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# Computing the Chow Variety of Quadratic Space Curves

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**Abstract.** Quadrics in the Grassmannian of lines in 3-space form a 19-dimensional projective space. We study the subvariety of coisotropic hypersurfaces. Following Gel'fand, Kapranov and Zelevinsky, it decomposes into Chow forms of plane conics, Chow forms of pairs of lines, and Hurwitz forms of quadric surfaces. We compute the ideals of these loci.

## Introduction

The Chow variety, introduced in 1937 by Chow and van der Waerden [3], parameterizes algebraic cycles of any fixed dimension and degree in a projective space, each given by its Chow form. The case of curves in  $\mathbb{P}^3$  goes back to an 1848 paper by Cayley [2]. A fundamental problem, addressed by Green and Morrison [7] as well as Gel'fand, Kapranov and Zelevinsky [5, §4.3], is to describe the equations defining Chow varieties. We present a definitive computational solution for the smallest non-trivial case, namely for cycles of dimension 1 and degree 2 in  $\mathbb{P}^3$ .

The *Chow form* of a cycle of degree 2 is a quadratic form in the Plücker coordinates of the Grassmannian  $G(2, 4)$  of lines in  $\mathbb{P}^3$ . Such a quadric in  $G(2, 4)$  represents the set of all lines that intersect the given cycle. Quadratic forms in Plücker coordinates form a projective space  $\mathbb{P}^{19}$ . The Chow variety we are interested in, denoted  $G(2, 2, 4)$ , is the set of all Chow forms in that  $\mathbb{P}^{19}$ . The aim of this note is to make the concepts in [2,3,7] and [5, §4.3] completely explicit.

We start with the 9-dimensional subvariety of  $\mathbb{P}^{19}$  whose points are the *coisotropic quadrics* in  $G(2, 4)$ . By [5, §4.3, Theorem 3.14], this decomposes as the Chow variety and the variety of Hurwitz forms [8], representing lines that are tangent to a quadric surface in  $\mathbb{P}^3$ . Section 1 studies the ideal generated by the coisotropy conditions. We work in a polynomial ring in 20 variables, one for each quadratic Plücker monomial on  $G(2, 4)$  minus one for the Plücker relation. We derive the coisotropic ideal from the differential characterization of coisotropy. Proposition 1 exhibits the decomposition of this ideal into three minimal primes.

Section 2 derives the radical ideal of the Chow variety  $G(2, 2, 4)$  in  $\mathbb{P}^{19}$ . Its two minimal primes represent Chow forms of plane conics and Chow forms of pairs of lines. We also study the characterization of Chow forms among all coisotropic quadrics by the vanishing of certain differential forms. These represent the integrability of the  $\alpha$ -distribution in [5, §4.3, Theorem 3.22]. After saturation by the irrelevant ideal, the integrability ideal is found to be radical.

## 1 Coisotropic Quadrics

The Grassmannian  $G(2, 4)$  is a quadric in  $\mathbb{P}^5$ . Its points are lines in  $\mathbb{P}^3$ . We represent these lines using dual Plücker coordinates  $\mathbf{p} = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$  subject to the Plücker relation  $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$ . Following [8, §2], by *dual* coordinates we mean that  $p_{ij}$  is the  $ij$ -minor of a  $2 \times 4$ -matrix whose rows span the line. The generic quadric in  $G(2, 4)$  is written as a generic quadratic form

$$Q(\mathbf{p}) = \mathbf{p} \cdot \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_1 & c_6 & c_7 & c_8 & c_9 & c_{10} \\ c_2 & c_7 & c_{11} & c_{12} & c_{13} & c_{14} \\ c_3 & c_8 & c_{12} & c_{15} & c_{16} & c_{17} \\ c_4 & c_9 & c_{13} & c_{16} & c_{18} & c_{19} \\ c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \end{pmatrix} \cdot \mathbf{p}^T. \quad (1)$$

The quadric  $Q(\mathbf{p})$  is an element in  $V := \mathbb{C}[\mathbf{p}]_2 / \mathbb{C}\{p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}\} \simeq \mathbb{C}^{21} / \mathbb{C}$ . Hence,  $\mathbf{c} = (c_0, c_1, \dots, c_{20})$  serves as homogeneous coordinates on  $\mathbb{P}^{19} = \mathbb{P}(V)$ , which – due to the Plücker relation – need to be understood modulo

$$c_5 \mapsto c_5 + \lambda, \quad c_9 \mapsto c_9 - \lambda, \quad c_{12} \mapsto c_{12} + \lambda. \quad (2)$$

The coordinate ring  $\mathbb{Q}[V]$  is a subring of  $\mathbb{Q}[c_0, c_1, \dots, c_{20}]$ , namely it is the invariant ring of the additive group action (2). Hence  $\mathbb{Q}[V]$  is the polynomial ring in 20 variables  $c_0, c_1, c_2, c_3, c_4, c_5 - c_{12}, c_6, c_7, c_8, c_9 + c_{12}, c_{10}, c_{11}, c_{13}, \dots, c_{20}$ .

We are interested in the  $\mathbf{c}$ 's that lead to *coisotropic* hypersurfaces of  $G(2, 4)$ . For these, the tangent space at any point  $\ell$ , considered as a subspace of  $T_\ell G(2, 4) = \text{Hom}(\ell, \mathbb{C}^4/\ell)$ , has the form  $\{\varphi \mid \varphi(a) = 0\} + \{\varphi \mid \text{im}(\varphi) \subset M\}$ , for some  $a \in \ell \setminus \{0\}$  and some plane  $M$  in  $\mathbb{C}^4/\ell$ . By [5, §4.3, (3.24)], the quadric hypersurface  $\{Q(\mathbf{p}) = 0\}$  in  $G(2, 4)$  is coisotropic if and only if there exist  $s, t \in \mathbb{C}$  such that

$$\frac{\partial Q}{\partial p_{01}} \cdot \frac{\partial Q}{\partial p_{23}} - \frac{\partial Q}{\partial p_{02}} \cdot \frac{\partial Q}{\partial p_{13}} + \frac{\partial Q}{\partial p_{03}} \cdot \frac{\partial Q}{\partial p_{12}} = s \cdot Q + t \cdot (p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}).$$

Equivalently, the vector  $(t, s, -1)^T$  is in the kernel of the  $21 \times 3$  matrix in Figure 1. The  $3 \times 3$  minors of this matrix are all in the subring  $\mathbb{Q}[V]$ . The *coisotropic ideal*  $I$  is the ideal of  $\mathbb{Q}[V]$  generated by these minors. The subscheme  $V(I)$  of  $\mathbb{P}^{19} = \mathbb{P}(V)$  represents all coisotropic hypersurfaces  $\{Q = 0\}$  of degree two in  $G(2, 4)$ . Using computations with Maple and Macaulay2 [6], we found that  $I$  has codimension 10, degree 92 and is minimally generated by 175 cubics. Besides,  $V(I)$  is the reduced union of three components, of dimensions nine, eight and five.

**Proposition 1.** *The coisotropic ideal is the intersection of three prime ideals:*

$$I = P_{\text{Hurwitz}} \cap P_{\text{ChowLines}} \cap P_{\text{Squares}}. \quad (3)$$

*So,  $I$  is radical. The prime  $P_{\text{Hurwitz}}$  has codimension 10 and degree 92, it is minimally generated by 20 quadrics, and its variety  $V(P_{\text{Hurwitz}})$  consists of Hurwitz forms of quadric surfaces in  $\mathbb{P}^3$ . The prime  $P_{\text{ChowLines}}$  has codimension 11 and*



means that  $G(2, 2, 4) = V(P_{\text{ChowConic}}) \cup V(P_{\text{ChowLines}})$ , where  $P_{\text{ChowConic}}$  is the homogeneous prime ideal in  $\mathbb{Q}[V]$  whose variety comprises the Chow forms of irreducible curves of degree two in  $\mathbb{P}^3$ . The ideal  $P_{\text{ChowConic}}$  has codimension 11 and degree 92, and it is minimally generated by 21 quadrics and 35 cubics. The radical ideal  $P_{\text{ChowConic}} \cap P_{\text{ChowLines}}$  has codimension 11, degree  $232 = 92 + 140$ , and it is minimally generated by 230 cubics.

Since  $G(2, 2, 4)$  should be contained in the coisotropic variety  $V(I)$ , it seems that  $P_{\text{ChowConic}}$  is missing from the decomposition (3). Here is the explanation:

**Proposition 2.** *Every Chow form of a plane conic in  $\mathbb{P}^3$  is also a Hurwitz form. In symbols,  $P_{\text{Hurwitz}} \subset P_{\text{ChowConic}}$  and thus  $V(P_{\text{ChowConic}}) \subset V(P_{\text{Hurwitz}})$ .*

Our first proof is by computer: just check the inclusion of ideals in Macaulay2. For a conceptual proof, we consider a  $4 \times 4$ -symmetric matrix  $M = M_0 + \epsilon M_1$ , where  $\text{rank}(M_0) = 1$ . By [8, eqn. (1)], the Hurwitz form of the corresponding quadric surface in  $\mathbb{P}^3$  is  $Q(\mathbf{p}) = \mathbf{p}(\wedge_2 M)\mathbf{p}^T$ . Divide by  $\epsilon$  and let  $\epsilon \rightarrow 0$ . The limit is the Chow form of the plane conic defined by restricting  $M_1$  to  $\ker(M_0) \simeq \mathbb{P}^2$ . This type of degeneration is familiar from the study of complete quadrics [4]. Proposition 2 explains why the locus of irreducible curves is not visible in (3).

Gel'fand, Kapranov and Zelevinsky [5, §4.3] introduce a class of differential forms in order to discriminate Chow forms among all coisotropic hypersurfaces. In their setup, these forms represent the integrability of the  $\alpha$ -distribution  $\mathcal{E}_{\alpha, Z}$ . We shall apply the tools of computational commutative algebra to shed some light on the characterization of Chow forms via integrability of  $\alpha$ -distributions.

For this, we use local affine coordinates instead of Plücker coordinates. A point in the Grassmannian  $G(2, 4)$  is represented as the row space of the matrix

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}. \quad (5)$$

We express the quadrics  $Q$  in (1) in terms of the local coordinates  $a_2, a_3, b_2, b_3$ , by substituting the Plücker coordinates with the minors of the matrix (5), i.e.,

$$p_{01} = 1, \quad p_{02} = b_2, \quad p_{03} = b_3, \quad p_{12} = -a_2, \quad p_{13} = -a_3, \quad p_{23} = a_2 b_3 - b_2 a_3. \quad (6)$$

We consider the following differential 1-forms on affine 4-space:

$$\begin{aligned} \alpha_1^1 &:= \frac{\partial Q}{\partial a_2} da_2 + \frac{\partial Q}{\partial a_3} da_3, & \alpha_2^1 &:= \frac{\partial Q}{\partial a_2} db_2 + \frac{\partial Q}{\partial a_3} db_3, \\ \alpha_1^2 &:= \frac{\partial Q}{\partial b_2} da_2 + \frac{\partial Q}{\partial b_3} da_3, & \alpha_2^2 &:= \frac{\partial Q}{\partial b_2} db_2 + \frac{\partial Q}{\partial b_3} db_3. \end{aligned} \quad (7)$$

By taking wedge products, we derive the 16 differential 4-forms

$$dQ \wedge d\alpha_j^i \wedge \alpha_l^k = q_{ijkl} \cdot da_2 \wedge da_3 \wedge db_2 \wedge db_3 \quad \text{for } i, j, k, l \in \{1, 2\}. \quad (8)$$

Here the expressions  $q_{ijkl}$  are certain polynomials in  $\mathbb{Q}[V][a_2, a_3, b_2, b_3]$ .

Theorems 3.19 and 3.22 in [5, §4.3] state that a squarefree coisotropic quadric  $Q$  is a Chow form if and only if all 16 coefficients  $q_{ijkl}$  are multiples of  $Q$ . By

taking normal forms of the polynomials  $q_{ijkl}$  modulo the principal ideal  $\langle Q \rangle$ , we obtain a collection of 720 homogeneous polynomials in  $\mathbf{c}$ . Among these, 58 have degree three, 340 have degree four, and 322 have degree five. The aforementioned result implies that these 720 polynomials cut out  $G(2, 2, 4)$  as a subset of  $\mathbb{P}^{19}$ .

The *integrability ideal*  $J \subset \mathbb{Q}[V]$  is generated by these 720 polynomials and their analogues from other affine charts of the Grassmannian, obtained by permuting columns in (5). We know that  $V(J) = G(2, 2, 4)$  holds set-theoretically. Maple, Macaulay2 and Magma verified for us that it holds scheme-theoretically:

**Proposition 3.** *The integrability ideal  $J$  is minimally generated by 210 cubics. Writing  $\mathbf{m}$  for the irrelevant ideal  $\langle c_0, c_1, \dots, c_{20} \rangle$  of  $\mathbb{Q}[V]$ , we have*

$$\sqrt{J} = (J : \mathbf{m}) = P_{\text{ChowConic}} \cap P_{\text{ChowLines}} \cap P_{\text{Squares}}. \quad (9)$$

## Conclusion

We reported on computational experiments with hypersurfaces in the Grassmannian  $G(2, 4)$  that are associated to curves and surfaces in  $\mathbb{P}^3$ . For degree 2, all relevant parameter spaces were described by explicit polynomials in 20 variables. All ideals and computations discussed in this note can be obtained at

[www3.math.tu-berlin.de/algebra/static/pluecker/](http://www3.math.tu-berlin.de/algebra/static/pluecker/)

Many possibilities exist for future work. Obvious next milestones are the ideals for the Chow varieties of degree 3 cycles in  $\mathbb{P}^3$ , and degree 2 cycles in  $\mathbb{P}^4$ . Methods from representation theory promise a compact encoding of their generators, in terms of irreducible  $\text{GL}(4)$ -modules. Another question we aim to pursue is motivated by the geometry of condition numbers [1]: express the volume of a tubular neighborhood of a coisotropic quadric in  $G(2, 4)$  as a function of  $\mathbf{c}$ .

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