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PERSISTENCE EXPONENT FOR RANDOM WALK ON DIRECTED VERSIONS OF \mathbb{Z}^2

NADINE GUILLOTIN-PLANTARD AND FRANÇOISE PÈNE

ABSTRACT. We study the persistence exponent for random walks in random sceneries (RWRS) with integer values and for some special random walks in random environment in \mathbb{Z}^2 including random walks in \mathbb{Z}^2 with random orientations of the horizontal layers.

1. INTRODUCTION AND MAIN RESULTS

Random walks in random sceneries were introduced independently by H. Kesten and F. Spitzer [15] and by A. N. Borodin [6]. Let $S = (S_n)_{n \geq 0}$ be a random walk in \mathbb{Z} starting at 0, i.e., $S_0 = 0$ and $X_n := S_n - S_{n-1}, n \geq 1$ is a sequence of i.i.d. (independent identically distributed) \mathbb{Z} -valued random variables. Let $\xi = (\xi_x)_{x \in \mathbb{Z}}$ be a field of i.i.d. \mathbb{Z} -valued random variables independent of S . The field ξ is called the random scenery. *The random walk in random scenery (RWRS)* $Z := (Z_n)_{n \geq 0}$ is defined by setting $Z_0 := 0$ and, for $n \in \mathbb{N}^*$,

$$Z_n := \sum_{i=1}^n \xi_{S_i}. \quad (1)$$

We will denote by \mathbb{P} the joint law of S and ξ . Limit theorems for RWRS have a long history, we refer to [14] for a complete review.

In the following, we consider the case when the common distribution of the scenery ξ_x is assumed to be symmetric with a third moment and with positive variance σ_ξ^2 . Concerning the random walk $(S_n)_{n \geq 1}$, the distribution of X_1 is assumed to be centered and square integrable with positive variance σ_X^2 . We assume without any loss of generality that neither the support of the distribution of X_1 nor the one of ξ_0 are contained in a proper subgroup of \mathbb{Z} .¹

Under the previous assumptions, the following weak convergence holds in the space of càdlàg real-valued functions defined on $[0, \infty)$, endowed with the Skorokhod topology (with respect to the classical J_1 -metric):

$$\left(n^{-\frac{1}{2}} S_{\lfloor nt \rfloor} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (\sigma_X Y(t))_{t \geq 0},$$

where Y is a standard real Brownian motion. We will denote by $(L_t(x))_{x \in \mathbb{R}, t \geq 0}$ a continuous version with compact support of the local time of the process $(\sigma_X Y(t))_{t \geq 0}$ (see [19]). In [15], Kesten and Spitzer proved the convergence in distribution of $((n^{-3/4} Z_{\lfloor nt \rfloor})_{t \geq 0})_n$, to a process

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¹If the subgroup of \mathbb{Z} generated by the support of the distribution of X_1 is $m\mathbb{Z}$ for some $m > 1$, then we replace $(S_n)_n$ by $(S_n/m)_n$ and $(\xi_x)_x$ by $(\xi_{mx})_x$ and we observe that these changes do not affect the RWRS Z .

If the greatest common divisor of the support of the distribution of ξ_0 is $\tilde{m} > 1$, we divide the RWRS Z by \tilde{m} .

$\Delta = (\Delta_t)_{t \geq 0}$ defined by

$$\Delta_t := \sigma_\xi \int_{\mathbb{R}} L_t(x) dW(x),$$

where $(W(x))_{x \geq 0}$ and $(W(-x))_{x \geq 0}$ are independent standard Brownian motions independent of Y . The process Δ is called Kesten-Spitzer process in the literature. We are interested in the persistence properties of the sum $Z_n, n \geq 1$. Our main result in this setup is the following one.

Theorem 1. *There exists a constant $c > 0$ such that for large enough T*

$$\mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \right] \leq T^{-1/4} (\log T)^c. \quad (2)$$

If moreover $\mathbb{E}[e^{\xi_1}] < \infty$, then there exist positive constants c', c'' and T_0 such that

$$T^{-1/4} (\log T)^{-c'} \left[H^{-1} \left(c'' T^{-1/4} \right) \right]^{-1} \leq \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \right] \quad (3)$$

for every $T > T_0$, where H is given by $H(t) := \mathbb{E}[e^{\xi_1} \mathbf{1}_{\{e^{\xi_1} > t\}}]$ and where $H^{-1}(x) := \inf\{t > 0 : H(t) < x\}$, for every $x > 0$.

In particular, if there exist $\eta > 1$, $A_1 > 0$ and $A_2 > 0$ such that

$$\forall x > 0, \quad \mathbb{P}(\xi_0 > x) \leq A_1 e^{-A_2 x^\eta},$$

then there exist $c' > 0$ and $T_0 > 0$ such that

$$T^{-1/4} e^{-c' (\log T)^{\frac{1}{\eta}}} \leq \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \right] \quad (4)$$

for every $T > T_0$.

If the distribution of ξ_1 has compact support, then there exist positive constants c' and T_0 such that

$$T^{-1/4} (\log T)^{-c'} \leq \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \right] \quad (5)$$

for every $T > T_0$.

The corresponding results for the continuous-time Kesten-Spitzer process Δ were obtained in [10], also cf. [18, 20, 11]. The case of random walk in random gaussian scenery was treated in [3] with a lower bound in $T^{-1/4} e^{-c' \sqrt{\log T}}$, coherent with our result. We would particularly like to stress that in all these results, the scenery was supposed to be gaussian.

Now we will state an analogous result for particular models of random walks $(M_n)_n$ in random environment on \mathbb{Z}^2 including random walks on \mathbb{Z}^2 with random orientation of the horizontal layers.

To the y -th horizontal line, we associate the \mathbb{Z} -valued random variable ξ_y , corresponding to the only authorized horizontal displacement of the walk $(M_n)_n$ on this horizontal line. We assume that $(\xi_y)_{y \in \mathbb{Z}}$ is a sequence of i.i.d. random variables the distribution of which is symmetric, has a moment of order 3 and a positive variance σ_ξ^2 . We consider a distribution ν on \mathbb{Z} admitting a variance and with null expectation (corresponding to the distribution of the vertical displacements when vertical displacement occur). We fix a parameter $\delta \in (0, 1)$. We consider a random walk in random environment $M = (M_n)_n$ on \mathbb{Z}^2 starting from the origin (i.e. $M_0 := (0, 0)$), moving horizontally (with respect to $(\xi_y)_y$) with probability δ and moving vertically (with respect to ν) with probability $1 - \delta$ as follows:

$$\mathbb{P}(M_{n+1} = (x + \xi_y, y) | M_n = (x, y)) = \delta \quad (\text{horizontal displacement})$$

$$\mathbb{P}(M_{n+1} = (x, y + z) | M_n = (x, y)) = (1 - \delta) \nu(\{z\}) \quad (\text{vertical displacement}).$$

Observe that if the ξ_y 's have Rademacher distribution (i.e. takes their values in $\{-1, 1\}$), then M is a walk on \mathbb{Z}^2 with random orientations of the horizontal layers, the y -th horizontal layer being oriented to the left if $\xi_y = -1$ and to the right if $\xi_y = 1$. Such models have been considered by Matheron and de Marsilly in [20], their transience has been established by Campanino and P  tritis in [8], see also [13] for their asymptotic behaviour and [9] for local limit theorem in this context.

The process M is strongly related to RWRS. Indeed it can be represented as follows

$$M_n = (\tilde{Z}_n, S_n) = \left(\sum_{k=1}^n \xi_{S_k} \varepsilon_k, S_n \right), \quad S_n := \sum_{k=1}^n \tilde{X}_k (1 - \varepsilon_k),$$

where $(\tilde{X}_k)_k$ is a sequence of i.i.d. random variables with distribution ν and $(\varepsilon_k)_k$ is a sequence of i.i.d. Bernoulli random variables with parameter δ (i.e. $\mathbb{P}(\varepsilon_k = 1) = \delta = 1 - \mathbb{P}(\varepsilon_k = 0)$). We assume that $(\xi_y)_y$, $(\tilde{X}_k)_k$ and $(\varepsilon_k)_k$ are independent. We then set $X_k := \tilde{X}_k (1 - \varepsilon_k)$. As for RWRS, we assume without any loss of generality that neither the support of ν nor the one of the distribution of ξ_0 are contained in a proper subgroup of \mathbb{Z} . Observe that the second coordinate S of M is a random walk. Hence we focus our study on the first coordinate \tilde{Z} of M , which is very similar to RWRS. Our second main result states that the conclusion of Theorem 1 is still valid for \tilde{Z} .

Theorem 2 (Persistence of \mathbf{M} on the leftside). *There exists a constant $c > 0$ such that for large enough T*

$$\mathbb{P} \left[\max_{k=1, \dots, T} \tilde{Z}_k \leq 1 \right] \leq T^{-1/4} (\log T)^{+c}. \quad (6)$$

If moreover $\mathbb{E}[e^{\xi_1}] < \infty$, then there exist positive constants c' , c'' and T_0 such that

$$T^{-1/4} (\log T)^{-c'} \left[H^{-1} \left(c'' T^{-\frac{1}{4}} \right) \right]^{-1} \leq \mathbb{P} \left[\max_{k=1, \dots, T} \tilde{Z}_k \leq 1 \right], \quad (7)$$

for every $T > T_0$. The function H is defined as in Theorem 1.

Let us recall that Z and \tilde{Z} are stationary but non-markovian processes with respect to the annealed distribution \mathbb{P} and that they are markovian but non-stationary given the scenery ξ .

In Section 2, we prove some useful technical lemmas concerning the random walk S as well as the random walk in random scenery Z and the analogous process \tilde{Z} . Section 3 is devoted to the proof of Theorems 1 and 2.

2. PRELIMINARY RESULTS

For every $y \in \mathbb{Z}$ and every integer $n \geq 1$, we write $N_n(y)$ for the number of visits of the walk S to site y before time n , i.e.

$$N_n(y) := \#\{k = 1, \dots, n : S_k = y\}.$$

Using this notation, we observe that Z can be rewritten as follows:

$$Z_n = \sum_{y \in \mathbb{Z}} \xi_y N_n(y).$$

Analogously

$$\tilde{Z}_n = \sum_{y \in \mathbb{Z}} \xi_y \tilde{N}_n(y),$$

with $\tilde{N}_n(y) := \#\{k = 1, \dots, n : S_k = y \text{ and } \varepsilon_k = 1\}$. The behaviour of $(\tilde{N}_n(y))_y$ will appear to be very similar to the behaviour of $(N_n(y))_y$, at least for our purpose.

2.1. Preliminary results on the random walk. We set $N_n^* = \sup_y N_n(y)$ and $R_n := \#\{y \in \mathbb{Z} : N_n(y) > 0\}$ for the number of sites that have been visited by the walk S before time n .

Lemma 3. *Let $\gamma \in (0, \frac{1}{2})$. We set*

$$\Omega_n^{(1)}(\gamma) := \{N_n^* \leq n^{\frac{1}{2}+\gamma}, R_n \leq n^{\frac{1}{2}+\gamma}\}$$

There exists $C_\gamma > 0$ such that

$$\mathbb{P}[\Omega_n^{(1)}(\gamma)] = 1 - O(\exp(-C_\gamma n^\gamma)).$$

Proof. Due to Lemma 34 of [9], we know that there exists $c_\gamma > 0$ such that $\mathbb{P}[R_n \leq n^{\frac{1}{2}+\gamma}] = 1 - O(\exp(-c_\gamma n^\gamma))$. Let us prove that the argument therein can be adapted to prove the same result for N_n^* instead of R_n . Observe first that $a \mapsto \mathbb{P}[N_n^* \geq a]$ is sub-multiplicative. Indeed, let a, b be two positive integers. Let us write $\tau_a := \min\{k \geq 1, N_k^* = a\}$.

$$\begin{aligned} \mathbb{P}[N_n^* \geq a + b] &= \sum_{j=1}^n \mathbb{P}[\tau_a = j, N_n^* - N_j^* \geq b] \\ &\leq \sum_{j=1}^n \mathbb{P}[\tau_a = j, \sup_y (N_n(y) - N_j(y)) \geq b] \\ &\leq \sum_{j=1}^n \mathbb{P}[\tau_a = j] \mathbb{P}[N_{n-j}^* \geq b] \\ &\leq \mathbb{P}[N_n^* \geq a] \mathbb{P}[N_n^* \geq b]. \end{aligned}$$

Hence $\mathbb{P}[N_n^* \geq ab] \leq (\mathbb{P}[N_n^* \geq a])^b$ and so

$$\begin{aligned} \mathbb{P}[N_n^* \geq \mathbb{E}[N_n^*] n^\gamma] &\leq \mathbb{P}[N_n^* \geq \lfloor 3\mathbb{E}[N_n^*] \rfloor]^{n^\gamma/3} \\ &\leq \left(\frac{\mathbb{E}[N_n^*]}{\lfloor 3\mathbb{E}[N_n^*] \rfloor} \right)^{n^\gamma/3} \leq 2^{-\lfloor n^\gamma/3 \rfloor}. \end{aligned}$$

We conclude by using the fact that $\mathbb{E}[N_n^*] \sim c' \sqrt{n}$. □

Lemma 4. *Let $\mu \in (0, 1]$ and $\gamma \in (0, 1/2)$ and $\vartheta > 0$ such that $\gamma > 2(1 - \mu)\vartheta$. For any $\delta \in (0, \frac{\gamma}{2} - (1 - \mu)\vartheta)$, we have*

$$\mathbb{P} \left[\sup_{y, z \in \mathbb{Z}, 0 < |y-z| < n^\vartheta} \frac{|N_n(y) - N_n(z)|}{|y-z|^\mu} > n^{\frac{1}{4}+\gamma} \right] = O \left(e^{-n^{\frac{\delta}{2}}} \right). \quad (8)$$

Under the assumptions of Theorem 2, we also have

$$\mathbb{P} \left[\sup_{y, z \in \mathbb{Z}, 0 < |y-z| < n^\vartheta} \frac{|\tilde{N}_n(y) - \tilde{N}_n(z)|}{|y-z|^\mu} > n^{\frac{1}{4}+\gamma} \right] = O \left(e^{-n^{\frac{\delta}{2}}} \right). \quad (9)$$

Proof. Due to Lemma 3, it is enough to prove that

$$\mathbb{P} \left[\sup_{k=1, \dots, n} |S_k| > e^{n^{\frac{\delta}{2}}} \right] = O \left(e^{-n^{\frac{\delta}{2}}} \right) \quad (10)$$

and that

$$\mathbb{P} \left[\sup_{y, z \in E_n, 0 < |y-z| < n^\vartheta} \frac{N_n(y) - N_n(z)}{|y-z|^\mu} > n^{\frac{1}{4}+\gamma} \right] = O \left(e^{-n^\delta} \right), \quad (11)$$

where $E_n := \{y \in \mathbb{Z} : |y| \leq e^{n^{\frac{\delta}{2}}}, N_n(y) \leq n^{\frac{1}{2}+\gamma}\}$ (and the analogous estimate with $N_n(\cdot)$ replaced by $\tilde{N}_n(\cdot)$ under the assumptions of Theorem 2). We start with the proof of the first estimate. From Doob's inequality, there exists some constant $C > 0$ such that

$$\mathbb{P}\left[\sup_{k=1,\dots,n} |S_k| > e^{n^{\frac{\delta}{2}}}\right] \leq C \mathbb{E}[S_n^2] e^{-2n^{\frac{\delta}{2}}} = O\left(ne^{-2n^{\frac{\delta}{2}}}\right)$$

so (10). Let us prove now the second estimate. Let $\tau_j(y)$ be the j -th visit time of $(S_n)_n$ to y , that is

$$\tau_0(y) := 0 \quad \text{and} \quad \forall j \geq 0, \tau_{j+1}(y) = \inf\{k > \tau_j(y) : S_k = y\}$$

(resp. $\tilde{\tau}_0(y) := 0, \tilde{\tau}_{j+1}(y) = \inf\{k > \tilde{\tau}_j(y) : S_k = y, \varepsilon_k = 1\}$). Let $y, z \in E_n$ be such that $N_n(y) - N_n(z) > 0$, then there exists $j \in \{1, \dots, \lfloor n^{\frac{1}{2}+\gamma} \rfloor\}$ such that $\tau_j(y) \leq n < \tau_{j+1}(y)$ (observe that $\tau_{\lfloor n^{\frac{1}{2}+\gamma} \rfloor + 1}(y) > n$). For this choice of j , we have

$$N_n(y) - N_n(z) \leq N_{\tau_j(y)}(y) - N_{\tau_j(y)}(z).$$

Therefore

$$\begin{aligned} \mathbb{P}\left[\sup_{y,z \in E_n, 0 < |y-z| < n^\vartheta} \frac{N_n(y) - N_n(z)}{|y-z|^\mu} > n^{\frac{1}{4}+\gamma}\right] \\ \leq \sum_{y,z \in E_n, 0 < |y-z| < n^\vartheta} \sum_{j=1}^{\lfloor n^{\frac{1}{2}+\gamma} \rfloor} \mathbb{P}\left[N_{\tau_j(y)}(y) - N_{\tau_j(y)}(z) \geq |y-z|^\mu n^{\frac{1}{4}+\gamma}\right], \\ \leq \sum_{y,z \in E_n, 0 < |y-z| < n^\vartheta} \sum_{j=1}^{\lfloor n^{\frac{1}{2}+\gamma} \rfloor} \mathbb{P}\left[1 + \sum_{k=1}^{j-1} (1 - M_k(y, z)) \geq |y-z|^\mu n^{\frac{1}{4}+\gamma}\right], \quad (12) \end{aligned}$$

since $N_{\tau_j(y)}(y) - N_{\tau_j(y)}(z) = j - \sum_{k=0}^{j-1} M_k(y, z) \leq j - \sum_{k=1}^{j-1} M_k(y, z)$, where, following [15], we write $M_k(y, z)$ for the number of visits of $(S_n)_n$ to z between its k -th and $(k+1)$ -th visit to y , i.e.

$$M_k(y, z) := \sum_{\tau_k(y) < n \leq \tau_{k+1}(y)} \mathbf{1}_{\{S_n = z\}}.$$

Under assumptions of Theorem 2, (12) still holds for $\tilde{N}_n(\cdot)$ instead of $N_n(\cdot)$ if we replace $\tau_j(y)$ by $\tilde{\tau}_j(y)$ and $M_k(y, z)$ by $\tilde{M}_k(y, z) := \sum_{\tilde{\tau}_k(y) < n \leq \tilde{\tau}_{k+1}(y)} \mathbf{1}_{\{S_n = z, \varepsilon_n = 1\}}$.

Due to the strong Markov property, $(M_k(y, z))_{k \geq 1}$ is a sequence of i.i.d. random variables. Let us recall (see pages 13-14 in [15] for more details) that its common law is given by

$$\mathbb{P}[M_k(y, z) = 0] = 1 - p(|y - z|), \quad \forall \ell \geq 1, \quad \mathbb{P}[M_k(y, z) = \ell] = (1 - p(|y - z|))^{\ell-1} (p(|y - z|))^2,$$

with $p(x) = p(-x) \sim \tilde{c}|x|^{-1}$. Observe also that, under the assumptions of Theorem 2, $(\tilde{M}_k(y, z))_{k \geq 1}$ is a sequence of i.i.d. random variables with $\mathbb{P}[\tilde{M}_k(y, z) = 0] = 1 - \tilde{p}(z - y)$ and $\mathbb{P}[\tilde{M}_k(y, z) = \ell] = \tilde{p}(z - y)(1 - \tilde{p}(y - z))^{\ell-1} \tilde{p}(y - z)$ if $\ell \geq 1$ where $\tilde{p}(x)$ denotes the probability that $(S_n, \varepsilon_n)_{n \geq 1}$ visits $(x, 1)$ before $(0, 1)$. Observe that ²

$$\tilde{p}(x) = \sum_{k \geq 0} (1 - \delta - p(x))^k p(x) (1 - \tilde{p}(-x)) = \frac{p(x)}{\delta + p(x)} (1 - \tilde{p}(-x)).$$

²The fact that $(S_n, \varepsilon_n)_{n \geq 1}$ visits $(x, 1)$ before $(0, 1)$ means that S visits 0 several times (let us say k times, with $k \geq 0$) before its first visit at x but that $\varepsilon_n = 0$ at each of these visits to 0 (this happens with probability $(1 - \delta - p(x))^k$), that S goes to x before coming back to 0 (this happens with probability $p(x)$) and finally that, starting from $S = x$, $(S_n, \varepsilon_n)_{n \geq 1}$ visits $(x, 1)$ before $(0, 1)$ (this happens with probability $1 - \tilde{p}(-x)$).

Iterating this formula we obtain that $\tilde{p}(x) = \frac{p(x)}{\delta+p(x)} \left(1 - \frac{p(x)}{\delta+p(x)}(1 - \tilde{p}(x))\right) = \frac{p(x)(\delta+p(x)\tilde{p}(x))}{(\delta+p(x))^2}$ which leads to

$$\frac{p(x)}{2+\delta} \leq \tilde{p}(x) = \frac{p(x)}{\delta+2p(x)} \leq \frac{p(x)}{\delta}.$$

There exists $C_0 > 1$ such that

$$\forall x \neq 0, \quad C_0^{-1}|x|^{-1} \leq p(x) \leq C_0|x|^{-1} \quad (13)$$

and

$$\forall x \neq 0, \quad C_0^{-1}|x|^{-1} \leq \tilde{p}(x) \leq C_0|x|^{-1}. \quad (14)$$

Observe that $M_1(y, z)$ has expectation 1 and admits exponential moment of every order:

$$\forall t > 0, \quad G_{|y-z|}(t) := \mathbb{E} \left[e^{t(1-M_1(0, |y-z|))} \right] = \frac{(1-p(|y-z|))e^t - 1 + 2p(|y-z|)}{1 - (1-p(|y-z|))e^{-t}}.$$

Hence, for every positive integer $J \leq n^{\frac{1}{2}+\gamma}$, due to the Markov inequality, we obtain that for every $t > 0$,

$$\begin{aligned} \mathbb{P} \left[1 + \sum_{k=1}^J (1 - M_k(y, z)) \geq |y-z|^\mu n^{\frac{1}{4}+\gamma} \right] &= \mathbb{P} \left[\exp \left(t + t \sum_{k=1}^J (1 - M_k(y, z)) \right) \geq \exp \left(t |y-z|^\mu n^{\frac{1}{4}+\gamma} \right) \right] \\ &\leq \exp \left(-t |y-z|^\mu n^{\frac{1}{4}+\gamma} \right) \mathbb{E} \left[\exp \left(t + t \sum_{k=1}^J (1 - M_k(y, z)) \right) \right] \\ &\leq \exp \left(-t |y-z|^\mu n^{\frac{1}{4}+\gamma} \right) (G_{|y-z|}(t))^J e^t \\ &\leq \exp \left(-t |y-z|^\mu n^{\frac{1}{4}+\gamma} \right) \left(\frac{(1-C_0^{-1}|y-z|^{-1})e^t - 1 + 2C_0^{-1}|y-z|^{-1}}{1 - (1-C_0^{-1}|y-z|^{-1})e^{-t}} \right)^J e^t \\ &\leq \exp \left(-t |y-z|^\mu n^{\frac{1}{4}+\gamma} \right) \left(\frac{(1-C_0^{-1}|y-z|^{-1})e^t - 1 + 2C_0^{-1}|y-z|^{-1}}{1 - (1-C_0^{-1}|y-z|^{-1})e^{-t}} \right)^{n^{\frac{1}{2}+\gamma}} e^t \end{aligned}$$

since the function $f : p \mapsto \frac{(1-p)e^t - 1 + 2p}{1 - (1-p)e^{-t}}$ is decreasing on $(0, 1)$ such that $f(0) = e^t$ and $f(1) = 1$. Now using the Taylor expansion of e^t at 0, we observe that

$$\frac{(1-p)e^t - 1 + 2p}{1 - (1-p)e^{-t}} = \frac{1 + \frac{q}{p}t + \frac{q}{p}\frac{t^2}{2} + \frac{q}{p}O(t^3)}{1 + \frac{q}{p}t - \frac{q}{p}\frac{t^2}{2} + \frac{q}{p}O(t^3)}$$

with $p = C_0^{-1}|y-z|^{-1}$ and $q = 1-p$ where $O(t^3)$ is uniform in p . Taking $t = pn^{-\frac{1}{4}-\frac{\gamma}{2}}$, we obtain

$$\frac{(1-p)e^t - 1 + 2p}{1 - (1-p)e^{-t}} = \frac{1 + qn^{-\frac{1}{4}-\frac{\gamma}{2}} + qp\frac{n^{-\frac{1}{2}-\gamma}}{2} + p^2O(n^{-\frac{3}{4}-\frac{3\gamma}{2}})}{1 + qn^{-\frac{1}{4}-\frac{\gamma}{2}} - qp\frac{n^{-\frac{1}{2}-\gamma}}{2} + p^2O(n^{-\frac{3}{4}-\frac{3\gamma}{2}})} = 1 + qp n^{-\frac{1}{2}-\gamma} + O(n^{-\frac{3}{4}-\frac{3\gamma}{2}}).$$

and so

$$\mathbb{P} \left[1 + \sum_{k=1}^J (1 - M_k(y, z)) \geq |y-z|^\mu n^{\frac{1}{4}+\gamma} \right] = O \left(e^{-C_0^{-1}|y-z|^\mu n^{\frac{\gamma}{2}}} \right) = O \left(e^{-C_0^{-1}n^{-(1-\mu)\vartheta+\frac{\gamma}{2}}} \right).$$

Taking $\delta \in (0, \frac{\gamma}{2} - (1-\mu)\vartheta)$ and combining this with (12), we deduce (11) and the analogous estimate for $\tilde{N}_n(\cdot)$ instead of $N_n(\cdot)$ under the assumptions of Theorem 2 (replacing M_k by \tilde{M}_k and $p(\cdot)$ by $\tilde{p}(\cdot)$ in the above argument). \square

2.2. A conditional local limit Theorem for the RWRS. Let φ_ξ be the characteristic function of ξ_1 . Since ξ_1 takes integer values, $e^{2i\pi\xi_1} = 1$ a.s. and so $\varphi_\xi(u) = 1$ for every $u \in 2\pi\mathbb{Z}$. Let us consider the positive integer d such that $d\{u : |\varphi_\xi(u)| = 1\} = 2\pi\mathbb{Z}$. Another characterization of d is that it is the positive generator of the subgroup of \mathbb{Z} generated by the $b - c$, with b and c in the support of the distribution of ξ_1 (i.e. by the support of the distribution of $\xi_0 - \xi_1$). Since the support of ξ_1 is not contained in a proper subgroup of \mathbb{Z} , we also have $d = \inf\{n \geq 1 : e^{2i\pi n\xi_1/d} = 1 \text{ a.s.}\}$. Observe that $e^{\frac{2i\pi}{d}\xi_1}$ is almost surely constant and so $(e^{\frac{2i\pi}{d}\xi_1})^2 = \varphi_\xi(\frac{2\pi}{d})^2 = e^{\frac{2i\pi}{d}(\xi_0+\xi_1)}$ almost surely. Since the distribution of ξ_1 is symmetric, $\mathbb{P}(\xi_0 + \xi_1 = 0) > 0$ and so $(e^{\frac{2i\pi}{d}\xi_1})^2 = 1$ almost surely. Hence either $d = 1$ (and $e^{\frac{2i\pi}{d}\xi_1} = 1$ a.s.) or $d = 2$ (and $e^{\frac{2i\pi}{d}\xi_1} = -1$ a.s.). The following lemma relates the conditional probability $\mathbb{P}[Z_n = 0 | S]$ to the self-intersection local time V_n of the random walk S up to time n . Let us recall that V_n is given by

$$V_n := \sum_{k,\ell=1}^n \mathbf{1}_{\{S_k=S_\ell\}} = \sum_{k,\ell=1}^n \sum_{y \in \mathbb{Z}} \mathbf{1}_{\{S_k=S_\ell=y\}} = \sum_y (N_n(y))^2.$$

Under the assumptions of Theorem 2, we set $\tilde{V}_n := \sum_y (\tilde{N}_n(y))^2$.

Lemma 5. *Let $\gamma \in (0, 1/48)$. There exists a sequence of S -measurable sets $(\Omega_n^{(0)}(\gamma))_n$, an integer $n_0 > 0$ and a positive constant c such that $\mathbb{P}(\Omega_n^{(0)}(\gamma)) = 1 + O\left(e^{-n^{\frac{\delta}{2}}}\right)$ for any $\delta < \frac{\gamma}{2}$ and such that, for every $n \geq n_0$ such that $n \in d\mathbb{N}$, the following inequalities hold on $\Omega_n^{(0)}(\gamma)$:*

$$\mathbb{P}[Z_n = 0 | S] \geq \frac{c}{\sqrt{V_n}},$$

$$n^{\frac{1}{2}-\gamma} \leq N_n^* \leq n^{\frac{1}{2}+\gamma}, \quad R_n \leq n^{\frac{1}{2}+\gamma} \quad \text{and} \quad n^{\frac{3}{2}-\gamma} \leq V_n \leq n^{\frac{3}{2}+\gamma}.$$

Under the assumptions of Theorem 2, there exists a sequence of $(S, (\varepsilon_k)_k)$ -measurable sets $(\tilde{\Omega}_n^{(0)}(\gamma))_n$, an integer $n_0 > 0$ and a positive constant c such that $\mathbb{P}(\tilde{\Omega}_n^{(0)}(\gamma)) = 1 + O\left(e^{-n^{\frac{\delta}{2}}}\right)$ for any $\delta < \frac{\gamma}{2}$ and such that, for every $n \geq n_0$ such that the following inequalities hold on $\tilde{\Omega}_n^{(0)}(\gamma)$:

$$\mathbb{P}\left[\tilde{Z}_n = 0 \mid (S, (\varepsilon_k)_k)\right] \geq \frac{c}{\sqrt{\tilde{V}_n}} \mathbf{1}_{\{\sum_{k=1}^n \varepsilon_k \in d\mathbb{N}^*\}},$$

$$n^{\frac{1}{2}-\gamma} \leq \sup \tilde{N}_n(\mathbb{Z}) \leq N_n^* \leq n^{\frac{1}{2}+\gamma}, \quad R_n \leq n^{\frac{1}{2}+\gamma} \quad \text{and} \quad n^{\frac{3}{2}-\gamma} \leq \tilde{V}_n \leq n^{\frac{3}{2}+\gamma}.$$

Remark: If we assume that φ_ξ is non negative (in this case $\mathbb{P}(\xi_1 = 0) > 0$), then there exists $c > 0$ such that for every $n \geq 1$

$$\mathbb{P}[Z_n = 0 | S] \geq \frac{c}{\sqrt{V_n}}.$$

Indeed, observe that

$$\mathbb{P}[Z_n = 0 | S] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E}[e^{itZ_n} | S] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_y \varphi_\xi(tN_n(y)) dt. \quad (15)$$

Remark that for every $y \in \mathbb{Z}$, $N_n(y) \leq \sqrt{V_n}$. We know that $\varphi_\xi(t) - 1 \sim -\frac{\sigma_\xi^2}{2} t^2$. Let $\beta > 0$ be such that, for every real number u satisfying $|u| < \beta$, we have $\varphi_\xi(u) \geq e^{-\sigma_\xi^2 u^2}$. Since φ_ξ is non

negative, we have

$$\begin{aligned}
\mathbb{P}[Z_n = 0|S] &\geq \frac{1}{2\pi} \int_{-\beta/\sqrt{V_n}}^{\beta/\sqrt{V_n}} \prod_y [\varphi_\xi(tN_n(y))] dt \\
&\geq \frac{1}{2\pi} \int_{-\beta/\sqrt{V_n}}^{\beta/\sqrt{V_n}} \prod_y e^{-\sigma_\xi^2 t^2 (N_n(y))^2} dt \\
&= \frac{1}{2\pi} \int_{-\beta/\sqrt{V_n}}^{\beta/\sqrt{V_n}} e^{-\sigma_\xi^2 t^2 V_n} dt \\
&\geq \frac{1}{2\pi\sigma_\xi\sqrt{V_n}} \int_{|u|<\sigma_\xi\beta} e^{-u^2} du.
\end{aligned}$$

The proof of Lemma 5 is based on the same idea. The fact that φ_ξ can take negative values complicates the proof.

Proof of Lemma 5. We have

$$\mathbb{P}[Z_n = 0|S] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E}[e^{itZ_n}|S] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_y \varphi_\xi(tN_n(y)) dt. \quad (16)$$

Observe that $e^{2i\pi\xi_1/d} = \mathbb{E}[e^{2i\pi\xi_1/d}]$ almost surely and so $\mathbb{E}[e^{2i\pi\xi_1/d}]^d = \mathbb{E}[e^{2i\pi\xi_1}] = 1$. Hence, for any integer $m \geq 0$ and any $u \in \mathbb{R}$, we have

$$\varphi_\xi\left(\frac{2m\pi}{d} + u\right) = \left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^m \varphi_\xi(u)$$

and so

$$\begin{aligned}
\mathbb{P}[Z_n = 0|S] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E}[e^{itZ_n}|S] dt \\
&= \frac{1}{2\pi} \sum_{k=0}^{d-1} \int_{-\pi/d}^{\pi/d} \prod_y \left[\left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^{kN_n(y)} \varphi_\xi(tN_n(y)) \right] dt \\
&= \frac{1}{2\pi} \sum_{k=0}^{d-1} \left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^{kn} \int_{-\pi/d}^{\pi/d} \prod_y [\varphi_\xi(tN_n(y))] dt \\
&= \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \prod_y [\varphi_\xi(tN_n(y))] \mathbf{1}_{\{n \in d\mathbb{N}^*\}} dt.
\end{aligned} \quad (17)$$

Under the assumptions of Theorem 2, proceeding analogously we obtain

$$\mathbb{P}[\tilde{Z}_n = 0|(S, (\varepsilon_k)_k)] = \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \prod_y [\varphi_\xi(t\tilde{N}_n(y))] \mathbf{1}_{\{\sum_{k=1}^n \varepsilon_k \in d\mathbb{N}^*\}} dt, \quad (18)$$

since $\sum_{y \in \mathbb{Z}} \tilde{N}_n(y) = \sum_{k=1}^n \varepsilon_k$. We know that $\varphi_\xi(t) - 1 \sim -\frac{\sigma_\xi^2}{2} t^2$. Let $\beta > 0$ be such that, for every real number u satisfying $|u| < \beta$, we have $e^{-\sigma_\xi^2 u^2} \leq \varphi_\xi(u) \leq e^{-\frac{\sigma_\xi^2}{4} u^2}$ (observe that the fact that the distribution of ξ is symmetric implies that φ_ξ takes real values). Using the fact that

$N_n(y) \leq N_n^* \leq \sqrt{V_n}$, we have

$$\begin{aligned} \frac{d}{2\pi} \int_{-\beta/N_n^*}^{\beta/N_n^*} \prod_y [\varphi_\xi(tN_n(y))] dt &\geq \frac{d}{2\pi} \int_{-\beta/\sqrt{V_n}}^{\beta/\sqrt{V_n}} \prod_y e^{-\sigma_\xi^2 t^2 (N_n(y))^2} dt \\ &= \frac{d}{2\pi} \int_{-\beta/\sqrt{V_n}}^{\beta/\sqrt{V_n}} e^{-\sigma_\xi^2 t^2 V_n} dt \\ &\geq \frac{d}{2\pi\sigma_\xi\sqrt{V_n}} \int_{|u|<\sigma_\xi\beta} e^{-u^2} du. \end{aligned}$$

This gives

$$\frac{d}{2\pi} \int_{-\beta/N_n^*}^{\beta/N_n^*} \prod_y \varphi_\xi(tN_n(y)) dt \geq \frac{c}{\sqrt{V_n}}, \quad (19)$$

for some positive constant c , and analogously

$$\frac{d}{2\pi} \int_{-\beta/\tilde{N}_n^*}^{\beta/\tilde{N}_n^*} \prod_y \varphi_\xi(tN_n(y)) dt \geq \frac{c}{\sqrt{\tilde{V}_n}}, \quad (20)$$

under the assumptions of Theorem 2 if $\tilde{V}_n \neq 0$.

Let $\Omega_n(\gamma)$ be the set defined by

$$\Omega_n(\gamma) = \left\{ R_n \leq n^{\frac{1}{2}+\gamma}, N_n^* \leq n^{\frac{1}{2}+\gamma}, \sup_{y \neq z; |y-z| \leq n} \frac{|N_n(y) - N_n(z)|}{|y-z|} \leq n^{\frac{1}{4}+\gamma} \right\}.$$

Due to Lemmas 3 and 4 (applied with $\mu = 1$ and $\vartheta = 1$), $\mathbb{P}(\Omega_n(\gamma)) = 1 + O\left(e^{-n^{\frac{\delta}{2}}}\right)$ for any $\delta < \frac{\gamma}{2}$. On $\Omega_n(\gamma)$, due to the Cauchy-Schwartz inequality, we have $n = \sum_y N_n(y) \mathbf{1}_{\{N_n(y) > 0\}} \leq \left(V_n \sum_y \mathbf{1}_{\{N_n(y) > 0\}}\right)^{\frac{1}{2}} \leq \sqrt{R_n V_n}$ and so $V_n \geq n^{\frac{3}{2}-\gamma}$. Observe also that $V_n \leq N_n^* \sum_y N_n(y) = n N_n^* \leq n^{\frac{3}{2}+\gamma}$. Moreover $n = \sum_y N_n(y) \leq R_n N_n^*$. Hence $N_n^* \geq n^{\frac{1}{2}-\gamma}$. This gives the three last inequalities in the first case.

Under the assumptions of Theorem 2, we set analogously

$$\tilde{\Omega}_n(\gamma) = \left\{ R_n \leq n^{\frac{1}{2}+\frac{3\gamma}{4}}, N_n^* \leq n^{\frac{1}{2}+\frac{3\gamma}{4}}, \sup_{y \neq z; |y-z| \leq n} \frac{|\tilde{N}_n(y) - \tilde{N}_n(z)|}{|y-z|} \leq n^{\frac{1}{4}+\gamma}, \sum_{k=1}^n \varepsilon_k > \frac{n(1-\gamma)\delta}{4} \right\}.$$

If n is large enough, on $\tilde{\Omega}_n(\gamma)$, we also obtain that $n^{\frac{3}{2}-\gamma} \leq \tilde{V}_n \leq n^{\frac{3}{2}+\gamma}$ and $\sup \tilde{N}_n(\mathbb{Z}) \geq n^{\frac{1}{2}-\gamma}$ using the same arguments as above and the fact that $\sum_y \tilde{N}_n(y) = \sum_{k=1}^n \varepsilon_k$.

To end the proof of the lemma, it remains to prove the first inequality. Due to (17) and (19), it remains to prove that there exists $n_1 > 0$ such that, for every $n \geq n_1$, on $\Omega_n(\gamma)$, we have

$$\int_{\frac{\beta}{N_n^*} \leq |t| \leq \frac{\pi}{d}} \prod_y |\varphi_\xi(tN_n(y))| dt \leq \int_{\beta n^{-\frac{1}{2}-\gamma} \leq |t| \leq \frac{\pi}{d}} \prod_y |\varphi_\xi(tN_n(y))| dt \leq \frac{c}{2\sqrt{V_n}}, \quad (21)$$

and the analogous inequality obtained by replacing $N_n(\cdot)$ by $\tilde{N}_n(\cdot)$ and V_n by \tilde{V}_n , under the assumptions of Theorem 2. To this end, we will use elements of the proof of [9] and more precisely the proofs of Propositions 9 and 10 therein.

We fix $\varepsilon > 3\gamma$ such that $3\gamma + 3\varepsilon < \frac{1}{4}$ (this is possible since $\gamma < \frac{1}{48}$). We first follow the proof of Proposition 9 in [9] (here $\varepsilon_0 = \beta$) and more precisely of Lemma 14 therein. Let $y_1 \in \mathbb{Z}$ be such that $N_n(y_1) = N_n^*$ and set $y_0 := \min\{y \geq y_1 : N_n(y) \leq \frac{\beta}{2} n^{\frac{1}{2}-\varepsilon}\}$. On $\Omega_n(\gamma)$, for n

large enough, $y_0 > y_1$ (since $\varepsilon > \gamma$) and so $N_n(y_0 - 1) > \frac{\beta}{2}n^{\frac{1}{2}-\varepsilon} \geq N_n(y_0)$. Moreover, still on $\Omega_n(\gamma)$, $N_n(y_0 - 1) - N_n(y_0) \leq n^{\frac{1}{4}+\gamma}$ which is smaller than $\frac{\beta}{4}n^{\frac{1}{2}-\varepsilon}$ for n large enough so that $\frac{\beta}{4}n^{\frac{1}{2}-\varepsilon} \leq N_n(y_0) \leq \frac{\beta}{2}n^{\frac{1}{2}-\varepsilon}$. Now, on $\Omega_n(\gamma)$, for every $z \in \mathbb{Z}$ such that $|y_0 - z| \leq e_n := \frac{\beta}{10}n^{\frac{1}{4}-\varepsilon-\gamma}$, then

$$|N_n(z) - N_n(y_0)| \leq |y_0 - z|n^{\frac{1}{4}+\gamma} \leq \frac{\beta}{10}n^{\frac{1}{2}-\varepsilon}$$

and so

$$\frac{\beta}{10}n^{\frac{1}{2}-\varepsilon} < N_n(z) < \beta n^{\frac{1}{2}-\varepsilon},$$

hence $|tN_n(z)| \leq \beta$ if $n^{-\frac{1}{2}-\gamma} < |t| < n^{-\frac{1}{2}+\varepsilon}$ and so, on $\Omega_n(\gamma)$,

$$\begin{aligned} \prod_{y \in \mathbb{Z}} |\varphi_\xi(tN_n(y))| &\leq \exp\left(-\frac{\sigma_\xi^2}{4}t^2 \sum_{z=y_0-e_n}^{y_0+e_n} (N_n(z))^2\right) \\ &\leq \exp\left(-\frac{\sigma_\xi^2}{4}n^{-1-2\gamma}2e_n\frac{\beta^2}{100}n^{1-2\varepsilon}\right) \\ &\leq \exp\left(-\frac{\sigma_\xi^2}{2}n^{\frac{1}{4}-3\gamma-3\varepsilon}\frac{\beta^3}{10^3}\right). \end{aligned}$$

Hence we have proved that, for n large enough, on $\Omega_n(\gamma)$,

$$\int_{\beta n^{-\frac{1}{2}-\gamma} \leq |t| \leq n^{-\frac{1}{2}+\varepsilon}} \prod_y |\varphi_\xi(tN_n(y))| dt \leq \frac{c}{4\sqrt{V_n}} \quad (22)$$

since $3\gamma + 3\varepsilon < \frac{1}{4}$.

Under the assumptions of Theorem 2, the same argument gives

$$\int_{\beta n^{-\frac{1}{2}-\gamma} \leq |t| \leq n^{-\frac{1}{2}+\varepsilon}} \prod_y |\varphi_\xi(t\tilde{N}_n(y))| dt \leq \frac{c}{4\sqrt{\tilde{V}_n}} \quad (23)$$

Now, Lemma 15 of [9] still holds with our set $\Omega_n(\gamma)$ since the proof only uses the fact that $N_n^* \leq n^{\frac{1}{2}+\gamma}$ and that $R_n \leq n^{\frac{1}{2}+\gamma}$. Due to this remark and using the notations and results contained in Section 2.8 of [9], we take for $\Omega_n^{(0)}(\gamma)$ the subset of $\Omega_n(\gamma) \cap \mathcal{D}_n$ on which $\#\{z : N_n(z) \in \mathcal{I}\} \geq n^{\frac{1}{2}-2\gamma}/4$ (with \mathcal{D}_n and \mathcal{I} being defined in Section 2.8 of [9] applied with $\alpha = 2$). Since $\gamma < \frac{1}{8}$ and $3\gamma < \varepsilon < \frac{1}{2}$, we obtain that $\mathbb{P}(\Omega_n(\gamma) \setminus \Omega_n^{(0)}(\gamma)) = o(e^{-cn})$ for some $c > 0^3$. Moreover, there exists an integer n_2 such that if $n \geq n_2$, on $\Omega_n^{(0)}(\gamma)$ we have

$$\forall t \in [n^{-\frac{1}{2}+\varepsilon}, \frac{\pi}{d}], \quad \prod_y |\varphi_\xi(tN_n(y))| \leq \exp(-n^\gamma) \leq \frac{c}{4\sqrt{V_n}} \quad (24)$$

(see the lines before the proof of Lemma 17 of [9]). The same argument (with the flat peaks instead of the peaks as explained in Section 5.4 of [9] gives also (for every n large enough)

$$\forall t \in [n^{-\frac{1}{2}+\varepsilon}, \frac{\pi}{d}], \quad \prod_y |\varphi_\xi(t\tilde{N}_n(y))| \leq \exp(-n^\gamma) \leq \frac{c}{4\sqrt{\tilde{V}_n}} \quad (25)$$

on some set $\tilde{\Omega}_n^{(0)}(\gamma)$ such that $\mathbb{P}(\tilde{\Omega}_n(\gamma) \setminus \tilde{\Omega}_n^{(0)}(\gamma)) = o(e^{-cn})$. \square

³Indeed, using the notations \mathcal{D}_n , \mathcal{E}_n and \mathcal{I} of Section 2.8 of [9], $\mathbb{P}(\mathcal{D}_n) = 1 - o(e^{-cn})$; moreover following the proof of Lemma 15 of [9] we obtain that $\Omega_n(\gamma) \cap \mathcal{D}_n \subset \mathcal{E}_n$, and finally, due to the remark following Lemma 17 of [9], $p_2(n) := \mathbb{P}(\mathcal{E}_n, \#\{z : N_n(z) \in \mathcal{I}\} < n^{\frac{1}{2}-2\gamma}/4) = o(e^{-cn})$. Therefore $\mathbb{P}(\Omega_n(\gamma) \setminus \Omega_n^{(0)}(\gamma)) \leq \mathbb{P}(\Omega_n(\gamma) \setminus \mathcal{D}_n) + p_2(n) = o(e^{-cn})$.

2.3. A conditional Berry-Esseen bound for RWRS. Let Φ be the distribution function of the standard gaussian distribution, i.e.

$$\forall u \in \mathbb{R}, \quad \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-\frac{x^2}{2}} dx. \quad (26)$$

For $p \geq 1$, let us define the p -fold self-intersection local time of the random walk up to time n

$$Q_n^{(p)} := \sum_{y \in \mathbb{Z}} N_n(y)^p.$$

Under the assumptions of Theorem 2, we define

$$\tilde{Q}_n^{(p)} := \sum_{y \in \mathbb{Z}} \tilde{N}_n(y)^p.$$

Lemma 6. *There exists a positive constant C such that for every $n \geq 1$*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left[\frac{Z_n}{\sigma_\xi \sqrt{V_n}} \leq x \middle| S \right] - \Phi(x) \right| \leq C \frac{\mathbb{E}[|\xi_0|^3]}{\mathbb{E}[|\xi_0|^2]^{3/2}} \frac{Q_n^{(3)}}{V_n^{3/2}}$$

and such that, under the assumptions of Theorem 2,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left[\frac{\tilde{Z}_n}{\sigma_\xi \sqrt{\tilde{V}_n}} \leq x \middle| (S, (\varepsilon_k)_k) \right] - \Phi(x) \right| \leq C \frac{\mathbb{E}[|\xi_0|^3]}{\mathbb{E}[|\xi_0|^2]^{3/2}} \frac{\tilde{Q}_n^{(3)}}{\tilde{V}_n^{3/2}}.$$

Proof. This result directly follows from Berry-Esseen theorem since conditionally on the random walk, Z_n (resp. \tilde{Z}_n) is the sum of centered, independent random variables $\xi_y N_n(y)$ (resp. $\xi_y \tilde{N}_n(y)$ under the assumptions of Theorem 2). \square

3. PROOF OF THEOREMS 1 AND 2

3.1. Relation to exponential functionals. The main idea is to relate the persistence probability to the exponential functional $\sum_{\ell=\ell_0}^T e^{Z_\ell}$ (with $\ell_0 \in \{0, 1\}$), cf. [22, 1, 10, 2, 11]. In [22] it is shown that the continuous-time analog of this quantity behaves as cT^{H-1} for any *continuous-time* H -self-similar process with stationary increments and a certain other time-reversibility property. Further, certain moment conditions are assumed in [22] (also see [21]). We will apply the following lemma which does not have these moment conditions and in which H -self-similarity (which does not make sense in discrete time) is replaced by (27) extracting the "natural scaling" of the process Z .

Lemma 7. [see Lemma 5 in [3]] *Let $Z = (Z_n)_{n \in \mathbb{N}}$ be a stochastic process with*

$$\lim_{T \rightarrow +\infty} \frac{1}{T^H \ell(T)} \mathbb{E} \left[\sup_{t \in [0, 1]} Z_{[tT]} \right] = \kappa, \quad (27)$$

for some $H \in (0, 1)$, $\kappa \in (0, \infty)$, and with ℓ being a slowly varying function at infinity. Further assume that Z is time-reversible in the sense that for any $T \in \mathbb{N}$, the vectors $(Z_{T-k} - Z_T)_{k=0, \dots, T}$ and $(Z_k)_{k=0, \dots, T}$ have the same law. Then,

$$\limsup_{x \rightarrow +\infty} \frac{x^{1-H}}{\ell(x)} \mathbb{E} \left[\left(\sum_{l=0}^{[x]} e^{Z_l} \right)^{-1} \right] \leq \kappa H$$

and

$$\liminf_{x \rightarrow +\infty} \frac{x^{1-H}}{\ell(x)} \mathbb{E} \left[\left(\sum_{l=1}^{[x]} e^{Z_l} \right)^{-1} \right] \geq \kappa H.$$

Note the difference in the summation $l = 0, \dots$ vs. $l = 1, \dots$, which complicates the use of this lemma to prove the lower bounds in Theorems 1 and 2. Our additional assumptions for the lower bounds of Theorems 1 and 2 come from the fact that the sum starts from 1 in the second inequality of Lemma 7.

3.2. Verification of Lemma 7 for RWRS. The goal of this subsection is to verify that Lemma 7 holds with $H := 3/4$ and $\ell \equiv 1$ for the RWRS Z and for \tilde{Z} .

We first show that RWRS is time-reversible. Note that

$$Z_{T-k} - Z_T = \sum_{j=1}^{T-k} \xi_{S_j} - \sum_{j=1}^T \xi_{S_j} = - \sum_{j=T-k+1}^T \xi_{S_j}, \quad k = 0, \dots, T.$$

By conditioning on the random walk and using the symmetry of the environment as well as the fact that the environment is i.i.d. (and thus spatially homogeneous), the above vector has the same distribution as

$$\sum_{j=T-k+1}^T \xi_{S_j - S_{T+1}} = \sum_{j=1}^k \xi_{S_{T-j+1} - S_{T+1}}, \quad k = 0, \dots, T.$$

Since $(\xi_y)_y$ and $(\xi_{-y})_y$ have the same distribution, the above vector has the same distribution as

$$\sum_{j=1}^k \xi_{S_{T+1} - S_{T-j+1}}, \quad k = 0, \dots, T. \quad (28)$$

Now we condition on the environment and use that

$$S_{T+1} - S_{T-j+1} = \sum_{i=T-j+2}^{T+1} X_i = \sum_{i=1}^j X_{T+2-i}, \quad j = 0, \dots, T,$$

has the same law as $(S_j)_{j=0, \dots, T}$, which in connection with (28) shows the claim that Z is time-reversible.

Under the assumptions of Theorem 2, using the fact that that $\tilde{Z}_k = \sum_{\ell=1}^k \xi_{S_{\ell-1}} \varepsilon_{\ell} = \sum_{\ell=1}^k \xi_{S_{\ell}} \varepsilon_{\ell}$ for every positive integer T , the vector

$$(\tilde{Z}_{T-k} - \tilde{Z}_T)_{k=0, \dots, T} = \left(- \sum_{\ell=T-k+1}^T \xi_{S_{\ell-1}} \varepsilon_{\ell} \right)$$

has the same distribution as

$$\left(\sum_{\ell=T-k+1}^T \xi_{S_T - S_{\ell-1}} \varepsilon_{\ell} \right)_{k=0, \dots, T} = \left(\sum_{\ell=T-k+1}^T \xi_{\tilde{X}_{\ell}(1-\varepsilon_{\ell}) + \dots + \tilde{X}_T(1-\varepsilon_T)} \varepsilon_{\ell} \right)_{k=0, \dots, T}$$

(since $(\xi_y)_y$ and $(-\xi_{S_T-y})_y$ have the same distribution given $(S, (\varepsilon_k)_k)$), which has the same distribution as

$$\left(\sum_{\ell=T-k+1}^T \xi_{\tilde{X}_{T-\ell+1}(1-\varepsilon_{T-\ell+1}) + \dots + \tilde{X}_1(1-\varepsilon_1)} \varepsilon_{T-\ell+1} \right)_{k=0, \dots, T} = \left(\sum_{\ell=1}^k \xi_{S_{\ell}} \varepsilon_{\ell} \right)_{k=0, \dots, T} = (\tilde{Z}_k)_{k=0, \dots, T}$$

(since $(\tilde{X}_{\ell}, \varepsilon_{\ell})_{\ell=1, \dots, T}$ and $(\tilde{X}_{T-\ell+1}, \varepsilon_{T-\ell+1})_{\ell=1, \dots, T}$ have the same distribution given ξ). Hence we have proved the time-reversibility of \tilde{Z} .

Now let us verify (27). Note that the sequence of random variables $T^{-3/4} \max_{k=1, \dots, T} Z_k$ is uniformly bounded in L^2 : Indeed, given S , the random variable Z_n is a sum of associated random variables with zero mean and finite variance, so from Theorem 2 in [23],

$$\mathbb{E}[(\max_{k=1, \dots, T} Z_k)^2 | S] \leq \mathbb{E}[Z_T^2 | S] = V_T.$$

By integrating with respect to the random walk, we get

$$\mathbb{E}[(\max_{k=1, \dots, T} Z_k)^2] \leq \mathbb{E}[V_T] \sim CT^{3/2},$$

cf. (2.13) in [15]. Since the sequence of processes $(Z_{[tT]}/T^{3/4})_{t \geq 0}$ weakly converges for the Skorokhod topology to the process $(\Delta_t)_{t \geq 0}$ (see [15] and the remark following Theorem 2 of [9]), we get

$$\lim_{T \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0,1]} \left(\frac{Z_{[tT]}}{T^{3/4}} \right) \right] = \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right] =: \kappa,$$

which is known to be finite using Proposition 2.1 in [10].

Under assumptions of Theorem 2, we proceed analogously to prove that (27) holds for \tilde{Z} . We obtain

$$\mathbb{E}[(\max_{k=1, \dots, T} \tilde{Z}_k)^2] \leq \mathbb{E}[\tilde{V}_T] \leq \mathbb{E}[V_T] \sim CT^{3/2},$$

cf. (2.13) in [15]. The fact that the sequence of processes $(\tilde{Z}_{[tT]}/T^{3/4})_{t \geq 0}$ weakly converges for the Skorokhod topology to the process $(K_\delta \Delta_t)_{t \geq 0}$ where $K_\delta = \frac{\delta}{(1-\delta)^{1/4}}$ has been proved in [13] and so

$$\lim_{T \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0,1]} \left(\frac{\tilde{Z}_{[tT]}}{T^{3/4}} \right) \right] = K_\delta \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right] =: \tilde{\kappa}.$$

3.3. Proof of the upper bound. As in [22] and [1], the main idea in the proof of the upper bound in (2), is to bound the exponential functionals $\left(\sum_{k=0}^T e^{Z_k} \right)^{-1}$ of Lemma 7 from below by restricting the expectation to a well-chosen set of paths.

Conditionally on S , Z_k is the sum of centered and positively associated random variables. It follows that for every $0 \leq u < v < w$ and all real numbers a, b ,

$$\mathbb{P} \left[\max_{k=u, \dots, v} Z_k \leq a, \max_{k=v+1, \dots, w} Z_k \leq b \middle| S \right] \geq \mathbb{P} \left[\max_{k=u, \dots, v} Z_k \leq a \middle| S \right] \mathbb{P} \left[\max_{k=v+1, \dots, w} Z_k \leq b \middle| S \right] \quad (29)$$

$$\mathbb{P} \left[\max_{k=u, \dots, v} Z_k \leq a, \max_{k=v+1, \dots, w} (Z_k - Z_v) \leq b \middle| S \right] \geq \mathbb{P} \left[\max_{k=u, \dots, v} Z_k \leq a \middle| S \right] \mathbb{P} \left[\max_{k=v+1, \dots, w} (Z_k - Z_v) \leq b \middle| S \right], \quad (30)$$

and analogously, under the assumptions of Theorem 2,

$$\begin{aligned} \mathbb{P} \left[\max_{k=u, \dots, v} \tilde{Z}_k \leq a, \max_{k=v+1, \dots, w} \tilde{Z}_k \leq b \middle| (S, (\varepsilon_k)_k) \right] &\geq \mathbb{P} \left[\max_{k=u, \dots, v} \tilde{Z}_k \leq a \middle| (S, (\varepsilon_k)_k) \right] \\ &\quad \mathbb{P} \left[\max_{k=v+1, \dots, w} \tilde{Z}_k \leq b \middle| (S, (\varepsilon_k)_k) \right] \end{aligned} \quad (31)$$

$$\begin{aligned} \mathbb{P} \left[\max_{k=u, \dots, v} \tilde{Z}_k \leq a, \max_{k=v+1, \dots, w} (\tilde{Z}_k - \tilde{Z}_v) \leq b \middle| (S, (\varepsilon_k)_k) \right] &\geq \mathbb{P} \left[\max_{k=u, \dots, v} \tilde{Z}_k \leq a \middle| (S, (\varepsilon_k)_k) \right] \\ &\quad \mathbb{P} \left[\max_{k=v+1, \dots, w} (\tilde{Z}_k - \tilde{Z}_v) \leq b \middle| (S, (\varepsilon_k)_k) \right]. \end{aligned} \quad (32)$$

Let us precise that these inequalities will play the role of the Slepian Lemma in [10, 11, 3]. Let $a_T \geq (\log T)^6$ and set $\beta_T := \sigma_\xi \sqrt{V_{a_T}}$. Let us define the random function

$$\phi(k) := \begin{cases} 1 & \text{for } 0 \leq k < a_T, \\ 1 - \beta_T & \text{for } a_T \leq k \leq T, \end{cases}$$

which is S -measurable. Clearly, we have

$$\mathbb{E} \left[\left(\sum_{k=0}^T e^{Z_k} \right)^{-1} \middle| S \right] \geq \left(\sum_{k=0}^T e^{\phi(k)} \right)^{-1} \mathbb{P}[\forall k \in \{0, \dots, T\}, Z_k \leq \phi(k) | S]. \quad (33)$$

From (29), we have

$$\mathbb{P}[\forall k \in \{0, \dots, T\}, Z_k \leq \phi(k) | S] \geq \mathbb{P}[\max_{k=0, \dots, a_T} Z_k \leq 1 | S] \mathbb{P}[\max_{k=a_T, \dots, T} Z_k \leq 1 - \beta_T | S].$$

Note that

$$\begin{aligned} \mathbb{P}[\max_{k=a_T, \dots, T} Z_k \leq 1 - \beta_T | S] &\geq \mathbb{P}[Z_{a_T} \leq -\beta_T; \max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 | S] \\ &\geq \mathbb{P}[Z_{a_T} \leq -\beta_T | S] \cdot \mathbb{P}[\max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 | S], \end{aligned}$$

by (30). Moreover, it is easy to check that for every $T > 1$

$$\sum_{k=0}^T e^{\phi(k)} \leq e(a_T + 1 + Te^{-\beta_T}).$$

In the following, C is a constant whose value may change but does not depend on T . Then, summing up (33) and the succeeding estimates, we can write that for T large enough

$$(a_T + Te^{-\beta_T}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{Z_k} \right)^{-1} \middle| S \right] \geq C \mathbb{P}[Z_{a_T} \leq -\beta_T | S] \mathbb{P}[\max_{k=0, \dots, a_T} Z_k \leq 1 | S] \mathbb{P}[\max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 | S]. \quad (34)$$

The two first probabilities in the right hand side of (34) can be approximated with the distribution function of the standard Gaussian law $\mathcal{N}(0, 1)$. The error by doing this approximation will be controlled by using Lemma 6. Indeed, we have

$$\mathbb{P}[Z_{a_T} \leq -\beta_T | S] = \mathbb{P}[Z_{a_T} \leq -\beta_T | S] - \Phi(-1) + \Phi(-1) \quad (35)$$

$$\begin{aligned} &\geq \Phi(-1) - \tilde{C} \frac{\mathbb{E}[|\xi_0|^3]}{\mathbb{E}[|\xi_0|^2]^{3/2}} \frac{Q_{a_T}^{(3)}}{V_{a_T}^{3/2}} \\ &\geq \Phi(-1) - \tilde{C} \frac{V_{a_T} N_{a_T}^*}{V_{a_T}^{3/2}} \\ &\geq \Phi(-1) - \tilde{C} \frac{N_{a_T}^*}{\sqrt{V_{a_T}}}, \end{aligned} \quad (36)$$

Moreover since the law of the random scenery is symmetric, from Lévy's inequality (see for instance Theorem 2.13.1 in [29]), we get

$$\mathbb{P}[\max_{k=0, \dots, a_T} Z_k \leq 1 | S] = 1 - \mathbb{P}[\max_{k=0, \dots, a_T} Z_k > 1 | S] \geq 1 - 2\mathbb{P}[Z_{a_T} > 1 | S] = \mathbb{P}[|Z_{a_T}| \leq 1 | S]. \quad (37)$$

Now, let $\gamma \in (0, 1/48)$, due to Lemma 5, for T large enough such that $a_T \in d\mathbb{N}$,

$$\begin{aligned} \mathbb{P}[|Z_{a_T}| \leq 1 | S] &\geq \mathbb{P}[Z_{a_T} = 0 | S] \\ &\geq \frac{c}{\sqrt{V_{a_T}}}, \end{aligned} \quad (38)$$

holds a.s. on a sequence of S -measurable sets $\Omega_{a_T}^{(0)}(\gamma)$. Due to (34), (36), (38), on $\Omega_{a_T}^{(0)}(\gamma)$, we have

$$\begin{aligned} \mathbb{P} \left[\max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 \middle| S \right] &\leq C \frac{(a_T + T e^{-\beta_T}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{Z_k} \right)^{-1} \middle| S \right]}{\mathbb{P}[Z_{a_T} \leq -\beta_T | S] \mathbb{P}[\max_{k=0, \dots, a_T} Z_k \leq 1 | S]} \\ &\leq C \frac{a_T^{\frac{3}{4} + \frac{\gamma}{2}} (a_T + T e^{-\sigma_\xi a_T^{\frac{3}{4} - \frac{\gamma}{2}}}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{Z_k} \right)^{-1} \middle| S \right]}{(\Phi(-1) - \tilde{C} a_T^{-\frac{1}{4} + \frac{3\gamma}{2}})} \\ &\leq \tilde{C}_\gamma a_T^{\frac{3}{4} + \frac{\gamma}{2}} (a_T + T e^{-\sigma_\xi a_T^{\frac{3}{4} - \frac{\gamma}{2}}}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{Z_k} \right)^{-1} \middle| S \right] \end{aligned}$$

for T large enough, where we used the facts that $\beta_T = \sigma_\xi \sqrt{V_{a_T}} \geq \sigma_\xi a_T^{\frac{3}{4} - \frac{\gamma}{2}}$ and that $\frac{N_{a_T}^*}{\sqrt{V_{a_T}}} \leq a_T^{-\frac{1}{4} + \frac{3\gamma}{2}}$. It comes

$$\mathbb{P} \left[\max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 \right] \leq \mathbb{P}((\Omega_{a_T}^{(0)}(\gamma))^c) + \tilde{C}_\gamma a_T^{\frac{3}{4} + \frac{\gamma}{2}} (a_T + T e^{-\sigma_\xi a_T^{\frac{3}{4} - \frac{\gamma}{2}}}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{Z_k} \right)^{-1} \right] \quad (39)$$

if $a_T \in d\mathbb{N}$. Let $\delta_0 \in (0, \frac{\gamma}{2})$, we take $a_T = d \left\lceil \frac{1}{d} ((\log T)/4)^{\frac{2}{\delta_0}} \right\rceil$, so that $\mathbb{P}((\Omega_{a_T}^{(0)}(\gamma))^c) = o\left(e^{-(a_T)^{\frac{\delta_0}{2}}}\right) \leq T^{-\frac{1}{4}}$. We observe that $a_T^{\frac{3}{4} - \frac{\gamma}{2}} \geq c_0 (\log T)^{(\frac{3}{\gamma} - 2)} > c_0 (\log T)^{142}$. Applying Lemma 7, we conclude that there exists some constant $c > 0$ such that

$$\mathbb{P} \left[\max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 \right] = O\left((\log T)^c T^{-\frac{1}{4}}\right). \quad (40)$$

The left hand side of (40) is greater than the quantity we want to bound from above, since by stationarity of increments,

$$\mathbb{P} \left[\max_{k=a_T, \dots, T} (Z_k - Z_{a_T}) \leq 1 \right] = \mathbb{P} \left[\max_{k=0, \dots, T-a_T} Z_k \leq 1 \right] \geq \mathbb{P} \left[\max_{k=0, \dots, T} Z_k \leq 1 \right]. \quad (41)$$

Let us make now the assumptions of Theorem 2. Analogously, on $\tilde{\Omega}_{a_T}^{(0)}(\gamma)$, if $\sum_{k=0}^{a_T} \varepsilon_k \in d\mathbb{N}$, the following inequality holds

$$\mathbb{P} \left[\max_{k=a_T, \dots, T} (\tilde{Z}_k - \tilde{Z}_{a_T}) \leq 1 \middle| (S, (\varepsilon_k)_k) \right] \leq \tilde{C}_\gamma a_T^{\frac{3}{4} + \frac{\gamma}{2}} (a_T + T e^{-\sigma_\xi a_T^{\frac{3}{4} - \frac{\gamma}{2}}}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{\tilde{Z}_k} \right)^{-1} \middle| (S, (\varepsilon_k)_k) \right]. \quad (42)$$

We take $0 < 2\delta_0 < \tilde{\gamma} < \gamma < \frac{1}{48}$ and $\tilde{a}_T := \left\lceil ((\log T)/4)^{\frac{2}{\delta_0}} \right\rceil$. We define $a_T := \min\{k \geq \tilde{a}_T : \sum_{\ell=1}^k \varepsilon_\ell \in d\mathbb{N}\}$ (here a_T is a $(S, (\varepsilon_k)_k)$ -measurable random variable). Since $d \leq 2$, we observe that

$$\mathbb{P}(a_T - \tilde{a}_T > \tilde{a}_T^{\delta_0}) \leq \mathbb{P} \left(\sum_{\ell=1}^{\tilde{a}_T^{\delta_0}} \varepsilon_\ell = 0 \right) = (1 - \delta)^{\tilde{a}_T^{\delta_0}} = o\left(e^{-(\tilde{a}_T)^{\frac{\delta_0}{2}}}\right) = o(T^{-\frac{1}{4}}).$$

Moreover, there exists \tilde{T}_0 such that, for every $T \geq \tilde{T}_0$, $\tilde{\Omega}_{\tilde{a}_T}^{(0)}(\tilde{\gamma}) \cap \{a_T \leq \tilde{a}_T + \tilde{a}_T^{\delta_0}\} \subseteq \tilde{\Omega}_{a_T}^{(0)}(\gamma)$. Hence

$$\begin{aligned} \mathbb{P} \left[\max_{k=a_T, \dots, T} (\tilde{Z}_k - \tilde{Z}_{a_T}) \leq 1 \right] &\leq \mathbb{P}((\tilde{\Omega}_{\tilde{a}_T}^{(0)}(\tilde{\gamma}))^c) + \mathbb{P}(a_T - \tilde{a}_T > \tilde{a}_T^{\delta_0}) \\ &\quad + \tilde{C}_\gamma a_T^{\frac{3}{4} + \frac{\gamma}{2}} (a_T + T e^{-\sigma_\xi a_T^{\frac{3}{4} - \frac{\gamma}{2}}}) \mathbb{E} \left[\left(\sum_{k=0}^T e^{\tilde{Z}_k} \right)^{-1} \right] \end{aligned} \quad (43)$$

and we conclude as above.

3.4. Proof of the lower bound. Fix $\beta > 1/4$ and define $Z_T^* := \max_{k=1, \dots, T} Z_k$. Observe that

$$\mathbb{E} \left[\left(\sum_{k=1}^T e^{Z_k} \right)^{-1} \right] = \mathbb{E} \left[\left(\sum_{k=1}^T e^{Z_k} \right)^{-1} \mathbf{1}_{Z_T^* \geq \beta \log T} \right] + \mathbb{E} \left[\left(\sum_{k=1}^T e^{Z_k} \right)^{-1} \mathbf{1}_{Z_T^* < \beta \log T} \right] \quad (44)$$

$$=: I_1(T) + I_2(T). \quad (45)$$

First, we clearly have

$$\begin{aligned} I_1(T) &\leq \mathbb{E}[e^{-Z_T^*} \mathbf{1}_{Z_T^* \geq \beta \log T}] \\ &\leq T^{-\beta}. \end{aligned}$$

We observe that

$$I_2(T) \leq \mathbb{E} \left[e^{-Z_1} \mathbf{1}_{Z_T^* < \beta \log T} \right].$$

Let us fix a parameter $\theta \in (0, 1)$ and let us define the event $A := \{Z_1 \geq -\log H^{-1} \left(\frac{3\kappa}{4} \theta T^{-\frac{1}{4}} \right)\}$. Then,

$$I_2(T) \leq H^{-1} \left(\frac{3\kappa}{4} \theta T^{-\frac{1}{4}} \right) \mathbb{P}[Z_T^* < \beta \log T] + \mathbb{E}[e^{-Z_1} \mathbf{1}_{A^c}]. \quad (46)$$

Since Z_1 has the same distribution as ξ_1 , its distribution is symmetric and so

$$\mathbb{E}[e^{-Z_1} \mathbf{1}_{A^c}] = H \left(H^{-1} \left(\frac{3\kappa}{4} \theta T^{-\frac{1}{4}} \right) \right) \leq \frac{3\kappa}{4} \theta T^{-\frac{1}{4}}.$$

But

$$\frac{3\kappa}{4} \theta < \liminf_{x \rightarrow +\infty} x^{\frac{1}{4}} (I_1(x) + I_2(x))$$

So we have shown that for T large,

$$\mathbb{P}[Z_T^* \leq \beta \log T] \geq c^{-1} T^{-1/4} \left[H^{-1} \left(\frac{3\kappa}{4} \theta T^{-\frac{1}{4}} \right) \right]^{-1}. \quad (47)$$

Let $\gamma \in (0, 1/48)$, $\delta_0 \in (0, \frac{\gamma}{2})$ and $a_T = d \lceil (\beta \log T)^{2/\delta_0} / d \rceil$. Note that from inequalities (29) and (30), we have

$$\begin{aligned} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \middle| S \right] &\geq \mathbb{P} \left[\max_{k=1, \dots, a_T} Z_k \leq 1; Z_{a_T} \leq -\beta \log T; \right. \\ &\quad \left. \max_{k=a_T+1, \dots, T} Z_k - Z_{a_T} \leq \beta \log T \middle| S \right] \\ &\geq \mathbb{P} \left[\max_{k=1, \dots, a_T} Z_k \leq 1 \middle| S \right] \cdot \mathbb{P}[Z_{a_T} \leq -\beta \log T | S] \\ &\quad \cdot \mathbb{P} \left[\max_{k=a_T+1, \dots, T} Z_k - Z_{a_T} \leq \beta \log T \middle| S \right] \end{aligned} \quad (48)$$

From Lemma 5, (37), (38) and Lemma 6, for T large enough, on $\Omega_{a_T}^{(0)}(\gamma)$,

$$\begin{aligned}
& \mathbb{P} \left[\max_{k=a_T+1, \dots, T} Z_k - Z_{a_T} \leq \beta \log T \middle| S \right] \\
& \leq c\sqrt{V_{a_T}} \left(\Phi \left(-\frac{\beta \log T}{\sigma_\xi \sqrt{V_{a_T}}} \right) - \tilde{C} \frac{N_{a_T}^*}{\sqrt{V_{a_T}}} \right)^{-1} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \middle| S \right] \\
& \leq ca_T^{\frac{3+2\gamma}{4}} \left(\Phi \left(-\frac{\beta \log T}{\sigma_\xi a_T^{\frac{3-2\gamma}{4}}} \right) - \tilde{C} a_T^{-\frac{1-6\gamma}{4}} \right)^{-1} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \middle| S \right] \\
& \leq ca_T^{\frac{3+2\gamma}{4}} \left(\Phi(-1) - \tilde{C} a_T^{-\frac{1-6\gamma}{4}} \right)^{-1} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \middle| S \right] \\
& \leq c(\log T)^{\frac{3+2\gamma}{2\delta_0}} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \middle| S \right]
\end{aligned} \tag{49}$$

$$\leq c(\log T)^{\frac{3+2\gamma}{2\delta_0}} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \middle| S \right] \tag{50}$$

Thus, using the last inequality and the stationarity of the increments, we get

$$\begin{aligned}
\mathbb{P}[Z_T^* \leq \beta \log T] & \leq \mathbb{E} \left[\mathbb{P} \left[\max_{k=a_T+1, \dots, T} Z_k - Z_{a_T} \leq \beta \log T \middle| S \right] \right] \\
& \leq \mathbb{E} \left[\mathbb{P} \left[\max_{k=a_T+1, \dots, T} Z_k - Z_{a_T} \leq \beta \log T \middle| S \right] \mathbf{1}_{\Omega_{a_T}^{(0)}(\gamma)} \right] + \mathbb{P}[(\Omega_{a_T}^{(0)}(\gamma))^c] \\
& \leq c(\log T)^{\frac{3+2\gamma}{2\delta_0}} \mathbb{P} \left[\max_{k=1, \dots, T} Z_k \leq 1 \right] + \mathbb{P}[(\Omega_{a_T}^{(0)}(\gamma))^c].
\end{aligned} \tag{51}$$

Since $\mathbb{P}((\Omega_{a_T}^{(0)}(\gamma))^c) = \mathcal{O} \left(e^{-(a_T)^{\frac{\delta_0}{2}}} \right) = \mathcal{O}(T^{-\beta})$, by combining (47) and (51), we get the lower bound.

Under the assumptions of Theorem 2, we proceed analogously by replacing Z by \tilde{Z} (and V by \tilde{V}) and we obtain, for T large enough,

$$\mathbb{P} \left[\tilde{Z}_{T-a_T}^* \leq \beta \log T \right] \geq \mathbb{P} \left[\tilde{Z}_T^* \leq \beta \log T \right] \geq c^{-1} T^{-1/4} \left[\tilde{H}^{-1} \left(\frac{3\kappa}{4} \theta T^{-\frac{1}{4}} \right) \right]^{-1}. \tag{52}$$

where \tilde{H} is given by $\tilde{H}(t) := \mathbb{E}[e^{\xi_1 \varepsilon_1} \mathbf{1}_{\{e^{\xi_1 \varepsilon_1} > t\}}] = \delta H(t) + (1-\delta) \mathbf{1}_{\{t < 1\}}$ (hence $\tilde{H}^{-1}(u) = H^{-1}(u/\delta)$) as soon as $u < \delta H(1)$) and

$$\mathbb{P} \left[\max_{k=a_T+1, \dots, T} \tilde{Z}_k - \tilde{Z}_{a_T} \leq \beta \log T \middle| (S, (\varepsilon_k)_k) \right] \leq c(\log T)^{\frac{3+2\gamma}{2\delta_0}} \mathbb{P} \left[\tilde{Z}_T^* \leq 1 \middle| (S, (\varepsilon_k)_k) \right] \tag{53}$$

on $\tilde{\Omega}_{a_T}^{(0)}$ provided $\sum_{k=1}^{a_T} \varepsilon_k \in d\mathbb{N}$. We proceed now as for the upper bound. We take $\tilde{a}_T := \left\lceil ((\log T)/4)^{\frac{2}{\delta_0}} \right\rceil$ and $0 < 2\delta_0 < \tilde{\gamma} < \gamma < \frac{1}{48}$ and $\tilde{a}_T = \lceil (\beta \log T)^{2/\delta_0} \rceil$. We define again $a_T := \min\{k \geq \tilde{a}_T : \sum_{\ell=1}^k \varepsilon_\ell \in d\mathbb{N}\}$. Using the stationarity of $(\tilde{Z}_k)_k$, we obtain

$$\begin{aligned}
c^{-1} T^{-1/4} \left[H^{-1} \left(\frac{3\kappa}{4\delta} \theta T^{-\frac{1}{4}} \right) \right]^{-1} & \leq \mathbb{P} \left[\tilde{Z}_{T-a_T}^* \leq 1 \right] \\
& \leq \mathbb{E} \left[\mathbb{P} \left[\max_{k=a_T+1, \dots, T} \tilde{Z}_k - \tilde{Z}_{a_T} \leq \beta \log T \middle| (S, (\varepsilon_k)_k) \right] \right] \\
& \leq \mathbb{P}((\tilde{\Omega}_{a_T}^{(0)}(\tilde{\gamma}))^c) + \mathbb{P}(a_T - \tilde{a}_T > \tilde{a}_T^{\delta_0}) \\
& \quad + c(\log T)^{\frac{3+2\gamma}{2\delta_0}} \mathbb{P} \left[\tilde{Z}_T^* \leq 1 \right]
\end{aligned}$$

for T large enough, from which we conclude.

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