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► **To cite this version:**

Jean-Bernard Lasserre. Moments and Legendre-Fourier Series for Measures Supported on Curves. Symmetry, Integrability and Geometry : Methods and Applications, 2015, Special Issue on Orthogonal Polynomials, Special Functions and Applications, 11 (077), 10 p. 10.3842/SIGMA.2015.077 . hal-01187750v2

**HAL Id: hal-01187750**

**<https://hal.science/hal-01187750v2>**

Submitted on 29 Sep 2015

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# Moments and Legendre–Fourier Series for Measures Supported on Curves<sup>\*</sup>

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Received August 28, 2015, in final form September 26, 2015; Published online September 29, 2015  
<http://dx.doi.org/10.3842/SIGMA.2015.077>

**Abstract.** Some important problems (e.g., in optimal transport and optimal control) have a relaxed (or weak) formulation in a space of appropriate measures which is much easier to solve. However, an optimal solution  $\mu$  of the latter solves the former if and only if the measure  $\mu$  is supported on a “trajectory”  $\{(t, x(t)) : t \in [0, T]\}$  for some measurable function  $x(t)$ . We provide necessary and sufficient conditions on moments  $(\gamma_{ij})$  of a measure  $d\mu(x, t)$  on  $[0, 1]^2$  to ensure that  $\mu$  is supported on a trajectory  $\{(t, x(t)) : t \in [0, 1]\}$ . Those conditions are stated in terms of Legendre–Fourier coefficients  $\mathbf{f}_j = (\mathbf{f}_j(i))$  associated with some functions  $f_j : [0, 1] \rightarrow \mathbb{R}$ ,  $j = 1, \dots$ , where each  $\mathbf{f}_j$  is obtained from the moments  $\gamma_{ji}$ ,  $i = 0, 1, \dots$ , of  $\mu$ .

*Key words:* moment problem; Legendre polynomials; Legendre–Fourier series

*2010 Mathematics Subject Classification:* 42C05; 42C10; 42A16; 44A60

## 1 Introduction

This paper is in the line of research concerned with the following issue: *which type and how much of information on the support of a measure can be extracted from its moments* (a research issue outlined in a *Problem session* at the 2013 Oberwolfach meeting on *Structured Function Systems and Applications* [2]). In particular, a highly desirable result is to obtain necessary and/or sufficient conditions on moments of a given measure to ensure that its support has certain geometric properties. For instance there is a vast literature on the old and classical  $L$ -moment problem, which asks for moment conditions to ensure that the underlying measure  $\mu$  is absolutely continuous with respect to some reference measure  $\nu$ , and with a density in  $L_\infty(\nu)$ . See, for instance, [3, 10, 11], more recently [7], and the many references therein.

Here we are interested in a problem that is somehow “orthogonal” to the  $L$ -moment problem. Namely, we consider the following generic problem: Let  $d\mu(x, t)$  be a probability measure on  $[0, 1] \times [0, 1]$ . Provide necessary and/or sufficient conditions on the moments of  $\mu$  to ensure that  $\mu$  is singular with respect to the Lebesgue measure  $d(x, t)$  on  $[0, 1]^2$ . In fact, and more precisely, suppose that:

- one knows all moments  $\gamma_i(j) = \int x^i t^j d\mu(x, t)$ ,  $i, j = 0, 1, \dots$ , of the measure  $\mu$ , and
- the marginal of  $\mu$  with respect to the “ $t$ ” variable is the Lebesgue measure  $dt$  on  $[0, 1]$ .

Then provide necessary and/or sufficient conditions on the moments  $(\gamma_i(j))$  of  $\mu$  to ensure that  $\mu$  is supported on a trajectory  $\{(t, x(t)) : t \in [0, 1]\} \subset [0, 1]^2$ , for some measurable function  $x : [0, 1] \rightarrow [0, 1]$ .

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<sup>\*</sup>This paper is a contribution to the Special Issue on Orthogonal Polynomials, Special Functions and Applications. The full collection is available at <http://www.emis.de/journals/SIGMA/OPSPA2015.html>

In contrast to the  $L$ -moment problem, and to the best of our knowledge, the above problem stated in this form has not received a lot of attention in the past even though it is crucial in some important applications (two of them having motivated our interest).

**Motivation.** In addition of being of independent interest, this investigation is motivated by at least two important applications:

- *The mass transfer (or optimal transport) problem.* In the weak (or relaxed) Monge–Kantorovich formulation of the mass transport problem originally stated by Monge, one searches for a measure  $d\mu(x, t)$  with prescribed marginals  $\nu_x$  and  $\nu_t$ , and which minimizes some cost functional  $\int c(x, t) d\mu(x, t)$ . However in the original Monge formulation, ultimately one would like to obtain an optimal solution  $\mu^*$  of the form  $d\mu^*(x, t) = \delta_{x(t)} d\nu_t(t)$  for some measurable function  $t \mapsto x(t)$  (the transportation plan) and a crucial issue is to provide conditions for this to happen<sup>1</sup>. For more details the interested reader is referred, e.g., to [13, pp. 1–5] and [9]. There exist some characterizations of the support of an optimal measure for the weak formulation. For instance, *c-cyclical monotonicity* relates optimality with the support of solutions, and more recently [1] have shown in the (more general) context of the generalized moment problem that under some weak conditions the support of optimal solutions is *finitely minimal / c-monotone*. (As defined in [1] a set  $\Gamma$  is called finitely minimal /  $c$ -monotone if each finite measure  $\alpha$  concentrated on finitely many atoms of  $\Gamma$  is cost minimizing among its competitors; in the optimal transport context, a competitor of  $\alpha$  is any finite measure  $\alpha'$  with same marginals as  $\alpha$ .) For more details the interested reader is referred to [1] and the references therein. But such a characterization does not say when this support is a trajectory.

- *Deterministic optimal control.* Using the concept of *occupation measures*, a *weak formulation* of deterministic optimal control problems replaces the original control problem with an infinite-dimensional optimization problem  $\mathcal{P}$  on a space of appropriate (occupation) measures on a Borel space  $\mathcal{X} \times \mathcal{U} \times [0, 1]$  with  $\mathcal{X} \subset \mathbb{R}^n$ ,  $\mathcal{U} \subset \mathbb{R}^m$ . For more details the interested reader is referred, e.g., to [8, 14], and the many references therein. An important issue is to provide conditions on the problem data under which the optimal value of the relaxed problem  $\mathcal{P}$  is the same as that of the original problem; see, e.g., [14]. Again this is the case if some optimal solution  $\mu^*$  (or every element of a minimizing sequence) of the relaxed problem is such that every marginal  $\mu_j^*$  of  $\mu^*$  with respect to  $(x_j, t)$ ,  $j = 1, \dots, n$ , and every marginal  $\mu_\ell^*$  of  $\mu^*$  with respect to  $(u_\ell, t)$ ,  $\ell = 1, \dots, m$ , is supported on a trajectory  $\{(t, x_j(t)) : t \in [0, 1]\}$  and on a trajectory  $\{(t, u_\ell(t)) : t \in [0, 1]\}$  for some measurable functions  $t \mapsto x_j(t)$  and  $t \mapsto u_\ell(t)$  on  $[0, 1]$ .

**Contribution.** Of course there is a particular case where one may conclude that  $\mu$  is singular with respect to the Lebesgue measure on  $[0, 1]^2$ . If there is a polynomial  $p \in \mathbb{R}[x, t]$  of degree say  $d$ , such that its vector of coefficients  $\mathbf{p}$  is in the kernel of the moment matrix  $\mathbf{M}_s$  (where  $\mathbf{M}_s[(i, j), (k, \ell)] = \gamma_{i+k, j+\ell}$ ,  $i + j, k + \ell \leq s$ , with  $d \leq s$ ), then  $\mu$  is supported on the variety  $\{(x, t) \in [0, 1]^2 : p(x, t) = 0\}$  and therefore is singular with respect to the Lebesgue measure on  $[0, 1]^2$ . But it may happen that  $p(x, t) = p(y, t) = 0$  for some  $t$  and some  $x \neq y$  and so even in this case additional conditions are needed to ensure existence of a trajectory  $\{(t, x(t)) : t \in [0, 1]\}$ .

We provide a set of explicit necessary and sufficient conditions on the moments  $\gamma_i = (\gamma_i(j))$  which state that for every fixed  $i$ , the moments  $\gamma_i(j)$ ,  $j = 0, 1, \dots$ , are limits of certain  $i$ -powers of the moments  $\gamma_1$ .

More precisely, an explicit linear transformation  $\Delta\gamma_1$  of the infinite vector  $\gamma_1$  is the vector of (shifted) Legendre–Fourier coefficients associated with the function  $t \mapsto x(t)$ . Then the conditions state that for each fixed  $i = 2, 3, \dots$ , the vector  $\Delta\gamma_i$  should be the vector of (shifted) Legendre–Fourier coefficients associated with the function  $t \mapsto x(t)^i$ , which in turn are expressible in terms of limits of “ $i$ -powers” of coefficients of  $\Delta\gamma_1$ .

At last but not least, it should be noted that all results of this paper are easily transposed to the multi-dimensional case of a measure  $d\mu(\mathbf{x}, t)$  on  $[0, 1]^n \times [0, 1]$  and supported on a trajectory

<sup>1</sup>Here  $\delta_z$  denote the Dirac measure at the point  $z$ .

$\{(t, \mathbf{x}(t)) : t \in [0, 1]\} \subset [0, 1]^{n+1}$  for some measurable mapping  $\mathbf{x} : [0, 1] \rightarrow [0, 1]^n$ . Indeed by proceeding *coordinate-wise* for each function  $t \mapsto x_i(t)$ ,  $i = 1, \dots, n$ , one is reduced to the case  $[0, 1]^2$  investigated here.

## 2 Notation, definitions and preliminary results

### 2.1 Notation and definitions

Given a Borel probability measure  $\mu$  on  $[0, 1]^2$ , define

$$\gamma_i(j) = \int_{[0,1]^2} t^j x^i d\mu(x, t), \quad i, j = 0, 1, \dots, \quad (2.1)$$

and for every fixed  $i \in \mathbb{N}$ , denote by  $\gamma_i$  the vector of moments  $(\gamma_i(j))$ ,  $j = 0, 1, \dots$ .

Let  $(\mathcal{L}_j)$ ,  $j = 0, 1, \dots$ , be the family of orthonormal polynomials with respect to the Lebesgue measure on  $[0, 1]$ . They can be deduced from the Legendre polynomials<sup>2</sup> via the change of variable  $t' = (2t-1)$ ; the  $(\mathcal{L}_j)$  are called the *shifted Legendre polynomials*. The polynomials  $(\mathcal{L}_j)$  can be also computed exactly from the moments  $\gamma_0$  of the Lebesgue measure  $dt$  on  $[0, 1]$  by computing some determinants of modified Hankel moment matrices. For instance,  $\mathcal{L}_0 = 1$ , and

$$\mathcal{L}_1(t) = a \det \begin{pmatrix} 1 & 1/2 \\ 1 & t \end{pmatrix} = a(t - 1/2) \quad \text{with} \quad a^2(1/3 - 1/2 + 1/4) = 1,$$

i.e.,  $a = \sqrt{12}$  and  $\mathcal{L}_1(t) = 2\sqrt{3}t - \sqrt{3}$ , and

$$\mathcal{L}_2(t) = b \det \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1 & t & t^2 \end{pmatrix} = b[t^2/12 - t/12 + 1/72],$$

with  $b > 0$  such that  $\int_0^1 \mathcal{L}_2(t)^2 dt = 1$ . See, e.g., [4] and [6].

The Lebesgue space  $L_2([0, 1]) := \{f : [0, 1] \rightarrow \mathbb{R}, \int_0^1 f^2 dx < \infty\}$ , equipped with the scalar product  $\langle f, g \rangle = \int_0^1 fg dx$  and the associated norm  $\|\cdot\|$ , is a Hilbert space and the polynomials are dense in  $L_2([0, 1])$ . In particular the family  $(\mathcal{L}_j)$ ,  $j = 0, 1, \dots$ , form an orthonormal basis of  $L_2([0, 1])$ . Let  $\ell^2$  denotes the space of square-summable sequences with norm also denoted by  $\|\cdot\|$ .

Finally, let  $\|f\|_\infty := \operatorname{ess\,sup}_{x \in [0,1]} |f(x)|$ , and similarly, for  $p \in \mathbb{R}[x]$  let  $\|p\|_\infty := \sup_{x \in [0,1]} |p(x)|$ . Then  $L_\infty([0, 1]) := \{f : \|f\|_\infty < \infty\}$ .

### 2.2 Some preliminary results

We next state some useful auxiliary results, some of them being standard in Real Analysis.

**Proposition 2.1.** *Let  $t \mapsto f(t)$  be an element of  $L_2([0, 1])$  and define  $\mathbf{f} = (\mathbf{f}(j))$  by*

$$\mathbf{f}(j) := \int_0^1 \mathcal{L}_j(t) f(t) dt, \quad j = 0, 1, \dots$$

*Then one has*

$$\sum_{j=0}^{\infty} \mathbf{f}(j) \mathcal{L}_j \left( := \lim_{n \rightarrow \infty} \sum_{j=0}^n \mathbf{f}(j) \mathcal{L}_j \right) = f \quad \text{in } L_2([0, 1]), \quad (2.2)$$

*and this decomposition is unique. Moreover  $\mathbf{f} \in \ell^2$  and  $\|f\| = \|\mathbf{f}\|$ .*

<sup>2</sup>The Legendre polynomials are orthonormal w.r.t. to the Lebesgue measure on  $[-1, 1]$ .

When the interval is  $[-1, 1]$  (instead of  $[0, 1]$  here) (2.2) is called the Legendre (or Legendre–Fourier) series expansion of the function  $f$  and  $\mathbf{f} = (\mathbf{f}(j))$  is called the vector of Legendre–Fourier coefficients.

The notation  $f^k$  stands for the function  $t \mapsto f(t)^k$ ,  $k \in \mathbb{N}$ . If  $f^k \in L_2([0, 1])$  we denote by  $\mathbf{f}_k = (\mathbf{f}_k(j)) \in \ell^2$  its (shifted) Legendre–Fourier coefficients so that  $\|f^k\| = \|\mathbf{f}_k\|$  (where again the latter norm is that of  $\ell^2$ ). Notice that we also have:

**Proposition 2.2.** *Let  $f, f^k, g \in L_2([0, 1])$  with  $k \in \mathbb{N}$ . Then  $g = f^k$  if and only if  $\hat{\mathbf{g}} = \hat{\mathbf{f}}_k$ .*

This follows from the uniqueness of the decomposition in the basis  $(\mathcal{L}_j)$ .

We also have the following helpful results:

**Lemma 2.3.** *Let  $f \in L_2([0, 1])$  with Legendre–Fourier coefficients  $\mathbf{f} = (\mathbf{f}(j))$ , and let  $(p_n) \subset \mathbb{R}[t]$  be a sequence of polynomials such that  $\|p_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ .*

*If  $\hat{\mathbf{p}}_n = (\hat{\mathbf{p}}_n(j))$  denotes the (shifted) Legendre–Fourier coefficients of  $p_n$ , for all  $n = 1, 2, \dots$ , then  $\|\hat{\mathbf{p}}_n - \mathbf{f}\| \rightarrow 0$  in  $\ell^2$  as  $n \rightarrow \infty$ .*

**Proof.** As  $\|p_n - f\|^2 = \int_0^1 (p_n - f)^2 dt$  and with  $d_n = \deg(p_n)$ ,

$$\begin{aligned} \int_0^1 (p_n - f)^2 dt &= \int_0^1 \left( \sum_{j=0}^{d_n} \hat{\mathbf{p}}_n(j) \mathcal{L}_j - f \right)^2 dt \\ &= \sum_{j=0}^{d_n} (\hat{\mathbf{p}}_n(j))^2 \underbrace{\int_0^1 \mathcal{L}_j^2 dt}_{=1} + 2 \sum_{k < j} \hat{\mathbf{p}}_n(j) \hat{\mathbf{p}}_n(k) \underbrace{\int_0^1 \mathcal{L}_j \mathcal{L}_k dt}_{=0} \\ &\quad - 2 \sum_{j=0}^{d_n} \hat{\mathbf{p}}_n(j) \underbrace{\int_0^1 \mathcal{L}_j f dt}_{\mathbf{f}(j)} + \underbrace{\|f\|^2}_{=\|\mathbf{f}\| = \sum_j \mathbf{f}(j)^2} \\ &\geq \sum_{j=0}^{d_n} \hat{\mathbf{p}}_n(j)^2 - 2 \hat{\mathbf{p}}_n(j) \mathbf{f}(j) + \mathbf{f}(j)^2 = \sum_{j=0}^{d_n} (\hat{\mathbf{p}}_n(j) - \mathbf{f}(j))^2, \end{aligned}$$

and the result follows because  $\|p_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ . ■

**Lemma 2.4.** *Let  $f \in L_\infty([0, 1])$  (hence  $f^k \in L_2([0, 1])$  for every  $k \in \mathbb{N}$ ). If a sequence  $(p_n) \subset \mathbb{R}[x]$  is such that  $\sup_n \|p_n\|_\infty < \infty$  and  $\|p_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ , then for every  $k \in \mathbb{N}$ ,*

$$\lim_{n \rightarrow \infty} \|p_n^k - f^k\| = 0.$$

*In addition, if  $\hat{\mathbf{p}}_n^k$  denotes the (shifted) Legendre–Fourier coefficients of  $p_n^k$ ,  $n = 1, \dots$ , then  $\|\hat{\mathbf{p}}_n^k - \mathbf{f}_k\| \rightarrow 0$  in  $\ell^2$  as  $n \rightarrow \infty$ .*

**Proof.** Let  $M > \max[\|f\|_\infty, \sup_n \|p_n\|_\infty]$  and fix  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|p_n^k - f^k\|^2 &= \int_0^1 (p_n^k - f^k)^2 dx = \int_0^1 (p_n - f)^2 \left( \sum_{\ell=0}^{k-1} p_n^{k-1-\ell} f^\ell \right)^2 dx \\ &\leq \int_0^1 (p_n - f)^2 \left( \sum_{\ell=1}^k |p_n^{k-\ell} f^{\ell-1}| \right)^2 dx \leq (kM^{k-1})^2 \int_0^1 (p_n - f)^2 dx \\ &= (kM^{k-1})^2 \|p_n - f\|^2 \quad (\rightarrow 0 \text{ as } n \rightarrow \infty.) \end{aligned}$$

Then the last statement follows from Lemma 2.3. ■

**Definition 2.5.** Let  $f \in L_2([0, 1])$  with Legendre–Fourier coefficients  $\mathbf{f}$ . For every  $k, n \in \mathbb{N}$ , define the polynomial  $f_n^{(k)} \in \mathbb{R}[x]$  and the vector  $\mathbf{f}_n^{(k)} \in \mathbb{R}^{kn+1}$  by

$$t \mapsto f_n^{(k)}(t) := \left( \sum_{j=0}^n \mathbf{f}(j) \mathcal{L}_j(t) \right)^k = \sum_{j=0}^{nk} \mathbf{f}_n^{(k)}(j) \mathcal{L}_j(t).$$

Observe that each entry  $\mathbf{f}_n^{(k)}(j)$ ,  $j = 0, \dots, nk$ , is a degree- $k$  form of the first  $n + 1$  Legendre–Fourier coefficients of  $\hat{\mathbf{f}}$ . Completing with zeros, consider  $\mathbf{f}_n^{(k)}$  to be an element of  $\ell^2$  and if  $\mathbf{f}_n^{(k)}$  converges in  $\ell^2$  as  $n \rightarrow \infty$ , call  $\mathbf{f}^{(k)} \in \ell^2$  its limit.

The limit  $\mathbf{f}^{(k)}$  can also be denoted  $\mathbf{f} \star \dots \star \mathbf{f}$ , the limit of the  $k$  times “ $\star$ -product” in  $\ell^2$  of the vector  $\mathbf{f} \in \ell^2$  by itself. Equivalently one may write  $\mathbf{f}^{(k)} = \mathbf{f}^{(k-1)} \star \mathbf{f}$  since  $f_n^{(k)}(t) = f_n^{(k-1)}(t) f_n^{(1)}(t)$  for all  $t \in [0, 1]$ , and  $\mathbf{f}_n^{(k-1)} \rightarrow \mathbf{f}^{(k-1)}$  as  $n \rightarrow \infty$ , as well as  $\mathbf{f}_n^{(1)} \rightarrow \mathbf{f}$ .

**Lemma 2.6.** Let  $f \in L_\infty([0, 1])$ , hence  $f^k \in L_2([0, 1])$  with (shifted) Legendre–Fourier coefficients  $\mathbf{f}_k \in \ell^2$  for every  $k \in \mathbb{N}$ , and assume that

$$\sup_n \|f_n^{(1)}\|_\infty \left( = \sup_n \left\| \sum_{j=0}^n \mathbf{f}(j) \mathcal{L}_j \right\|_\infty \right) < \infty.$$

Then  $\mathbf{f}_k = \mathbf{f}^{(k)} = \mathbf{f} \star \dots \star \mathbf{f}$  ( $k$  times) for every  $k = 1, 2, \dots$ , meaning that for every fixed  $k \in \mathbb{N}$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \left( \sum_{j=0}^n \mathbf{f}(j) \mathcal{L}_j \right)^k - f^k \right\| &= \lim_{n \rightarrow \infty} \left\| \sum_{j=0}^{kn} \mathbf{f}_n^{(k)}(j) \mathcal{L}_j - f^k \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{j=0}^n \mathbf{f}_k(j) \mathcal{L}_j - f^k \right\| = 0. \end{aligned}$$

Equivalently,  $\mathbf{f}_k = \mathbf{f}_{k-1} \star \mathbf{f}$  for every  $k = 2, 3, \dots$

**Proof.** The result is a direct consequence of Lemmas 2.3 and 2.4 with  $p_n = f_n^{(1)}$  (and the definition of the limit “ $\star$ -product” in Definition 2.5).  $\blacksquare$

### 3 Main result

Assume that we are given all moments of a nonnegative measure  $d\mu(x, t)$  on a box  $[a, b] \times [c, d] \subset \mathbb{R}^2$ . After a re-scaling of its moments we may and will assume that  $\mu$  is a probability measure supported on  $[0, 1]^2$  with associated moments

$$\gamma_i(j) = \int_{[0,1]^2} x^i t^j d\mu(x, t), \quad i, j = 0, 1, \dots$$

We further assume that the marginal measure  $\mu_t$  with respect to the variable  $t$ , is the Lebesgue measure on  $[0, 1]$ , that is,  $\gamma_0(j) = 1/(j + 1)$ ,  $j = 0, 1, \dots$

A standard disintegration of the measure  $\mu$  yields

$$\gamma_i(j) = \int_{[0,1]^2} t^j x^i d\mu(x, t) = \int_0^1 t^j \underbrace{\left( \int_{[0,1]} x^i \psi(dx|t) \right)}_{=: f_i(t)} dt, \quad i, j = 0, 1, \dots, \quad (3.1)$$

where the stochastic kernel  $\psi(\cdot|t)$  is the conditional probability on  $[0, 1]$  given  $t \in [0, 1]$ . Observe that the measurable function  $f_i$  in (3.1) is nonnegative and uniformly bounded by 1 because  $|x^i| \leq 1$  on  $[0, 1]$  for every  $i$ , and so  $f_i \in L_\infty([0, 1])$  for every  $i = 1, \dots$

The vector  $\gamma_i = (\gamma_i(j))$ ,  $j = 0, 1, \dots$ , is the vector of moments of the measure  $d\mu_i(t) = f_i(t)dt$  on  $[0, 1]$ , for every  $i = 1, 2, \dots$ . The (shifted) Legendre–Fourier vector of coefficients  $\hat{\mathbf{q}}_i$  of  $f_i$  are obtained easily from the (infinite) vector  $\gamma_i$  via a triangular linear system. Indeed write

$$\mathcal{L}_j(t) = \sum_{k=0}^j \Delta_{jk} t^k, \quad \forall t \in [0, 1], \quad j = 0, 1, \dots,$$

where  $\Delta_{jj} > 0$ , or in compact matrix form

$$\begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \cdot \\ \mathcal{L}_n \\ \cdot \end{bmatrix} = \Delta \begin{bmatrix} 1 \\ t \\ \cdot \\ t^n \\ \cdot \end{bmatrix}, \quad (3.2)$$

for some infinite lower triangular matrix  $\Delta$  with all diagonal elements being strictly positive. Therefore

$$\Delta \gamma_i = \Delta \int_0^1 \begin{bmatrix} 1 \\ t \\ \cdot \\ t^n \\ \cdot \end{bmatrix} f_i(t) dt = \int_0^1 \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \cdot \\ \mathcal{L}_n \\ \cdot \end{bmatrix} f_i(t) dt = \hat{\mathbf{q}}_i.$$

Suppose that the measure  $\mu$  is supported on a trajectory  $\{(t, x(t)) : t \in [0, 1]\} \subset [0, 1]^2$  for some measurable (density) function  $x : [0, 1] \rightarrow [0, 1]$ . The measurable function  $t \mapsto x(t)$  is an element of  $L_\infty([0, 1])$  because  $\|x\|_\infty \leq 1$ . Then by Proposition 2.1,

$$x = \sum_{j=0}^{\infty} \hat{\mathbf{x}}(j) \mathcal{L}_j \quad \text{in } L_2([0, 1]), \quad (3.3)$$

where  $\hat{\mathbf{x}} = (\hat{\mathbf{x}}(j)) \in \ell^2$  is its vector of (shifted) Legendre–Fourier coefficients (with  $\|\hat{\mathbf{x}}\| = \|x\|$ ). Similarly, for every  $k = 2, 3, \dots$ , the function  $t \mapsto x(t)^k$  is in  $L_\infty([0, 1])$  and

$$x^k = \sum_{j=0}^{\infty} \hat{\mathbf{x}}_k(j) \mathcal{L}_j \quad \text{in } L_2([0, 1]),$$

with vector of (shifted) Legendre–Fourier coefficients  $\hat{\mathbf{x}}_k \in \ell^2$  such that  $\|x^k\| = \|\hat{\mathbf{x}}_k\|$ .

We also recall the notation  $\hat{\mathbf{x}}_n^{(k)} \in \mathbb{R}^{kn+1}$  for the vector of coefficients in the basis  $(\mathcal{L}_j)$  of the polynomial  $t \mapsto \left( \sum_{j=0}^n \hat{\mathbf{x}}(j) \mathcal{L}_j(t) \right)^k$ , and when considered as an element of  $\ell^2$  (by completing with zeros) denote by  $\hat{\mathbf{x}}^{(k)} \in \ell^2$  its limit when it exists.

**Theorem 3.1.** *Let  $\mu$  be a Borel probability measure on  $[0, 1]^2$  and let  $\gamma_i(j)$ ,  $i, j = 0, 1, \dots$ , be the moments of  $\mu$  in (2.1).*

(a) If  $\mu$  is supported on a trajectory  $\{(t, x(t)) : t \in [0, 1]\}$  for some nonnegative measurable function  $t \mapsto x(t)$  on  $[0, 1]$  and if  $\sup_n \left\| \sum_{j=0}^n \hat{x}(j) \mathcal{L}_j \right\|_\infty < \infty$ , then

$$\hat{x}_i = \hat{x}^{(i)} = \underbrace{\hat{x} \star \cdots \star \hat{x}}_{i \text{ times}} = \hat{x}_{i-1} \star \hat{x}, \quad \forall i = 2, 3, \dots, \quad (3.4)$$

Equivalently

$$\Delta \gamma_i = (\Delta \gamma_1)^{(i)} = (\Delta \gamma_{i-1} \star \Delta \gamma_1)^{(i)}, \quad \forall i = 2, 3, \dots, \quad (3.5)$$

where  $\Delta$  is the non singular triangular matrix defined in (3.2).

(b) Conversely, if (3.5) holds then  $\mu$  is supported on a trajectory  $\{(t, x(t)) : t \in [0, 1]\}$  for some measurable function  $t \mapsto x(t)$  on  $[0, 1]$ , and (3.4) also holds.

**Proof.** The (a) part. As  $\mu$  is supported on  $[0, 1]^2$  one has  $\|x\|_\infty \leq 1$  and so the function  $t \mapsto x(t)^i$  is in  $L_2([0, 1])$  for every  $i = 1, 2, \dots$ . So let  $t \mapsto x(t)$  be written as in (3.3). Consider the function  $t \mapsto x(t)^i$ , for every fixed  $i \in \mathbb{N}$ , so that

$$x^i = \sum_{j=0}^{\infty} \hat{x}_i(j) \mathcal{L}_j \quad \text{in } L_2([0, 1]),$$

where the (shifted) Legendre–Fourier vector of coefficients  $\hat{x}_i$  is obtained by  $\hat{x}_i = \Delta^{-1} \gamma_i$ . But by Lemma 2.6, we also have

$$x^i = \left( \sum_{j=0}^{\infty} \hat{x}(j) \mathcal{L}_j \right)^i \left( = \lim_{n \rightarrow \infty} \left( \sum_{j=0}^n \hat{x}(j) \mathcal{L}_j \right)^i \right) \quad \text{in } L_2([0, 1]),$$

with  $\|x^i\| = \|\hat{x}_i\|$  and  $\hat{x}_i = \Delta \gamma_i$ . In other words,  $\hat{x}^{(i)} = \hat{x}_i$  or equivalently,  $\hat{x}^{(i)} = \Delta \gamma_i = (\Delta \gamma_1)^{(i)}$ , which is (3.4).

We next prove the (b) part. By the disintegration (3.1) of the measure  $\mu$ ,

$$\gamma_i(j) = \int_0^1 t^j f_i(t) dt, \quad i, j = 0, 1, \dots$$

for some nonnegative measurable functions  $f_i \in L_\infty([0, 1])$ ,  $i = 1, 2, \dots$

As (3.5) holds one may conclude that  $\hat{q}_i = \hat{q}_1^{(i)}$  where  $\hat{q}_i$  is the (shifted) Legendre–Fourier vector of coefficients associated with  $f_i \in L_2([0, 1])$ ,  $i = 1, \dots$ . Hence by Proposition 2.2,  $f_i(t) = f_1(t)^i$  a.e. on  $[0, 1]$ , for every  $i = 1, 2, \dots$ . That is, for every  $i = 1, 2, \dots$ , there exists a Borel set  $B_i \subset [0, 1]$  with Lebesgue measure zero such that  $f_i(t) = f_1(t)^i$  for all  $t \in [0, 1] \setminus B_i$ . Therefore the Borel set  $B = \cup_{i=1}^{\infty} B_i$  has Lebesgue measure zero and for all  $i = 1, 2, \dots$ ,

$$f_i(t) = f_1(t)^i, \quad \forall t \in [0, 1] \setminus B.$$

Hence for every  $t \in [0, 1] \setminus B$ ,

$$\int_{[0,1]} x^i \psi(dx|t) = f_1(t)^i = \int_{[0,1]} x^i d\delta_{f_1(t)}, \quad \forall i = 1, 2, \dots$$

where  $\delta_{f_1(t)}$  is the Dirac measure at the point  $f_1(t) \in [0, 1]$ . As measures on compact sets are moment determinate, one must have  $\psi(dx|t) = \delta_{f_1(t)}$ , for all  $t \in [0, 1] \setminus B$ . Therefore  $d\mu(x, t) = \delta_{f_1(t)} dt$ , i.e., the measure  $\mu$  is supported on the trajectory  $\{(t, x(t)) : t \in [0, 1]\}$ , where  $x(t) = f_1(t)$  for almost all  $t \in [0, 1]$ .  $\blacksquare$



**Remark 3.2.** If the trajectory  $t \mapsto x(t)$  is a polynomial of degree say  $d$ , then the vector of Legendre–Fourier coefficients  $\hat{\mathbf{x}} \in \ell^2$  has at most  $d + 1$  non-zero elements. Therefore for every  $j = 2, \dots$ ,  $\hat{\mathbf{x}}_j \in \ell^2$  also has at most  $jd + 1$  non-zero elements and the condition (3.5) can be checked easily.

In Theorem 3.1(a) one assume that  $\sup_n \left\| \sum_{j=0}^n \hat{\mathbf{x}}(j) \mathcal{L}_j \right\|_\infty < \infty$  which is much weaker than, e.g., assuming the uniform convergence  $\left\| \sum_{j=0}^n \hat{\mathbf{x}}(j) \mathcal{L}_j - x \right\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . The latter (which is also much stronger than the a.e. pointwise convergence) can be obtained if the function  $x(t)$  has some smoothness properties. For instance if  $x$  belongs to some Lipschitz class of order larger than or equal to  $1/2$ , then uniform convergence takes place and one may even obtain rates of convergence; see, e.g., [12] and also [15] for a comparison (in terms of convergence) of Legendre and Chebyshev expansions. In fact, quoting the authors of [5], “... knowledge of the partial spectral sum of an  $L_2$  function in  $[-1, 1]$  furnishes enough information such that an exponential convergent approximation can be constructed in any subinterval in which  $f$  is analytic”.

**Example 3.3.** To illustrate Theorem 3.1 consider the following toy example with  $\mu$  on  $[0, 1]^2$  and with marginal w.r.t. “ $t$ ” being the uniform distribution on  $[0, 1]$  and conditional  $\psi(dx|t) = \delta_{\exp(-t)}$  for all  $t \in [0, 1]$ . That is,  $t \mapsto x(t) = \exp(-t)$ .

Then the first 11 Legendre–Fourier coefficients  $\hat{\mathbf{x}}(j)$ ,  $j = 0, \dots, 10$  of  $x$  read

$$\hat{\mathbf{x}} = [0.63212055 \quad -0.1795068 \quad 0.0230105 \quad -0.0019370 \quad 0.0001217 \\ -0.0000061 \quad 0.0000002 \quad -0.00000001 \quad -0.00000004 \quad -0.0000015 \quad -0.0000625].$$

Similarly the first 11 Legendre–Fourier coefficients of  $t \mapsto x(t)^2 = \exp(-2t)$  read

$$\hat{\mathbf{x}}_2 = [0.4323323 \quad -0.2344075 \quad 0.0588678 \quad -0.0097965 \quad 0.0012219 \\ -0.0001219 \quad 0.0000101 \quad -0.0000007 \quad 0.00000004 \quad -0.000000004 \quad -0.000000004].$$

With  $n = 5$  the polynomial  $t \mapsto x_5^{(2)}(t) := \left( \sum_{k=0}^5 \hat{\mathbf{x}}(k) \mathcal{L}_k(t) \right)^2$  reads

$$x_5^{(2)}(t) = \sum_{k=0}^{10} \hat{\mathbf{x}}_5^{(2)}(k) \mathcal{L}_k(t), \quad t \in \mathbb{R}, \quad \text{with} \\ \hat{\mathbf{x}}_5^{(2)} = [0.4323336 \quad -0.2344129 \quad 0.0588626 \quad -0.0097976 \quad 0.0012219 \\ -0.0001218 \quad 0.0000098 \quad -0.0000006 \quad 0.00000003 \quad -0.000000001 \quad 0.0000000],$$

and we can observe that  $\hat{\mathbf{x}}_5^{(2)} - \hat{\mathbf{x}}_2 \approx O(10^{-5})$ .

In fact the curves  $t \mapsto x_5^{(2)}(t)$  and  $t \mapsto \exp(-2t)$  are almost indistinguishable on the interval  $[0, 1]$ .

### 3.1 A more general case

We have considered a measure  $\mu$  on  $[0, 1]^2$  whose marginal with respect to  $t \in [0, 1]$  is the Lebesgue measure. The conditions of Theorem 3.1 are naturally stated in terms of the (shifted) Legendre–Fourier coefficients associated with the functions  $t \mapsto f_i(t)$  of  $L_2([0, 1])$  defined in (3.1).

However, the same conclusions also hold if the marginal of  $\mu$  with respect to  $t \in [0, 1]$  is some measure  $d\nu = h(t)dt$  for some nonnegative function  $h \in L_1([0, 1])$  with all moments finite. The

only change is that now we have to consider the orthonormal polynomials  $t \mapsto \mathcal{H}_j(t)$ ,  $j = 0, 1, \dots$ , with respect to  $\nu$ . Recall that all the  $\mathcal{H}_j$ 's can be computed from the moments

$$\gamma_0(j) = \int t^j d\mu(x, t) = \int_0^1 t^j d\nu(t) = \int_0^1 t^j h(t) dt, \quad j = 0, 1, \dots$$

Then proceeding as before, for every  $i = 1, 2, \dots$ ,

$$\int t^j x^i d\mu(x, t) = \int_0^1 t^j \left( \underbrace{\int_{[0,1]} x^i \psi(dx|t)}_{=f_i(t) \in L_2([0,1], \nu)} \right) h(t) dt, \quad j = 0, 1, \dots,$$

and we now consider the vector of coefficients  $\hat{\mathbf{f}}_{hi} = (\hat{\mathbf{f}}_{hi}(j))$  defined by

$$\hat{\mathbf{f}}_{hi}(j) = \int_0^1 \mathcal{H}_j(t) f_i(t) h(t) dt, \quad j = 0, 1, \dots$$

the analogues for the measure  $d\nu = h(t)dt$  and the function  $f_i \in L_2([0, 1], \nu)$ , of the (shifted) Legendre–Fourier coefficients  $\hat{\mathbf{f}}_i(j)$  of  $f_i$  in (3.1) for the Lebesgue measure on  $[0, 1]$ . Then the conditions in Theorem 3.1 would be exactly the same as before, excepted that now,  $\hat{\mathbf{x}} = (\hat{\mathbf{x}}(j))$  with

$$\hat{\mathbf{x}}(j) = \int_0^1 \mathcal{H}_j(t) x(t) h(t) dt, \quad j = 0, 1, \dots$$

### 3.2 Discussion

Theorem 3.1 may have some practical implications. For instance consider the *weak* formulation  $\mathcal{P}$  of an optimal control problem  $\mathbf{P}$  as an infinite-dimensional optimization problem on an appropriate space of (occupation) measures, as described, e.g., in [14]. In [8] the authors propose to solve a hierarchy of semidefinite relaxations  $(\mathcal{P}_k)$ ,  $k = 1, 2, \dots$ , of  $\mathcal{P}$ . Each optimal solution of  $\mathcal{P}_k$  provides with a finite sequence  $\mathbf{z}^k = (z_{j,\alpha,\beta}^k)$  such that when  $k \rightarrow \infty$ ,  $\mathbf{z}^k \rightarrow \mathbf{z}^*$  where  $\mathbf{z}^*$  is the infinite sequence of some measure  $d\mu(t, \mathbf{x}, \mathbf{u})$  on  $[0, 1] \times \mathbf{X} \times \mathbf{U}$ , where  $\mathbf{X} \subset \mathbb{R}^n$ ,  $\mathbf{U} \subset \mathbb{R}^m$ , are compact sets.

Under some conditions both problems  $\mathbf{P}$  and its relaxation  $\mathcal{P}$  have same optimal value. If  $\mu$  is supported on feasible trajectories  $\{(t, \mathbf{x}(t), \mathbf{u}(t)) : t \in [0, 1]\}$  then these trajectories are optimal for the initial optimal control problem  $\mathbf{P}$ . So it is highly desirable to check whether indeed  $\mu$  is supported on trajectories from the only knowledge of its moments  $\mathbf{z}^* = (z_{j,\alpha,\beta}^*)$ .

By construction of the moment sequences  $\mathbf{z}^k$  one already knows that the marginal of  $\mu$  with respect to the variable “ $t$ ” is the Lebesgue measure on  $[0, 1]$ . Therefore we are typically in the situation described in the present paper. Indeed to check whether  $\mu$  is supported on trajectories  $\{(t, (x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t))) : t \in [0, 1]\}$ , one considers each coordinate  $x_i(t)$  or  $u_j(t)$  separately. For instance, for  $x_i(t)$  one considers the subset of moments  $\gamma_k(j) = (z_{j,\alpha,0}^*)$  with  $j = 0, 1, \dots$ ,  $\alpha = (0, \dots, 0, k, 0, \dots, 0) \in \mathbb{N}^n$ ,  $k = 0, 1, \dots$ , with  $k$  in position  $i$ . If (3.4) holds then indeed the marginal  $\mu_{t,x_i}$  of  $\mu$  on  $(t, x_i)$ , with moments  $(\gamma_k(j))$  is supported on a trajectory  $\{(t, x_i(t)) : t \in [0, 1]\}$ .

Of course, in (3.4) there are countably many conditions to check whereas in principle only finitely many moments of  $\mathbf{z}^*$  are available (and with some inaccuracy due to (a) solving numerically a truncation  $\mathcal{P}_k$  of  $\mathcal{P}$ , and (b) the convergence  $\mathbf{z}^k \rightarrow \mathbf{z}^*$  has not taken place yet). So an issue of future investigation is to provide necessary (or sufficient?) conditions based only on finitely many (approximate) moments of  $\mu$ .

## Acknowledgements

Research funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement ERC-ADG 666981 TAMING).

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