An Algebra of Granular Temporal Relations for Qualitative Reasoning

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Abstract

In this paper, we propose a qualitative formalism for representing and reasoning about time at different scales. It extends the results of Euzenat [1995] concerning time and overcomes their major limitations, allowing one to reason about relations between points and intervals. Our approach is more expressive than most algebras of temporal relations: for instance, some relations are more relaxed than those in Allen's [1983] algebra, while others are stricter. In particular, it enables the modeling of imprecise, gradual, or intuitive relations, such as "just before" or "almost meet". In addition, we give several results about how a relation changes when considered at different granularities. Finally, we provide an algorithm to compute the algebraic closure of a temporal constraint network in our formalism, which can be used to check its consistency.

1 Introduction

Autonomous agents must often be able to reason about time; the study of qualitative formalisms is a way to achieve this concern. The most popular of such formalisms are algebras of temporal relations, in which the time location of events (called *primitives* or *entities*) are defined relatively to each other, without quantification or measurement. The *point algebra* of Vilain *et al.* [1986] (further studied by Ladkin and Maddux [1994] and Hirsch [1996]) deals with the three elementary relations between points. The *interval algebra* of Allen [1983] characterizes and allows one to reason about the 13 elementary relations between two intervals. Vilain's [1982] *point and interval algebra*, developed by Meiri [1996] and Krokhin and Jonsson [2002], simultaneously extends these two algebras by adding relations between a point and an interval, for a total of 26 elementary relations. In each of these three algebras, *operators* allow one to deduce new relations between entities, and thus to reason about events.

However, these formalisms consider neither data with different precisions nor the concept of multiscale representation and reasoning. This concept allows for the representation of events at various levels of detail, called *granularities*, such as days, weeks, or months. Thanks to granularities, one can organize knowledge hierarchically, and reason about it at different scales. For instance, within the time relations, one can say that an interval A meets another interval B at a coarse granularity (i.e., looking at it with a general point of view), but that A is before B at a fine granularity (i.e., with a closer point of view). The usual algebras cannot process this knowledge without leading to an inconsistency. To solve this problem, we must first determine how relations change when viewed at different granularities. Euzenat [2001] thus defined granular conversion operators describing the transformation of relations, in particular for the point algebra and for the interval algebra.

Nevertheless, the conversion operators of interval relations are not always relevant: if an interval becomes a point when seen at a coarser granularity, no upward conversion is possible, since the relations of the conversion are only between intervals (not between an interval and a point). For instance, consider two intervals $A = [01{:}30, 03{:}10]$ and $B = [03{:}15, 05{:}07]$ during the same day. At the granularity of minutes, A is before B, whereas at the granularity of hours, A meets B. At the granularity of days, A and B are indistinguishable, since they are both included in the same day; the relation between A and B is thus the "equality of points", which exists neither in Allen's algebra nor in the interval conversion table of Euzenat. Consequently, the conditions in which upward conversion is applicable depend, in particular, on the duration of intervals, which is not always known. Hence, Euzenat's upward conversion operator of the interval algebra does not adequately describe relations at all possible granularities: reasoning is impossible when granularities become too coarse.

To overcome the shortcomings of this approach, we propose, in this paper, new conversion operators for point and interval relations. Moreover, we define a multi-scale version of Vilain's [1982] point and interval algebra[1] based on these conversions, with the goal of representing and reasoning about imprecise temporal knowledge, e.g., as expressed by humans, or coming from heterogeneous sources. From

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[1]In fact, we extend a more expressive version of the point and interval algebra, in which the type of an entity is not fixed (and can thus be ambiguous).

this point of view, our formalism can be seen as an alternative to the fuzzy interval algebra of Schockaert *et al.* [2006], which also makes it possible to model imprecise relations. In particular, we are interested in qualitative granularities, i.e., granularities for which no quantitative information (such as the duration of granules) is known, only their relative scales. These qualitative granularities allow more flexibility in the modeling process, and are for example necessary for analyzing a temporal information expressed in natural language, as the granularities are then not exactly known and thus cannot be quantitatively modeled.

The next section recalls the main concepts about temporal granularity that we consider in this paper. Then, in the third section, we introduce granular relations between points and intervals. The fourth section presents our algebra of granular temporal relations and its operators, including the granular conversion ones. In Section 5, we propose a polynomial algorithm to compute the algebraic closure of a granular temporal constraint network, which can be used to check its consistency, in the case of qualitative granularities. The sixth section is a discussion about related work. Finally, the conclusion and future work are presented in Section 7.

2 Preliminaries

2.1 Temporal Granularities

There are many possible approaches to formalize time granularities. Our work is based on the set-theoretic framework of Bettini *et al.* [2000] as presented by Euzenat and Montanari [2005]. Let us recall the basic concepts that we need to build our formalism.

Time is modeled by a *time domain* T, which is a discrete or dense, totally ordered set. Its elements are called *time points*.

Definition 1. A granularity g is a function from a discrete ordered set I_g to the power set of the time domain T such that:

$$\forall i,j,k \in I_g, \quad i<k<j \wedge g_i \neq \varnothing \wedge g_j \neq \varnothing \implies g_k \neq \varnothing$$

$$\forall i,j \in I_g, \quad i<j \implies \forall x \in g_i, \forall y \in g_j,\, x<y$$

Thus, each granularity is a sequence of subsets of the time domain that preserves the natural order of the time points. Sets g_i are called *granules* of the granularity g. By abuse of notation, we consider that T is itself a granularity, viewing it as the function $T : i \in T \mapsto \{i\}$; i.e., $I_T = T$ (even if it is not discrete) and $T_i = \{i\}$.

Bettini *et al.* have defined several relations between granularities; we are interested in two of them in particular.

Definition 2. Let g,h be two granularities over T.

- g is *finer* than h, denoted $g \preceq h$, if and only if $\forall i \in I_g, \exists j \in I_h \colon g_i \subseteq h_j$, i.e., each granule of g is included in a granule of h.
- h is *coarser* than g,[2] denoted $g \sqsubseteq h$, if and only if $\forall j \in I_h, \exists S \subseteq I_g \colon h_j = \bigcup_{i \in S} g_i$, i.e., each granule of h is composed of granules of g.

[2]This relation is called "g groups into h" by Bettini *et al.*

Relations $\preceq$ and $\sqsubseteq$ are partial orders; they give two different characterizations of the intuitive notion of "being more precise", which both require that the two granularities be aligned. If both $g \preceq h$ and $g \sqsubseteq h$ hold, then the set of granules of g is a partition of the set of granules of h.

Bettini *et al.* have also defined two conversion operators between granularities. They associate a granule of a given granularity with the corresponding granules of another granularity. The upward conversion $\uparrow$, which converts the granule to a "less precise" granularity, is defined as:

$$\forall i \in I_g, \quad \uparrow_g^h i = \{\, j \in I_h : g_i \subseteq h_j \,\}$$

The downward conversion $\downarrow$, which converts the granule to a "more precise" granularity, is defined as:

$$\forall j \in I_h, \quad \downarrow_g^h j = \left\{ S \subseteq I_g : h_j = \bigcup_{i \in S} g_i \right\}$$

Proposition 3. *Let g and h be two granularities over T, and let $i \in I_g$ and $j \in I_h$. If $g \preceq h$ (resp. $g \sqsubseteq h$), then $\uparrow_g^h i$ (resp. $\downarrow_g^h j$) contains exactly one element.*

2.2 Gregorian Calendar

To represent relations between events, one first has to choose a specific set of granularities. Among the most useful is the Gregorian calendar, in which the month, year, week, and day scales are granularities (the hour, minute, and second scales can easily be added). For each pair (g,h) of these granularities such that g is (intuitively) "more precise" than h, both $g \preceq h$ and $g \sqsubseteq h$ hold (e.g., we have "months $\preceq$ years" and "months $\sqsubseteq$ years"), except for the pairs weeks-months and weeks-years: since their granules are not aligned, weeks are not finer than months and years, nor are months and years coarser than weeks.

Note that a granularity is not necessarily a partition of the time domain, as there may be gaps. An example is the granularity of business weeks, of which granules are Monday to Friday periods (weekends are excluded); we have "business weeks $\preceq$ weeks" but not "business weeks $\sqsubseteq$ weeks", and "days $\sqsubseteq$ business weeks" but not "days $\preceq$ business weeks" [Bettini *et al.*, 2000].

3 Formalizing Temporal Relations with Granularities

In this section, we define point relations in the context of time granularities, building upon the formal framework of Bettini *et al.* [2000]. Then, we introduce our definitions of granular relations between points and intervals, and finally discuss on the modeling possibilities offered by our framework.

3.1 Granular Point Relations

We start by formally defining granular relations between points, which was not done in the original work of Euzenat [1995]. Thanks to this formalization, we can exhibit sufficient conditions to use our granular conversion tables (and incidentally, Euzenat's). There are three possible *elementary relations* between two time points at a given granularity.

r_{AB}	$a^- R\, a^+$	$b^- R\, b^+$	$a^- R\, b^-$	$a^- R\, b^+$	$a^+ R\, b^-$	$a^+ R\, b^+$
$<$	$=$	$=$	$<$	$<$	$<$	$<$
$\cdot a$	$=$	$<$	$>$	$>$	$>$	$>$
$\cdot b$	$=$	$<$	$<$	$<$	$<$	$<$
b	$<$	$<$	$<$	$<$	$<$	$<$
o	$<$	$<$	$<$	$<$	$>$	$<$
m	$<$	$<$	$<$	$<$	$=$	$<$
d	$<$	$<$	$>$	$<$	$>$	$<$
$\cdot d$	$=$	$<$	$>$	$<$	$>$	$<$
s	$<$	$<$	$=$	$<$	$>$	$<$
$\cdot s$	$=$	$<$	$=$	$<$	$=$	$<$
f	$<$	$<$	$>$	$<$	$>$	$=$
$\cdot f$	$=$	$<$	$>$	$=$	$>$	$=$
e	$<$	$<$	$=$	$<$	$>$	$=$
$=$	$=$	$=$	$=$	$=$	$=$	$=$

Table 1: Definitions of the granular relations between intervals and points from the granular point relations at the same granularity (superscripts are omitted for clarity).

Definition 4. Let $a, b \in T$. The granular point relations "before at granularity g" $<^g$, "equals at granularity g" $=^g$, and "after at granularity g" $>^g$ are defined as:

$$a <^g b \iff \exists i, j \in I_g : i < j \wedge a \in g_i \wedge b \in g_j$$
$$a =^g b \iff \exists i \in I_g : a, b \in g_i$$
$$a >^g b \iff \exists i, j \in I_g : i > j \wedge a \in g_i \wedge b \in g_j$$

We say that a time point a is *representable* at granularity g if $a \in \bigcup_{i \in I_g} g_i$, and we denote it by $a \propto g$. If two time points are representable at granularity g, then the three relations are exhaustive: if $a \propto g$ and $b \propto g$, then $a <^g b$ or $a =^g b$ or $a >^g b$. They are also exclusive, since we cannot have more than one granular elementary relation between two points.

3.2 Granular Point and Interval Relations

We define a time entity E as an ordered pair $(e^-, e^+) \in T^2$ of time points such that $e^- \leq e^+$. This pair represents the set of points that are between e^- and e^+ in T, that is, $\{t \in T : e^- \leq t \leq e^+\}$. A time entity E is said to be representable at a granularity g if $e^- \propto g$ and $e^+ \propto g$; this is denoted by $E \propto g$. At granularity g, E can be of two types: if $e^- =^g e^+$, then E is a point, whereas if $e^- <^g e^+$, then E is an interval.

There are thus three categories of possible elementary relations between two time entities at a given granularity g. First, there are the interval relations: before, meets, overlaps, starts, during, and finishes, denoted by b^g, m^g, o^g, s^g, d^g, and f^g; their inverses, denoted by $\bar{r}^g$ (with r^g one of these relations); and the equals relation, denoted by e^g. Second, there are the point relations: before $<^g$, equals $=^g$, and after $>^g$. Third, there are the point-interval relations: before $\cdot b^g$, during $\cdot d^g$, starts $\cdot s^g$, finishes $\cdot f^g$, and after $\cdot a^g$; and their inverses, the interval-point relations, that we also denote by $\bar{r}^g$, with r^g a point-interval relation.

The definitions of these relations between time entities at the considered granularity are given in Table 1; they are the same as in the non-granular framework, i.e., on the time domain [Krokhin and Jonsson, 2002]. For instance, let $A = (a^-, a^+)$ and $B = (b^-, b^+)$ be two time entities; $A\ m^g\ B$ holds if $a^- <^g a^+$, $b^- <^g b^+$, and $a^+ =^g b^-$. To obtain the definitions of the inverse relations, substitute "$<$" for "$>$" and "$>$" for "$<$" in columns 4 to 7, and swap columns 2 and 3 as well as columns 5 and 6. In the following, we denote by $\mathcal{R}$ the set of all 26 elementary relations.

In many situations, the actual bounds of the entities are unknown. In fact, even the actual elementary relation between two entities is generally unknown: often, several elementary relations are possible, and there is no way to choose. The next definition is used to represent such ambiguities.

Definition 5. A *general relation* R is a set of elementary relations: $R = \{r_1, \ldots, r_n\} \subseteq \mathcal{R}$. We say that R holds at granularity g between two time entities A, B, denoted $A\ R^g\ B$ or $A\ (r_1 \cdots r_n)^g\ B$, if and only if $\bigvee_{i=1}^{n} A\ r_i^g\ B$ holds.

For example, "at granularity h, A is an interval and is before or meets B, or A is a point and is before B which is an interval" is denoted by $A\,(\cdot b\, b\, m)^h\, B$.

3.3 Modeling Temporal Imprecise Relations

Granular relations are generally imprecise. In fact, when granularity changes, relations become ambiguous (see 4.1). Indeed, we can interpret "granular meets" as "almost meets", with a precision that depends on the granularity. For example, we can model "nearly meets" using some granularity and "practically meets" using a finer granularity. Granularities allow one to relax the point and interval relations, which is useful to model real phenomena, such as imperfect synchronizations, and relations expressed in natural, informal language.

Moreover, by combining different granular relations between the same two entities at different granularities, one can model new intuitive and imprecise relations. Most of these combined relations are more restricted than the classic relations. Let g and h be granularities such that g is more precise than h; the intuitive relation "just before" can be modeled by "$A\ (\cdot b)^g\ B$ and $A\ (=)^h\ B$", or by "$A\ (b)^g\ B$ and $A\ (m)^h\ B$", or by "$A\ (\cdot \bar{a})^g\ B$ and $A\ (\cdot \bar{f})^h\ B$", etc., depending on the type of the temporal entities at the two granularities (if the types are not known, a general relation can be used: "$A\ (\cdot b b)^g\ B$ and $A\ (= e)^h\ B$", etc). Depending on the intended application, it may be important to keep in mind that this modeling of "just before" has a stronger meaning than that of common sense. Yet this modeling is relevant, because with natural language, the associated precision of the relations is unknown, as it can vary from person to person. This can be represented in our formalism using qualitative granularities, i.e., not specifying any information about their structure, except for their relative precision with relations $\preceq$ and $\sqsubseteq$. Qualitative granularities allow one to model and use a concept of non-metric proximity.

4 Algebra of Granular Temporal Relations

In this section, we define the operators of our algebra and their rules, which enable the deduction of new relations. Then, we explain the advantages of granular point and interval relations for reasoning.

4.1 Conversion between Granularities

Knowing that a relation R holds at a granularity g between A and B can give information about the relation between A and B at another granularity. For instance, intuitively, if A is before B at some granularity, B cannot be before A at another. However, this information often lacks precision, since

r	$\uparrow r$	$\downarrow r$
b	$b\cdot\bar{a}\cdot b < \cdot\bar{f}\cdot sm =$	b
$\cdot a$	$\cdot a > \cdot f =$	$\cdot a\bar{b}$
$\cdot b$	$\cdot b < \cdot s =$	$\cdot bb$
$<$	$<=$	$< \cdot b\cdot\bar{a}b$
o	$om\cdot ss\cdot\bar{f}\bar{f}e =$	o
m	$m\cdot s\cdot\bar{f} =$	mob
d	$\cdot ss\cdot dd\cdot ff e =$	d
$\cdot d$	$\cdot s\cdot d\cdot f =$	$\cdot dd$
s	$s\cdot se =$	sod
$\cdot s$	$\cdot s =$	$mo\cdot ss\cdot dd\cdot bb$
f	$f\cdot fe =$	$fd\bar{o}$
$\cdot f$	$\cdot f =$	$\bar{m}\bar{o}\cdot dd\cdot ff\cdot a\bar{b}$
e	$e =$	$eo\bar{o}s\bar{s}f\bar{f}d\bar{d}$
$=$	$=$	$\mathcal{R}$ (all)

Table 2: Conversion of relations between points and intervals.

the real relation between the entities is ambiguous. Hence, converting an elementary relation to a different granularity can yield a general relation. We first introduce the conversion table, which shows what information can be deduced from a granularity to another, then present sufficient conditions for this table to be applied.

Definition 6. For each non-inverse elementary relation $r \in \mathcal{R}$, we define the *upward conversion* $\uparrow r$ and the *downward conversion* $\downarrow r$ of r as general relations using Table 2. The conversion of an inverse elementary relation is then defined by $\downarrow \bar{r} = \bigcup_{s\in\downarrow r} \bar{s}$ and $\uparrow \bar{r} = \bigcup_{s\in\uparrow r} \bar{s}$.

Finally, the conversion of a general relation is given by $\uparrow R = \bigcup_{r\in R} \uparrow r$ and $\downarrow R = \bigcup_{r\in R} \downarrow r$.

Theorem 7. *Let A and B be two time entities, g and h two granularities, and R a general relation.*

- *If $g \sqsubseteq h$, then $A\ R^h\ B \implies A\ (\downarrow R)^g\ B$.*
- *If $g \preceq h$, then $A\ R^g\ B \implies A\ (\uparrow R)^h\ B$.*

Proof. We represent each elementary relation between points and intervals as a conjonction of point relations between the entity bounds, using Table 1. Then, combining the property "g is finer than h" or "g is coarser than h" with the definition of granularities, we get a conjunction of disjunctions of point relations between entity bounds, that we can convert back to a set of mutually consistent point and interval relations, thanks to the distributivity between conjunction and disjunction.

For example, if $A\ s^g\ B$, then $b^- =^g a^- <^g a^+ <^g b^+$ and thus there exist $i, j, k \in I_g$ such that $a^-, b^- \in g_i$, $a^+ \in g_j$, $b^+ \in g_k$, and $i < j < k$. According to Definition 1, as $i < j < k$, we find $a^-, b^- < a^+ < b^+$. Since $g \preceq h$, there exist $i', j', k' \in I_h$ such that $g_i \subseteq h_{i'}$, $g_j \subseteq h_{j'}$, and $g_k \subseteq h_{k'}$. Therefore, $a^-, b^- \in h_{i'}$ and thus $a^- =^h b^-$; moreover, by contraposition of the second point of Definition 1, we deduce that $i' \le j' \le k'$. Consequently, $a^- =^h b^- \le^h a^+ \le^h b^+$, which corresponds to 4 possible rows in Table 1. We finally conclude that $A\ (s\cdot se =)^h\ B$.

As for general relations, since $A\ R^g\ B \iff \bigvee_{r\in R} A\ r^g\ B$, using the mechanism described above and noting that disjunction is commutative and associative, it can be shown that the final result is the same. ☐

Operator	Rule
$\circ$	$\exists C \in \mathcal{E}: A\ R^g\ C \wedge C\ S^g\ B \iff A\ (R\circ S)^g\ B$
$\uparrow$	$g \preceq h \wedge A\ R^g\ B \implies A\ (\uparrow R)^h\ B$
$\downarrow$	$g \sqsubseteq h \wedge A\ R^h\ B \implies A\ (\downarrow R)^g\ B$
$\cap$	$A\ R^g\ B \wedge A\ S^g\ B \iff A\ (R\cap S)^g\ B$
$\bar{\cdot}$	$A\ R^g\ B \iff B\ \bar{R}^g\ A$

Table 3: Rules to deduce new relations; A, B are time entities, g, h are granularities, and $R, S \subseteq \mathcal{R}$ are general relations.

As an example of use of this theorem, consider the relation $A\ (\cdot bbm)^h\ B$ mentioned earlier: if $g \sqsubseteq h$, we can conclude that $A\ (mob\cdot b)^g\ B$ by downward conversion.

This theorem shows that granular conversions of relations between time entities have sufficient conditions that are easy to verify.[3] Moreover, we could also define a table of conversion between granularities that are not aligned. However, this is not necessary, because such a conversion would be equivalent to applying an aligned downward conversion followed by an aligned upward conversion, which is almost always possible by using the time domain as an intermediary, thanks to the following property.

Proposition 8. *Let g be a granularity over T.*

- *$T \sqsubseteq g$ always holds.*
- *If $\bigcup_{i\in I_g} g_i = T$ then $T \preceq g$.*

So in the Gregorian calendar, or in granularities that are generally used in natural language, both $T \preceq g$ and $T \sqsubseteq g$ always hold. Consequently, for the majority of applications we envision, our table can be useful even if the conditions are not satisfied, making it possible to perform, for example, a conversion between weeks and months, or weeks and years.

4.2 Reasoning in our Temporal Algebra

Time algebras allow one to make deductions about the relations between time entities, thanks to their operators. Classical temporal algebras have three operators: composition $\circ$, intersection $\cap$, and inversion $\bar{\cdot}$. In our algebra, these operators are only used on relations at the same granularity (see the definition of Meiri [1996]). Moreover, classical composition allows one to deduce the type of time entities (point or interval), if there is an ambiguity. We add to these the operators of granular conversion, $\uparrow$ and $\downarrow$, which allow one to deduce relations from another granularity. Rules to deduce new relations using the operators are summed up in Table 3.

[3] As a side note, these sufficient conditions also apply to Euzenat's conversion table of point relations, and likewise to his conversion table of interval relations—but only as long as the intervals do not become points at a coarser granularity; yet this latter condition requires the knowledge of granule and interval durations.

4.3 Reasoning about Imprecise Relations

As explained in Section 3.3, thanks to the relations defined on several granularities, we can represent, for example, the relation "A is just before B and B finishes soon after" by $A\ b^{mn}\ B$ and $A\ (=)^{hour}\ B$, and the relation "A is really before C" by $A\ (\cdot b\, b)^{hour}\ C$. It is interesting to notice that using our operators, this entails that $B\ b^{mn}\ C$, whereas this conclusion cannot be deduced if we only have $A\ b\ B$ and $A\ b\ C$, as would be the case without granularities. With the point relations or point-interval relations, we can compare the "distances" and use them when reasoning.

Moreover, contrary to what one might think, it is not enough to convert all relations at the finest or the desired granularity and then to compose them only at this single granularity. For instance, let g be a granularity such that $T \preceq g$, and A,B,C time entities such that $A\ (<)^T\ B$, $B\ (>)^T\ C$, $A\ (<>)^T\ C$, and $A\ (=)^g\ B$, $B\ (<>)^g\ C$, $A\ (<>)^g\ C$. With a downward conversion and a composition, we deduce nothing. However, with an upward conversion, we find $B\ (>)^g\ C$; by composition, we have $A\ (>)^g\ C$; and by downward conversion, we deduce $A\ (>)^T\ C$.

By reasoning with coarse granularities, we can take into account the fact that some entity bounds become indistinguishable at a coarse granularity (i.e., they are equal at this granularity) while others do not; this can allow us to deduce their order. For example, if, at a given granularity, a and b are equal but c is not equal to a and b, then c cannot be between a and b at any finer granularity. Therefore, reasoning with all granularities at the same time provides more information and detects more inconsistencies.

5 Consistency

In this section, we are interested in checking the consistency of *constraint networks* of time entities connected by granular temporal relations in the context of qualitative granularities. More precisely, a constraint network is consistent if we can find an instantiation of its variables that satisfies all the constraints. In the context of time algebras, this problem appears, for instance, in planning with temporal constraints [Allen, 1991] or in plan recognition [Song, 1994].

We define what a constraint network is in our context, and give a polynomial algorithm to compute the algebraic closure of a network, which can be used to check its consistency. In this section, we assume that $T \preceq g$ for every granularity g; i.e., we are not interested in granularities with gaps (see Prop. 8), thus all granularities g,h verify $g \preceq h \iff g \sqsubseteq h$.

5.1 Constraint Networks, Algebraic Closure

A constraint network in our framework consists of a set $\mathcal{E} = \{E_1,\dots,E_n\}$ of time entity variables, a set $\mathcal{G} = \{g_1,\dots,g_m\}$ of granularity variables (where g_1 is always the time domain T), and constraints of the form "$E_i\ R^{g_k}\ E_j$" (with $R \subseteq \mathcal{R}$) and of the form "$g_i \preceq g_j$". If the relationship between two granularities is unknown, it can be omitted; but all granularities are necessarily coarser and less fine than the time domain, so the granularities form a half-lattice. The constraint network is a *scenario* if there is an elementary relation as constraint between any two variables at any granularity. A *solution* of a constraint network is an assignment of all variables that satisfies all the constraints, i.e., an assignment of granularities to the g_i such that all relationships $\preceq$ are satisfied, together with an assignment of entities to the E_i such that all granular relations between them are satisfied. A constraint network is said to be *consistent* if it has at least one solution.

Note that these constraint networks use *qualitative granularities*: no quantitative information is used—it is not possible to check the consistency of the constraints for a specific set of granularities, such as the Gregorian calendar, although an inconsistency in the qualitative case implies an inconsistency in the quantitative case. This is notably intended to be used whenever granularities are not known sufficiently precisely (in particular from natural language), and when one needs to relax or approximate the quantitative frame.

Without granularities, the algebraic closure of a constraint network is a constraint network that has the exact same set of solutions and satisfies the following property: $\forall A,B,C \in \mathcal{E}, R_{AC} \subseteq R_{AB} \circ R_{BC}$, where R_{XY} denotes the set of elementary relations between entities X and Y. Generalizing this notion, we say that a granular constraint network is *algebraically closed* if it satisfies $\forall A,B,C \in \mathcal{E}, \forall g \in \mathcal{G}, R^g_{AC} \subseteq R^g_{AB} \circ R^g_{BC}$, and for all $A,B \in \mathcal{E}$ and all $g,h \in \mathcal{G}$, if $g \preceq h$ then $R^h_{AB} \subseteq \uparrow R^g_{AB}$ and $R^g_{AB} \subseteq \downarrow R^h_{AB}$. In other words, applying any of the operators cannot provide additional information.

5.2 Consistency and Algebraic Closure

Checking the consistency of a constraint network is NP-complete for the interval algebra and for the point and interval algebra [Krokhin and Jonsson, 2002]. In this context, a classic way to check consistency is to explore all scenarios of a constraint network while pruning inconsistent cases by applying the algebraic closure, since a scenario is consistent if and only if its algebraic closure does not contain the empty set. This also holds for constraint networks that are not scenarios, for instance, in the case of the point algebra or in the ORD-Horn subclass [Nebel and Bürckert, 1995]. In more general cases, the algebraic closure method detects some inconsistencies, but not all of them. However, in some algebras, computing the algebraic closure is not even sufficient to check the consistency of a scenario [Renz and Ligozat, 2005].

Fortunately, it does not happen in our algebra: while the consistency problem is also NP-complete—since with only one granularity, our algebra boils down to Vilain's point and interval algebra—the following property holds nevertheless.

Proposition 9. *A scenario is consistent if and only if its algebraic closure does not contain the empty set.*

Proof. The main idea is that the bounds of entities are totally ordered at each granularity, since there is no empty set in the algebraic closure. We start by expressing the constraints of the scenario using the granular point algebra. Next, we use the following algorithm to instantiate variables: First, it instantiates each entity bound so that the constraints on the time domain are satisfied. Second, for each granularity, it constructs the granules as intervals $[a,b[$ such that all entity bounds that are equal at this granularity, and only them, are in the same granule, where a is the earliest of these bounds and b is the earliest bound that is after at this granularity. Using the fact that the conversion table is respected and that the algorithm has constructed a correct alignment, we can prove that the granularities respect the relation $\preceq$. □

Thanks to this property, the algebraic closure can notably be used in a search to check the consistency of a constraint network (exhibiting a consistent scenario) [Renz and Ligozat, 2005]. One can use, e.g., the algorithm of Ladkin and Reinefeld [1992], replacing their path consistency algorithm by the algebraic closure algorithm presented in the next section.

5.3 Algebraic Closure Algorithm

Algorithm 1 computes the algebraic closure of a constraint network, and can detect inconsistencies. It takes as input a list L of tuples (R, A, B, g), the constraints, where A and B are entities, g is a granularity, and R is the set of authorized elementary relations between A and B at granularity g. The current knowledge about the relation between A and B at granularity g is registered in variables R^g_{AB}; initially, each one is set to $\mathcal{R}$ (i.e., all elementary relations), and at the end, R^g_{AB} contains the relation between A and B at granularity g in the algebraic closure of the initial constraint network.

The algorithm refines the current R^g_{AB} with the constraints in L, and propagates this new knowledge by calling the procedures `convert` and `compose`. The former converts the current relation to every aligned granularity, while the latter composes the current relation with every other relation at the same granularity. The deduced constraints are then added to L. Note that while we only apply this algorithm to our multi-scale version of the point and interval algebra, it can actually be used with any sub-formalism, such as the corresponding multi-scale version of the point algebra.

Algorithm 1: Computation of the algebraic closure of a granular temporal constraint network.

while *L is not empty* **do**
  $(R, A, B, g) \leftarrow$ `pop` (L)
  $r \leftarrow R \cap R^g_{AB}$
  if $r \neq R^g_{AB}$ **then**
    if $r = \varnothing$ **then**
      There is an inconsistency
    $R^g_{AB} \leftarrow r$
    append $(\overline{R^g_{AB}}, B, A, g)$ to L
    `compose` (A, B, g, L)
    `convert` (A, B, g, L)

Procedure `convert` (A, B, g, L)
  foreach $h \in \mathcal{G}$ *such that* $g \preceq h$ **do**
    append $(\uparrow R^g_{AB}, A, B, h)$ to L
  foreach $h \in \mathcal{G}$ *such that* $h \preceq g$ **do**
    append $(\downarrow R^g_{AB}, A, B, h)$ to L

Procedure `compose` (A, B, g, L)
  foreach $C \in \mathcal{E} \setminus \{A, B\}$ **do**
    append $(R^g_{AB} \circ R^g_{BC}, A, C, g)$ to L

Let us show that Algorithm 1 is polynomial. If there are n time entities, the total number of relations at each granularity is $n(n-1)$. The idea is that a relation R^g_{AB} cannot be modified more than 26 times, since its size can only decrease. Thus, procedures `compose` and `convert` will be called, in the worst case, $26 \cdot mn(n-1)$ times. The former procedure performs $n-2$ compositions, and the latter, $m-1$ conversions in the worst case. Consequently, the number of operations is bounded by $26 \cdot mn(n-1)(n-2+m-1)$. Thus, Algorithm 1 is in $O(mn^2(m+n))$. If m is constant, then the algorithm is in $O(n^3)$, which is the complexity of the algebraic closure algorithm in the classical setting.

5.4 An Example of Reasoning

In this section, we show how to deduce the relation between A and C from "A is overlapped by B, but they are indistinguishable at a coarser point of view", and "B starts C, and they are not equal at the same coarser point of view". The corresponding constraints are $g \preceq h$, $A\ (\bar{o})^g\ B$, $A\ (=)^h\ B$, $B\ (s)^g\ C$, and $B\ (\mathcal{R} \setminus \{e, =\})^h\ C$. By upward conversion of $B\ (s)^g\ C$ and intersection, we find $B\ (\cdot s s)^h\ C$. By composition of $A\ (=)^h\ B$ and $B\ (\cdot s s)^h\ C$, we deduce $A\ (\cdot s)^h\ C$. Next, by downward conversion, we conclude $A\ (m o \cdot s s \cdot d d \cdot b b)^g\ C$. Then by composition between $A\ (\bar{o})^g\ B$ and $B\ (s)^g\ C$, we deduce that $A\ (\bar{o} d f)^g\ C$. Finally, by intersection, we find $A\ (d)^g\ C$. Suppose now that we also know that $A\ (f)^g\ C$; we would deduce $A\ (\varnothing)^g\ C$, and hence prove that the network is inconsistent.

6 Related Work and Discussion

Unlike Euzenat's conversion table of interval relations, we do not need quantitative information, namely the duration of the time entities, to know whether we can apply the conversion operators to a granularity that is *finer* or *coarser*: they can always be used. Since by granularity change an interval can become a point and conversely, it is impossible to reason with granularities that are too coarse using only Euzenat's conversions of interval relations.

Our multi-scale formalism allows for a dense time domain and non-aligned granularities, whose conversions enable a fusion of different precisions. This is more flexible than Euzenat's conversion operators: for instance, an "equals" that becomes a "before" at a finer granularity is inconsistent with the conversion table for intervals, but not with our conversion table, since an "equality" can be a "point equality". The combination of point-interval relations and interval relations at different granularities reduces ambiguity when reasoning (see 4.3). We have the possibility to not specify the type of a time entity (point or interval) and deduce this type. Moreover, even though we do not detail this for space reasons, our conversion operators satisfy the list of desirable properties that any system of granularity conversion operators should verify according to Euzenat [1995].

There are several other approaches of granular time with symbolic constraints. That of Badaloni and Berati [1994] combines numeric and symbolic constraints, but only numeric constraints are converted by granularity change. The intervals are removed from the temporal network if they are points at a coarse granularity. In addition, the network can become inconsistent if the associated granularity is too coarse.

Becher *et al.* [2000] offer a formalism in which granularities are defined implicitly. Indeed, it features relations describing, in particular, that the granularity of the first entity is finer than (resp. coarser than, equal to) the granularity of the second entity, without specifying the two granularities.

Bittner [2002] presents an approach based on the concept of approximation. It features a table of conversion, but it is less precise and less general than ours. In fact, at the coarsest granularity, the type of an entity is not defined, and at the finest granularity, the entity can only be an interval. Another table of conversion is given for non-convex periods. However, the results are limited to two granularities.

Finally, Bettini *et al.* [2000] offer a granular approach with numeric constraints; and anchored time formalisms with granularities, where the time entities are precisely located, have also been proposed and studied [Franceschet and Montanari, 2001; Bettini *et al.*, 2000; Montanari, 1996].

7 Conclusion and Future Work

The addition of granularities to time algebras enables one to combine information with different precisions, to model imprecise relations and to reason about them. Thanks to our generalization of the granular conversion table of interval relations of Euzenat [1995], in which relations between points and intervals are allowed, granular conversions can always be applied with finer or coarser granularities. Moreover, representation is more expressive, since new intuitive relations are available, and deductions are more flexible and precise: the qualitative information of different granularities are fully used. Within our formalism, we can check the consistency of a set of relations between points and intervals defined on several scales and using a concept of qualitative proximity as in natural language. In addition, our work theoretically completes Euzenat's in setting forth sufficient conditions to use his tables of granular conversions (as well as ours), building on the definition of time granularities by Bettini *et al.* [2000].

There are several extension tracks: we are currently searching for a subclass of temporal networks in our algebra for which consistency checking is polynomial. Next, we plan to analyze whether the algebraic closure enforces path consistency when the time domain is dense. We also intend to generalize these ideas to qualitative spatial formalisms. Moreover, in a more general framework, we plan to study the conditions under which polynomial consistency checking can be preserved when adding qualitative granularities to any algebra of relations.

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