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TABLEAU SEQUENCES, OPEN DIAGRAMS, AND BAXTER FAMILIES

SOPHIE BURRILL, JULIEN COURTIEL, ERIC FUSY, STEPHEN MELCZER, AND MARNI MISHNA

Abstract. Walks on Young's lattice of integer partitions encode many objects of algebraic and combinatorial interest. Chen *et al.* established connections between such walks and arc diagrams. We show that walks that start at $\varnothing$, end at a row shape, and only visit partitions of bounded height are in bijection with a new type of arc diagram – open diagrams. Remarkably two subclasses of open diagrams are equinumerous with well known objects: standard Young tableaux of bounded height, and Baxter permutations. We give an explicit combinatorial bijection in the former case.

1. Introduction

The lattice of partition diagrams, where domination is given by inclusion of Ferrers diagrams, is known as Young's lattice. Walks on this lattice are of significant importance since they encode many objects of combinatorial and algebraic interest.

At a first level, a walk on Young's lattice is a sequence of Ferrers diagrams such that at most a box is added or deleted at each step. A class of such sequences is also known as a *tableau family*. There are several combinatorial classes in explicit bijection with tableau families ending in an empty shape, in particular when there are restrictions on the height of the tableaux which appear.

In this work we study tableau families that encode walks in Young's lattice that start at the empty partition and end with a partition composed of a single part: $\lambda = (m)$, $m \geq 0$. Additionally, they are bounded, meaning that they only visit partitions that have at most k parts, for some fixed k. In particular, we generalize the results of Chen *et al.* [12], and Bousquet-Mélou and Xin [7] to prove that two classic combinatorial classes –Young tableaux of bounded height and Baxter permutations – are in bijection with bounded height tableau families.

1.1. Part 1. Oscillating tableaux and Young tableaux of bounded height.

The first tableau family that we study is the set of *oscillating tableaux* with height bounded by k. These appear in the study of partitions avoiding certain nesting and crossing patterns [12]. Our first main result is a bijection connecting oscillating tableaux to the class of standard Young tableau of bounded height. Young tableaux are more commonly associated with oscillating tableau, and ours is a very different connection. This result demonstrates a new facet of the ubiquity of Young tableaux.

Theorem 1. *The set of oscillating tableaux of size n with height bounded by k, which start at the empty partition and end in a row shape $\lambda = (m)$, is in bijection with the set of standard Young tableaux of size n with height bounded by $2k$, with m odd columns.*

The proof of Theorem 1 is an explicit bijection between the two classes. One consequence of the bijective map is the symmetric joint distribution of two kinds of nesting patterns inside the class of involutions.

Enumerative formulas for Young tableaux of bounded height have been known for almost half a century [18, 19, 4], but new enumerative formulas can be derived from Theorem 1, notably an expression which can be written as a diagonal of a multivariate rational function. The new generating function expressions are the subject of Section 4.3.

1.2. Part 2. Hesitating tableaux and Baxter Permutations.

In the second part, we consider the family of *hesitating tableaux*. These tableau sequences appear in studies of set partitions avoiding so-called *enhanced* nesting and crossing patterns. The work of [12] again serves to describe bijections between lattice paths and arc diagrams. Using a lattice path interpretation, we make a generating function argument

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to connect this combinatorial class to Baxter permutations. This connection was recognized by Xin and Zhang [26]. Here we offer an explicit proof, using formulas of Bousquet-Mélou and Xin [7], of the following result.

Theorem 2. *The number of hesitating tableaux of length $2n$ of height strictly less than three is equal to the number B_{n+1} of Baxter permutations of length $n+1$, where*

$$B_n = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}.$$

Baxter numbers have been described as the "big brother" of the well known Catalan numbers: they are the counting series for many combinatorial classes, and these classes often contain natural subclasses which are counted by Catalan numbers. For example, doubly alternating Baxter permutations have a Catalan number counting sequence [21]. One consequence of Theorem 2 is a new two variable generating tree construction for Baxter numbers.

Unlike the results in Part 1, our proof of Theorem 2 is not a combinatorial bijection. One impediment to a bijective proof is a lack of a certain symmetry in the class of hesitating tableaux that is present in most known Baxter classes. A bijection would certainly be of interest, and in fact we conjecture a refinement of Theorem 2, in Conjecture 1, which could guide a combinatorial bijection.

We begin with definitions in Section 2, and some known bijections. Then we focus on the standard Young Tableaux of bounded height in Section 4, followed by our study of Baxter objects in Section 5.

2. The combinatorial classes

We begin with precise definitions for the combinatorial classes that are used in our results.

2.1. Tableaux families.

As mentioned above, a common encoding of walks on Young's lattice is given by sequences of Ferrers diagrams. We consider three variants. Each sequence starts from the empty shape, and has a specified ending shape; the difference between them is the limitations they impose on when one can add or remove a box. The *length* of a sequence is the number of elements, minus one. (It is the number of steps in the corresponding walk.)

A vacillating tableau is an *even* length sequence of Ferrers diagrams $(\lambda^{(0)}, \ldots, \lambda^{(2n)})$ where consecutive elements in the sequence are either the same or differ by one square, under the restriction that $\lambda^{(2i)} \geq \lambda^{(2i+1)}$ and $\lambda^{(2i-1)} \leq \lambda^{(2i)}$.

A hesitating tableau is an even length sequence of Ferrers diagrams $(\lambda^{(0)}, \ldots, \lambda^{(2n)})$ where consecutive differences of elements in the sequence fall under one of the following categories[1] :

- $\lambda^{(2i)} = \lambda^{(2i+1)}$ and $\lambda^{(2i+1)} < \lambda^{(2i+2)}$ (do nothing; add a box)
- $\lambda^{(2i)} > \lambda^{(2i+1)}$and $\lambda^{(2i+1)} = \lambda^{(2i+2)}$ (remove a box; do nothing)
- $\lambda^{(2i)} < \lambda^{(2i+1)}$ and $\lambda^{(2i+1)} > \lambda^{(2i+2)}$ (add a box; remove a box).

An oscillating tableau is simply a sequence of Ferrers diagrams such that at every stage a box is either added or deleted. Remark that the length of the sequence is not necessarily even.

In each case, if no diagram in the sequence is of height $k+1$, we say that the tableau has its *height bounded by k*. Figure 1 shows examples of the different tableaux.

2.2. Lattice walks.

Each integer partition represented as a Ferrers diagram in a tableau sequence can also be represented by a vector of its parts. If the tableau sequence is bounded by k, then a k-tuple is sufficient.

The sequence of vectors defines a lattice path. For example, each of the three tableau families above each directly corresponds to a lattice path family in the region

$$W_k = \{(x_1, x_2, \ldots, x_k) : x_i \in \mathbb{Z}, x_1 \geq x_2 \geq \cdots \geq x_k \geq 0\}$$

starting at the origin $(0, \ldots, 0)$. We can explicitly define three classes of lattice paths by translating the constraints on the tableau families.

[1]Recall $\lambda \leq \mu$ means that $\lambda_i \leq \mu_i$ for all i

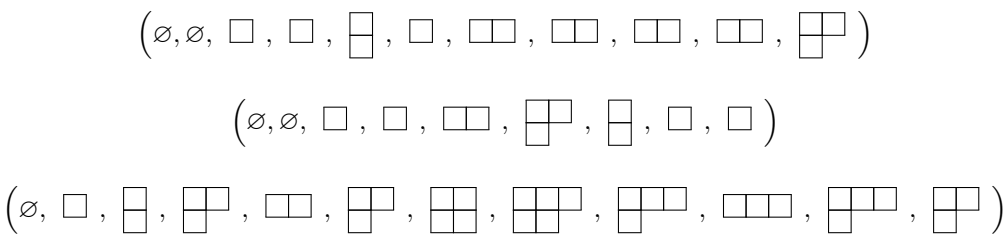

FIGURE 1. *From top to bottom.* A vacillating tableau of length 10, a hesitating tableau of length 8, an oscillating tableau of length 11. In each case, the height is bounded by 2.

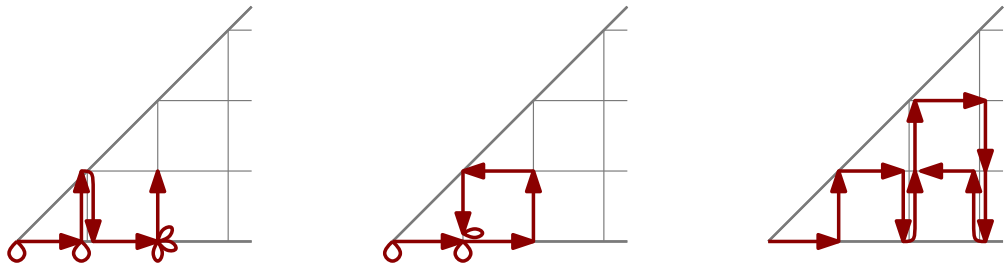

FIGURE 2. *From left to right.* A W_2-vacillating walk, a W_2-hesitating walk, a W_2-oscillating walk. The stay steps are drawn as loops. (These walks correspond to the tableaux of FIGURE 1.)

Remark. Twice in this article, in order to relate previous results, we use a translation of this region and still identify it as W_k. The translated regions are identical to the original up to a small shift of coordinates. This change is detailed explicitly in the text (the allowed sets of steps are never changed).

Let e_i be the elementary basis vector with a 1 at position i and 0 elsewhere. The steps in our lattice model are all elementary vectors, with possibly one exception: the zero vector, also called *stay step*. The length of the walk increases with a stay step, but the position does not change.

A W_k-vacillating walk is a walk of *even* length in W_k using (i) two consecutive stay steps; (ii) a stay step followed by an e_i step; (iii) a $-e_i$ step followed by a stay step; (iv) a $-e_i$ step followed by an e_j step.

A W_k-hesitating walk has even length and steps occur in the following pairs: (i) a stay step followed by an e_i step; (ii) a $-e_i$ step followed by a stay step; (iii) an e_i step follow by $-e_j$ step.

A W_k-oscillating walk starts at the origin and takes steps of type e_i or $-e_i$, for $1 \leq i \leq k$. It does not permit stay steps.

Some examples are depicted in Figure 2.

2.3. **Open arc diagrams.** Arc diagrams are a useful way to provide a graphical representation of a combinatorial class. They are particularly useful to detect certain patterns. Matchings and set partitions are examples of classes that have natural representations using arc diagrams. In the arc diagram representation of a set partition of $\{1, 2, \ldots, n\}$, a row of dots is labelled from 1 to n. A partition block $\{a_1, a_2, \ldots, a_j\}$, ordered $a_1 < a_2 < \ldots < a_j$, is represented by the set of arcs $\{(a_1, a_2), (a_2, a_3), \ldots, (a_{j-1}, a_j)\}$ which are always drawn above the row of dots. We adopt the convention that a part of size one, say $\{i\}$, contributes a loop, that is a trivial arc (i, i). In this work, we do not draw the loops, although some authors do. The set partition $\pi = \{\{1, 3, 7\}, \{2, 8\}, \{4\}, \{5, 6\}\}$ is depicted as an arc diagram in Figure 3. Matchings are represented similarly, with each pair contributing an arc.

A set of k distinct arcs $(i_1, j_1), \ldots, (i_k, j_k)$ forms a *k-crossing* if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. They form an *enhanced k-crossing* if $i_1 < i_2 < \cdots < i_k \leq j_1 < j_2 < \cdots < j_k$. (By convention, an isolated dot of the partition forms an enhanced 1-crossing.) They form a *k-nesting* if $i_1 < i_2 < \cdots < i_k < j_k < \cdots <$

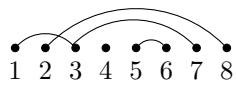

FIGURE 3. The set partition $\pi = \{1,3,7\},\{2,8\},\{4\},\{5,6\}$

$j_2 < j_1$. They form an *enhanced k-nesting* if $i_1 < i_2 < \cdots < i_k \leq j_k < \cdots < j_2 < j_1$ (As previously, $i_k = j_k$ means that i_k is an isolated element in the set partition.). Figure 4 illustrates a 3-nesting, an enhanced 3-nesting, and a 3-crossing.

FIGURE 4. Patterns in arc diagrams. *From left to right:* a 3-crossing, an enhanced 3-crossing, a 3-nesting, an enhanced 3-nesting.

Recently, Burrill, Elizalde, Mishna and Yen [10] generalized arc diagrams by permitting *open arcs*: in these diagrams each arc has a left endpoint but not necessarily a right endpoint. The open arcs can be viewed as arcs "under construction". An *open partition* (resp. an *open matching*) is a set partition (resp. a matching) diagram with open arcs. In open matchings, the left endpoint of an open arc is never the right endpoint of another arc. Figure 5 shows examples of such diagrams.

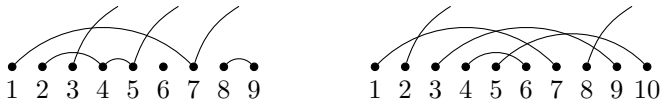

FIGURE 5. An open partition and an open matching.

We are also interested in crossing and nesting patterns in open diagrams. Here we simplify the notation of [10]. A *k-crossing* in an open diagram is either a set of k mutually crossing arcs (as before), or the union of $k-1$ mutually crossing arcs and an open arc whose left endpoint is to the right of the last left endpoint and to the left of the first right endpoint of the $k-1$ crossing arcs. A *k-nesting* in an open diagram is either a set of k mutually nesting arcs, or a set of $k-1$ mutually nesting arcs, and an open arc whose left endpoint is to the left of the $k-1$ nesting arcs. We generalize enhanced k-crossings and enhanced k-nestings in an open diagram similarly. Examples are given in Figure 6. If we want to point out that a crossing (or nesting) has no open arc, we say that it is a *plain* k-crossing (or k-nesting).

FIGURE 6. Patterns in open diagrams. *From left to right:* a 3-crossing, an enhanced 3-crossing, a 3-nesting, and an enhanced 3-nesting.

3. BIJECTIONS

3.1. **Description of Chen, Deng, Du, Stanley, Yan's bijection.** The work of Chen, Deng, Du, Stanley and Yan [12] describes bijections between arc diagram families and tableau families. In this section we summarize a selection of their results, and adapt it to our needs. Their main bijection maps a set partition π to a sequence of Young tableaux[2], the shapes of which form a vacillating tableau, denoted by $\phi(\pi)$. We do

[2]A *Young tableau* is defined here as the filling of a Ferrers diagram with positive integers, such that the entries in each row and in each column are strictly decreasing (usually the entries are increasing; the reason for this change is explained later). The set of entries does not need to form an interval of the form $\{1,\ldots,n\}$.

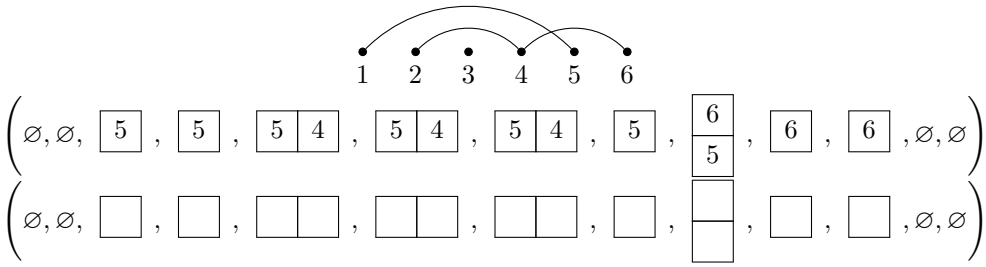

FIGURE 7. *Top.* The set partition $\pi = \{1,5\},\{2,4,6\},\{3\}$. *Middle.* The corresponding Young tableau sequence. *Bottom.* The vacillating tableau given by $\phi(\pi)$.

not describe the generalization of their construction to hesitating tableaux and oscillating tableaux (still due to Chen *et al.*), but it exists and it will be used for the proof of Propositions 6 and 7.

We describe here their bijection ϕ but with a slight difference: we read the arc diagrams from left to right, instead from right to left as it was done originally. In concrete terms, it means that the image of a partition π under ϕ, as we write it, is the mirror image of the actual $\phi(\pi)$. Our approach is justified by the fact that natural properties emerge when the reading direction is swapped. This can be particularly seen through Proposition 3, where the size of the crossings and nestings around the ith dot is linked to the height and the width of the ith Ferrers diagram.

Let π be a set partition of size n. We are going to build from π a sequence of Young tableaux where the entries are decreasing in each row and each column – the fact that we use decreasing order instead of increasing order is a direct consequence of the change of the reading direction. The first entry is the empty Young tableau. We increment a counter i by one from 1 to n. A given step in the algorithm proceeds as follows. If i is the right-hand endpoint of an arc in π, then delete i from the previous tableau (it turns out that i must be in a corner). Otherwise, replicate the previous tableau. Then, after this move, if i is a left-hand endpoint of an arc (i,j) in π, insert j by the Robinson-Schensted-Knuth (RSK) insertion algorithm for the decreasing order into the previous tableau. If i is not a left-hand endpoint, replicate the previous tableau.

The output of this process is a sequence of Young Tableaux starting from and ending at the empty Young tableau. The sequence of shapes is given by a vacillating tableau and is denoted $\phi(\pi)$.

Example. Consider the partition π from Figure 7. The number 1 is the left-hand endpoint of the arc $(1,5)$, but not the right-hand endpoint of any arc, so the first three Young tableaux are $\varnothing, \varnothing, \boxed{5}$. Similarly, 2 is the left-hand endpoint of $(2,4)$ but not a right-hand endpoint, so the two following Young Tableaux are $\boxed{5}$, $\boxed{5}\boxed{4}$. The number 3 is an isolated point, so the tableau $\boxed{5}\boxed{4}$ is repeated twice. The number 4 being the right-hand endpoint of $(2,4)$ and the left-hand endpoint of $(4,6)$, we delete 4, then we add 6: we obtain $\boxed{5}$, $\begin{smallmatrix}\boxed{6}\\ \boxed{5}\end{smallmatrix}$. The rest of the sequence is given in Figure 7.

Given a vacillating tableau $(\varnothing, \lambda_1, \ldots, \lambda_{2n-1}, \varnothing)$, there exists a unique way to fill the entries of the Ferrers diagrams into Young tableaux so that it corresponds to an image of a set partition. This has been proved in [12], and implies that ϕ is a bijection.

In an arc diagram, we say that the *segment* $[i, i+1]$ *is below* a k-crossing if the arc diagram contains k arcs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $i_1 < i_2 < \cdots < i_k \leq i$ and $i+1 \leq j_1 < j_2 < \cdots < j_k$. Similarly, the *segment* $[i, i+1]$ *is below* a k-nesting if there exist k arcs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $i_1 < i_2 < \cdots < i_k \leq i$ and $i+1 \leq j_k < \cdots < j_2 < j_1$. For instance, in Figure 7, the segment $[3,4]$ is below a 2-nesting but not below a 2-crossing, while the segment $[4,5]$ is below a 2-crossing but not below a 2-nesting. With this definition we can formulate and prove a stronger version of [12, Theorem 3.2] (this property can also easily be seen in the growth diagram formulation of the bijection – see [23]).

Proposition 3. *Let π be a partition of size n and $\phi(\pi) = (\lambda_0, \dots, \lambda_{2n})$. For every $i \in \{1, \dots, n\}$, the segment $[i, i+1]$ of π is below a k-crossing (resp. k-nesting) if and only if λ_{2i} in $\phi(\pi)$ has at least k rows (resp. k columns).*

Example. We continue our example and verify that $\lambda^{(6)} = \square\square$ has 2 columns but not 2 rows, and accordingly $[3, 4]$ is below a 2-nesting, but not a 2-crossing.

Proof. Let $(T_0, \dots, T_{2n})$ be the sequence of Young tableaux corresponding to the partition π. We use some ingredients from the proof of Theorem 3.2 of [12, p. 1562][3]:

(1) A pair (i, j) is an arc in the representation of π if and only if j is an entry in $T_{2i}, T_{2i+1}, \dots, T_{2(j-1)}$;
(2) Let $\sigma_i = w_1 w_2 \dots w_r$ denote the permutation of the entries of T_i such that $w_1, w_2, \dots, w_r$ have been inserted in $(T_0, \dots, T_{2n})$ in this order;
(3) The permutation σ_i has an increasing subsequence of length k if and only if the partition λ_i has at least k rows.

The following statements are then equivalent:

- The segment $[i, i+1]$ is below a k-crossing.
- There exist k arcs $(i_1, j_1), \dots, (i_k, j_k)$ in π such that
$$i_1 < i_2 < \dots < i_k \leq i \text{ and } i+1 \leq j_1 < j_2 < \dots < j_k.$$
- There exist k numbers $j_1 < j_2 < \dots < j_k$ that are entries of T_{2i} such that $j_1, j_2, \dots, j_k$ have been inserted in this order in $(T_0, \dots, T_{2n})$.
- There exist k numbers $j_1 < j_2 < \dots < j_k$ such that $j_1 j_2 \dots j_k$ is a subsequence of σ_{2i}.
- The diagram λ_{2i} has at least k rows.

The proof for k-crossings is similar. □

Considering all intervals $[i, i+1]$ for $1 \leq i \leq n$, we recover the statement of Theorem 3.2 from [12].

Corollary 4. *A set partition π has no $(k+1)$-crossing (resp. no $(k+1)$-nesting) if and only if no Ferrers diagram in the sequence $\phi(\pi)$ has $k+1$ rows (resp. columns).*

Remark. The *crossing level* of a set partition π, denoted $cr(\pi)$, is the maximal k such that π has a k-crossing. Similarly, the *nesting level* of a set partition π, denoted $ne(\pi)$, is the maximal k such that π has a k-nesting. Chen *et al.* conclude from the previous corollary that the joint distribution of cr and ne over all the set partition diagrams of fixed size is symmetric. That is,

$$\sum_{\substack{\pi \text{ set partition diagram} \\ \text{of size } n}} x^{cr(\pi)} y^{ne(\pi)} = \sum_{\substack{\pi \text{ set partition diagram} \\ \text{of size } n}} y^{cr(\pi)} x^{ne(\pi)}.$$

Let τ denote transposition, the operation that transposes every Ferrers diagram inside a vacillating tableau. Then $\phi^{-1} \circ \tau \circ \phi$ swaps the crossing level and the nesting level of a set partition. Moreover, note that $\phi^{-1} \circ \tau \circ \phi$ preserves the opener/closer sequence, i.e., if the number i is an isolated point (resp. a left endpoint, a right endpoint, a left and right endpoint at the same time) in a partition π, then i is an isolated point (resp. a left endpoint, a right endpoint, a left and right endpoint at the same time) in $\phi^{-1} \circ \tau \circ \phi(\pi)$.

3.2. **Bijections with open partitions.** Next we describe a generalization of the bijection of Chen *et al.* to the class of tableaux ending at a row shape. We thereby link to the classes of Section 2.

Proposition 5. *A bijection can be constructed between any two of the following classes:*

(1) the set of open partition diagrams of length n with no $(k+1)$-crossing, with m open arcs;
(2) the set of open partition diagrams of length n with no $(k+1)$-nesting, with m open arcs;
(3) the set of vacillating tableaux of length $2n$, with maximum height bounded by k, ending in a row of length m;
(4) the set of W_k-vacillating walks of length $2n$ ending at $(m, 0, \dots, 0)$.

[3]Recall that one bijection is the mirror image of the other. So the indices differ between [12] and here.

Proof. **Bijection (1) ⇔ (3).** Since we would like to use the aforementioned bijection ϕ, the idea here simply consists in closing every open arc in a very natural way – we thereby recover classic closed diagrams.

Let us be more precise. Let π be an open partition diagram of length n with m open arcs and no $(k+1)$-crossing. We build a new partition diagram $\overline{\pi}$ of length $n+m$ without open arcs by closing the m open arcs of π in decreasing order. That is, if $i_1 < i_2 < \cdots < i_m$ denote the positions of the m open arcs of π, the partition $\overline{\pi}$ is the closure, obtained by replacing the m open arcs with the arcs $(i_1, n+m), (i_2, n+m-1), \ldots, (i_m, n+1)$, as shown in Figure 8. Note that the open arcs are closed in such a way that no new crossing is created.

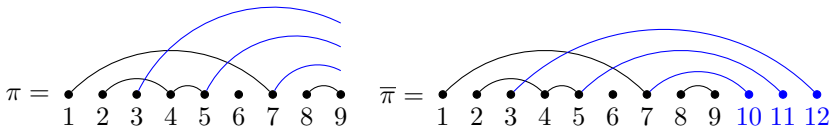

FIGURE 8. *Left.* An open partition diagram, π, with 3 open arcs. *Right.* The corresponding closed partition diagram $\overline{\pi}$ ending with a 3-nesting, obtained by closing the 3 open arcs of π in reverse order.

The m last elements of $\overline{\pi}$ form the end of an m-nesting. Consequently, each crossing of $\overline{\pi}$ has at most one element inside $\{n+1, \ldots, n+m\}$; so the preimage of any ℓ-crossing of $\overline{\pi}$ is also an ℓ-crossing. As π has no $(k+1)$-crossing, the diagram $\overline{\pi}$ has no $(k+1)$-crossing.

Let $\phi(\overline{\pi}) = (\lambda_0, \ldots, \lambda_{2(n+m)})$ be the image of $\overline{\pi}$ under ϕ. By Corollary 4, the height of this vacillating tableau is bounded by k. Moreover, the segment $[n, n+1]$ in $\overline{\pi}$ is below an m-nesting but not below a 2-crossing. By Proposition 3, it means that λ_{2n} is a column with at least m rows. Since $\phi(\overline{\pi})$ ends with an empty diagram and one can delete at most one cell every two steps, λ_{2n} has exactly m rows. Thus, $(\lambda_0, \ldots, \lambda_{2n})$ is a vacillating tableau of length $2n$, with maximum height bounded by k, ending in a column of length m.

The transformation is bijective: a vacillating tableau $(\lambda_0, \ldots, \lambda_{2n})$ from the set (3) can be concatenated with $((m-2), (m-2), \ldots, (1), \varnothing, \varnothing)$, where (j) denotes the partition of i only composed of a single part of size j. If we change its preimage under ϕ by opening the arcs ending in $\{n+1, \ldots, n+m\}$ into m open arcs, we recover the initial open diagram π.

Bijection (2) ⇔ (3). The previous bijection is adapted with an additional application of the transposition operator τ.

Bijection (3) ⇔ (4). This is a straightforward consequence of the encoding. As the vacillating tableaux end at a row of length m, the endpoints of the walks must be the point $(m, 0, \ldots, 0)$ □

The open diagram case inherits many properties from the closed diagram case. For example, the statistics of crossing level and nesting level are equidistributed. Also, the problem of finding a direct bijection between open partitions with no k-crossing and open partitions with no k-nesting without going through the vacillating tableaux seems to be as difficult as the closed case.

However, the nesting level and the crossing level do not have symmetric joint distribution for open partitions. This constitutes a difference with the (closed) partition diagrams.

Furthermore, the other generalizations of Chen *et al.* – specifically the ones that concern the hesitating and oscillating tableaux – can also be extended to tableaux ending at a row shape, and open partitions. The proofs are similar.

Proposition 6. *The following classes are in bijection:*

(1) the set of open matching diagrams of length n with no $(k+1)$-crossing, with m open arcs;
(2) the set of open matching diagrams of length n with no $(k+1)$-nesting, with m open arcs;
(3) the set of oscillating tableaux of length n, with height bounded by k, ending in a row of length m;
(4) the set of W_k-oscillating walks of length n ending at $(m, 0, \ldots, 0)$.

Proposition 7. *The following classes are in bijection:*

(1) the set of open partition diagrams of length n with no enhanced $(k+1)$-crossing, with m open arcs;
(2) the set of open partition diagrams of length n with no enhanced $(k+1)$-nesting, with m open arcs;

(3) the set of hesitating tableaux of length $2n$, with height bounded by k, ending in a row of length m;
(4) the set of W_k-hesitating walks of length $2n$ ending at $(m, 0, \dots, 0)$.

4. Young tableaux, involutions and open matchings

4.1. **Bijections.** We can now prove our first main result, namely Theorem 1. Our strategy is to use Proposition 6, and prove the following result, from which Theorem 1 is a straightforward consequence.

Proposition 8. *The set of standard Young tableaux of size n with height bounded by $2k$ and m odd columns are in bijection with the set of open matching diagrams of length n, with m open arcs and with no $(k+1)$-crossing.*

As far as we can tell, this theorem was first conjectured by Burrill [9][4]. Our proof uses the Robinson-Schensted-Knuth (RSK) correspondence, and the bijection of Chen *et al.*.

A different proof was communicated to us by Christian Krattenthaler [24]. It relies on the RSK correspondence like our proof, but also on *jeu de taquin* (an operation on Young tableaux invented by Schützenberger [25]). We note that the two bijections differ: our bijection has the advantage of preserving – just like the Chen *et al.* construction – the "opener/closer" sequence (in a formulation using diagrams on both sides of the bijection; cf Lemma 11 for more details), a strong property which does not clearly appear in Krattenthaler's alternative. His proof passes through growth diagrams [23].

The following lemma presents a classic property of the RSK correspondence.

Lemma 9. *(Robinson-Schensted-Knuth correspondence) The set of standard Young tableaux of size n with height bounded by k and m odd columns is in bijection with involutions of size n with m fixed points and no decreasing subsequence of length $k+1$.*

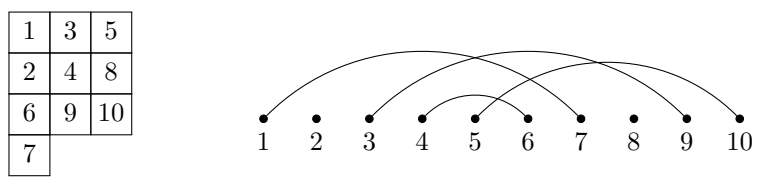

Figure 9. *Left.* A standard Young tableau Y of size 10. *Right.* The arc diagram representation of the involution $(1\ 7)(3\ 9)(4\ 6)(5\ 10)$. This involution is the image of (Y, Y) under the RSK correspondence.

As a first step, Lemma 9 yields combinatorial objects that are close to open matchings. Indeed, involutions have a very natural arc diagram representation: cycles $(i\,j)$ are represented by an arc, and fixed points are isolated dots. An example is shown in Figure 9. We can map involutions into the set of open matchings by simply changing every isolated point into an open arc. Under this map, there is a simple correspondence between decreasing sequences in an involution and nestings in the open diagram.

Lemma 10. *Let $k \in \mathbb{Z}_{\geq 1}$. An involution has no decreasing subsequence of length $2k+1$ if and only if there is no enhanced k-nesting in its arc diagram representation.*

Proof. Let α be an involution. If its arc diagram has an enhanced k-nesting then α contains k cycles $(i_1\, j_1), \dots, (i_k\, j_k)$ that satisfy $i_1 < i_2 < \cdots < i_k \leq j_k < \cdots < j_1$, which clearly induces a decreasing subsequence of length $2k-1$.

Conversely, assume that there exist $2k-1$ numbers $i_1 < i_2 < \cdots < i_{2k-1}$ such that $\alpha(i_{2k-1}) < \cdots < \alpha(i_1)$. If $\alpha(i_k) - i_k \geq 0$, then $i_1 < \cdots < i_k \leq \alpha(i_k) < \cdots < \alpha(i_1)$: this means that $(i_1, \alpha(i_1)), \dots, (i_k, \alpha(i_k))$ form an enhanced k-nesting. Otherwise, $\alpha(i_k) - i_k \leq 0$. Thus $\alpha(i_{2k-1}) < \cdots < \alpha(i_k) \leq i_k < \cdots < i_{2k-1}$: the arcs $(\alpha(i_{2k-1}), i_{2k-1}), \dots, (\alpha(i_k), i_k)$ form an enhanced k-nesting. □

By the two preceding lemmas, the proof of Proposition 8 is reduced to the proof that involution diagrams of length n with m fixed points and no enhanced $(k+1)$-nesting are in bijection with open matching diagrams of length n with m open arcs and no $(k+1)$-crossing. This is established by the following lemma.

[4]More precisely, this conjecture used open matchings with no $(k+1)$-nesting.

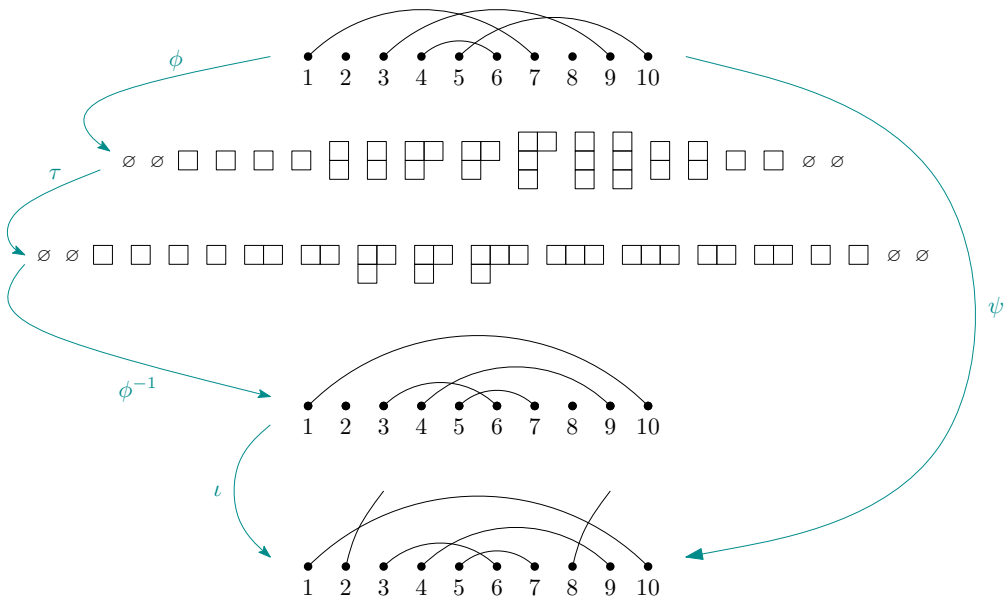

FIGURE 10. Image of the involution of Figure 9 under ψ.

Lemma 11. *There is a bijection ψ between involution diagrams and open matching diagrams, such that for α an involution diagram and $\beta = \psi(\alpha)$, the diagrams α and β have same length, the number of fixed points in α is the number of open arcs in β, and for any $\ell \geq 1$ there is an enhanced ℓ-nesting in α if and only if there is an ℓ-crossing in β. In addition the opener/closer sequence of α (seeing fixed points as openers) is the same as the opener/closer sequence of β.*

Proof of Proposition 8. We describe ψ, a bijective map between involutions and open matchings. It is formed as a composition of other maps. We have already defined ϕ, the bijection from set partition diagrams to vacillating tableaux from Section 3.1, and τ, the transpose action which can be applied to any tableau sequence. We add ι, the operation that changes every isolated dot in an involution diagram into an open arc. Let ψ be the composition $\iota \circ \phi^{-1} \circ \tau \circ \phi$. Figure 10 shows an example of the action of ψ.

Since ϕ, τ and ι can all be reversed, the mapping ψ is bijective. Moreover, recall from the end of Section 3.1, the mapping $\phi^{-1} \circ \tau \circ \phi$ preserves the opener-closer sequence. Therefore, every involution of size n with m fixed points is mapped under ψ to an open matching diagram of size n with m open arcs.

Assume that an involution α has an enhanced ℓ-nesting $(i_1, j_1) \dots (i_\ell, j_\ell)$. If $i_\ell \neq j_\ell$, this enhanced nesting is also a plain ℓ-nesting. By the remark at the end of Section 3.1, we know that $\phi^{-1} \circ \tau \circ \phi(\alpha)$ has a ℓ-crossing, so the same holds for $\psi(\alpha)$.

If $i_\ell = j_\ell$, then i_ℓ is a fixed point of α and hence an open arc in $\psi(\alpha)$. Moreover, the segment $[i_\ell, i_\ell + 1]$ is below the $(\ell - 1)$-nesting $(i_1, j_1) \dots (i_{\ell-1}, j_{\ell-1})$. So by Proposition 3, the $2i_\ell$th diagram of $\phi(\alpha)$ has at least ℓ columns. The $2i_\ell$th diagram of $\tau \circ \phi(\alpha)$ has then k rows, and so $[i_\ell, i_\ell + 1]$ is below a $(\ell - 1)$-crossing in $\psi(\alpha)$. Thus, i_ℓ is in $\psi(\alpha)$ an open arc below a $(\ell - 1)$-crossing: the open matching $\psi(\alpha)$ has a ℓ-crossing. The converse is proved similarly.

In summary, ψ is a bijection between involution diagrams of size n with m fixed points and no enhanced k-nesting and open matchings diagrams of size n with m open arcs and no k-crossing. □

The standard Young tableau in Figure 9 is mapped to the open arc diagram at the bottom of Figure 10. Here, the parameter m takes value 2.

Remark that standard Young tableaux with height bounded by odd numbers are also characterized in terms of open matching diagrams (but this time constrained by the plain nestings or crossings).

Proposition 12. *The following classes are in bijection:*

(i) the set of standard Young tableaux of size n with m odd columns and height bounded by $2k-1$;
(ii) the set of involutions of size n with m fixed points and no decreasing subsequence of length $2k$;
(iii) the set of open matching diagrams of length n with no plain *k-crossing and with m open arcs;*
(iv) the set of open matching diagrams of length n with no plain *k-nesting and with m open arcs.*

Proof. The RSK correspondence (specifically, the property described in Lemma 9) gives a straightforward bijection between (i) and (ii). Then, seeing isolated points as open arcs, it is easy to adapt Lemma 10 in order to show the correspondence between (ii) and (iv). Finally the bijection between (iii) and (iv) is given by $\phi^{-1}\circ\tau\circ\phi$ (still by interpreting isolated points as open arcs), where ϕ and τ are defined in Section 3.1. □

4.2. **A new symmetric joint distribution for involutions.** While looking for the previous bijection we found a surprising symmetry property for involutions, which is now presented. Section 3.1 contained the definition of *nesting level*; in the context of involution diagrams, the notion can be refined in two different ways, depending on whether we regard involution diagrams as enhanced set partition diagrams or as open matchings.

The *enhanced nesting level* of an involution α, denoted $\mathrm{ne}_{\downarrow}(\alpha)$, is the maximal number of *dots* in an enhanced nesting of α (note that k marks the number of dots not number of arcs). Pursuant to Lemma 10, the number $\mathrm{ne}_{\downarrow}(\alpha)$ is also the length of the longest decreasing subsequence of α. Similarly, we define the *open nesting level* of an involution α, denoted $\mathrm{ne}_{\leftarrow}(\alpha)$: after transforming the diagram of α into an open matching by changing every isolated point into an open arc, the open nesting level of α is the maximal number of dots inside a nesting.

Remark that an enhanced nesting and a nesting in an open diagram are identical if they both have an even number of dots; these are then plain nestings. The difference is made when the number of dots is odd, say $2k+1$. In this case, an enhanced nesting is made of a dot *below* a plain k-nesting, while a nesting in an open matching is made of an open arc to the *left* of a plain k-nesting. This justifies the notation $\mathrm{ne}_{\downarrow}$ and $\mathrm{ne}_{\leftarrow}$. Figure 11 compares the two patterns.

Example 13. *The open nesting level of the involution* $(1\ 7)(3\ 9)(4\ 6)(5\ 10)$*, depicted in Figure 9, is* 5*: the numbers* $2,3,4,6,9$ *form a nesting if we transform the dot* 2 *into an open arc. However the enhanced nesting level of the same involution is* 4*: there is no dot below any* 2*-nesting.*

FIGURE 11. *Left.* An enhanced 3-nesting with respectively 5 and 6 dots. *Right.* A 3-nesting with respectively 5 and 6 dots (in an open matching).

A (weak) link between the two statistics can be easily derived from the preceding study, as stated in the following proposition.

Proposition 14. *There is a bijection θ from involution diagrams to involution diagrams such that for α any involution diagram, $\beta=\theta(\alpha)$, and $\ell\geq 1$, there is an enhanced ℓ-nesting in α if and only if there is an ℓ-nesting in the open matching obtained by changing every fixed point in β by an open arc. In other words, there exists a bijection θ between involutions α such that $\mathrm{ne}_{\downarrow}(\alpha)=2k-1$ or $2k$ and involutions β such that $\mathrm{ne}_{\leftarrow}(\beta)=2k-1$ or $2k$.*

In addition, θ preserves the length, the number of fixed points and the opener/closer sequence (viewing fixed points as openers).

Proof. We define θ as the composition of

(1) the mapping ϕ described by Lemma 11;
(2) the operation that changes isolated points into open arcs;
(3) the mapping from open matchings to oscillating tableaux with bounded width ending a column (see Subsection 3.2 and more precisely Proposition 6);

(4) the transposition of the Ferrers diagrams;
(5) the mapping from oscillating tableaux with bounded height ending at a column to open matchings;
(6) the operation that changes open arcs into isolated points.

All the properties of θ presented in the statement of this proposition are direct consequences of Lemma 11 and Proposition 6. □

Note that an enhanced nesting is preserved when the diagram is reflected. This is not true for an odd nesting in an open diagram, because the isolated point must be to the left of the nesting. Despite the fact they do not share this property, the enhanced nesting level and the open nesting level have symmetric distribution, as stated in the following theorem.

Theorem 15. *The statistics* $\mathrm{ne}_{\leftarrow}$ *and* $\mathrm{ne}_{\downarrow}$ *have a symmetric joint distribution over all the involutions of size* n *with* m *fixed points, i.e.,*

$$\sum_{\substack{\alpha \text{ involution} \\ \text{of size } n \\ \text{with } m \text{ fixed points}}} x^{\mathrm{ne}_{\leftarrow}(\alpha)} y^{\mathrm{ne}_{\downarrow}(\alpha)} = \sum_{\substack{\alpha \text{ involution} \\ \text{of size } n \\ \text{with } m \text{ fixed points}}} y^{\mathrm{ne}_{\leftarrow}(\alpha)} x^{\mathrm{ne}_{\downarrow}(\alpha)}.$$

Remark. The bijection θ from Proposition 14 does not swap the statistics $\mathrm{ne}_{\downarrow}$ and $\mathrm{ne}_{\leftarrow}$. For instance, the involution $\alpha_1 = (1\ 5)(2\ 3)$ of size 5 is mapped to the involution $\alpha_2 = (2\ 3)(4\ 5)$: we have $\mathrm{ne}_{\downarrow}(\alpha_1) = \mathrm{ne}_{\leftarrow}(\alpha_1) = 4$ but $\mathrm{ne}_{\leftarrow}(\alpha_2) = 3$ and (even worse!) $\mathrm{ne}_{\downarrow}(\alpha_2) = 2$. Nonetheless, the existence of the function θ (and more particularly the fact that an involution with enhanced nesting level $2k-1$ or $2k$ is mapped under θ to an involution with open nesting level $2k-1$ or $2k$) is sufficient to prove Theorem 15.

Proof. Consider all involutions of fixed size, with a fixed number of fixed points. Let $a_{i,j}$ be the number of involutions α in this class of such that $\mathrm{ne}_{\downarrow}(\alpha) = i$ and $\mathrm{ne}_{\leftarrow}(\alpha) = j$. By Proposition 14, the bijection θ maps involutions α such that $\mathrm{ne}_{\downarrow}(\alpha) = 2k-1$ or $2k$ to involutions β such that $\mathrm{ne}_{\leftarrow}(\beta) = 2k-1$ or $2k$; hence

$$\sum_{j\geq 0} a_{2k-1,j} + a_{2k,j} = \sum_{i\geq 0} a_{i,2k-1} + a_{i,2k}. \tag{1}$$

We can simplify the expression in Equation (1) as the values $\mathrm{ne}_{\downarrow}(\alpha)$ and $\mathrm{ne}_{\leftarrow}(\alpha)$ can only differ by at most one for a given involution α. Indeed, if ℓ denotes the maximal number of arcs inside a nesting of an involution, the open nesting level and the enhanced nesting level must equal either 2ℓ or $2\ell+1$. Therefore, $a_{i,j} = 0$ except for pairs (i,j) of the form $(2\ell, 2\ell)$, $(2\ell, 2\ell+1)$, $(2\ell+1, 2\ell)$ or $(2\ell+1, 2\ell+1)$. Equation (1) can be thus rewritten as:

$$a_{2k-1,2k-1} + a_{2k-1,2k-2} + a_{2k,2k} + a_{2k,2k+1} = a_{2k-2,2k-1} + a_{2k-1,2k-1} + a_{2k,2k} + a_{2k+1,2k},$$

or after simplification

$$a_{2k,2k+1} - a_{2k+1,2k} = a_{2k-2,2k-1} - a_{2k-1,2k-2}.$$

In other words, the sequence $(a_{2k,2k+1} - a_{2k+1,2k})$ is constant over all $k \geq 0$. But since it equals 0 for $k = 0$, we have for every $k \geq 0$,

$$a_{2k,2k+1} = a_{2k+1,2k}.$$

The other terms $a_{i,j}$ such that $i \neq j$ vanish, so the last equality is sufficient to conclude the proof. □

The previous proof is simple but not constructive: can we describe an involution (on involutions) that swaps the statistics $\mathrm{ne}_{\leftarrow}$ and $\mathrm{ne}_{\downarrow}$? The answer is yes, and a description can be given in terms of iterations of θ, where θ is the mapping defined by Proposition 14.

Lemma 16. *Let* $\theta^{(\ell)}$ *be the* ℓ*th iteration of* θ *and* $A_{i,j}$ *be the set of involutions* α *such that* $\mathrm{ne}_{\downarrow}(\alpha) = i$ *and* $\mathrm{ne}_{\leftarrow}(\alpha) = j$.

For every α *in* $A_{2k,2k+1}$ *with* $k \geq 0$*, there exists* $m \geq 1$ *such that* $\theta^{(\ell)}(\alpha) \notin A_{2k,2k+1} \cup A_{2k+1,2k}$ *for* $\ell \in \{1, \ldots, m-1\}$*, and* $\theta^{(m)}(\alpha) \in A_{2k+1,2k}$*. Moreover, for every* α' *in* $A_{2k+1,2k}$*, there exists* $m' \geq 1$ *such that* $\theta^{(\ell')}(\alpha') \notin A_{2k,2k+1} \cup A_{2k+1,2k}$ *for* $\ell' \in \{1, \ldots, m'-1\}$*, and* $\theta^{(m')}(\alpha') \in A_{2k,2k+1}$.

In other words, in the orbit of any involution under θ *(this orbit is cyclic since* θ *is bijective and the set of involutions of fixed size is finite), the elements of* $A_{2k,2k+1} \cup A_{2k+1,2k}$ *alternate between* $A_{2k,2k+1}$ *and* $A_{2k+1,2k}$.

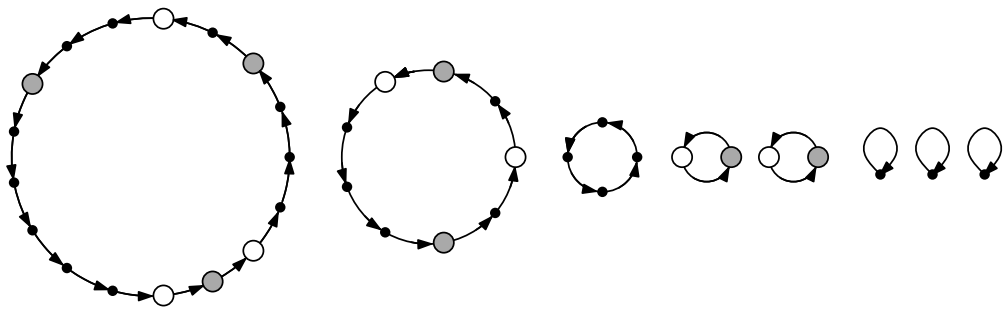

FIGURE 12. Schematic representation of typical orbits under θ. The white circles represent the elements of $A_{2k,2k+1}$, the gray circles the elements of $A_{2k+1,2k}$ and the small points are the remaining elements.

An example of this correspondence is illustrated in Figure 12.

Proof. Consider $i > 0$ such that $\theta^{(i)}(\alpha) = \alpha$ (such an i exists as θ acts bijectively on the finite set of involutions of a fixed length). Since $\mathrm{ne}_{\leftarrow}(\alpha) = 2k+1$, we have $\mathrm{ne}_{\downarrow}(\theta^{(i-1)}(\alpha)) > 2k$.

Let m denote the smallest $j > 0$ such $\mathrm{ne}_{\downarrow}(\theta^{(j)}(\alpha)) > 2k$. We have then $\mathrm{ne}_{\downarrow}(\theta^{(m-1)}(\alpha)) \leq 2k$. Using the properties of θ we know that $\mathrm{ne}_{\leftarrow}(\theta^{(m)}(\alpha)) \leq 2k$, hence $\mathrm{ne}_{\downarrow}(\theta^{(m)}(\alpha)) \leq 2k+1$. As $\mathrm{ne}_{\downarrow}(\theta^{(m)}(\alpha)) > 2k$, we must have $\mathrm{ne}_{\downarrow}(\theta^{(m)}(\alpha)) = 2k+1$ and $\mathrm{ne}_{\leftarrow}(\theta^{(m)}(\alpha)) = 2k$.

We have just showed that for α and $\theta^{(i)}(\alpha)$ belonging to $A_{2k,2k+1}$, there exists $m \in \{1, \ldots, i-1\}$ such that $\theta^{(m)}(\alpha) \in A_{2k+1,2k}$. The proof is over if we manage to show the statement concerning α'. This can be done either by using the same reasoning or by an argument of cardinality (with Theorem 15). $\square$

The previous lemma sets out how to build the desired involution. Essentially, from an involution of $A_{2k+1,2k}$, we iterate θ until obtaining an involution of $A_{2k,2k+1}$. If, on the other hand, the involution belongs to $A_{2k,2k+1}$ we want to go backward, so we iterate θ^{-1} until obtaining an involution of $A_{2k+1,2k}$. If an involution does not fall under one of the previous forms, it necessarily belongs to a set of the form $A_{\ell,\ell}$, and we can then set this involution as a fixed point.

Proposition 17. *Let α be an involution and $A_{i,j}$ be the set of involutions α such that $\mathrm{ne}_{\downarrow}(\alpha) = i$ and $\mathrm{ne}_{\leftarrow}(\alpha) = j$. If $\alpha \in A_{2k,2k+1}$, set m_α as the smallest integer m such that $\theta^{(m)}(\alpha) \in A_{2k+1,2k}$. If $\alpha \in A_{2k+1,2k}$, set m_α as the opposite of smallest integer m such that $\theta^{(-m)}(\alpha) \in A_{2k,2k+1}$. Otherwise, set m_α as 0.*

The mapping $\alpha \mapsto \theta^{(m_\alpha)}(\alpha)$ is an involution on the class of involutions that exchanges the statistics $\mathrm{ne}_{\downarrow}$ and $\mathrm{ne}_{\leftarrow}$. It preserves the size of the involutions, the number of fixed points and the opener/closer sequence[5].

Remark. What about the *open crossing level* of an involution, that is to say the maximum number of dots contained in a k-crossing, when this involution is transformed into an open matching? It is easy to see that the open crossing level shares the same distribution as the open nesting level or the enhanced nesting level (in particular via the bijection $\phi^{-1} \circ \tau \circ \phi$). However, this statistic does not have a symmetric distribution, whether it is with the open nesting level or with the enhanced nesting level.

4.3. **Generating function expressions.** One consequence of Theorem 1 is a collection of new generating function expressions for standard Young tableaux of bounded height. They come from an application of enumeration results of Weyl chamber walks [17, 20].

4.3.1. *A Determinant Expression.* The generating functions for Young tableaux of bounded height and Weyl chamber walks can both be expressed in terms of Bessel functions. We denote by

$$b_j(x) = I_j(2x) = \sum_n \frac{(2x)^{2n+j}}{n!(n+j)!},$$

[5] if we consider isolated points as left endpoints of arcs

the *hyperbolic Bessel function of the first kind of order* j.

Let $\widetilde{Y}_k(t)$ be the exponential generating function for the class of standard Young tableaux with height bounded by k. Formulas for $\widetilde{Y}_k(t)$ follow from works of Gordon, Houten, Bender and Knuth [18, 19, 4], which depend on the parity of k. We are only interested in the even values here.

Theorem 18 ([18, 19, 4])**.** *The exponential generating function for the class of standard Young tableaux of height bounded by* $2k$ *is given by*

$$\widetilde{Y}_{2k}(t) = \det[b_{i-j}(t) + b_{i+j-1}(t)]_{1\le i,j\le k}.$$

Around the same time, Grabiner-Magyar [20] determined a formula for the exponential generating function of the W_k-oscillating walks of length n between two given points. Throughout this section, we translate the region W_k to apply the previous results. Namely, we use

$$W_k = \{(x_1, \dots, x_k) \in \mathbb{Z}^k : x_1 > \cdots > x_k > 0\}.$$

Theorem 19 (Grabiner-Magyar [20])**.** *For fixed* $\lambda, \mu \in W_k$*, the exponential generating function* $\widetilde{O}_{\lambda,\mu}(t)$ *of the* W_k*-oscillating walks from* λ *to* μ*, counted by their lengths, satisfies*

$$\widetilde{O}_{\lambda,\mu}(t) = \det\left(b_{\mu_i-\lambda_j}(2t) - b_{\mu_i+\lambda_j}(2t)\right)_{1\le i,j\le k}.$$

We specialize the start and end positions as $\lambda = \delta = (k, k-1, \dots, 1)$ and $\mu = \delta + me_1 = (k+m, k-1, \dots, 1)$. We are interested in the sum over all values of m, and define $\widetilde{O}_k(t) \equiv \sum_{m\ge 0} \widetilde{O}_{\delta, me_1+\delta}(t)$. We deduce the following.

Proposition 20. *The exponential generating function for the class of oscillating tableaux ending with a row shape is the finite sum*

$$\widetilde{O}_k(t) = \sum_{u=0}^{k-1} (-1)^u \sum_{\ell=u}^{2k-1-2u} b_\ell \det(b_{i-j} - b_{kd-i-j})_{0\le i\le k-1, i\ne u, 1\le j\le k-1}.$$

This follows from the fact that the infinite sum which arises from direct application of Grabiner and Magyar's formula telescopes after applying the identity $b_{-k} = b_k$. The proof is technical, and largely an exercise in tracking indices after applying co-factor expansion of the determinants.

The bijection between the classes implies $\widetilde{O}_k(t) = \widetilde{Y}_{2k}(t)$. Here are the first two values, which are well known:

$$\widetilde{O}_1(t) = \widetilde{Y}_2(t) = b_0 + b_1, \quad \widetilde{O}_2(t) = \widetilde{Y}_4(t) = b_0^2 + b_0b_1 + b_0b_3 - 2b_1b_2 - b_2^2 - b_1^2 + b_1b_3.$$

4.3.2. *A Diagonal Expression.* Standard Young tableaux can also be viewed as oscillating tableaux with no deleting steps. (The entries tell you which box was added at a given time.) This gives us an interesting correspondence between two lattice path classes.

Theorem 21. *The set of oscillating lattice walks of length* n *in* W_k *starting at* $\delta = (k, k-1, \dots, 1)$ *and ending at the boundary at the boundary* $\{me_1 + \delta : m \ge 0\}$ *is in bijection with the set of oscillating lattice walks of length* n *in* W_{2k}*, using only positive steps* (e_j)*, starting at* δ *and ending anywhere in the region.*

We next obtain a new diagonal expression for standard Young tableaux of bounded height. The expression is also a corollary of the bijection. We find the expression via the oscillating walks, and an application of Gessel and Zeilberger's Weyl chamber reflectable walk model. The advantage of these diagonal representations is potential access to asymptotic enumeration formulas, and possibly alternative combinatorial representations. All of the generating functions are D-finite, and we can use the work of [6] to determine bounds on the shape of the annihilating differential equation.

Theorem 22. *The ordinary generating function for oscillating walks starting at* δ *and ending on the boundary* $\{m\, e_1 + \delta : m \ge 0\}$*, is given by the formula*

$$O_k(t) = \Delta\left[\frac{t^{2k-1}(z_3z_4^2\cdots z_k^{k-2})(z_1+1)\prod_{1\le j<i\le k}(z_i - z_j)(z_iz_j - 1)\cdot\prod_{2\le i\le k}(z_i^2-1)}{1 - t(z_1\cdots z_k)(z_1 + \overline{z}_1 + \cdots z_k + \overline{z}_k)}\right].$$

The proof of Theorem 22 is a rather direct application of Gessel and Zeilberger's formula for reflectable walks in Weyl chambers (the reader is directed to [17, 20] for details).

Proof. As oscillating tableaux are counted by walks in the region $W_k := \{(x_1, \dots, x_d) : x_1 > \cdots > x_k > 0\}$, taking steps $\{\pm e_1, \dots, \pm e_k\}$, and starting at $\delta = (k, k-1, \dots, 1)$, they are a reflectable walk model in the Weyl chamber B_k. This Weyl chamber is generated by the simple roots

$$\{e_1 - e_2, \dots, e_{k-1} - e_k, e_k\},$$

and has a Weyl group G of order $2^k\, k!$: the group has the full action of $\mathfrak{S}_k$ on $\mathbb{R}^k$, along with negating any subset of coordinates. Any group element σ which acts on δ by permuting its entries (keeping all its entries positive) has its order in the Weyl group the same as its order in $\mathfrak{S}_k$. Furthermore, negating k entries of an element corresponds to an even element of the Weyl group if and only if k is even. Thus, if

$$E(\mathbf{z}) := (z_1 \cdots z_k) \prod_{i<j} (z_i - z_j),$$

then

$$\sum_{w \in G} (-1)^{l(w)} \mathbf{z}^{w(\delta)} = \sum_{I \subset \{1,\dots,k\}} (-1)^{|I|} E(\sigma_I(\mathbf{z})),$$

where σ_I sends the element z_j to $1/z_j$ if $j \in I$ and fixes it otherwise. This sum simplifies to

$$A(\mathbf{z}) = \sum_{w \in G} (-1)^{l(w)} \mathbf{z}^{w(\delta)} = \frac{1}{(z_1 \cdots z_k)^k} \prod_{1 \le j < i \le k} (z_i - z_j)(z_i z_j - 1) \prod_{1 \le i \le k} (z_i^2 - 1),$$

which can be proven by noting that both expressions represent $(z_1 \cdots z_k)^k A(\mathbf{z})$ as a polynomial of total degree $k(3k+1)/2$ with the same solutions (and then comparing the leading coefficients of both expressions).

We are interested in walks that end on the boundary $M = \{me_1 + \delta : m \in \mathbb{N}\}$. The generating function of the endpoints is

$$B(\mathbf{z}) = \sum_{\mathbf{b} \in M} \mathbf{z}^{-\mathbf{b}} = \frac{1}{z_1^{k-1} z_2^{k-1} z_3^{k-2} \cdots z_k (z_1 - 1)}.$$

Finally, the generating function of unrestricted walks starting at the origin is

$$C(\mathbf{z}) = \frac{1}{1 - t(z_1 \cdots z_k)(z_1 + \overline{z}_1 + \cdots z_k + \overline{z}_k)}.$$

So by the classical result of Gessel and Zeilberger, we have the generating function representation

$$F(t) := \Delta\left(A(\mathbf{z}) B(\mathbf{z}) C(\mathbf{z})\right),$$

which is exactly the formula given in the statement of this theorem. □

5. Tableau sequences as Baxter classes

The combinatorial class that came to be known as Baxter permutations was introduced in 1967 in a paper of Baxter [2] studying compositions of commuting functions. A Baxter permutation of size n is a permutation $\sigma \in \mathfrak{S}_n$ such that there are no indices $i < j < k$ satisfying $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$. We shall denote by B_n the number of Baxter permutations of size n is B_n. They constitute entry A001181 of the Online Encyclopedia of Integer Sequences (OEIS) [22].

Chung, Graham, Hoggart and Kleiman [13] found the explicit formula

$$B_n = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}. \tag{2}$$

Many combinatorial classes have subsequently been discovered to have the same counting sequence – for example triples of lattice paths [14] and plane bipolar orientations [3]. A recent comprehensive survey of Felsner, Fusy, Noy and Orden [15] finds many structural commonalities among these seemingly diverse families of objects. Remarkably, there are intuitive bijections connecting these classes, see for instance [5].

The generating function of hesitating tableaux (i) in Proposition 7 was determined by Xin and Zhang [26]. Baxter numbers appear in their Table 3, and they mention that the equivalence between the two could be proved by applying creative telescoping to a formula for B_n resembling the one given in Equation (2) above.

Our contribution to this area is an explicit proof of that equivalence, and an exploration of the connection between these classes and the other well known Baxter classes. Clearly, the classes of Proposition 7 have

combinatorial bijections between them, but they **do not** share many of the properties of the other known Baxter classes. However, each of them does have a natural subclass of objects enumerated by Catalan numbers, as many Baxter families also do. (For example, non-crossing partitions are counted by Catalan numbers.)

Proposition 23. *The following classes are in bijection:*

(i) the set of hesitating tableaux with height bounded by 2, starting with empty diagram, ending in a partition with a single part;
(ii) the set of open partition diagrams of length n with no enhanced 3-crossing;
(iii) W_2-hesitating walks of length n ending on the x-axis;
(iv) Baxter permutations of size $n+1$.

Remark that Theorem 2 is simply the implication that (i) and (iv) from Proposition 23 are in bijection. We prove this with a generating function argument, and deduce the other bijections using Proposition 7.

5.1. **Proof of Theorem 2.** We prove Theorem 2 and conjecture a stronger result which could be useful to prove the bijection combinatorially. This conjecture is partially verified using some of the intermediary computations, so it is useful to have them made explicit. We note that this is slightly different from both the proof that appears in a previous version of this work [11] and from the suggested proof of Xin and Zhang [26].

We first set up some notation. Let $\overline{x} = \frac{1}{x}$, and consider the ring of formal series $\mathbb{Q}[x, \overline{x}][\![t]\!]$. The operator CT_x extracts the constant term in x of series of $\mathbb{Q}[x, \overline{x}][\![t]\!]$.

We recall the work of Bousquet-Mélou and Xin [7]. Here, we only require the $k = 2$ case from their work, and have consequently eliminated some of the subscripts from the statements of their results. Also, note that their definition of W_2 is shifted one unit to the right, hence in the statement of their results, walks start at $(1, 0)$ rather than $(0, 0)$.

Let Q denote the first quadrant in the plane, $Q = \{(x, y) : x, y \geq 0\}$, and let W_2 denote the region $W_2 = \{(x, y) : x > y \geq 0\}$. Walks taking n steps that start at λ and end at μ and remain in Q and W_2 are, respectively, denoted by $q(\lambda, \mu, n)$ and $w(\lambda, \mu, n)$.

Bousquet-Mélou and Xin's Proposition 12 in [7], based on a classic reflection argument, implies the following. For any starting and ending points λ and μ in W_2, the number of W_2-hesitating walks going from λ to μ can be expressed in terms of the number of Q-hesitating walks:

$$w(\lambda, \mu, n) = q(\lambda, \mu, n) - q(\lambda, \overline{\mu}, n)$$

where $\overline{(x, y)} = (y, x)$. They define a simple sign reversing involution between pairs of walks; the walks restricted to W_2 appear as fixed points.

We consider the following two generating functions for Q-hesitating walks that start at $(1, 0)$ and end on an axis:

$$H(x; t) = \sum_{i \geq 1, n \geq 0} q((1, 0), (i, 0), 2n) x^i \, t^n \quad \text{and} \quad V(y; t) = \sum_{i \geq 1, n \geq 0} q((1, 0), (0, i), 2n) y^i \, t^n.$$

By applying the proposition we see immediately that the bivariate generating function $W(x; t)$ for W_2-hesitating walks that start at $(0, 1)$ and end on the x- axis satisfies the formula

$$W(x; t) = \sum_{i \geq 1, n \geq 0} w_2((1, 0), (i, 0), 2n) \, x^i t^n = H(x; t) - V(x; t). \tag{3}$$

Theorem 2 is equivalent to the statement

$$W(1; t) = \sum B_{n+1} t^n. \tag{4}$$

Proof of Theorem 2. For $i \geq 0$, Bousquet-Mélou and Xin [7] show the following

$$[x^{i+1}]H(x; t) = \mathrm{CT}_x \frac{Y}{t(1 + x)} \bar{x}^{2+i} (x^2 - \bar{x}^2 Y^2 + \bar{x}^3 Y) \tag{5}$$

$$[x^{i+1}]V(x; t) = \mathrm{CT}_x \frac{Y}{t(1 + x)} x^{3+i} (x^2 - \bar{x}^2 Y^2 + \bar{x}^3 Y). \tag{6}$$

Thus, we deduce

$$[x^{i+1}]W(x;t) = [x^{i+1}]H(x;t) - V(x;t) = \mathrm{CT}_x \frac{Y}{t(1+x)}(\bar{x}^{2+i} - x^{3+i})(x^2 - \bar{x}^2 Y^2 + \bar{x}^3 Y).$$

Hence we have

$$W(1;t) = T(t) + U(t) + V(t),$$

where

$$\begin{aligned} T(t) &= \mathrm{CT}_x \sum_{i\geq 0} \frac{Y}{t(1+x)}(\bar{x}^i - x^{5+i}), \\ U(t) &= -\mathrm{CT}_x \sum_{i\geq 0} \frac{Y^3}{t(1+x)}(\bar{x}^{4+i} - x^{1+i}), \\ V(t) &= \mathrm{CT}_x \sum_{i\geq 0} \frac{Y^2}{t(1+x)}(\bar{x}^{5+i} - x^i). \end{aligned}$$

Then we use the following identity from [7] (valid for $k \geq 1$ and $\ell \in \mathbb{Z}$)

$$\mathrm{CT}_x \frac{Y^k}{t(1+x)} \bar{x}^\ell = \mathrm{CT}_x \frac{Y^k}{t(1+x)} x^{\ell-k+1},$$

which gives the following simplifications:

$$\begin{aligned} T(t) &= \mathrm{CT}_x \sum_{i\geq 0} \frac{Y}{t(1+x)}(x^i - x^{5+i}) = \mathrm{CT}_x \frac{Y}{t(1+x)}(1 + x + x^2 + x^3 + x^4), \\ U(t) &= -\mathrm{CT}_x \sum_{i\geq 0} \frac{Y^3}{t(1+x)}(x^{2+i} - x^{1+i}) = \mathrm{CT}_x \frac{Y^3}{t(1+x)} x, \\ V(t) &= \mathrm{CT}_x \sum_{i\geq 0} \frac{Y^2}{t(1+x)}(x^{4+i} - x^i) = \mathrm{CT}_x \frac{Y^2}{t(1+x)}(-1 - x - x^2 - x^3). \end{aligned}$$

Hence, defining $A_{\ell,k}(t) = \mathrm{CT}_x \frac{Y^k}{t(1+x)} x^\ell$, we can collect terms to obtain

$$W(1;t) = \sum_{r=0}^{4} A_{r,1}(t) + A_{1,3}(t) - \sum_{r=0}^{3} A_{r,2}(t). \tag{7}$$

It is shown in [7] that the Lagrange inversion formula yields, for $n \in \mathbb{N}$,

$$[t^n]A_{\ell,k}(t) = \sum_{j\in\mathbb{Z}} a_n(\ell,k,j), \quad \text{with} \quad a_n(\ell,k,j) = \frac{k}{n+1}\binom{n+1}{j}\binom{n+1}{j+k}\binom{n}{j-\ell}.$$

Here we apply the convention $\binom{n}{j} = 0$ for $j < 0$ or $j > n$.

Next, it is straightforward to detect and check the linear relations (valid for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$)

$$a_n(4,1,n-j+2) + a_n(1,3,j-1) - a_n(2,2,n-j+1) - a_n(3,2,j) = 0,$$

$$a_n(1,1,n-j) + a_n(2,1,j+1) - a_n(0,2,j) = 0,$$

which respectively give $A_{4,1}(t)+A_{1,3}(t)-A_{2,2}(t)-A_{3,2}(t) = 0$ and $A_{1,1}(t)+A_{2,1}(t)-A_{0,2}(t) = 0$. Remarkably, expression (7) for $W(1;t)$ simplifies to

$$W(t) = A_{0,1}(t) + A_{3,1}(t) - A_{1,2}(t).$$

For $n \geq 1$, the Baxter number B_n is given by $B_n = \sum_{j\in\mathbb{Z}} b_{n,j}$, with $b_{n,j} = \frac{\binom{n+1}{j-1}\binom{n+1}{j}\binom{n+1}{j+1}}{\binom{n+1}{1}\binom{n+1}{2}}$, and again it is easy to detect and check that (for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$)

$$a_n(0,1,j) + a_n(3,1,j+1) - a_n(1,2,j) = b_{n+1,j+1},$$

so that $A_{0,1}(t) + A_{3,1}(t) - A_{1,2}(t) = \sum_{n\geq 0} B_{n+1} t^n$, and thus $[t^n]W(1;t) = B_{n+1}$. □

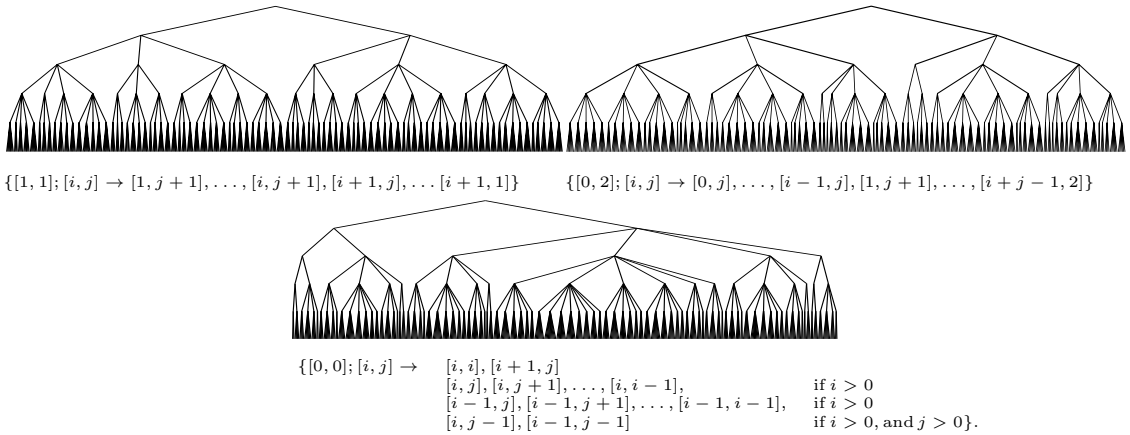

FIGURE 13. The first five levels of each of the Baxter generating trees. They are respectively from [5] [8] [10].

5.2. **Consequence: a new generating tree.** A *generating tree* for a combinatorial class expresses recursive structure in a rooted plane tree with labeled nodes. The objects of size n are each uniquely generated, and the set of objects of size n comprise the nth level of the tree. They are useful for enumeration, and for showing that two classes are in bijection. Theorem 2 yields a new generating tree construction for Baxter objects.

Several different formalisms exist for generating trees, notably [1]. The central properties are as follows. Every object γ in a combinatorial class $\mathcal{C}$ is assigned a label $\ell(\gamma) \in \mathbb{Z}^k$, for some fixed k. There is a rewriting rule on these labels with the property that if two nodes have the same label then the ordered list of labels of their children is also the same. We consider labels that are pairs of positive integers, specified by $\{\ell_{\text{Root}} : [i,j] \to \text{Succ}([i,j])\}$, where ℓ_{Root} is the label of the root.

Two generating trees for Baxter objects are known in the literature, and one consequence of Theorem 2 is a third, using the generating tree for open partitions given by Burrill *et al.* [10]. This tree differs from the other two already at the third level, illustrating a very different decomposition of the objects. For the three different systems we give the succession rules, and the first 5 levels of the tree (unlabelled), in Figure 13.

5.3. **A conjectured refinement.** We have proved that the coefficients $a(n,m)$ counting W_2-hesitating walks of length $2n$ from $(0,0)$ to $(m,0)$ satisfy $\sum_m a(n,m) = B_{n+1}$, with B_n the nth Baxter number. A bijective proof is yet to be found, and in that perspective a natural question is whether the parameter m corresponds to a simple parameter on another Baxter family.

Proposition 24. *The family of Q-hesitating excursions, that is to say the hesitating walks in the lattice $Q = \{(x,y) : x,y \geq 0\}$ starting and ending at the origin, is a Baxter family: the number of such walks of length $2n$ is equal to B_{n+1}.*

Proof. We show an easy bijection with the set $\mathcal{T}_n$ of non-intersecting triples of lattice paths each of length n with steps either $N = (0,1)$ (north steps) or $E = (1,0)$ (east steps), with respective starting points $(-1,1)$, $(0,0)$, $(1,-1)$ and respective ending points $(k-1, n-k+1)$, $(k, n-k)$, $(k+1, n-k-1)$ for some $k \in \{0,\ldots,n\}$. For 3 distinct points p_1, p_2, p_3 in $\mathbb{Z}^2$ on a same line of slope -1, ordered from top-left to bottom-right, define the *distance-pair* for (p_1,p_2,p_3) as the pair (i,j) of nonnegative integers such that $x(p_1) = x(p_2) - i - 1$ and $x(p_3) = x(p_2) + j + 1$. Let $(P_1,P_2,P_3) \in \mathcal{T}_n$. For $r \in \{0,\ldots,n\}$ and $i \in \{1,2,3\}$, let $p_i^{(r)}$ be the point on P_i after r steps, and let $d(r)$ be the distance-pair for $(p_1^{(r)}, p_2^{(r)}, p_3^{(r)})$; note that $d(0) = (0,0)$ and $d(n) = (0,0)$ and that $d(r) \in q$ for $0 \leq r \leq n$. Moreover, for $0 < r \leq n$, the vector $\delta(r) := d(r) - d(r-1)$ is in the set $\{(\pm 1,0),(0,\pm 1),(1,-1),(-1,1),(0,0)\}$, with two possibilities for being $(0,0)$ (whether the rth steps in P_1,P_2,P_3 are all north or all east). Hence the situation for the successive distance-pairs $d_0,\ldots,d_n$ is exactly

the same as for the successive points of even rank in a Q-excursion of length $2n$. Figure 14 (left and middle) illustrates this bijection. $\square$

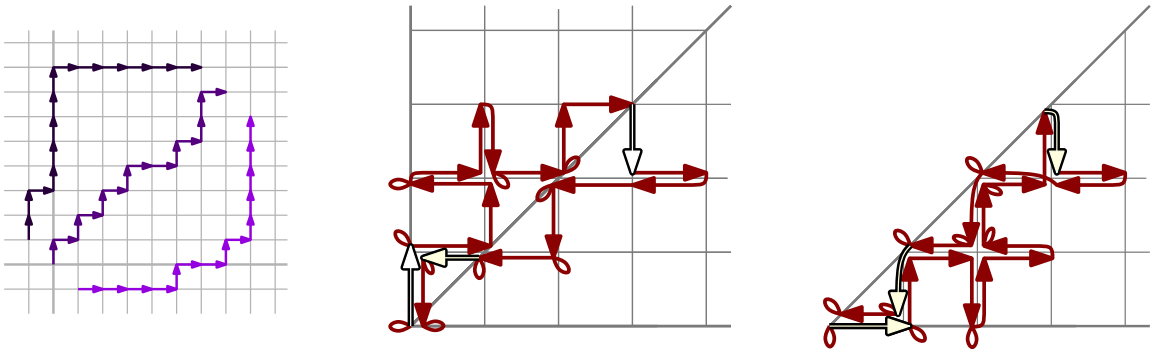

FIGURE 14. *Left.* A non-intersecting triple of lattice paths. *Middle.* A Q-hesitating excursion. The stay steps are drawn as loops. The switch-multiplicity of the walk is 3 (the white arrows indicate the marked steps). *Right.* A W_2-hesitating excursion with 3 marked steps each leaving the diagonal. These three objects are in correspondence.

We now define a secondary parameter m for Q-excursions. Let w be a Q-excursion of length $2n$ where e_r denotes the rth step, for $1 \leq r \leq 2n$. Consider, if any, the first step e_{i_1} that visits the region $x < y$. Then consider, if any, the first step e_{i_2} after e_{i_1} that visits the region $x > y$, and so on (switching between $x < y$ and $x > y$ each time). We have here a stopping iterative process yielding, for some $m \geq 0$, m marked steps $e_{i_1}, \ldots, e_{i_m}$ with $i_1 < \ldots < i_m$; m is called the *switch-multiplicity* of the excursion. For instance, the switch-multiplicity of the excursion at the middle of Figure 14 is 3. Note also that $m \leq n$ since two marked steps cannot be consecutive (the case $m = n$ is reached by the unique excursion where $i_1 = 1, i_2 = 3, i_3 = 5, \ldots$, i.e., the excursion that alternates pairs of steps $(0,1),(0,-1)$ with pairs of steps $(1,0),(-1,0)$). Denote by $q(n,m)$ the number of Q-hesitating excursions of length $2n$ and switch-multiplicity m, and $a(n,m)$ the number of W_2-hesitating walks of length $2n$ from $(0,0)$ to $(m,0)$.

Conjecture 1. *For $n, m \geq 0$, we have $q(n,m) = a(n,m)$.*

We have thought of the switch-multiplicity as a natural candidate because of the analogy with a well-known bijection between excursions of length $2n$ on the line $\mathbb{Z}$ and walks of length $2n$ starting at 0 on the half-line $\mathbb{Z}_{\geq 0}$ (with steps in ± 1 for both types of walks), where a similar switch-multiplicity parameter for excursions (this time switching between $\mathbb{Z}_{<0}$ and $\mathbb{Z}_{>0}$) corresponds to half the ending abscissa of positive walks. However, what surprises us is that, while we have strong evidence the conjecture is true, we do not even have a proof for $m = 1$ (the case $m = 0$ is trivial).

Let us now slightly reformulate the conjecture so that we have W_2-hesitating walks on both sides. Consider a Q-hesitating excursion w of length $2n$, with $e_{i_1}, \ldots, e_{i_m}$ the steps given by the stopping iterative process (switching between $x < y$ and $x > y$). Accordingly w splits into a concatenated sequence of $m+1$ parts $\pi_0, \ldots, \pi_m$, where π_0 is the part before e_{i_1} ($\pi_0 = w$ if $m = 0$), for $1 \leq h < m$, π_h is the part between e_{i_h} (included) and $e_{i_{h+1}}$ (excluded), and for $m \geq 1$, π_m is the ending part of W_2 starting from e_{i_m}. Each walk π_i starts and ends on the diagonal $x = y$ and stays in $x \geq y$ for i even and in $x \leq y$ for i odd. Hence, if we reflect each odd walk π_{2i+1} according to the diagonal $x = y$, we obtain a W_2-hesitating walk from $(0,0)$ to $(0,0)$ with m marked steps (the steps at positions $i_1, \ldots, i_m$) each leaving the diagonal $x = y$. In addition, due to recording the marked steps, there is no loss of information (the original excursion can be recovered). This correspondence is depicted by Figure 14 (middle and right). Hence, if we denote by $a(n; i, j, m)$ the number of W_2-hesitating walks of length $2n$ from $(0,0)$ to (i,j) with $2n$ steps and m marked steps each leaving the diagonal $x = y$, Conjecture 1 can be reformulated as:

Conjecture 2 (Reformulation)**.** *For $n, m \geq 0$, we have*

$$a(n; m, 0, 0) = a(n; 0, 0, m).$$

Actually, an even stronger symmetry seems to hold:

Conjecture 25. *For $n, i, j \geq 0$, we have*

$$a(n; i, 0, j) = a(n; j, 0, i).$$

Note that there is clearly a one-to-one correspondence between steps leaving the diagonal $x = y$ and steps ending at the diagonal $x = y$. Hence $a(n; i, j, m)$ is also the number of W_2-hesitating walks of length $2n$ from $(0, 0)$ to (i, j) with $2n$ steps and m marked steps each ending at the diagonal $x = y$. In that form it is easy to obtain a recurrence for the coefficients $a(n; i, j, m)$ by considering the effect of adding the last two steps (note that each of the two last steps has to be unmarked if empty or not ending at $x = y$, and might be either unmarked or marked if non-empty and ending at $x = 1$). Denote by $\mathcal{S}$ the set of steps $\{(\pm 1, 0), (0, \pm 1), (-1, 1), (1, -1)\}$ together with the two stationary steps s_1 and s_2, where s_1 simulates taking the pair $(1, 0)$ and $(-1, 0)$ as a single step, and s_2 simulates taking the pair $(0, 1)$ and $(0, -1)$ as a single step. We have the following recurrence (with δ the Kronecker symbol), from which we have been able to check that Conjecture 25 holds for all n, i, j with $n \leq 56$:

- for $n = 0$,

$$\begin{aligned} a(n; i, j, m) &= 1 \text{ if } i = j = m = 0, \\ a(n; i, j, m) &= 0 \text{ otherwise,} \end{aligned}$$

- for $n > 0$,

$$\begin{aligned} a(n; i, j, m) &= 0 \quad \text{for } (i, j, m) \notin \mathcal{D} := \{0 \leq j \leq i,\ 0 \leq m \leq n\}, \\ a(n; i, j, m) &= \delta_{i=j} \cdot \sum_{s \in \mathcal{S} \setminus s_1} a(n-1; i - x(s), j - y(s), m) \\ &\quad + \delta_{i>j} \cdot \sum_{s \in \mathcal{S}} a(n-1; i - x(s), j - y(s), m) \\ &\quad + \delta_{i=j} \cdot \sum_{s \in \mathcal{S} \setminus s_1} a(n-1; i - x(s), j - y(s), m-1) \\ &\quad + \delta_{i=j+1} \cdot a(n-1; i, j, m-1) \quad \text{for } (i, j, m) \in \mathcal{D}. \end{aligned}$$

6. Conclusion

We conclude with a few thoughts on future directions. Starting with the correspondences described here, we can easily give an expression for the generating function of standard Young tableaux of bounded height as a diagonal of a rational function. The original proofs of the D-finiteness of these generating functions (for arbitrary k) of Gessel [16] used a diagonal type operation over series with infinite variable sets. We are interested in the conjecture of Christol which states that globally bounded D-finite series can always be expressed as a diagonal of a rational function. Perhaps the key to understanding how to find such diagonal expressions lurks in Weyl chamber walk representations of combinatorial objects, since they easily yield diagonal expressions from the machinery of Gessel and Zeilberger.

Baxter numbers generalize, in some sense, Catalan numbers. Both are ubiquitous combinatorial sequences, and both are related to hesitating walk families. Are hesitating walks in higher dimensions similarly common?

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References

[1] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002. Formal power series and algebraic combinatorics (Barcelona, 1999).

[2] Glen Baxter. On fixed points of the composite of commuting functions. *Proc. Amer. Math. Soc.*, 15:851–855, 1964.

[3] R. J. Baxter. Dichromatic polynomials and Potts models summed over rooted maps. *Ann. Comb.*, 5(1):17–36, 2001.

[4] Edward A. Bender and Donald E. Knuth. Enumeration of plane partitions. *J. Combinatorial Theory Ser. A*, 13:40–54, 1972.

[5] Nicolas Bonichon, Mireille Bousquet-Mélou, and Éric Fusy. Baxter permutations and plane bipolar orientations. *Sém. Lothar. Combin.*, 61A:Art. B61Ah, 29, 2009/11.

[6] Alin Bostan, Pierre Lairez, and Bruno Salvy. Creative telescoping for rational functions using the Griffiths-Dwork method. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC)*, New York, NY, USA. ACM., pages 93–100, 2013.

[7] Mireille Bousquet-Mélou and Guoce Xin. On partitions avoiding 3-crossings. *Sém. Lothar. Combin.*, 54:Art. B54e, 21 pp. (electronic), 2005/07.

[8] Mathilde Bouvel and Olivier Guibert. Refined Enumeration of Permutations Sorted with Two Stacks and a D_8-Symmetry. *Ann. Comb.*, 18(2):199–232, 2014.

[9] Sophie Burrill. *A generating tree approach to k-nonnesting arc diagrams*. PhD thesis, Simon Fraser University, Burnaby, Canada, July 2014. http://summit.sfu.ca/system/files/iritems1/14390/etd8479_SBurrill.pdf.

[10] Sophie Burrill, Sergi Elizalde, Marni Mishna, and Lily Yen. A generating tree approach to k-nonnesting partitions and permutations. *arXiv preprint arXiv:1108.5615*, 2012.

[11] Sophie Burrill, Melczer, Stephen, and Marni Mishna. A Baxter class of a different kind. In *Proceedings of the 27th International Conference on Formal Power Series and Algebraic Combinatorics, Korea*, 2015.

[12] William Y. C. Chen, Eva Y. P. Deng, Rosena R. X. Du, Richard P. Stanley, and Catherine H. Yan. Crossings and nestings of matchings and partitions. *Trans. Amer. Math. Soc.*, 359(4):1555–1575 (electronic), 2007.

[13] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman. The number of Baxter permutations. *J. Combin. Theory Ser. A*, 24(3):382–394, 1978.

[14] Serge Dulucq and Olivier Guibert. Baxter permutations. In *Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995)*, volume 180, pages 143–156, 1998.

[15] Stefan Felsner, Éric Fusy, Marc Noy, and David Orden. Bijections for Baxter families and related objects. *J. Combin. Theory Ser. A*, 118(3):993–1020, 2011.

[16] Ira M. Gessel. Symmetric functions and P-recursiveness. *J. Combin. Theory Ser. A*, 53(2):257–285, 1990.

[17] Ira M. Gessel and Doron Zeilberger. Random walk in a Weyl chamber. *Proc. Amer. Math. Soc.*, 115(1):27–31, 1992.

[18] Basil Gordon. Notes on plane partitions. V. *J. Combinatorial Theory Ser. B*, 11:157–168, 1971.

[19] Basil Gordon and Lorne Houten. Notes on plane partitions. I, II. *J. Combinatorial Theory*, 4:72–80; 81–99, 1968.

[20] David J. Grabiner and Peter Magyar. Random walks in Weyl chambers and the decomposition of tensor powers. *J. Algebraic Combin.*, 2(3):239–260, 1993.

[21] Olivier Guibert and Svante Linusson. Doubly alternating Baxter permutations are Catalan. *Discrete Math.*, 217(1-3):157–166, 2000. Formal power series and algebraic combinatorics (Vienna, 1997).

[22] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org, 2011.

[23] Christian Krattenthaler. Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes. *Adv. in Appl. Math.*, 37(3):404–431, 2006.

[24] Christian Krattenthaler. Bijections between oscillating tableaux and (semi)standard tableaux via growth diagrams. arXiv:1412.5646 [math.CO], 2014.

[25] M.-P. Schützenberger. La correspondance de Robinson. In *Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976)*, pages 59–113. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.

[26] Guoce Xin and Terence Y. J. Zhang. Enumeration of bilaterally symmetric 3-noncrossing partitions. *Discrete Math.*, 309(8):2497–2509, 2009.

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