_{\mathcal{M}}, \forall \sigma \in \mathcal{E}^{(i)}, \ \forall i = 1, \ldots, d.$$

Let us then verify that the discrete gradient and divergence are dual.

Lemma 2.5 (Discrete div $-\boldsymbol{\nabla}$ duality)**.** *Let $q \in L_{\mathcal{M}}$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ then we have:*

$$\int_\Omega q \ \mathrm{div}_{\mathcal{M}}\boldsymbol{v} \ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}} q \cdot \boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = 0. \tag{16}$$

Proof. Let $q \in L_{\mathcal{M}}$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$. By the definition (7) of the discrete divergence operator, we have:

$$\int_\Omega q \ \mathrm{div}_{\mathcal{M}}\boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} q_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma| v_{K,\sigma}.$$

with $v_{K,\sigma} = v_\sigma \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}_i$ for $\sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K), i = 1, \ldots, d$. Thanks to the conservativity (9) of the flux we get that:

$$\begin{aligned}\int_\Omega q \ \mathrm{div}_{\mathcal{M}}\boldsymbol{v} \ \mathrm{d}\boldsymbol{x} &= \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}, \sigma = K|L} (q_K|\sigma|v_{K,\sigma} + q_L|\sigma|v_{L,\sigma}) \\ &= \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}, \sigma = K|L} |\sigma|(q_K - q_L)v_{K,\sigma}.\end{aligned}$$

Therefore, by the definition (14) of the discrete derivative of q, we get:

$$\int_\Omega q \ \mathrm{div}_{\mathcal{M}}\boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = -\sum_{i=1}^d \sum_{\sigma\in\mathcal{E}^{(i)}, \sigma = K|L} |D_\sigma| v_\sigma \eth_i q = -\int_\Omega \boldsymbol{\nabla}_{\mathcal{E}} q \cdot \boldsymbol{v} \ \mathrm{d}\boldsymbol{x},$$

which concludes the proof. □

Discrete Laplace operator - For $i = 1 \ldots, d$, we classically define the i^{th} component of the discrete Laplace operator by:

$$-\Delta_{\mathcal{E}}^{(i)} : \quad \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}} u_i = -\displaystyle\sum_{\sigma\in\mathcal{E}^{(i)}} (\Delta u)_\sigma \chi_{D_\sigma}, \ \text{with} \ -(\Delta u)_\sigma = \frac{1}{|D_\sigma|} \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} \phi_{\sigma,\epsilon}(u_i) \end{array} \right. \tag{17}$$

$\tilde{\mathcal{E}}(D_\sigma)$ denotes the faces of D_σ and

$$\phi_{\sigma,\epsilon}(u_i) = \begin{cases} \frac{|\epsilon|}{d_\epsilon}(u_\sigma - u_{\sigma'}), \ \text{if} \ \ \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}, \\ \frac{|\epsilon|}{d_\epsilon} u_\sigma, \ \text{if} \ \epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases}$$

where d_ϵ is defined by (5). Note that we have the usual finite volume property of local conservativity of the flux through an interface $\epsilon = \overrightarrow{\sigma \ |\sigma'}$:

$$\phi_{\sigma,\epsilon}(u_i) = -\phi_{\sigma',\epsilon}(u_i), \quad \forall \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}. \tag{18}$$

Then the discrete Laplace operator of the full velocity vector is defined by

$$\begin{array}{rl} -\Delta_{\mathcal{E}}: & \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ & \boldsymbol{u} \mapsto -\Delta_{\mathcal{E}}\boldsymbol{u} = (-\Delta_{\mathcal{E}}^{(1)}u_1, \ldots, -\Delta_{\mathcal{E}}^{(d)}u_d)^t. \end{array} \tag{19}$$

Discrete convection operator - Let us consider the momentum equation (1b) for the i^{th} component of the velocity, and integrate it on a cell D_σ, $\sigma \in \mathcal{E}^{(i)}$. By the Stokes formula we then need to discretise $\sum_{\epsilon\subset\partial D_\sigma} \int_\epsilon u_i \boldsymbol{u}\cdot\boldsymbol{n}_{\sigma,\epsilon}\ \mathrm{d}\gamma(\boldsymbol{x})$, where $\boldsymbol{n}_{\sigma,\epsilon}$ denotes the unit normal vector to ϵ outward D_σ and $\mathrm{d}\gamma(\boldsymbol{x})$ denotes the integration with respect to the $d-1$-dimensional Lebesgue measure. For $\epsilon = \sigma|\sigma'$, the convection flux $\int_\epsilon u_i\boldsymbol{u}\cdot\boldsymbol{n}_{\sigma,\epsilon}\ \mathrm{d}\gamma(\boldsymbol{x})$ is approximated by $|\epsilon|u_{\sigma,\epsilon}u_\epsilon$, where

$$u_\epsilon = (u_\sigma + u_{\sigma'})/2, \tag{20}$$

and $|\epsilon|u_{\sigma,\epsilon}$ is the numerical mass flux through ϵ outward D_σ; this flux must be chosen carefully to obtain the L^2 stability of the scheme. More precisely, we need that a discrete counterpart of the free divergence of $\boldsymbol{u}$ be satisfied also on the dual cells. We distinguish two cases:

- First case – The vector $\boldsymbol{e}_i$ is normal to ϵ, and ϵ is included in a primal cell K, with $\mathcal{E}^{(i)}(K) = \{\sigma, \sigma'\}$. Then the mass flux through $\epsilon = \sigma|\sigma'$ is given by:

$$|\epsilon|u_{\sigma,\epsilon} = \frac{1}{2}\,(-|\sigma|u_{K,\sigma} + |\sigma'|u_{K,\sigma'}). \tag{21}$$

- Second case – The vector $\boldsymbol{e}_i$ is tangent to ϵ, and ϵ is the union of the halves of two primal faces τ and τ' such that $\sigma = K|L$ with $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$|\epsilon|u_{\sigma,\epsilon} = \frac{1}{2}\,(|\tau|u_{K,\tau} + |\tau'|u_{L,\tau'}). \tag{22}$$

Note that with this definition, we again have the usual finite volume property of local conservativity of the flux through an interface $\overrightarrow{\sigma|\sigma'}$, *i.e.*

$$|\epsilon|u_{\sigma,\epsilon} = -|\epsilon|u_{\sigma',\epsilon} \tag{23}$$

together with the following discrete free divergence condition on the dual cells:

$$\sum_{\epsilon\in\mathcal{E}(D_\sigma)} |\epsilon|u_{\sigma,\epsilon} = \frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|u_{K,\sigma} + \frac{1}{2}\sum_{\sigma\in\mathcal{E}(L)} |\sigma|u_{L,\sigma} = 0. \tag{24}$$

Note that we have also $u_{\sigma,\epsilon} = 0$ if $\epsilon \subset \partial\Omega$, which is consistent with the boundary conditions (1c).

We now define the i-th component $C_{\mathcal{E}}^{(i)}(\boldsymbol{u})$ of the non linear convection operator by:

$$C_{\mathcal{E}}^{(i)}(\boldsymbol{u}): \quad \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ v \longmapsto C_{\mathcal{E}}^{(i)}(\boldsymbol{u})v = \displaystyle\sum_{\sigma\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}} \frac{1}{|D_\sigma|}\left(\sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=\sigma|\sigma'}} |\epsilon|u_{\sigma,\epsilon}\frac{v_\sigma + v_{\sigma'}}{2}\right)\chi_{D_\sigma}, \end{array} \right. \tag{25}$$

and the full discrete convection operator $\boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})$, $\mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0}$ by

$$\boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})\boldsymbol{v} = (C_{\mathcal{E}}^{(1)}(\boldsymbol{u})v_1, \ldots, C_{\mathcal{E}}^{(d)}(\boldsymbol{u})v_d)^t.$$

3. The steady case

With the notations introduced in the previous sections, the MAC scheme for the discretisation of Problem (1) on a MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ reads:

$$\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0},\ p \in L_{\mathcal{M},0}, \tag{26a}$$
$$-\Delta_{\mathcal{E}} \boldsymbol{u} + \boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})\boldsymbol{u} + \boldsymbol{\nabla}_{\mathcal{E}} p = \mathcal{P}_{\mathcal{E}} \boldsymbol{f}, \tag{26b}$$
$$\mathrm{div}_{\mathcal{M}} \boldsymbol{u} = 0, \tag{26c}$$

where $L_{\mathcal{M},0} = \{q \in L_{\mathcal{M}} \quad \int_\Omega q \ \mathrm{d}\boldsymbol{x} = 0\}$ and $\mathcal{P}_{\mathcal{E}}$ is the cell mean-value operator defined by

$$\begin{aligned}
&\mathcal{P}_{\mathcal{E}} \boldsymbol{v} = \left(\mathcal{P}_{\mathcal{E}}^{(1)} v_1, \cdots, \mathcal{P}_{\mathcal{E}}^{(d)} v_d\right) \in H_{\mathcal{E},0}^{(1)} \times \cdots \times H_{\mathcal{E},0}^{(d)}, \text{ where for } i = 1, \dots d, \\
&\mathcal{P}_{\mathcal{E}}^{(i)} : \quad L^1(\Omega) \longrightarrow H_{\mathcal{E},0}^{(i)} \\
&\qquad\qquad v_i \longmapsto \mathcal{P}_{\mathcal{E}} v_i \ ;\ i = 1, \cdots, d, \\
&\qquad\qquad \mathcal{P}_{\mathcal{E}}^{(i)} v_i = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \left(\frac{1}{|D_\sigma|} \int_{D_\sigma} v_i(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x} \right) \chi_{D_\sigma}.
\end{aligned} \tag{27}$$

3.1. **Weak form of the scheme**

We first recall the definition of the discrete H_0^1 inner product [10]; it is obtained by multiplying the discrete Laplace operator scalarly by a test function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (18):

$$\begin{aligned}
&\forall (\boldsymbol{u}, \boldsymbol{v}) \in {\mathbf{H}_{\mathcal{E},0}}^2, \qquad \int_\Omega -\Delta_{\mathcal{E}} \boldsymbol{u} \cdot \boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} = \sum_{i=1}^{d} [u_i, v_i]_{1,\mathcal{E}^{(i)},0}, \\
&\text{with } [u_i, v_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon = \overrightarrow{\sigma|\sigma'}}} \frac{|\epsilon|}{d_\epsilon} \ (u_\sigma - u_{\sigma'}) \ (v_\sigma - v_{\sigma'}) + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \\ \epsilon \subset \partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} \ u_\sigma \ v_\sigma.
\end{aligned} \tag{28}$$

The bilinear forms $\left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \times H_{\mathcal{E},0}^{(i)} \to \mathbb{R} \\ (u, v) \mapsto [u_i, v_i]_{1,\mathcal{E}^{(i)},0} \end{array} \right.$ and $\left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \to \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) \mapsto [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} \end{array} \right.$ are inner products on $H_{\mathcal{E},0}^{(i)}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H_0^1 norms:

$$\|u_i\|_{1,\mathcal{E}^{(i)},0}^2 = [u_i, u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon = \overrightarrow{\sigma|\sigma'}}} \frac{|\epsilon|}{d_\epsilon} \ (u_\sigma - u_{\sigma'})^2 + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \\ \epsilon \subset \partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} \ u_\sigma^2 \text{ for } i = 1, \dots, d, \tag{29a}$$

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0}^2 = [\boldsymbol{u}, \boldsymbol{u}]_{1,\mathcal{E},0} = \sum_{i=1}^{d} \|u_i\|_{1,\mathcal{E}^{(i)},0}^2. \tag{29b}$$

Since we are working on Cartesian grids, this inner product may be formulated as the L^2 inner product of discrete gradients. Indeed, consider the following discrete gradient of each velocity component u_i.

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\eth_1 u_i, \dots, \eth_d u_i) \text{ with } \eth_j u_i = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)} \\ \epsilon \perp \boldsymbol{e}_j}} (\eth_j u_i)_{D_\epsilon} \ \chi_{D_\epsilon}, \tag{30}$$

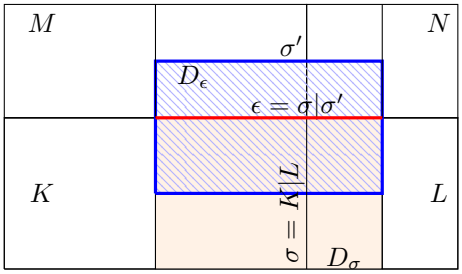


FIGURE 2. Full grid for definition of the derivative of the velocity.

where $(\eth_j u_i)_{D_\epsilon} = \dfrac{u_{\sigma'} - u_\sigma}{d_\epsilon}$ with $\epsilon = \overrightarrow{\sigma|\sigma'}$, and $D_\epsilon = \epsilon \times \boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma'}$ (see Figure 2). This definition is compatible with the definition of the discrete derivative $(\eth_i u_i)_K$ given by (11), since, if $\epsilon \subset K$ then $D_\epsilon = K$. With this definition, it is easily seen that

$$\int_\Omega \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \, \mathrm{d}\boldsymbol{x} = [u, v]_{1,\mathcal{E}^{(i)},0}, \forall u, v \in H^{(i)}_{\mathcal{E},0}, \forall i = 1, \ldots, d. \tag{31}$$

where $[u, v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H^1_0 inner product defined by (28). We may then define

$$\nabla_{\mathcal{E}} \boldsymbol{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \ldots, \nabla_{\mathcal{E}^{(d)}} u_d),$$

so that

$$\int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u} : \nabla_{\mathcal{E}} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0}.$$

With this formulation, the MAC scheme for the linear Stokes problem as a gradient scheme in the sense introduced in [12], see [14] and [8] for more details on the generalization of this formulation to other schemes. In the stationary case, we can show the (strong) convergence of this discrete gradient to the gradient of the exact velocity, and thus also show the strong convergence of the pressure, see section 4.4.

The weak form $b_{\mathcal{E}}$ of the nonlinear convection operator is defined by:

$$\begin{aligned} &\forall (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in {\mathbf{H}_{\mathcal{E},0}}^3, \qquad b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i=1}^{d} b^{(i)}_{\mathcal{E}}(\boldsymbol{u}, v_i, w_i), \\ &\text{where for } i = 1, \ldots, d, \quad b^{(i)}_{\mathcal{E}}(\boldsymbol{u}, v_i, w_i) = \int_\Omega C^{(i)}_{\mathcal{E}}(\boldsymbol{u}) v_i \; w_i \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{32}$$

We are now in position to introduce a weak formulation of the scheme, which reads:

$$\begin{aligned} &\text{Find } (\boldsymbol{u}, p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0} \text{ and, } \forall (\boldsymbol{v}, q) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}, \\ &\int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u} : \nabla_{\mathcal{E}} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega p \, \mathrm{div}_{\mathcal{M}}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \qquad (33a) \\ &\int_\Omega \mathrm{div}_{\mathcal{M}} \boldsymbol{u} \, q \, \mathrm{d}\boldsymbol{x} = 0, \qquad (33b) \end{aligned}$$

and which is equivalent to the MAC scheme (26).

3.2. **Existence and stability**

Lemma 3.1 (Estimate on $b_{\mathcal{E}}$)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid and let $b_{\mathcal{E}}$ be defined by* (32)*. There exists $C_{\eta_{\mathcal{M}}} > 0$, depending only on the regularity $\eta_{\mathcal{M}}$ of the mesh defined by* (6) *such that:*

$$\forall (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{E}_{\mathcal{E}} \times {\mathbf{H}_{\mathcal{E},0}}^2, \qquad |b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C_{\eta_{\mathcal{M}}} \; \|\boldsymbol{u}\|_{L^4(\Omega)^d} \|\boldsymbol{v}\|_{1,\mathcal{E},0} \; \|\boldsymbol{w}\|_{L^4(\Omega)^d} \tag{34}$$

and

$$\forall(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\in \boldsymbol{E}_{\mathcal{E}}\times {\mathbf{H}_{\mathcal{E},0}}^2,\qquad |b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|\le C_{\eta_{\mathcal{M}}}\ \|\boldsymbol{u}\|_{1,\mathcal{E},0}\ \|\boldsymbol{v}\|_{1,\mathcal{E},0}\|\boldsymbol{w}\|_{1,\mathcal{E},0}, \tag{35}$$

Proof. We closely follow the proof of the estimate in the continuous case, where the nonlinear term $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{w}\ \mathrm{d}\boldsymbol{x}$ is estimated thanks to the Hölder inequality and the Sobolev embedding: there exist $C_1\ge 0$ and $C_2\ge 0$ depending only on Ω such that

$$\begin{aligned}|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\le C_1\|\boldsymbol{u}\|_{L^4(\Omega)^d}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)^{d\times d}}\|\boldsymbol{w}\|_{L^4(\Omega)^d}\\ &\le C_2\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^{d\times d}}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)^{d\times d}}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)^{d\times d}}.\end{aligned}$$

Let $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\in \boldsymbol{E}_{\mathcal{E}}\times {\mathbf{H}_{\mathcal{E},0}}^2$. Thanks to (24), we have:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u},u_i,v_i) = \sum_{\sigma\in\mathcal{E}^{(i)}} w_\sigma \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} |\epsilon|(v_\epsilon - v_\sigma)u_{\sigma,\epsilon}.$$

From the definition (20) of u_ϵ and with a discrete integration by parts, we get that:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u},u_i,v_i) = -\frac{1}{2}\sum_{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}} (v_\sigma - v_{\sigma'})|\epsilon|u_{\sigma,\epsilon}(w_{\sigma'}+w_\sigma)$$

From the definition (21)-(22) of $u_{\sigma,\epsilon}$ we have for $\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}$:

$$|u_{\sigma,\epsilon}|\le\begin{cases}\frac{1}{2}(|u_\sigma|+|u_{\sigma'}|) & \text{if } \epsilon\perp\boldsymbol{e}_i,\\ \frac{1}{2}(|u_\tau|+|u_{\tau'}|) & \text{if } \epsilon\not\perp\boldsymbol{e}_i \text{ and } \epsilon\subset\tau\cup\tau',\end{cases}$$

where τ and τ' are the faces of $\mathcal{E}^{(j)}, j\neq i$ such that $\epsilon\subset\tau\cup\tau'$.

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u},u_i,v_i)\le \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_i}} |\epsilon|(|u_\sigma|+|u_{\sigma'}|)|v_\sigma-v_{\sigma'}||w_\sigma+w_{\sigma'}| + \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\not\perp\boldsymbol{e}_i,\epsilon\subset\tau\cup\tau'}} |\epsilon|(|u_\tau|+|u_{\tau'}|)|v_\sigma-v_{\sigma'}||w_\sigma+w_{\sigma'}|.$$

Using Hölder's inequality, we get:

$$\begin{aligned}\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_i}} |\epsilon||u_\sigma||v_\sigma-v_{\sigma'}||w_\sigma| &= \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_i}} |\epsilon|^{\frac14}d_\epsilon^{\frac14}|u_\sigma|\frac{\sqrt{|\epsilon|}}{\sqrt{d_\epsilon}}|v_\sigma-v_{\sigma'}||\epsilon|^{\frac14}d_\epsilon^{\frac14}|w_\sigma|\\ &\le \Big(\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_i}} |\epsilon|d_\epsilon|u_\sigma|^4\Big)^{\frac14}\Big(\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_i}} (v_\sigma-v_{\sigma'})^2\frac{|\epsilon|}{d_\epsilon}\Big)^{\frac12}\Big(\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_i}} |\epsilon|d_\epsilon|w_\sigma|^4\Big)^{\frac14}\\ &\le \|u_i\|_{L^4(\Omega)}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{L^4(\Omega)}.\end{aligned}$$

Therefore, with similar computations for the terms involving $u_{\sigma'}$, u_τ,$u_{\tau'}$, $u_{\sigma'}$ and $w_{\sigma'}$, we get:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u},u_i,v_i)\le C_{\mathcal{M}}\big[\|u_i\|_{L^4(\Omega)}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{L^4(\Omega)} + \sum_{\substack{j\in[\![1,d]\!]\\ j\neq i}}\|u_j\|_{L^4(\Omega)}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{L^4(\Omega)}\big], \forall i\in[\![1,d]\!],$$

where $C_{\mathcal{M}}$ only depends on $\eta_{\mathcal{M}}$ (6). We then deduce (34). By the discrete Sobolev inequality [10, Lemma 3.5], we also have

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u},u_i,v_i) \le C_{\eta_{\mathcal{M}}}\big[\|u_i\|_{1,\mathcal{E}^{(i)},0}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{1,\mathcal{E}^{(i)},0} + \sum_{\substack{j\in[\![1,d]\!]\\ j\neq i}} \|u_j\|_{1,\mathcal{E}^{(i)},0}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{1,\mathcal{E}^{(i)},0}\big], \forall i \in [\![1,d]\!],$$

from which we get (35).

□

Lemma 3.2 ($b_{\mathcal{E}}$ is skew-symmetric)**.** *Let $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E}_{\mathcal{E}} \times \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0}$ then ;*

$$b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \tag{36}$$

and therefore

$$\forall \boldsymbol{u} \in \boldsymbol{E}_{\mathcal{E}}, \qquad b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = 0. \tag{37}$$

Proof. The proof follows that of the continuous case, which is based on a integration by parts. Indeed

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\ \mathrm{d}\boldsymbol{x} = -\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}).$$

Let $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E}_{\mathcal{E}} \times \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0}$. By (24) we have:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u},v_i,w_i) = \sum_{\sigma\in\mathcal{E}^{(i)}} w_\sigma \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} |\epsilon|(v_\epsilon - v_\sigma)u_{\sigma,\epsilon}, \text{ for any } i \in [\![1,d]\!].$$

From the definition (20) of u_ϵ and with a discrete integration by parts, we get by conservativity of the flux (23) that:

$$\begin{aligned} b_{\mathcal{E}}^{(i)}(\boldsymbol{u},u_i,v_i) &= -\frac{1}{2}\sum_{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}} |\epsilon|(v_\sigma - v_{\sigma'})u_{\sigma,\epsilon}(w_{\sigma'}+w_\sigma) \\ &= \frac{1}{2}\sum_{\epsilon=\overrightarrow{\sigma'|\sigma}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}} |\epsilon|(v_\sigma + v_{\sigma'})u_{\sigma',\epsilon}(w_{\sigma'}-w_\sigma). \end{aligned}$$

which yields (36) thanks to another discrete integration by parts. □

In order to obtain an a priori estimate on the pressure, we introduce a so-called Fortin interpolation operator, which preserves the divergence. The following lemma is given in [17, Theorem 1, case $q=2$], and we re-state here with our notations for the sake of clarity.

Lemma 3.3 (Fortin interpolation operator)**.** *Let $\mathcal{D} = (\mathcal{M},\mathcal{E})$ be a MAC grid of Ω. For $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E}},$, we define $\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v}$ by*

$$\begin{aligned} &\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v} = \left(\widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)}v_1,\cdots,\widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)}v_d\right) \in \mathbf{H}_{\mathcal{E}}, \ \textit{where for } i=1,\dots d, \\ &\widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)}: \quad H_0^1(\Omega) \longrightarrow H_{\mathcal{E},0}^{(i)} \\ &\qquad\qquad v_i \longmapsto \widetilde{\mathcal{P}}_{\mathcal{E}}v_i\ ;\ i=1,\cdots,d, \\ &\qquad\qquad \widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)}v_i(\boldsymbol{x}) = \frac{1}{|\sigma|}\int_\sigma v_i(\boldsymbol{x})\ \mathrm{d}\gamma(\boldsymbol{x}),\ \forall \boldsymbol{x}\in D_\sigma,\ \sigma\in\mathcal{E}^{(i)}. \end{aligned} \tag{38}$$

For $q \in L^2(\Omega)$, we define $\mathcal{P}_{\mathcal{M}}q \in L_{\mathcal{M}}$ by:

$$\mathcal{P}_{\mathcal{M}}q(\boldsymbol{x}) = \frac{1}{|K|}\int_K q(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}. \tag{39}$$

Let $\eta_{\mathcal{M}} > 0$ be defined by (6). Let $\boldsymbol{\varphi} \in (H_0^1(\Omega))^d$, then

$$\mathrm{div}_{\mathcal{M}}(\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}) = \mathcal{P}_{\mathcal{M}}(\mathrm{div}\boldsymbol{\varphi}), \tag{40a}$$

$$\|\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}\|_{1,\mathcal{E},0} \leq C_{\eta_{\mathcal{M}}}\|\nabla\boldsymbol{\varphi}\|_{(L^2(\Omega))^d}, \tag{40b}$$

where $C_{\eta_{\mathcal{M}}}$ depends only on $\eta_{\mathcal{M}}$ and Ω. In particular, if $\mathrm{div}\boldsymbol{\varphi} = 0$, then $\mathrm{div}_{\mathcal{M}}(\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}) = 0$.

Theorem 3.4 (Existence and estimates)**.** *There exists a solution to (33), and there exists $C_{\eta_{\mathcal{M}}} > 0$ depending only on the regularity $\eta_{\mathcal{M}}$ of the mesh and Ω, such that any solution of (33) satisfies the following stability estimate:*

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0} + \|p\|_{L^2(\Omega)} \leq C_{\eta_{\mathcal{M}}}\ \|\boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{41}$$

Proof. Let us start by an *a priori* estimate on the approximate velocity. Assume that $(\boldsymbol{u}, p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ satisfies (26); taking $\boldsymbol{v} = \boldsymbol{u}$ in (33a) we get that:

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0}^2 = \int_\Omega p\ \mathrm{div}_{\mathcal{M}}\ \mathrm{d}\boldsymbol{x}\ \boldsymbol{u} - b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) + \int_\Omega \boldsymbol{f}_{\mathcal{E}} \cdot \boldsymbol{u}\ \mathrm{d}\boldsymbol{x}.$$

Since $\mathrm{div}_{\mathcal{M}}\boldsymbol{u} = 0$ and $b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = 0$ by (36) this yields that

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0} \leq \mathrm{diam}(\Omega)\|\boldsymbol{f}\|_{(L^2)^d}. \tag{42}$$

thanks to the fact that $\|\boldsymbol{f}_{\mathcal{E}}\|_{(L^2(\Omega))^d} \leq \|\boldsymbol{f}\|_{(L^2(\Omega))^d}$ and to the discrete Poincaré inequality [10, Lemma 9.1].

An *a priori* estimate on the pressure is obtained by remarking as in [27] that the MAC scheme is inf-sup stable. Indeed, since $p \in L_0^2(\Omega)$, there exists $\boldsymbol{\varphi} \in (H_0^1(\Omega))^d$ such that $\mathrm{div}\boldsymbol{\varphi} = p$ a.e. in Ω and

$$\|\boldsymbol{\varphi}\|_{(H_0^1(\Omega))^d} \leq c\|p\|_{L^2(\Omega)}, \tag{43}$$

where c depends only on Ω [23]. Taking $\boldsymbol{v} = \widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}$ (defined by (38)) as test function in (33a), we obtain thanks to Lemma 3.3 that

$$[\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0} + b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \int_\Omega p^2\ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}}\boldsymbol{f} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x}.$$

Thanks to the estimate (35) on $b_{\mathcal{E}}$ and the Cauchy-Schwarz inequality we get:

$$\|p\|_{L^2(\Omega)}^2 \leq \|\boldsymbol{u}\|_{1,\mathcal{E},0}\|\boldsymbol{v}\|_{1,\mathcal{E},0} + C_{\eta_{\mathcal{M}}}\|\boldsymbol{u}\|_{1,\mathcal{E},0}^2\|\boldsymbol{v}\|_{1,\mathcal{E},0} + \|\boldsymbol{f}\|_{L^2(\Omega)^d}\|\boldsymbol{v}\|_{L^2(\Omega)^d},$$

which yields

$$\|p\|_{L^2} \leq C_{\eta_{\mathcal{M}}}\|\boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{44}$$

thanks to (40b), (43) and to the estimate (42).

Let us now prove the existence of a solution to (33). Consider the continuous mapping

$$\begin{aligned} F : \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0} \times [0,1] &\longrightarrow \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0} \\ (\boldsymbol{u}, p, \zeta) &\mapsto F(\boldsymbol{u}, p, \zeta) = (\hat{\boldsymbol{u}}, \hat{p}) \end{aligned}$$

where $(\hat{\boldsymbol{u}}, \hat{p}) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ is such that

$$\int_\Omega \hat{\boldsymbol{u}} \cdot \boldsymbol{v} = [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0} + \zeta\ b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \int_\Omega p\ \mathrm{div}_{\mathcal{M}}\ \boldsymbol{v} - \int_\Omega \boldsymbol{f}_{\mathcal{E}} \cdot \boldsymbol{v},\ \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0} \tag{45a}$$

$$\int_\Omega \hat{p}\ q = \int_\Omega \mathrm{div}_{\mathcal{M}}\ \boldsymbol{u}\ q,\ \forall q \in L_{\mathcal{M}}. \tag{45b}$$

It is easily checked that F is indeed a one to one mapping, since the values of $\hat{u}^{(i)}; i = 1, \cdots, d$, and $\hat{p}$ are readily obtained by setting in this system $v_i = 1_{D_\sigma}$, $v_j = 0, j \neq i$ in (45a) and $q = 1_K$ in (45b). The mapping F is continuous; moreover, if $(\boldsymbol{u}, p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ is such that $F(\boldsymbol{u}, p, \zeta) = (0, 0)$, then for any $(\boldsymbol{v}, q) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$,

$$[\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} + \zeta\ b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega p \operatorname{div}_{\mathcal{M}}(\boldsymbol{v})\ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}} \boldsymbol{f} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x},$$
$$\int_\Omega \operatorname{div}_{\mathcal{M}}(\boldsymbol{u})\, q\ \mathrm{d}\boldsymbol{x} = 0.$$

The arguments used in the above estimates on possible solutions of (33) may be used in a similar way to show that $(\boldsymbol{u}, p)$ is bounded independently of ζ. Since $F(\boldsymbol{u}, p, 0) = 0$ is a bijective affine function by the stability of the linear Stokes problem (see [2]), the existence of at least one solution $(\boldsymbol{u}, p)$ to the equation $F(\boldsymbol{u}, p, 1) = 0$ which is exactly (33), follows by a topological degree argument (see [7] for the theory, [9] for the first application to a nonlinear scheme and [13, Theorem 4.3] for an easy formulation of the result which can be used here). □

3.3. **Convergence analysis**

In order to prove the convergence of the scheme, we introduce an alternate convection operator $b_{\mathcal{M}}$, defined on the pressure grid and easier to manipulate in the proofs. It relies on the reconstruction of each velocity component on all edges (or faces in 3D) of the mesh.

Lemma 3.5 (Full grid velocity interpolate)**.** *For a given MAC mesh $(\mathcal{M}, \mathcal{E})$, we define, for $i, j = 1, \ldots, d$, the i-th full grid velocity reconstruction operator by*

$$\begin{aligned} \mathcal{R}_{\mathcal{E}}^{(i,j)} :\ & H_{\mathcal{E},0}^{(i)} \to L^2(\Omega) \\ & v \mapsto \mathcal{R}_{\mathcal{E}}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}^{(j)}} \widehat{v}_\sigma \chi_{D_\sigma}, \end{aligned} \tag{47}$$

where

$$\widehat{v}_\sigma = v_\sigma \ \textit{if}\ \sigma \in \mathcal{E}^{(i)}, \quad \widehat{v}_\sigma = \frac{1}{\operatorname{card}(\mathcal{N}_\sigma)} \sum_{\sigma' \in \mathcal{N}_\sigma} v_{\sigma'}\ \textit{otherwise}, \tag{48}$$

$$\textit{where, for any } \sigma \in \mathcal{E} \setminus \mathcal{E}^{(i)},\ \mathcal{N}_\sigma = \{\sigma' \in \mathcal{E}^{(i)}, \overline{D_\sigma} \cap \sigma' \neq \emptyset\}. \tag{49}$$

Then there exists $C \geq 0$, depending only on the regularity of the mesh defined by (6)*, such that, for any $v \in L^2(\Omega)$, and any $i, j = 1, \ldots, d$, $\|\mathcal{R}_{\mathcal{E}}^{(i,j)} v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega)}$.*

Proof. Let us prove the bound on $\|\mathcal{R}_{\mathcal{E}}^{(i,j)}\|_{L^2(\Omega)}$ for $d = 2$, $i = 1$ and $j = 2$. Other cases are similar. Let $v \in H_{\mathcal{E},0}^{(i)}$. By definition of $\mathcal{R}_{\mathcal{E}}^{(i,j)} v$, retaining for each $\sigma \in \mathcal{E}_{\mathrm{int}}$ the cells where v_σ is involved and noting that $\left[\frac{1}{4}(a+b+c+d)\right]^2 \leq a^2 + b^2 + c^2 + d^2$, we have:

$$\begin{aligned} \|\mathcal{R}_{\mathcal{E}}^{(i,j)} v\|_{L^2(\Omega)}^2 &\leq \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}}^{(i)} \\ \sigma = K|L}} v_\sigma^2 (|D_{\sigma_K^t}| + |D_{\sigma_K^b}| + |D_{\sigma_L^t}| + |D_{\sigma_L^b}|) \\ &\leq 4\eta^2 \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}}^{(i)} \\ \sigma = K|L}} v_\sigma^2 |D_\sigma| \end{aligned}$$

where $D_{\sigma_K^t}$ (resp. $D_{\sigma_K^b}$ denotes the velocity cell associated to the top (resp. bottom) edge of K, with $\sigma = K|L$, see Figure 3.

□

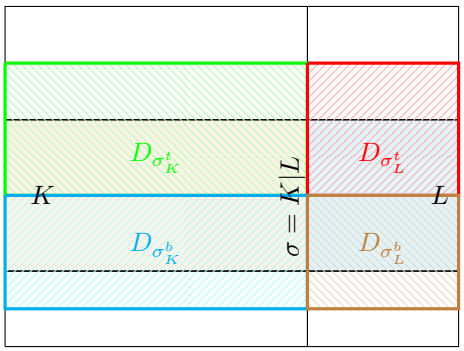


FIGURE 3. Full grid velocity interpolate.

Lemma 3.6 (Weak consistency of the nonlinear convection term)**.** *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$, with $\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n)$ be a sequence of meshes such that $h_{\mathcal{M}_n} = \max_{K\in\mathcal{M}_n} \mathrm{diam}(K) \to 0$ as $n \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_n} \leq \eta$ for any $n \in \mathbb{N}$ (with $\eta_{\mathcal{M}_n}$ defined by (6)). Let $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ be two sequences of functions such that*

- *$\boldsymbol{v}_n \in \mathbf{H}_{\mathcal{E}_n,0}$ and $\boldsymbol{w}_n \in \mathbf{H}_{\mathcal{E}_n,0}$,*
- *the sequences $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ converge in $L^2(\Omega)^d$ to $\bar{\boldsymbol{v}} \in L^2(\Omega)^d$ and $\bar{\boldsymbol{w}} \in L^2(\Omega)^d$ respectively.*

Let $(\Pi_{\mathcal{E}_n})_{n\in\mathbb{N}}$ be a family of interpolators satisfying (12) and let $\boldsymbol{\varphi} \in C_c^\infty(\Omega)^d$. Then $b_{\mathcal{E}}(\boldsymbol{v}_n, \boldsymbol{w}_n, \Pi_{\mathcal{E}_n}\boldsymbol{\varphi}) \to b(\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}}, \boldsymbol{\varphi})$ as $n \to +\infty$.

Proof. Let $i \in [\![1, d]\!]$. We have: $b_{\mathcal{E}_n}(\boldsymbol{v}_n, \boldsymbol{w}_n, \Pi_{\mathcal{E}_n}\boldsymbol{\varphi}) = \sum_{i=1}^d b_{\mathcal{E}}^{(i)}(\boldsymbol{v}, w_i, \Pi_{\mathcal{E}}^{(i)}\varphi_i)$, where we have omitted the sub- and superscripts n for the sake of clarity in the right hand side of the equality, with:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{v}, w_i, \Pi_{\mathcal{E}}^{(i)}\varphi_i) = \sum_{\sigma\in\mathcal{E}^{(i)}} \varphi_{i,\sigma} \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} |\epsilon| v_{\sigma,\epsilon} w_\epsilon = S_1 + S_2,$$

where $\varphi_{i,\sigma} = \varphi_i(\boldsymbol{x}_\sigma)$, with

$$S_1 = \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon\perp\boldsymbol{e}_i, \epsilon\subset K}} |\epsilon| \frac{v_\sigma + v_{\sigma'}}{2} \frac{w_\sigma + w_{\sigma'}}{2} \varphi_{i,\sigma}, \quad S_2 = \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon\not\perp\boldsymbol{e}_i, \epsilon\subset\tau\cup\tau'}} |\epsilon| \frac{v_\tau + v_{\tau'}}{2} \frac{w_\sigma + w_{\sigma'}}{2} \varphi_{i,\sigma},$$

where τ and τ' are the faces of $\mathcal{E}^{(j)}, j \neq i$ such that $\epsilon \subset \tau \cup \tau'$.

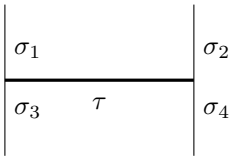
For $K \in \mathcal{M}$ and $\sigma, \sigma' \in \mathcal{E}(K) \cap \mathcal{E}^{(i)}$ we denote by $\tilde{v}_{K,i}$ the mean value $\frac{1}{2}(v_\sigma + v_{\sigma'})$. Reordering over the edges, we get that

$$\begin{aligned} S_1 &= \sum_{\substack{K\in\mathcal{M} \\ K=\overrightarrow{[\sigma\sigma']}}} |\sigma| \tilde{v}_{K,i} \tilde{w}_{K,i} (\varphi_{i,\sigma} - \varphi_{i,\sigma'}) \\ &= \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} |(D_{K,\sigma}\tilde{v}_{K,i} + D_{L,\sigma}\tilde{v}_{L,i}) \eth_i \Pi_{\mathcal{E}}^{(i)}\varphi_i \\ &\to -\int_\Omega \bar{v}_i \; \bar{w}_i \; \partial_i\varphi_i \; \mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty \end{aligned}$$

thanks to the fact that $\tilde{v}_{n,i} \to \bar{u}_i$ and $\tilde{w}_{n,i} \to \bar{w}_i$ in $L^2(\Omega)$, and thanks to Lemma 2.3. Now $S_2 = \sum_{\substack{j\in[1,d]\\ j\neq i}} S_{2,j}$ with

$$S_{2,j} = \sum_{\tau\in\mathcal{E}_{\rm int}^{(j)}} |\tau| \frac{v_\tau}{4} \left[\sum_{k=1}^{4} (w_{\sigma_3} + w_{\sigma_1})\, \varphi_{i,\sigma_1} + (w_{\sigma_4} + w_{\sigma_2})\, \varphi_{i,\sigma_2} - (w_{\sigma_1} + w_{\sigma_3})\, \varphi_{i,\sigma_3} - (w_{\sigma_2} + w_{\sigma_4})\, \varphi_{i,\sigma_4} \right]$$

where $(\sigma_k)_{k=1,\dots,4}$ are the four neighbouring faces (or edges) of τ belonging to $\mathcal{E}^{(i)}$, *i.e.* such that $\bar{\tau} \cap \bar{\sigma}_k \not\emptyset$, see figure on the right.



Thus,

$$S_{2,j} = \sum_{\tau\in\mathcal{E}_{\rm int}^{(j)}} |\tau| \frac{v_\tau}{4} \left[(w_{\sigma_3} + w_{\sigma_1})\, (\varphi_{i,\sigma_1} - \varphi_{i,\sigma_3}) + (w_{\sigma_4} + w_{\sigma_2})\, (\varphi_{i,\sigma_2} - \varphi_{i,\sigma_4}) \right]$$

$$= - \sum_{\tau\in\mathcal{E}^{(j)}} |D_\tau| v_\tau \hat{w}_\tau \partial_j \varphi_i(\boldsymbol{x}_\tau) + R$$

where $|R| \le C_{\boldsymbol{\varphi},\eta} \|v_{n,i}\|_{L^2(\Omega)} \|w_{n,j}\|_{L^2(\Omega)} h_n$, with $C_{\boldsymbol{\varphi},\eta} \ge 0$ depending only on $\boldsymbol{\varphi}$ and η. Hence

$$S_{2,j} \to - \int_\Omega \bar{v}_i\ \bar{w}_j\ \partial_j \varphi_i \ \mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty,$$

which concludes the proof. □

Theorem 3.7 (Convergence of the scheme)**.** *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$, with $\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n)$ be a sequence of meshes such that $h_{\mathcal{M}_n} = \max_{K\in\mathcal{M}_n} \mathrm{diam}(K) \to 0$ as $n \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_n} \le \eta$ for any $n \in \mathbb{N}$ (with $\eta_{\mathcal{M}_n}$ defined by (6)). Let $(\boldsymbol{u}_n, p_n)$ be a solution to the MAC scheme (26) or its weak form (33), for $\mathcal{D} = \mathcal{D}_n$. Then there exists $\bar{\boldsymbol{u}} \in H^1_0(\Omega)^d$ and $\bar{p} \in L^2(\Omega)$ such that, up to a subsequence:*

- *the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\bar{\boldsymbol{u}}$ in $L^2(\Omega)^d$,*
- *the sequence $(\nabla_n \boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\nabla\bar{\boldsymbol{u}}$ in $L^2(\Omega)^{d\times d}$,*
- *the sequence $(p_n)_{n\in\mathbb{N}}$ converges to $\bar{p}$ in $L^2(\Omega)$,*
- *$(\bar{\boldsymbol{u}}, \bar{p})$ is a solution to the weak formulation (2).*

Proof. Thanks to the estimate (42) on the velocity, we can apply the classical translate estimate [10, Theorem 14.2] and the estimates on the translates [10, Theorem 14.1] to obtain the existence of a subsequence of approximate solutions $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ which converges to some $\bar{\boldsymbol{u}} \in L^2(\Omega)^d$. From the estimates on the translates, we also get the regularity of the limit, that is $\bar{\boldsymbol{u}} \in H^1_0(\Omega)^d$. The estimate (44) on the pressure then yields the weak convergence of a subsequence of $(p_n)_{n\in\mathbb{N}}$ to some $\bar{p}$ in $L^2(\Omega)$. Let us then pass to the limit in the scheme in order to prove its (weak) consistency.

Passing to the limit in the mass balance equation: Let $\psi \in C^\infty_c(\Omega)$, taking $\psi_n = \Pi_{\mathcal{M}_n}\psi$ the point-wise interpolate defined by (15) as test function in (33b) and using (16), we get that:

$$0 = \int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \psi_n \ \mathrm{d}\boldsymbol{x} = - \int_\Omega \nabla_{\mathcal{M}_n}\ \psi_n \cdot \boldsymbol{u}_n \ \mathrm{d}\boldsymbol{x} = - \sum_{i=1}^{d} \int_\Omega u_n^{(i)} \eth_i \psi_n \ \mathrm{d}\boldsymbol{x}.$$

Therefore, thanks to Lemma 2.4,

$$0 = \lim_{n\to+\infty} \int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \psi_n \ \mathrm{d}\boldsymbol{x} = - \sum_{i=1}^{d} \int_\Omega \bar{u}^{(i)} \partial_i \psi \ \mathrm{d}\boldsymbol{x} = - \int_\Omega \bar{\boldsymbol{u}} \cdot \nabla\psi \ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}\bar{\boldsymbol{u}}\ \psi \ \mathrm{d}\boldsymbol{x}.$$

so that $\bar{\boldsymbol{u}}$ satisfies (33b).

Passing to the limit in the momentum balance equation: Let $\boldsymbol{\varphi} = (\varphi_1, \cdots, \varphi_d)^t \in (C_c^\infty(\Omega))^d$, and take $\boldsymbol{\varphi}_n = \Pi_{\mathcal{E}_n}\boldsymbol{\varphi} = (\varphi_{n,1}, \cdots, \varphi_{n,d})^t \in \boldsymbol{H}_{\mathcal{E}_n,0}$ as test function in (33a); this yields:

$$\int_\Omega \nabla_{\mathcal{E}_n}\boldsymbol{u}_n : \nabla_{\mathcal{E}_n}\boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\varphi}_n) - \int_\Omega p_n \,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}_n}\boldsymbol{f}\cdot\boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x}. \tag{50}$$

Thanks to the L^2 convergence of $\boldsymbol{u}_n$ to $\bar{\boldsymbol{u}}$, to the weak L^2 convergence of p_n to $\bar{p}$ and to the uniform convergence of $\mathcal{P}_{\mathcal{E}_n}\boldsymbol{f}$ to $\boldsymbol{f}$ and of $\mathrm{div}_{\mathcal{M}_n}\boldsymbol{\varphi}_n$ to $\mathrm{div}\boldsymbol{\varphi}$ (see Lemma 2.3) as $n \to +\infty$, we have

$$\int_\Omega \mathcal{P}_{\mathcal{E}_n}\boldsymbol{f}\cdot\boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{\varphi}} \,\mathrm{d}\boldsymbol{x} \text{ and } \int_\Omega p_n \,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \bar{p}\,\mathrm{div}\,\bar{\boldsymbol{\varphi}} \,\mathrm{d}\boldsymbol{x} \text{ as } n \to \infty$$

From [10, Proof of Theorem 9.1], we get that

$$\int_\Omega \nabla_{\mathcal{E}_n} u_{n,i} : \nabla_{\mathcal{E}_n}\varphi_{n,i} \,\mathrm{d}\boldsymbol{x} = [u_{n,i}, \varphi_{n,i}]_{1,\mathcal{E}_n^{(i)},0} \to -\int_\Omega \bar{u}_i \Delta\varphi_i \,\mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

and therefore

$$\int_\Omega \nabla_{\mathcal{E}_n}\boldsymbol{u}_n : \nabla_{\mathcal{E}_n}\boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x} \to -\sum_{i=1}^d \int_\Omega \bar{u}_i\Delta\varphi_i \,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\bar{\boldsymbol{u}} : \nabla\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

By Lemma 3.6, we have

$$\lim_{n\to+\infty} b_{\mathcal{E}_n}(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\varphi}_n) = \int_\Omega (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}}\cdot\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}. \tag{51}$$

Passing to the limit as $n \to +\infty$ in (50) thus yields that $\bar{\boldsymbol{u}}$ and $\bar{p}$ satisfy (2).

Let us now prove the strong convergence of $\nabla_{\mathcal{E}_n}\boldsymbol{u}_n$ to $\nabla\bar{\boldsymbol{u}}$ in $L^2(\Omega)$. The sequence $(\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)^{d\times d}$ and therefore, there exists $\xi \in L^2(\Omega)^{d\times d}$ and a subsequence still denoted by $(\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converging to ξ weakly in $L^2(\Omega)^{d\times d}$. Since $\int_\Omega \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{u}_n\ \boldsymbol{\varphi}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}_n\boldsymbol{\varphi}_n\ \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x}$, the uniqueness of the limit in the sense of distributions implies that $\nabla\bar{\boldsymbol{u}} = \xi$. Taking $\boldsymbol{\varphi}_n = \boldsymbol{u}_n$ in (50) this yields:

$$\int \nabla_{\mathcal{E}_n}\boldsymbol{u}_n : \nabla_{\mathcal{E}_n}\boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}_n}\boldsymbol{f}\cdot\boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x}.$$

Passing to the limit as $n \to \infty$ we get that:

$$\|\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{u}_n\|^2_{L^2(\Omega)^{d\times d}} = \|\boldsymbol{u}_n\|^2_{1,\mathcal{E}_n,0} \to \int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{u}} \,\mathrm{d}\boldsymbol{x} = \|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|^2_{L^2(\Omega)^{d\times d}},$$

which implies the strong convergence of the discrete gradient of the velocity.

Let us finally prove the strong convergence of the pressure. Let $\boldsymbol{\varphi}_n \in (H_0^1(\Omega))^d$ be such that $\mathrm{div}\boldsymbol{\varphi}_n = p_n$ a.e. in Ω and

$$\|\boldsymbol{\varphi}_n\|_{H_0^1(\Omega)^d} \le c\|p_n\|_{L^2(\Omega)},$$

where c depends only on Ω. Let $\boldsymbol{\psi}_n = \widetilde{\mathcal{P}}_{\mathcal{E}_n}\boldsymbol{\varphi}_n$; thanks to Lemma 3.3, we have $\|\boldsymbol{\psi}_n\|_{1,\mathcal{E}_n,0} \le c\ C_{\eta_n}\|p_n\|_{L^2(\Omega)}$, and since p_n is piecewise constant, we get that $\mathrm{div}_{\mathcal{M}_n}\boldsymbol{\psi}_n = p_n$, Therefore, taking $\boldsymbol{\psi}_n = \widetilde{\mathcal{P}}_{\mathcal{E}_n}\boldsymbol{\varphi}_n$ as test function in (33a), we obtain:

$$\int_\Omega p_n^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla_{\mathcal{E}_n}\boldsymbol{u}_n : \nabla_{\mathcal{E}_n}\boldsymbol{\psi}_n \,\mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}_n) - \int_\Omega \mathcal{P}_{\mathcal{E}_n}\boldsymbol{f}\cdot\boldsymbol{\psi}_n \,\mathrm{d}\boldsymbol{x},$$
$$\text{and } \|\boldsymbol{\psi}_n\|_{1,\mathcal{E},0} \le c\ C_{\eta_n}\|p_n\|_{L^2(\Omega)}.$$

From the bound on $\|\boldsymbol{\psi}_n\|_{1,\mathcal{E},0}$ we know that $\boldsymbol{\psi}_n$ converges to some $\boldsymbol{\psi} \in H^1_0(\Omega)^d$ in $L^2(\Omega)$ and thanks to (31) that $\nabla_{\mathcal{E}_n}\boldsymbol{\psi}_n \to \nabla\boldsymbol{\psi}$ weakly in $(L^2(\Omega)^{d\times d})$ as $n \to +\infty$. Passing to the limit as $n \to \infty$ we get that

$$\|p_n\|^2_{L^2(\Omega)} \to \int_\Omega \nabla\bar{\boldsymbol{u}} : \nabla\boldsymbol{\psi}\ \mathrm{d}\boldsymbol{x} + b(\bar{\boldsymbol{u}},\bar{\boldsymbol{u}},\boldsymbol{\psi}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\psi}\ \mathrm{d}\boldsymbol{x}.$$

Since $(\bar{\boldsymbol{u}},\bar{p})$ satisfies (2), this implies that $\|p_n\|_{L^2(\Omega)} \to \|\bar{p}\|_{L^2(\Omega)}$, which in turn yields that $p_n \to \bar{p}$ in $L2(\Omega)$ as $n \to +\infty$. □

Lemma 3.8 (Convergence of the full grid velocity interpolate)**.** *Let $(\mathcal{M}_n,\mathcal{E}_n)_{n\in\mathbb{N}}$ be a sequence MAC meshes such that $h_{\mathcal{M}_n} \to 0$ as $n \to +\infty$, and $(\eta_{\mathcal{M}_n})_{n\in\mathbb{N}}$ remains bounded. Let $\bar{v} \in L^2(\Omega)$, and let $(v_n)_{n\in\mathbb{N}}$ be such that $v_n \in \mathcal{E}_n^{(i)}$ and v_n converges to $\bar{v}$ as $n \to +\infty$ in $L^2(\Omega)$. Let $i,j = 1,\dots,d$ and $\mathcal{R}^{(i,j)}_{\mathcal{E}_n}$ be the full grid velocity reconstruction operator defined by* (47)*. Then $\mathcal{R}^{(i,j)}_{\mathcal{E}_n}v_n \to \bar{v}$ in $L^2(\Omega)$ as $n \to +\infty$.*

Proof. Let $\varphi \in C^\infty_c(\Omega)$. Denoting $\mathcal{R}^{(i,j)}_{\mathcal{E}_n}$ by $\mathcal{R}_n$ and $\mathcal{P}^{(i)}_{\mathcal{E}_n}$ by $\mathcal{P}_n$ for short (recall that $\mathcal{P}^{(i)}_{\mathcal{E}_n}$ is defined by (27)) we have:

$$\|\mathcal{R}_nv_n-\bar{v}\|_{L^2(\Omega)} \le \|\mathcal{R}_nv_n-\mathcal{R}_n\circ\mathcal{P}_n\bar{v}\|_{L^2(\Omega)} + \|\mathcal{R}_n\circ\mathcal{P}_n\bar{v}-\mathcal{R}_n\circ\mathcal{P}_n\varphi\|_{L^2(\Omega)} + \|\mathcal{R}_n\circ\mathcal{P}_n\varphi-\varphi\|_{L^2(\Omega)} + \|\varphi-\bar{v}\|_{L^2(\Omega)}.$$

Since $\mathcal{R}_nv_n = \mathcal{R}_n\circ\mathcal{P}_nv_n$, and thanks to the fact that $\|\mathcal{R}_n\|_{L^2(\Omega)}$ is bounded (see Lemma 3.5) and that $\|\mathcal{P}_n\|_{L^2(\Omega)} \le 1$, we get that there exists $C \ge 0$ such that

$$\|\mathcal{R}_nv_n-\bar{v}\|_{L^2(\Omega)} \le C\|v_n-\bar{v}\|_{L^2(\Omega)} + C\|\bar{v}-\varphi\|_{L^2(\Omega)} + \|\mathcal{R}_n\circ\mathcal{P}_n\varphi-\varphi\|_{L^2(\Omega)} + \|\varphi-\bar{v}\|_{L^2(\Omega)}.$$

Let $\varepsilon > 0$. Let us choose $\varphi \in C^\infty_c(\Omega)$ so that $\|\varphi-\bar{v}\|_{L^2(\Omega)} \le \frac{\varepsilon}{C+1}$. There exists n_1 such that $C\|v_n-\bar{v}\|_{L^2(\Omega)} \le \varepsilon$, $\forall n \ge n_1$, and there exists n_2 such that $\|\mathcal{R}_n\circ\mathcal{P}_n\varphi-\varphi\|_{L^2(\Omega)} \le \varepsilon$, $\forall n \ge n_2$. Therefore, for $n \ge \max(n_1,n_2)$, we get:

$$\|\mathcal{R}_nv_n-\bar{v}\|_{L^2(\Omega)} \le 3\varepsilon,$$

which concludes the proof. □

4. Unsteady case

4.1. **Time discretization**

Let us now turn to the time discretization of the problem (3); we consider a MAC grid $\mathcal{D} = (\mathcal{M},\mathcal{E})$ of Ω in the sense of Definition 2.1, and a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $(0,T)$, and, for the sake of simplicity, a constant time step $\delta t = t_{n+1} - t_n$; hence $t_n = n\delta t$ for $n \in \{0,\cdots,N-1\}$. Let $\{u_\sigma^{(n+1)}, \sigma \in \mathcal{E}^{(i)}, n \in \{0,\cdots,N-1\}\}$ and $\{p_K^{(n+1)}, K \in \mathcal{M}, n \in \{0,\cdots,N-1\})$ be the sets of discrete velocity and pressure unknowns; we define the corresponding piecewise constant functions $\boldsymbol{u} = (u_1,\dots,u_d)$ and p. For the velocities, these constant functions are of the form:

$$u_i = \sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}} u_\sigma^{(n+1)}\chi_{D_\sigma}\chi_{]t_n,t_{n+1}]},$$

where $\chi_{]t_n,t_{n+1}]}$ is the characteristic function of the interval $]t_n,t_{n+1}]$. We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\boldsymbol{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. For the pressure, the constant functions are of the form:

$$p(\boldsymbol{x},t) = p_K^{(n+1)} \text{ for } \boldsymbol{x} \in K \qquad \text{and } t \in]t_n,t_{n+1}].$$

and we denote by $Y_{\mathcal{M},\delta t}$ the space of such piecewise constant functions.

We look for an approximation $(\boldsymbol{u},p)\in \boldsymbol{X}_{\mathcal{E},\delta t}\times Y_{\mathcal{M},\delta t}$ of $(\bar{\boldsymbol{u}},\bar{p})$ solution of the problem (3). For $\sigma\in\mathcal{E}^{(i)}$, $i\in\{1,\cdots,d\}$ the value $u_\sigma^{(n+1)}$ is an expected approximation of $u_i(\boldsymbol{x},t_{n+1})$, for $\boldsymbol{x}\in D_\sigma$, and the value $p_K^{(n+1)}$ is an expected approximation of $p(\boldsymbol{x},t_{n+1})$ for $\boldsymbol{x}\in K$. For a given $\boldsymbol{u}\in\boldsymbol{X}_{\mathcal{E},\delta t}$ associated to the set of discrete velocity unknowns $\{u_\sigma^{(n+1)},\sigma\in\mathcal{E}^{(i)},n\in\{0,\cdots,N-1\}\}$, and for $n\in\{0,\cdots,N-1\}$, we denote by $u_i^{(n)}\in H_{\mathcal{E},0}^{(i)}$ the piecewise constant function defined by $u_i^{(n)}(\boldsymbol{x})=u_\sigma^{(n)}$ for $\boldsymbol{x}\in D_\sigma,\sigma\in\mathcal{E}^{(i)}$, and set $\boldsymbol{u}^{(n)}=(u_1^{(n)},\ldots,u_d^{(n)})^t\in\mathbf{H}_{\mathcal{E}}$. Setting

$$\boldsymbol{u}(\cdot,0)=\sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}}u_\sigma^{(0)}\chi_{D_\sigma}=\mathcal{P}_{\mathcal{E}}\boldsymbol{u}_0,$$

we define the discrete derivative $\eth_t\boldsymbol{u}\in\boldsymbol{X}_{\mathcal{E},\delta t}$ by

$$\eth_t\boldsymbol{u}=\sum_{n=0}^{N-1}\frac{1}{\delta t}(\boldsymbol{u}^{(n+1)}-\boldsymbol{u}^{(n)})\chi_{]t_n,t_{n+1}]}$$

Denoting by $\boldsymbol{u}^{(n)}=\boldsymbol{u}(\cdot,t_n)$ and $p^{(n)}=p(\cdot,t_n)$, the time-implicit MAC scheme for the transient Navier-Stokes reads:

Initialization

$$\boldsymbol{u}^{(0)}=\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{u}_0 \tag{52a}$$

Step $n\geq 0$. Solve for $\boldsymbol{u}^{(n+1)}$ and $p^{(n+1)}$:

$$\boldsymbol{u}^{(n+1)}\in\mathbf{H}_{\mathcal{E},0},\ p^{(n+1)}\in L_{\mathcal{M},0}, \tag{52b}$$

$$\eth_t\boldsymbol{u}^{(n+1)}-\Delta_{\mathcal{E}}\boldsymbol{u}^{(n+1)}+\boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u}^{(n+1)})\boldsymbol{u}^{(n+1)}+\boldsymbol{\nabla}_{\mathcal{E}}\ p^{(n+1)}=\boldsymbol{f}_{\mathcal{E}}^{(n+1)}, \tag{52c}$$

$$\mathrm{div}_{\mathcal{M}}\boldsymbol{u}^{(n+1)}=0, \tag{52d}$$

where for all $n\in\{0,\ldots,N-1\}$, $\boldsymbol{f}_{\mathcal{E}}^{(n+1)}=\mathcal{P}_{\mathcal{E}}\boldsymbol{f}(\cdot,t^{(n+1)})$ (recall that $\mathcal{P}_{\mathcal{E}}$ is the mean value operator defined by (27)). A weak formulation of Step n of the scheme (52) reads:

$$\begin{aligned}&\text{Find } \boldsymbol{u}^{(n+1)}\in\boldsymbol{E}_{\mathcal{E}}\ ;\ n\in\{0,\cdots,N-1\},\ \text{such that, for any } \boldsymbol{v}\in\boldsymbol{E}_{\mathcal{E}},\\ &\int_\Omega\eth_t\boldsymbol{u}^{(n+1)}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}+\int_\Omega\nabla\boldsymbol{u}^{(n+1)}\cdot\nabla\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}+b_{\mathcal{E}}(\boldsymbol{u}^{(n+1)},\boldsymbol{u}^{(n+1)},\boldsymbol{v})=\int_\Omega\boldsymbol{f}_{\mathcal{E}}^{(n+1)}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{53}$$

4.2. **Existence and estimates on the approximation solution**

Lemma 4.1 (Existence and first estimates on the velocity)**.** *There exists at least a solution $\boldsymbol{u}\in\boldsymbol{X}_{\mathcal{M},\delta t}$ satisfying* (52)*. Furthermore, there exists $C>0$ depending only on $\boldsymbol{u}_0$ and $\boldsymbol{f}$ such that any function $\boldsymbol{u}\in\boldsymbol{X}_{\mathcal{M},\delta t}$ satisfying* (52) *satisfies:*

$$\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})}\leq C, \tag{54}$$

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)}\leq C, \tag{55}$$

where $\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})}=\displaystyle\sum_{n=0}^{N-1}\delta t\|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0}$, $\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)}=\max\{\|\boldsymbol{u}^{(n+1)}\|_{L^2(\Omega)^d},\ n\in[\![0,N-1]\!]\}$, *and* $\boldsymbol{u}^{(n)}=\boldsymbol{u}(\cdot,t_n)$.

Proof. We prove the a priori estimates (54) and (55). The existence of a solution then follows by a topological degree argument as for the stationary case.

Let $M \leq N-1$; taking $\boldsymbol{v} = \boldsymbol{u}^{(n+1)}$ in (53), multiplying by δt and summing the result over $n \in \{0, \cdots, M\}$, we obtain thanks to Lemma 3.2 and to the Cauchy-Schwarz inequality:

$$\sum_{n=0}^{M} \sum_{\sigma \in \mathcal{E}^{(i)}} |D_\sigma| u_\sigma^{(n+1)} (u_\sigma^{(n+1)} - u_\sigma^{(n)}) + \sum_{n=0}^{M} \delta t \|u_i^{(n+1)}\|_{1,\mathcal{E}^{(i)},0}^2 \leq \sum_{n=0}^{M} \delta t \|f_i(.,t_{n+1})\|_{L^2(\Omega)} \|u_i(.,t_{n+1})\|_{L^2(\Omega)}.$$

Using the fact that for all $a, b \in \mathbb{R}$, $a(a-b) = \frac{1}{2}(a-b)^2 + \frac{1}{2}a^2 - \frac{1}{2}b^2$ for the first term of the left hand-side and the discrete Poincaré and Young inequalities for the right and side, we get that

$$\|u_i^{(M+1)}\|_{L^2(\Omega)}^2 + \sum_{n=0}^{M} \delta t \|u_i^{(n+1)}\|_{1,\mathcal{E}^{(i)},0}^2 \leq \|u_i^{(0)}\|_{L^2(\Omega)}^2 + C_P^2 \|f^{(i)}\|_{L^2(0,T;L^2(\Omega))}^2,$$

where $C_P > 0$ depends only on Ω. On one hand, this inequality yields the L^∞ estimate (55); on the other hand, taking $M = N-1$ and summing for $i = 1, \ldots, d$, we get the L^2 estimate (54).

□

Next we turn to an estimate on the discrete time derivative. To this end, we introduce the following discrete dual norms on $\mathbf{H}_{\mathcal{E},0}$ and $\boldsymbol{X}_{\mathcal{E},\delta t}$.

$$\begin{aligned} &\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0} \mapsto \|\boldsymbol{v}\|_{\boldsymbol{E}_\mathcal{E}'} = \max\{\left| \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \right| \ ; \ \boldsymbol{\varphi} \in \boldsymbol{E}_\mathcal{E} \text{ and } \|\boldsymbol{\varphi}\|_{1,\mathcal{E},0} \leq 1\}, \\ &\boldsymbol{v} \in \boldsymbol{X}_{\mathcal{E},\delta t} \mapsto \|\boldsymbol{v}\|_{L^{4/3}(0,T;\boldsymbol{E}_\mathcal{E}')} = \left(\sum_{n=0}^{N-1} \delta t \|\boldsymbol{f}^{n+1)}\|_{\boldsymbol{E}_\mathcal{E}'}^{4/3} \right)^{3/4}. \end{aligned} \tag{56}$$

Lemma 4.2 (Estimate on the dual norm of the discrete time derivative)**.** *Let $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution to* (52)*. Then there exists $C > 0$ depending only on $\boldsymbol{u}_0$, Ω, $\eta_\mathcal{M}$ and $\boldsymbol{f}$ such that:*

$$\|\eth_t \boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}_\mathcal{E}')} \leq C.$$

Proof. If $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ is a solution to (52) then $\boldsymbol{u}^{(n+1)} = \boldsymbol{u}(\cdot, t_{n+1}) \in \boldsymbol{E}_\mathcal{E}$ is a solution to (53); taking $\boldsymbol{v} \in \boldsymbol{E}_\mathcal{E}$ such that $\|\boldsymbol{v}\|_{1,\mathcal{E},0} \leq 1$ as test function in (53) we have $\forall n \in \{0, \cdots, N-1\}$:

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla \boldsymbol{u}^{(n+1)} \cdot \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \ b_\mathcal{E}(\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}_\mathcal{E}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

By Lemma 3.2 and thanks to the estimate (34) we have

$$|b_\mathcal{E}(\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v})| \leq C_{\eta_\mathcal{M}} \|\boldsymbol{u}^{(n+1)}\|_{L^4(\Omega))^d}^2 \|\boldsymbol{v}\|_{1,\mathcal{E},0}.$$

Using the Cauchy-Schwarz inequality, we note that

$$\|\boldsymbol{u}^{(n+1)}\|_{L^4(\Omega)^d}^4 = \int_\Omega |\boldsymbol{u}^{(n+1)}| |\boldsymbol{u}^{(n+1)}|^3 \, \mathrm{d}\boldsymbol{x} \leq \|\boldsymbol{u}^{(n+1)}\|_{L^2(\Omega)^d} \|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^3.$$

Therefore, thanks to the estimate (55) of Lemma 4.1,

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \leq C_{\eta_\mathcal{M}} C \|\boldsymbol{u}^{(n+1)}\|_{(L^6(\Omega))^d}^{3/2} + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0} + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{(L^2(\Omega))^d},$$

Hence

$$\begin{aligned} \|\eth_t \boldsymbol{u}^{(n+1)}\|_{\boldsymbol{E}_\mathcal{E}'}^{4/3} &\leq 9 \left((C_{\eta_\mathcal{M}} C)^{\frac{4}{3}} \|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^{4/3} + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{L^2(\Omega)^d}^{4/3} \right) \\ &\leq 9 \left((C_{\eta_\mathcal{M}} C)^{\frac{4}{3}} \|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^2 + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{L^2(\Omega)^d}^2 + 2 \right). \end{aligned}$$

Multiplying this latter inequality by δt and summing for $n = 0, \dots, N-1$, we get

$$\|\boldsymbol{v}\|^{\frac{4}{3}}_{L^{4/3}(0,T;\boldsymbol{E}'_{\mathcal{E}})} \leq 9\left((C_{\eta_{\mathcal{M}}}C)^{\frac{4}{3}}\|\boldsymbol{u}\|^2_{L^2(0,T,L^6(\Omega)^d)} + \|\boldsymbol{u}\|^2_{L^2(0,T,\mathbf{H}_{\mathcal{E},0})} + \|\boldsymbol{f}\|^2_{L^2(0,T,L^2(\Omega)^d)} + 2T\right).$$

We conclude by the discrete Sobolev inequality [10, Lemma 3.5] and thanks to the $L^2(0,T;\mathbf{H}_{\mathcal{E},0})$ estimate on $\boldsymbol{u}$ given in (54). □

4.3. **Convergence analysis**

Theorem 4.3 (Convergence of the scheme)**.** *Let $(\delta t_m)_{m\in\mathbb{N}}$ and $(\mathcal{D}_m)_{m\in\mathbb{N}} = (\mathcal{M}_m, \mathcal{E}_m)_{m\in\mathbb{N}}$ be a sequence of time steps and MAC grids (in the sense of Definition 2.1) such that $\delta t_m \to 0$ and $h_{\mathcal{M}_m} \to 0$ as $m \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_m} \leq \eta$ for any $m \in \mathbb{N}$ (with $\eta_{\mathcal{M}_m}$ defined by (6)). Let $\boldsymbol{u}_m$ be a solution to (53) for $\delta t = \delta t_m$ and $\mathcal{D} = \mathcal{D}_m$. Then there exists $\bar{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$ such that, up to a subsequence:*

- *the sequence $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ converges to $\bar{\boldsymbol{u}}$ in $L^{4/3}(0,T;L^2(\Omega)^d)$,*
- *$\bar{\boldsymbol{u}}$ is a solution to the weak formulation (4).*
- *$\partial_t\bar{\boldsymbol{u}} \in L^{4/3}(0,T;E'(\Omega))$.*

Proof. We proceed in four steps.

First step: compactness in $L^{4/3}(0,T;L^2(\Omega)^d)$.

The first step consists in applying the discrete Aubin-Simon theorem 5.3 in order obtain the existence of subsequence of $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ which converges to $\bar{\boldsymbol{u}}$ in $L^{4/3}((0,T);L^2(\Omega)^d)$. In our setting, we apply Theorem 5.3 with $p = \frac{4}{3}$; the Banach space B of is $L^2(\Omega)^d$, and the spaces X_m and Y_m consist in the space $\mathbf{H}_{\mathcal{E}_m,0}$ endowed with the norms defined respectively in (29) and (56). By [10, Theorem 14.2] and the Kolmogorov compactness theorem (see e.g. [10, Theorem 14.1]) we obtain that $(X_m, Y_m)_{m\in\mathbb{N}}$ is compactly embedded in B in the sense of Definition 5.1. Let us then show that the sequence $(X_m, Y_m)_{m\in\mathbb{N}}$ is compact-continuous in $L^2(\Omega)^d$ in the sense of Definition 5.2. Let $\boldsymbol{u}_m \in \mathbf{H}_{\mathcal{E}_m,0}$ such that $(\|\boldsymbol{u}_m\|_{1,\mathcal{E}_m,0})_{m\in\mathbb{N}}$ is bounded and $\|\boldsymbol{u}_m\|_{\boldsymbol{E}'_m} \to 0$ as $m \to +\infty$. Assume that $\boldsymbol{u}_m \to \widehat{\boldsymbol{u}}$ in $(L^2(\Omega))^d$; by definition (56) of the dual norm, we have

$$\int_\Omega \boldsymbol{u}_m \cdot \boldsymbol{u}_m \, \mathrm{d}\boldsymbol{x} \leq \|\boldsymbol{u}_m\|_{1,\mathcal{E}_m,0}\|\boldsymbol{u}_m\|_{\boldsymbol{E}'_m}.$$

Passing to the limit in this inequality as $m \to \infty$, we get that $\widehat{\boldsymbol{u}} = 0$, so that the sequence $(X_m, Y_m)_{m\in\mathbb{N}}$ is compact-continuous in $L^2(\Omega)^d$. We now check the three assumptions (H1), (H2) and (H3) of Theorem 5.3: By Lemma 4.1, the sequence $\|\boldsymbol{u}_m\|_{L^1(0,T;\mathbf{H}_{\mathcal{E},0})}$ is bounded, and thanks to the discrete Poincaré inequality, the sequence $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ is also bounded in $L^{4/3}(0,T;(L^2(\Omega)^d))$; furthermore, the sequence $\|\delta_t\boldsymbol{u}_m\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\mathcal{E}})}$ is bounded by Lemma 4.2. Hence, Theorem 5.3 applies and there exists $\bar{\boldsymbol{u}} \in L^{4/3}(0,T;L^2(\Omega)^d)$ such that, up to a subsequence,

$$\boldsymbol{u}_m \to \bar{\boldsymbol{u}} \text{ in } L^{4/3}\left(0,T;L^2(\Omega)^d\right) \text{ as } m \to +\infty.$$

Step 2: Convergence in $L^2(\Omega\times(0,T))$.

By Lemma 4.1, the sequence $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T,L^2(\Omega)^d)$, and therefore, there exists $\hat{\boldsymbol{u}} \in L^\infty(0,T;L^2(\Omega)^d)$ and a subsequence $(\boldsymbol{u}_{\phi(m)})_{m\in\mathbb{N}}$ converging to $\hat{\boldsymbol{u}}$ $\star$-weakly in $L^\infty(0,T;L^2(\Omega)^d)$. Since $\boldsymbol{u}_{\phi(m)} \to \bar{\boldsymbol{u}}$ in $L^{4/3}(0,T;L^2(\Omega)^d)$, the uniqueness of the limit in the sense of distributions implies that $\bar{\boldsymbol{u}} = \hat{\boldsymbol{u}}$ so that $\bar{\boldsymbol{u}} \in L^\infty(0,T;L^2(\Omega)^d)$. By a classical interpolation result on $L^p(0,T)$ spaces, we have

$$\|\bar{\boldsymbol{u}} - \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega)^d)} \leq \|\bar{\boldsymbol{u}} - \boldsymbol{u}_m\|^{2/3}_{L^{4/3}(0,T;L^2(\Omega)^d)}\|\bar{\boldsymbol{u}} - \boldsymbol{u}_m\|^{1/3}_{L^\infty(0,T;L^2(\Omega)^d)},$$

which implies that $\boldsymbol{u}_m$ converges towards $\bar{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega)^d)$ as m tends to infinity.

Step 3: Weak consistency of the scheme

The notion of weak consistency that we use here is the Lax-Wendroff notion: we show that if a sequence of approximate solutions of the scheme converges to some limit, then this limit is a weak solution to the original problem. Let us then show that $\bar{\boldsymbol{u}}$ satisfies (4). Let $\boldsymbol{\varphi} \in C^\infty_c(\Omega\times[0,T))^d$, such that $\mathrm{div}\boldsymbol{\varphi} = 0$. By Lemma

3.3, we have $\mathrm{div}_{\mathcal{M}_m}\widetilde{\mathcal{P}}_{\mathcal{E}_m}\boldsymbol{\varphi}(\cdot,t_n)=0$, and so we can take $\boldsymbol{\varphi}_m^{(n)}=\widetilde{\mathcal{P}}_{\mathcal{E}_m}\boldsymbol{\varphi}(\cdot,t_n)\in\boldsymbol{E}_{\mathcal{E}}$ as test function in (53) ; multiplying by δt_m and summing for $n=\{0,\dots,N_m-1\}$ (with $N_m\delta t_m=T$), we then get:

$$\sum_{n=0}^{N_m-1}\delta t_m\left(\int_\Omega \eth_t\boldsymbol{u}_m^{(n+1)}\cdot\boldsymbol{\varphi}_m^{(n)}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}\boldsymbol{\varphi}_m^{(n)}\,\mathrm{d}\boldsymbol{x}+b_{\mathcal{E}_m}(\boldsymbol{u}_m^{(n+1)},\boldsymbol{u}_m^{(n+1)},\boldsymbol{\varphi}_m^{(n)})\right.$$
$$\left.-\int_\Omega\mathcal{P}_{\mathcal{E}_m}\boldsymbol{f}^{(n+1)}\cdot\boldsymbol{\varphi}_m^{(n)}\,\mathrm{d}\boldsymbol{x}\right)=0.$$

The first term of the left handside reads $T_{1m}=\sum_{i=1}^d T_{1m,i}$ with

$$\begin{aligned}T_{1m,i}&=\sum_{n=0}^{N_m-1}\sum_{\sigma\in\mathcal{E}^{(i)}}|D_\sigma|\,(u_{m,\sigma}^{(n+1)}-u_{m,\sigma}^{(n)})\varphi_{m,\sigma}^{(n)}\\&=-\sum_{n=0}^{N_m-1}\delta t\sum_{\sigma\in\mathcal{E}^{(i)}}|D_\sigma|u_{m,\sigma}^{(n+1)}\frac{\varphi_{m,\sigma}^{(n+1)}-\varphi_{m,\sigma}^{(n)}}{\delta t}-\sum_{\sigma\in\mathcal{E}^{(i)}}|D_\sigma|u_{m,\sigma}^{(0)}\,\varphi_{m,\sigma}^{(0)}\\&=-\int_0^T\int_\Omega u_{m,i}(\boldsymbol{x},t)\eth_t\varphi_{m,i}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_\Omega\mathcal{P}_{\mathcal{E}_m}^{(i)}u_{0,i}(\boldsymbol{x})\varphi_m^{(0)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

We know that $u_{m,i}\to\overline{u}_i$ in $L^2(0,T;L^2(\Omega))$ as $m\to+\infty$. By definition, the discrete partial derivative $\eth_t\varphi_{m,i}$ converges uniformly to $\partial\varphi_i$ as $m\to+\infty$. Moreover, $\mathcal{P}_{\mathcal{E}_m}u_{0,i}$ converges to $(\bar{u}_{0,i})$ in $L^q(\Omega)$ for all q in $[1,2]$, and $\varphi_{m,\sigma}^{(0)}$ converges to $\bar{\varphi}_i(\cdot,0)$ in $L^q(\Omega)$ for all q in $[1,\infty]$. Hence

$$T_{1m}\to-\int_0^T\int_\Omega\bar{\boldsymbol{u}}(\boldsymbol{x},t)\cdot\partial_t\boldsymbol{\varphi}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_\Omega\bar{\boldsymbol{u}}_0(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}\text{ as }m\to\infty.\tag{57}$$

Let us then study the second term of the left hand side. We have

$$\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}\boldsymbol{\varphi}_m^{(n)}\,\mathrm{d}\boldsymbol{x}=\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}\boldsymbol{\varphi}_m^{(n+1)}\,\mathrm{d}\boldsymbol{x}+\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}(\boldsymbol{\varphi}_m^{(n)}-\boldsymbol{\varphi}_m^{(n+1)})\,\mathrm{d}\boldsymbol{x}.$$

As in the stationary case, we get that

$$\sum_{n=0}^{N_m-1}\delta t_m\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}\boldsymbol{\varphi}_m^{(n+1)}\,\mathrm{d}\boldsymbol{x}\to\int_0^T\int_\Omega\nabla\bar{\boldsymbol{u}}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\text{ as }m\to+\infty.$$

Moreover, thanks to the regularity of $\boldsymbol{\varphi}$,

$$\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}(\boldsymbol{\varphi}_m^{(n+1)}-\boldsymbol{\varphi}_m^{(n)})\,\mathrm{d}\boldsymbol{x}\le\delta t_m C_{\boldsymbol{\varphi}}\|\boldsymbol{u}_m^{(n+1)}\|_{1,\mathcal{E},0}$$

where C_φ depends only on $\boldsymbol{\varphi}$. We thus get that

$$\sum_{n=0}^{N_m-1}\delta t_m\int_\Omega\nabla_{\mathcal{E}_m}\boldsymbol{u}_m^{(n+1)}:\nabla_{\mathcal{E}_m}(\boldsymbol{\varphi}_m^{(n+1)}-\boldsymbol{\varphi}_m^{(n)})\,\mathrm{d}\boldsymbol{x}\to0\text{ as }m\to+\infty.$$

Similarly, we have

$$\int_\Omega\mathcal{P}_{\mathcal{E}_m}\boldsymbol{f}^{(n+1)}\cdot(\boldsymbol{\varphi}_m^{(n)}-\boldsymbol{\varphi}_m^{(n+1)})\,\mathrm{d}\boldsymbol{x}\le\delta tC_{\boldsymbol{\varphi}}\|\boldsymbol{f}(\cdot,t_{n+1})\|_{L^2(\Omega)}\to0\text{ as }m\to+\infty,$$

so that

$$\sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \mathcal{P}_{\mathcal{E}_m} \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} \to \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \text{ as } m \to +\infty.$$

The convection term is easily dealt with by remarking that Lemma 3.6 implies that

$$\sum_{m=0}^{N-1} \delta t_m b_{\mathcal{E}}(\boldsymbol{u}_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}) \to \int_0^T b(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{\varphi}) \, \mathrm{d}t \text{ as } n \to +\infty.$$

Therefore, $\bar{u}$ is indeed a solution of (4).

Step 4: Regularity of the limit
Thanks to [10, theorems 14.1 and 14.2] the sequence of normed vector spaces $(\mathbf{H}_{\mathcal{E}_m,0}, \| \cdot \|_{1,\mathcal{E}_m,0})_{m\in\mathbb{N}}$ is $L^2(\Omega)^d$-limit-included in $H_0^1(\Omega)^d$ in the sense of Definition 5.4. We have $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ in $L^2(0,T,L^2(\Omega))$ as $m \to \infty$ and $(\|\boldsymbol{u}_m\|_{L^2(0,T;\mathbf{H}_{\mathcal{E}_m,0})})_{m\in\mathbb{N}}$ is bounded thanks to Lemma 4.1. Therefore Theorem 5.5 applies and $\bar{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$.

Let us finally show that $\partial_t \bar{\boldsymbol{u}} \in L^{4/3}(0,T;E'(\Omega))$. Let $\boldsymbol{\varphi} \in C_c^\infty(\Omega\times(0,T))$ such that $\mathrm{div}\boldsymbol{\varphi} = 0$. Let $\boldsymbol{\varphi}_m \in \boldsymbol{E}_{\mathcal{E}_m}$ be defined by

$$\boldsymbol{\varphi}_m(\cdot,t) = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{\varphi}(\cdot,t) \, \mathrm{d}t \text{ for } t \in [t_n, t_{n+1}[.$$

Thanks to Lemma 4.2, there exists $C \geq 0$ depending only on $\boldsymbol{u}_0$, Ω, η and $\boldsymbol{f}$ such that:

$$\int_0^T \int_\Omega \eth_t \boldsymbol{u}_m \cdot \boldsymbol{\varphi}_m \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \leq C \|\boldsymbol{\varphi}_m\|_{L^4(0,T;\mathbf{H}_{\mathcal{E},0})}.$$

By Lemma 3.3, there exists C_2 depending only on η and Ω, such that $\|\boldsymbol{\varphi}_m\|_{L^4(0,T;\mathbf{H}_{\mathcal{E},0})} \leq C_2 \|\boldsymbol{\varphi}\|_{L^4(0,T;\boldsymbol{E}(\Omega)}$, where $\boldsymbol{E}(\Omega)$ is endowed with the H_0^1 norm. Hence passing to the limit as $m \to +\infty$ in a similar way as for T_{1m} in Step 1, we get that

$$\int_0^T \int_\Omega \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \leq C C_2 \|\boldsymbol{\varphi}\|_{L^4(0,T;\boldsymbol{E}(\Omega))}.$$

We then get that $\partial_t \bar{\boldsymbol{u}} \in L^{4/3}(0,T;\boldsymbol{E}'(\Omega))$ by density. □

4.4. **Case of the unsteady Stokes equations**

In the case of the unsteady Stokes equations, that is Problem (3) where we omit the nonlinear convection term in (3b), stronger estimates can be obtained, which entail the weak convergence of the pressure. We assume in this section that $\boldsymbol{u}_0 \in H^1(\Omega)^d$ and that $\mathrm{div}\boldsymbol{u}_0 = 0$, and consider the following weak formulation of the unsteady Stokes problem:

Find $(\bar{\boldsymbol{u}}, \bar{p}) \in L^2(0,T;\boldsymbol{E}(\Omega)) \times L^2(0,T;L^2(\Omega))$ such that $\forall \boldsymbol{\varphi} \in C_c^\infty([0,T[\times\Omega)^d$

$$-\int_0^T \int_\Omega \bar{\boldsymbol{u}}(\boldsymbol{x},t) \cdot \partial_t \boldsymbol{\varphi}(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x},0) \, \mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega \nabla \bar{\boldsymbol{u}}(\boldsymbol{x},t) : \nabla \boldsymbol{\varphi}(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$
$$- \int_0^T \int_\Omega \bar{p} \, \mathrm{div}\boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{f}(\boldsymbol{x},t) \cdot \boldsymbol{\varphi}(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \quad (58)$$

Note that this formulation does not use divergence free test functions as in (4). Indeed, in the case of the Stokes equations, we are able to show the (weak) convergence of the pressure and we thus consider a formulation in which the pressure is present. Note that the two formulations are in fact equivalent.

The scheme We look for an approximation $(\boldsymbol{u}, p) \in \boldsymbol{X}_{\mathcal{E},\delta t} \times Y_{\mathcal{M},\delta t}$ of $(\boldsymbol{u}, p)$ solution to the problem (58); we

consider the time-implicit MAC scheme which reads:

Initialization

$$\boldsymbol{u}^{(0)} = \widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{u}_0 \tag{59a}$$

Step $n \geq 0$. Solve for $\boldsymbol{u}^{(n+1)}$ and $p^{(n+1)}$:

$$\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0},\ p^{(n+1)} \in L_{\mathcal{M}},\ \int_\Omega p^{(n+1)}\ \mathrm{d}\boldsymbol{x} = 0, \tag{59b}$$

$$\eth_t \boldsymbol{u}^{(n+1)} - \Delta_{\mathcal{E}}\boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_{\mathcal{E}}\ p^{(n+1)} = \boldsymbol{f}_{\mathcal{E}}^{(n+1)}, \tag{59c}$$

$$\mathrm{div}_{\mathcal{M}}\boldsymbol{u}^{(n+1)} = 0, \tag{59d}$$

Note that the choice of the discretization of the initial condition in (59a), together with the assumption $\mathrm{div}\boldsymbol{u}_0 = 0$ implies that $\mathrm{div}_{\mathcal{M}}\ \boldsymbol{u}^{(0)} = 0$; this fact is important for the obtention of the estimates. A weak formulation of (59b)–(59d) reads:

$$\text{Find } (\boldsymbol{u}^{(n+1)}, p^{(n+1)}) \in \boldsymbol{E}_{\mathcal{E}} \times L_{\mathcal{M}}\ ;\ \int_\Omega p^{(n+1)}\ \mathrm{d}\boldsymbol{x} = 0,\ \text{and } \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}, \tag{60}$$

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)}\ \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}}\boldsymbol{u}^{(n+1)} : \nabla_{\mathcal{E}}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} - \int_\Omega p^{(n+1)}\mathrm{div}_{\mathcal{M}}\ \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x}. \tag{61}$$

The estimates of Lemma 4.1 on the approximate solutions obtained in the case of the Navier-Stokes equations are of course still valid. However we get stronger estimates on $\eth_t\boldsymbol{u}$ and on p, as we proceed to show.

Lemma 4.4 (Estimates on the discrete time derivative)**.** *Let $\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0}$ be a solution to* (59)*; then there exists $C > 0$ depending only on $\boldsymbol{u}_0$, Ω, $\eta_{\mathcal{M}}$ and $\boldsymbol{f}$ such that:*

$$\|\eth_t\ \boldsymbol{u}\|_{L^2(0,T;L^2(\Omega)^d)} \leq C, \tag{62}$$

$$\|\eth_t\ \boldsymbol{u}\|_{L^\infty(0,T;\mathbf{H}_{\mathcal{E},0})} \leq C. \tag{63}$$

Proof. Let $\boldsymbol{u}^{(n+1)} \in \boldsymbol{E}_{\mathcal{E}}$ be a solution to (59b)–(59d). Taking $\boldsymbol{v} = \eth_t\boldsymbol{u}^{(n+1)}$ in (61) we get:

$$\int_\Omega (\eth_t\boldsymbol{u}^{(n+1)})^2\ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}}\boldsymbol{u}^{(n+1)} : \nabla_{\mathcal{E}}(\eth_t\boldsymbol{u}^{(n+1)})\ \mathrm{d}\boldsymbol{x} - \int_\Omega p^{(n+1)}\ \mathrm{div}_{\mathcal{M}}(\eth_t\boldsymbol{u}^{(n+1)})\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \eth_t\boldsymbol{u}^{(n+1)}\ \mathrm{d}\boldsymbol{x}. \tag{64}$$

By linearity of the discrete time derivative discrete divergence operators, and thanks to by (59d), we get that $\mathrm{div}_{\mathcal{M}}(\eth_t\boldsymbol{u}^{(n+1)}) = \eth_t(\mathrm{div}_{\mathcal{M}}\boldsymbol{u}^{(n+1)}) = 0$. Multiplying (64) by δt and summing the result over $n \in \{0, \cdots, M\}$; $M \leq N-1$ we obtain $T_1 + T_2 = T_3$ where

$$T_1 = \sum_{n=0}^{M} \delta t \int_\Omega (\eth_t\boldsymbol{u}^{(n+1)})^2\ \mathrm{d}\boldsymbol{x},\ T_2 = \sum_{n=0}^{M} \delta t \int_\Omega \nabla_{\mathcal{E}}\boldsymbol{u}^{(n+1)} : \eth_t(\nabla_{\mathcal{E}}\boldsymbol{u}^{(n+1)})\ \mathrm{d}\boldsymbol{x},\ \text{and } T_3 = \sum_{n=0}^{M} \delta t \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \eth_t\boldsymbol{u}^{(n+1)}\ \mathrm{d}\boldsymbol{x}.$$

We have:

$$T_2 = \sum_{n=0}^{M} (\frac{1}{2}\|\boldsymbol{u}^{(n+1)}\|_{1\mathcal{E},0}^2 - \frac{1}{2}\|\boldsymbol{u}^{(n)}\|_{1,\mathcal{E},0}^2 + \frac{1}{2}\|\boldsymbol{u}^{(n+1)} - \boldsymbol{u}^{(n)}\|_{1,\mathcal{E},0}^2) \geq \frac{1}{2}\|\boldsymbol{u}^{M+1}\|_{1,\mathcal{E},0}^2 - \frac{1}{2}\|\boldsymbol{u}_0\|_{1,\mathcal{E},0}^2.$$

By the Cauchy-Schwarz and the Young inequalities we obtain:

$$T_3 \leq \sum_{n=0}^{M} \delta t \big(\int_\Omega |\boldsymbol{f}(.,t_{n+1})|^2\ \mathrm{d}\boldsymbol{x}\big)^{\frac{1}{2}} \big(\int_\Omega (\eth_t\boldsymbol{u}^{(n+1)})^2\ \mathrm{d}\boldsymbol{x}\big)^{\frac{1}{2}} \leq \frac{1}{2}\|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega)^d)}^2 + \frac{1}{2}\sum_{n=0}^{M} \delta t \int_\Omega (\eth_t\boldsymbol{u}^{(n+1)})^2\ \mathrm{d}\boldsymbol{x}.$$

Gathering the above inequalities, we get that:

$$\sum_{n=0}^{M} \delta t \int_\Omega (\eth_t \boldsymbol{u}^{(n+1)})^2 \, \mathrm{d}\boldsymbol{x} + \|\boldsymbol{u}^{M+1}\|_{1,\mathcal{E},0}^2 \leq \|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega)^d)}^2 + \|\boldsymbol{u}_0\|_{1,\mathcal{E},0}^2. \tag{65}$$

This in turn yields the $L^\infty(L^2)$ estimate (63) and the $L^2(L^2)$ estimate (62) (taking $M = N-1$) on the discrete derivative, with $C = \sqrt{2}(\|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega)^d)} + \|\boldsymbol{u}_0\|_{(H^1)^d})$.

□

Lemma 4.5 (Estimate on the pressure)**.** *Let $(\boldsymbol{u}, p) \in \boldsymbol{X}_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t}$ be a solution to* (59)*. There exists $C \geq 0$ depending only on Ω, $\eta_\mathcal{M}$ and $\boldsymbol{f}$ such that:*

$$\|p\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{66}$$

Proof. With the same arguments as in the proof of the pressure estimate in Proposition 3.4, we choose $\boldsymbol{v} = \widetilde{\mathcal{P}}_\mathcal{E} \boldsymbol{\varphi}$ as test function in (61), where $\boldsymbol{\varphi} \in H_0^1(\Omega)^d$ is such that $\mathrm{div}\boldsymbol{\varphi} = p^{(n+1)}$ and $\|\nabla \boldsymbol{\varphi}\|_{L^2(\Omega)^{d\times d}} \leq c\|p^{(n+1)}\|_{L^2(\Omega)}$, with c depending only on Ω. Thanks to (40a) we then obtain:

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_\mathcal{E} \boldsymbol{u}^{(n+1)} : \nabla_\mathcal{E} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} - \|p^{(n+1)}\|_{L^2(\Omega)}^2 = \int_\Omega \boldsymbol{f}_\mathcal{E}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x},$$

Thanks to the Cauchy-Schwarz and Poincaré inequalities and to the estimate (40a) we then get that there exists $C_{\eta_\mathcal{M}}$ depending on Ω and on the regularity of the mesh such that

$$\|p^{(n+1)}\|_{L^2(\Omega)}^2 \leq C_{\eta_\mathcal{M}} \left(\|\eth_t \boldsymbol{u}^{(n+1)}\|_{(L^2(\Omega))^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^2 + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{L^2(\Omega)^d}^2 \right).$$

Summing (4.4) over $n \in \{0, \cdots, N-1\}$ and multiplying by δt yields the result thanks to (54), and (62).

□

Theorem 4.6 (Convergence of the scheme)**.** *Let $(\delta t)_m \in (0,T)$ and $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of meshes such that $(\delta t)_m \to 0$ and $\max_{K\in\mathcal{M}_m} \mathrm{diam}(K) \to 0$ as $m \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_m} \leq \eta$ for any $m \in \mathbb{N}$ (with $\eta_{\mathcal{M}_m}$ defined by* (6)*). Let $(\boldsymbol{u}^{(m)}, p^m)$ be a solution to* (59) *for $(\delta t)_m = \delta t$ and $\mathcal{D} = \mathcal{D}_m$. Then there exists $(\bar{\boldsymbol{u}}, \bar{p}) \in L^2(0,T;\boldsymbol{E}(\Omega)) \times L^2(0,T;L^2(\Omega))$ such that, up to a subsequence:*

- *the sequence $(\boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ converges to $\bar{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega)^d)$,*
- *the sequence $(p_m)_{m\in\mathbb{N}}$ weakly converges to $\bar{p}$ in $\in L^2(0,T;L^2(\Omega))$,*
- *$(\bar{\boldsymbol{u}}, \bar{p})$ is a solution to the weak formulation* (58)*.*

Proof. The convergence of the sequence of discrete solutions of the velocity follow from the Theorem 4.3 and the convergence of the sequence of discrete solutions of the pressure in $L^2(0,T;L^2(\Omega))$ follow from the estimate (66). Let us then show that $(\bar{\boldsymbol{u}}, \bar{p})$ satisfies (58). Let $\boldsymbol{\varphi} \in C_c^\infty(\Omega \times [0,T))^d$. Taking $\boldsymbol{\varphi}_m^{(n)} = \widetilde{\mathcal{P}}_{\mathcal{E}_m} \boldsymbol{\varphi}(\cdot, t_n) \in \mathbf{H}_{\mathcal{E}_m,0}$ as test function in (61), multiplying by δt_m and summing for $n = \{0, \ldots, N_m - 1\}$ (with $N_m \delta t_m = T$), we obtain:

$$\sum_{n=0}^{N_m-1} \delta t_m \left(\int_\Omega \eth_t \boldsymbol{u}_m^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} \right.$$
$$\left. - \int_\Omega \mathcal{P}_{\mathcal{E}_m} \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} \right) = 0.$$

Let us deal with the pressure term, (all other terms of the equation can be dealt with as in the proof of Theorem 4.3). We have:

$$\int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} = \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \boldsymbol{\varphi}_m^{(n+1)} \, \mathrm{d}\boldsymbol{x} + \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} (\boldsymbol{\varphi}_m^{(n)} - \boldsymbol{\varphi}_m^{(n+1)}) \, \mathrm{d}\boldsymbol{x}.$$

By Lemma 3.3 and thanks to the regularity of $\boldsymbol{\varphi}$,

$$\begin{aligned}\int_\Omega p_m^{(n+1)}\mathrm{div}_{\mathcal{M}_m}\ (\boldsymbol{\varphi}^{(n)}-\boldsymbol{\varphi}^{(n+1)})\ \mathrm{d}\boldsymbol{x} &\leq \|p_m^{(n+1)}\|_{L^2(\Omega)}\|\mathrm{div}\ (\boldsymbol{\varphi}^{(n)}-\boldsymbol{\varphi}^{(n+1)})\|_{L^2(\Omega)}\\ &\leq \delta t_m\ C_{\boldsymbol{\varphi}}\|p_m^{(n+1)}\|_{L^2(\Omega)}\\ &\to 0 \text{ as } m\to+\infty.\end{aligned}$$

We proved in the stationary case (see the proof of Theorem 3.7) that

$$\int_\Omega p_m^{(n+1)}\mathrm{div}_{\mathcal{M}_m}\ \boldsymbol{\varphi}_m^{(n+1)}\ \mathrm{d}\boldsymbol{x}\to\int_\Omega \bar{p}\ \mathrm{div}\boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x} \text{ as } m\to\infty,$$

and this concludes the proof that $(\bar{u},\bar{p})$ is indeed a solution of (58). □

5. Appendix: Discrete functional analysis

Definition 5.1 (Compactly embedded sequence of spaces)**.** Let B be a Banach space; a sequence $(X_m)_{m\in\mathbb{N}}$ of Banach spaces included in B is compactly embedded in B if any sequence $(u_m)_{m\in\mathbb{N}}$ satisfying:

- $u_m\in X_m$ $(\forall m\in\mathbb{N})$,
- the sequence $(\|u_m\|_{X_m})_{m\in\mathbb{N}}$ is bounded,

is relatively compact in B.

Definition 5.2 (Compact-continuous sequence of spaces)**.** Let B be a Banach space, and let $(X_m)_{m\in\mathbb{N}}$ and $(Y_m)_{m\in\mathbb{N}}$ be sequences of Banach spaces such that $X_m\subset B$ for $m\in\mathbb{N}$. The sequence $(X_m,Y_m)_{m\in\mathbb{N}}$ is compact-continuous in B if the following conditions are satified:

- The sequence $(X_m)_{m\in\mathbb{N}}$ is compactly embedded in B (see Definition 5.1),
- $X_m\subset Y_m$ (for all $m\in\mathbb{N}$),
- if the sequence $(u_m)_{m\in\mathbb{N}}$ is such that $u_m\in X_m$ (for all $m\in\mathbb{N}$), $(\|u_m\|_{X_m})_{m\in\mathbb{N}}$ is bounded and $\|u_m\|_{Y_m}\to 0$ as $m\to+\infty$, then any subsequence of $(u_m)_{m\in\mathbb{N}}$ converging in B converges to 0 (in B).

The following theorem is proved [4] and is a generalization of a previous work carried out in [15].

Theorem 5.3 (Aubin-Simon Theorem with a sequence of subspaces and a discrete derivative.)**.** *Let $1\leq p<\infty$, let B be a Banach space, and let $(X_m)_{m\in\mathbb{N}}$ and $(Y_m)_{m\in\mathbb{N}}$ be sequences of Banach spaces such that $X_m\subset B$ for $m\in\mathbb{N}$. We assume that the sequence $(X_m,Y_m)_{m\in\mathbb{N}}$ is compact-continuous in B. Let $T>0$ and $(u^{(m)})_{m\in\mathbb{N}}$ be a sequence of $L^p(0,T;B)$ satisfying the following conditions:*

- *(H1) the sequence $(u^{(m)})_{m\in\mathbb{N}}$ is bounded in $L^p(0,T;B)$.*
- *(H2) the sequence $(\|u^{(m)}\|_{L^1(0,T;X_m)})_{m\in\mathbb{N}}$ is bounded.*
- *(H3) the sequence $(\|\partial_t u^{(m)}\|_{L^p(0,T;Y_m)})_{m\in\mathbb{N}}$ is bounded.*

Then there exists $u\in L^p(0,T;B)$ such that, up to a subsequence, $u^{(m)}\to u$ in $L^p(0,T;B)$.

Definition 5.4 (B-limit-included)**.** Let B be a Banach space, $(X_m)_{m\in\mathbb{N}}$ be a sequence of Banach spaces included in B and X be a Banach space included in B. The sequence $(X_m)_{m\in\mathbb{N}}$ is B-limit-included in X if there exists $C\in\mathbb{R}$ such that if u is the limit in B of a subsequence of a sequence $(u_m)_{m\in\mathbb{N}}$ verifying $u_m\in X_m$ and $\|u_m\|_{X_m}\leq 1$, then $u\in X$ and $\|u\|_X\leq C$.

The regularity of a possible limit of approximate solutions may be proved thanks to the theorem which we recall below [16, Theorem B1].

Theorem 5.5 (Regularity of the limit)**.** *Let $1\leq p<\infty$ and $T>0$. Let B be a Banach space, $(X_m)_{m\in\mathbb{N}}$ be a sequence of Banach spaces included in B and B-limit-included in X (where X is a Banach space included in B). Let $T>0$ and, for $m\in\mathbb{N}$, Let $u_m\in L^p(0,T;X_m)$. We assume that the sequence $(\|u_m\|_{L^p(0,T;X_m)})_{m\in\mathbb{N}}$ is bounded and that $u_m\to u$ a.e. as $m\to\infty$. Then $u\in L^p(0,T;X)$.*

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