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# FIXED POINTS FOR MONOTONE ITERATIVELY LOCAL CONTRACTIONS

Let the quasi-ordered metric space  $(X, d, \leq)$  and the increasing self-mapping  $T$  of  $X$  be such that: for each  $x \in X$  with  $x \leq Tx$ , there exists a rank  $n(x) \in N$  and an increasing function  $f(x) : R_+^{2n(x)+1} \rightarrow R_+$  with

$$d(T^{n(x)}x, T^{n(x)}y) \leq f(x)(d(x, Tx), \dots, d(x, T^{n(x)}x); d(x, y), \dots, d(x, T^{n(x)}y)),$$

for all  $y \in X$ ,  $x \leq y \leq Ty$ ; then, under some additional assumptions involving these elements,  $T$  has at least one fixed point in  $X$ . A number of related contributions in this direction due to Sehgal, Guseman and Matkowski are obtained as corollaries.

Let  $(X, d)$  be a complete metric space and  $T$ , a self-mapping of  $X$ . Assume that for each  $x \in X$  there exists a  $n(x) \in N$ , such that  $T^{n(x)}$  is (metrically) contractive at  $x$ ; then, under what additional conditions does  $T$  possess a fixed point in  $X$ ?

A first answer to this question was given in 1969 by Sehgal [12] - for continuous  $T$  - through a specific iterative procedure; a reformulation of it for mappings which are not necessarily continuous was performed in the 1970 Guseman's paper [5]. During the last decade, a number of technical extensions (in the sense of the contractivity condition) of these basic results were obtained by Ćirić [2], Khazanchi [7], Iseki [6], Rhoades [11] and Singh [13]. The most general statement of this kind, obtained in 1977 by Matkowski [9], reads as follows.

**Theorem 1.** *Suppose that there is a function  $f : R_+^5 \rightarrow R_+$ , increasing in each variable, such that, denoting*

$$g(t) = f(t, t, t, 2t, 2t), \quad t \geq 0,$$

*the following conditions are fulfilled*

$$(c_1) \quad \lim_{n \rightarrow \infty} g^n(t) = 0, \text{ for all } t > 0$$

$$(c_2) \quad t - g(t) \rightarrow \infty \text{ as } t \rightarrow \infty;$$

*and suppose that for each  $x \in X$  there is a positive integer  $n(x) \in N$  such that*

$$(1) \quad \begin{aligned} & d(T^{n(x)}x, T^{n(x)}y) \leq \\ & f(d(x, T^{n(x)}x), d(x, y), d(x, T^{n(x)}y), d(T^{n(x)}x, y), d(T^{n(x)}y, y)) \end{aligned}$$

for all  $y \in X$ . Then,  $T$  has a unique fixed point  $z \in X$  and  $T^n x \rightarrow z$ , for any  $x \in X$ .

A close analysis of the above conditions shows that, by virtue of the evident relations

$$d(T^{n(x)}x, y) \leq d(x, T^{n(x)}x) + d(x, y), \quad x, y \in X,$$

$$d(T^{n(x)}y, y) \leq d(x, T^{n(x)}y) + d(x, y), \quad x, y \in X,$$

a slight extension of Theorem 1 might be reached if one replaces (1) by the contractivity condition

$$(1') \quad \begin{aligned} & d(T^{n(x)}x, T^{n(x)}y) \leq \\ & F(d(x, T^{n(x)}x), d(x, y), d(x, T^{n(x)}y)), \quad y \in X, \end{aligned}$$

where  $F : R_+^3 \rightarrow R_+$  is defined as

$$F(\xi, \eta, \zeta) = f(\xi, \eta, \zeta, \xi + \eta, \zeta + \eta), \quad \xi, \eta, \zeta \geq 0.$$

A natural question appearing at this moment is that of determining what happens when the right-hand side of (1') depends on the variables

$$((d(x, T^i x); 1 \leq i \leq n(x)), (d(x, T^j y); 0 \leq j \leq n(x)));$$

or, in other words, when the function  $F = F(x)$  acts from  $R_+^{2n(x)+1}$  to  $R_+$ . At the same time, observe that, from a "relational" viewpoint, the result we just recorded may be deemed as being expressed modulo the *trivial quasi-ordering* on  $X$ ; so that, a more adequate formulation of it in terms of *genuine quasi-orderings* would be of interest. It is precisely our main aim to get a generalization - under the above lines - of the fixed point Matkowski theorem; some further extensions to contractivity conditions involving all iterates of  $T$  in the right-hand side of (1') (Browder [1]) or different iterates of  $T$  in the left-hand side (Fisher [4]) will be given elsewhere.

Let  $X$  be a nonempty set,  $d$  a *metric* on  $X$  and  $\leq$  a *quasi-ordering* (i.e., a reflexive and transitive relation) on  $X$ . A sequence  $(x_n; n \in N)$  in  $X$  will be said to be *increasing* when

$$x_i \leq x_j \text{ for } i \leq j;$$

in this context, the notation  $x_n \uparrow x$  will mean:

$$(x_n; n \in N) \text{ is increasing and convergent to } x.$$

The ambient space  $X$  will be termed *quasi-order complete* when any increasing Cauchy sequence converges; of course, any complete metric space is quasi-order complete too, but the converse is not in general valid as simple examples show. It is supposed further that

$X$  is a quasi-order complete metric space

the self-mapping  $T$  of  $X$  is increasing ( $x \leq y$  implies  $Tx \leq Ty$ )

and the conditions below are satisfied

- (i)  $Y := X_T^i := \{x \in X; x \leq Tx\}$  is not empty
  - (ii) to any  $x$  in  $Y$  there corresponds a rank  $n(x) \in N$  and a function  $f(x) : R_+^{2n(x)+1} \rightarrow R_+$ , increasing in each variable, with
- $$(2) \quad \begin{aligned} & d(T^{n(x)}x, T^{n(x)}y) \leq \\ & f(x)(d(x, Tx), \dots, d(x, T^{n(x)}x); d(x, y), \dots, d(x, T^{n(x)}y)), \quad y \in Y, x \leq y. \end{aligned}$$

Let  $g(x)$  indicate, for each  $x \in Y$ , the function from  $R_+$  to itself, given by

$$g(x)(t) = f(x)(t, \dots, t; t, \dots, t), \quad t \geq 0.$$

The following technical result will be largely used in the sequel.

**Lemma 1.** *Suppose that, for each  $x_0 \in Y$ ,*

$$(iii) \quad g(x_0)(t) < t, \quad t > 0 \quad (\text{hence } g(x_0)(0) = 0)$$

and

$$t - g(x_0)(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty$$

$$(iv) \quad \lim_{k \rightarrow \infty} g(x_k) \circ \dots \circ g(x_0)(t) = 0, \quad t > 0,$$

where  $n_0 = n(x_0)$ ,  $x_1 = T^{n_0}x_0$  and, inductively,  
 $n_i = n(x_i)$ ,  $x_{i+1} = T^{n_i}x_i$ ,  $i \geq 1$ .

Then, for each  $x \in Y$ ,

$$(T^m x; m \in N) \text{ converges (in } X);$$

and, moreover,

$$d(T^m x, T^m y) \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad \text{for any } y \in Y, x \leq y.$$

*Proof.* Let  $x \in Y$  be given. We firstly claim that

$$(3) \quad d(x, T^m x) \leq t, \quad m \in N, \quad \text{for some } t = t(x) > 0.$$

Indeed, it follows by (iii) that, given  $\alpha > 0$ , there exists  $\beta = \beta(\alpha, x) \geq \alpha$ , such that

$$(4) \quad t \leq \alpha + g(x)(t) \text{ implies } t \leq \beta.$$

Put  $\alpha = \max(d(x, Tx), \dots, d(x, T^{n(x)}x))$ . We want to show that (3) holds, with  $t = \beta$ . To this end, suppose that the considered assertion would be false; and let  $m$  denote the infimum of those ranks for which the reverse of (3) takes place. Of course,

$$m > n(x), d(x, T^k x) \leq \beta, 1 \leq k \leq m-1, \text{ and } d(x, T^m x) > \beta;$$

so that, by (2), we get the relation

$$\begin{aligned} d(x, T^m x) &\leq d(x, T^{n(x)} x) + d(T^{n(x)} x, T^m x) \leq \\ &\alpha + f(x)(d(x, Tx), \dots, d(x, T^{n(x)} x); d(x, T^{m-n(x)} x), \dots, d(x, T^m x)) \leq \\ &\alpha + f(x)(\alpha, \dots, \alpha; \beta, \dots, \beta, d(x, T^m x)) \leq \alpha + g(x)(d(x, T^m x)); \end{aligned}$$

contradicting (4) and proving our assertion. In this case, letting  $x = x_0$  in  $Y$ , put

$$n_0 = n(x_0) = m_0, x_1 = T^{n_0} x_0 = T^{m_0} x_0$$

and, inductively,

$$n_i = n(x_i), m_i = n_0 + \dots + n_i, x_{i+1} = T^{n_i} x_i = T^{m_i} x_0, i \geq 1.$$

By the above claim,

$$d(x_0, T^m x_0) \leq t_0, m \in N, \text{ for some } t_0 > 0;$$

so that, combining with (2), we have

$$\begin{aligned} d(x_1, T^m x_1) &= d(T^{n_0} x_0, T^{n_0} T^m x_0) \leq f(x_0)(d(x_0, Tx_0), \dots, d(x_0, T^{n_0} x_0); \\ &d(x_0, T^m x_0), \dots, d(x_0, T^{n_0+m} x_0)) \leq g(x_0)(t_0), m \in N; \end{aligned}$$

or equivalently,

$$d(T^{m_0} x_0, T^m x_0) \leq g(x_0)(t_0), m \geq m_0;$$

this fact, again combined with (2), gives us

$$\begin{aligned} d(x_2, T^m x_2) &= d(T^{n_1} x_1, T^{n_1} T^m x_1) \leq f(x_1)(d(x_1, Tx_1), \dots, d(x_1, T^{n_1} x_1); \\ &d(x_1, T^m x_1), \dots, d(x_1, T^{n_1+m} x_1)) \leq g(x_1) \circ g(x_0)(t_0), m \in N; \end{aligned}$$

or equivalently,

$$d(T^{m_1} x_0, T^m x_0) \leq g(x_1) \circ g(x_0)(t_0), m \geq m_1;$$

and so on. By a finite induction procedure one gets

$$d(x_{k+1}, T^m x_{k+1}) \leq g(x_k) \circ \dots \circ g(x_0)(t_0), m, k \in N;$$

or equivalently,

$$d(T^{m_k} x_0, T^m x_0) \leq g(x_k) \circ \dots \circ g(x_0)(t_0), m \geq m_k, k \in N;$$

from which, combining with (iv), one arrives at the conclusion:

$$(T^n x_0; n \in N) \text{ is an increasing Cauchy (hence convergent) sequence.}$$

Finally, given the element  $y_0 \in Y$  with  $x_0 \leq y_0$ , put

$$y_1 = T^{n_0}y_0 \text{ and, inductively, } y_{i+1} = T^{n_i}y_i = T^{m_i}y_0, i \geq 1;$$

and observe that, again by the above claim (3), one has

$$d(x_0, T^m x_0), d(x_0, T^m y_0) \leq t_0, m \in N, \text{ for some } t_0 > 0.$$

This fact, combined with (2), leads us by the same procedure as before, at the estimate

$$d(x_{k+1}, T^m x_{k+1}), d(x_{k+1}, T^m y_{k+1}) \leq g(x_k) \circ \dots \circ g(x_0)(t_0), m, k \in N;$$

or equivalently

$$d(T^{m_k} x_0, T^m x_0), d(T^{m_k} x_0, T^m y_0) \leq g(x_k) \circ \dots \circ g(x_0)(t_0), m \geq m_k, k \in N,$$

proving that  $(T^m x_0; m \in N)$  and  $(T^m y_0; m \in N)$  have the same limit when it exists; and so, completing the argument. **Q.E.D.**

Let  $X$ ,  $d$  and  $\leq$  be endowed with their previous meaning. Given the mapping  $T$  from  $X$  to itself, let us call it *continuous at the left* when

$$x_n \uparrow x \text{ and } x_n \leq x, n \in N, \text{ imply } Tx_n \rightarrow Tx.$$

Also, the ambient quasi-ordering  $\leq$  will be said to be *self-closed*, when

$$x \leq y_n, n \in N \text{ and } y_n \uparrow y \text{ imply } x \leq y;$$

note that any semi-closed quasi-ordering in Nachbin's sense [10, Appendix] is necessarily self-closed.

As an immediate application of Lemma 1, the first main result of the present note is the following one.

**Theorem 2.** *Under the general hypotheses (i)-(iv), assume in addition that  $T$  is continuous at the left and  $\leq$  is self-closed.*

*Then, the following conclusions will be valid*

$$(C_1) \quad Z := X_T^c := \{x \in X; x = Tx\} \text{ is not empty}$$

$$(C_2) \quad \text{for every } x \in Y, (T^n x; n \in N) \text{ converges to an element of } Z$$

$$(C_3) \quad \text{if } x, y \in Y \text{ are comparable, } (T^n x; n \in N) \text{ and } (T^n y; n \in N) \text{ have the same limit (in } Z).$$

*Proof.* Indeed, it follows by Lemma 1 that, given  $x \in Y$ ,

$T^n x \uparrow z$  (and so  $T^n x \leq z$ ,  $n \in N$ ), for some  $z \in X$ ;

in which case, by the left continuity property,  $T^n x \uparrow Tz$  and the proof is complete. **Q.E.D.**

Let us call the considered quasi-ordering  $\leq$ ,

*anti self-closed* when  $y_n \leq x$ ,  $n \in N$ , and  $y_n \uparrow y$  imply  $y \leq x$ ;  
(observe at this moment that a sufficient condition for  $\leq$  to be anti self-closed is that  $\geq$  (its *dual*) be semi-closed)

*normal* when it is both self-closed and anti self-closed.

The following theorem (as another application of Lemma 1) is the second main result of the present note.

**Theorem 3.** *Under the conditions (i)-(iv), assume in addition that  $\leq$  is a normal ordering on  $X$ . Then, conclusions  $(C_1)$ – $(C_3)$  above continue to hold; and, moreover,*

$(C_4)$  for each  $x \in Y$  the element  $z = \lim_{n \rightarrow \infty} T^n x$  has the properties  
 $(P_1)$ :  $x \leq z$ ,  $(P_2)$ : if  $z \leq y \in Y$  then  $z = y$ .

*Proof.* Let  $x \in Y$  be arbitrary fixed. By Lemma 1,

$T^n x \uparrow z$  (hence  $x \leq T^n x \leq z$ ,  $n \in N$ ) for some  $z \in X$ .

It immediately follows that

$T^n x \leq Tz$ ,  $n \in N$ ;

so, by normality,  $z \leq Tz$ . Now,  $x \leq z \in Y$  gives, again by Lemma 1,

$T^n z \uparrow z$  (hence  $Tz \leq T^n z \leq z$ ,  $n \in N$ )

and therefore (as  $\leq$  is ordering)  $z \in Z$ . The remaining part is evident. **Q.E.D.**

A technical inspection of the additional conditions involved in the statements above shows that it is possible in some concrete cases that

neither  $T$  is continuous at the left

nor  $\leq$  is a normal ordering.

To discuss this eventuality, assume that, for any  $x \in Y$ , the function  $f(x)$  from  $R_+^{2n(x)+1}$  to  $R_+$  has the properties

(v) for each  $(\alpha_1, \dots, \alpha_{n(x)}) \in R_+^{n(x)}$  with  $\alpha_{n(x)} > 0$ , there exists  $\beta > 0$  with  
 $\beta + f(x)(\alpha_1, \dots, \alpha_{n(x)}; \beta, \dots, \beta) < \alpha_{n(x)}$

(vi) for each  $(\alpha_1, \dots, \alpha_{n(x)}) \in R_+^{n(x)}$  with  $\alpha_1 > 0$ ,  $\alpha_{n(x)} = 0$ , we have  $f(x)(\alpha_1, \dots, \alpha_{n(x)}; \alpha_1, \dots, \alpha_{n(x)}, \alpha_1) < \alpha_1$ .

Now, as a completion of the above results, we have

**Theorem 4.** *Again under the hypotheses (i)-(iv), assume in addition that (v)+(vi) hold and that  $\leq$  is a normal quasi-ordering. Then, conclusions  $(C_1)$ - $(C_3)$  still remain valid.*

*Proof.* Let  $x \in Y$  be a given element. By the above reasoning,

$$T^n x \uparrow z \text{ (hence } x \leq T^n x \leq z, n \in N) \text{ for some } z \in Y;$$

this fact, in conjunction with Lemma 1, gives us  $T^n z \uparrow z$ . Let  $\varepsilon > 0$  be arbitrary fixed; there exists  $k(\varepsilon) \in N$  such that

$$d(z, T^k z) \leq \varepsilon, \forall k \geq k(\varepsilon);$$

and this gives

$$\begin{aligned} d(z, T^{n(z)} z) &\leq d(z, T^m z) + d(T^{n(z)} z, T^m z) \leq d(z, T^m z) + \\ &f(z)(d(z, Tz), \dots, d(z, T^{n(z)} z); d(z, T^{m-n(z)} z), \dots, d(z, T^m z)) \leq \\ &\varepsilon + f(z)(d(z, Tz), \dots, d(z, T^{n(z)} z); \varepsilon, \dots, \varepsilon), \forall m \geq k(\varepsilon) + n(z); \end{aligned}$$

so, by (v),  $z = T^{n(z)} z$ . Furthermore,

$$\begin{aligned} d(z, Tz) &= d(T^{n(z)} z, T^{n(z)} Tz) \leq \\ &f(z)(d(z, Tz), \dots, d(z, T^{n(z)} z); d(z, Tz), \dots, d(z, T^{n(z)} z), d(z, T^{n(z)} Tz)) = \\ &f(z)(d(z, Tz), \dots, d(z, T^{n(z)-1} z), 0; d(z, Tz), \dots, d(z, T^{n(z)-1} z), 0, d(z, Tz)); \end{aligned}$$

and therefore, if we suppose  $z \neq Tz$ , (vi) will be contradicted. This completes the proof of Theorem 4. **Q.E.D.**

**Remark.** Theorem 2 may be viewed as a quasi-order extension of Sehgal's result quoted before (cf. also Dugundji and Granas [3, Ch 1, Sect 3]) while Theorem 4 is a quasi-order "functional" version of Matkowski's contribution stated as Theorem 1; remark at this point the essential role of the variables of rank

$$1, n(x), n(x) + 1, 2n(x), 2n(x) + 1$$

in the conditions (v)+(vi) involving  $f(x)$ . At the same time, Theorem 3 – although formulated as a fixed point result – may be deemed in fact as a maximality principle in  $(Y, \leq)$ , being comparable under this perspective to a related author's one [14], obtained by means of a "compactness" procedure (cf. Krasnoselskii and Sobolev [8]).



To complete our exposition, it would be desirable to show by a concrete example that our theorems are effectively independent from the other contributions in this area subsumed to Theorem 1.

Let  $A$  denote the subset of all  $(x, y) \in \mathbb{R}^2$  with  $x \geq 0$ , endowed with the Euclidean metric and with the ordering:

$$(x, y) \leq (x', y') \text{ if and only if } x \geq x', y \geq y';$$

and let  $X$  indicate the subset of  $A$  defined as

$$X = \{(\frac{1}{n}, \frac{1}{n}); n \geq 1\} \cup \{(0, 0), (2, -2), (3, -3)\},$$

endowed with the induced metric and ordering; clearly,  $X$  is a complete (hence order complete) metric space and  $\leq$  is a normal ordering on  $X$ . Define a self-mapping  $T$  of  $X$  by

$$T(\frac{1}{n}, \frac{1}{n}) = (\frac{1}{n+1}, \frac{1}{n+1}), n \geq 1,$$

$$T(0, 0) = (0, 0), T(2, -2) = (3, -3), T(3, -3) = (2, -2).$$

Of course, (i) holds with

$$Y = \{(\frac{1}{n}, \frac{1}{n}); n \geq 1\} \cup \{(0, 0)\}.$$

At the same time, given  $x = (\frac{1}{n}, \frac{1}{n}) \in Y$ , we have, for any  $y = (\frac{1}{m}, \frac{1}{m}) \in Y$  with  $m \geq n$ , the relations

$$d(Tx, Ty) \leq \frac{n}{n+1}d(x, y),$$

proving (ii) holds too with

$$f(x)(t) = g(x)(t) = \frac{n}{n+1}t, t \geq 0.$$

Finally, observe that (iii) is trivially satisfied in our case; while (iv) reduces to the evident relation

$$\lim_{k \rightarrow \infty} \frac{n}{n+k} = 0, n \in \mathbb{N}.$$

Consequently, Theorem 3 is applicable to this context, the only fixed point of  $T$  being the origin. However, conditions of Theorem 1 are not fulfilled; because, e.g.,  $(T^n(2, -2); n \in \mathbb{N})$  does not converge to  $(0, 0)$ . This proves our claim.

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