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ABSTRACT

Management of the portfolios containing low liquidity assets is a tedious problem. The buyer proposes the price that can differ greatly from the paper value estimated by the seller, so the seller can not liquidate his portfolio instantly and waits for a more favorable offer. To minimize losses and move the theory towards practical needs one can take into account the time lag of the liquidation of an illiquid asset. Working in the Merton's optimal consumption framework with continuous time we consider an optimization problem for a portfolio with an illiquid, a risky and a risk-free asset. While a standard Black-Scholes market describes the liquid part of the investment the illiquid asset is sold at an exogenous random moment with prescribed liquidation time distribution. The investor has the logarithmic utility function as a limit case of a HARA-type utility. Different distributions of the liquidation time of the illiquid asset are under consideration - a classical exponential distribution and Weibull distribution that is more practically relevant. Under certain conditions we show the existence of the viscosity solution in both cases. Applying numerical methods we compare classical Merton's strategies and the optimal consumption-allocation strategies for portfolios with different liquidation time distributions of an illiquid asset.

KEYWORDS

portfolio optimization — illiquidity — viscosity solutions — random income

1. Introduction

Financial crises of 2008 helped practitioners to understand the difficulties connected with a management of portfolios with illiquid assets and showed a significant need for solid mathematical models addressing this problem. Though financial institutes deal with illiquid assets on a regularly basis there is no general framework for such portfolios especially if they provide stochastic incomes or down payments.

The most challenging task one faces defining such framework is to incorporate the illiquidity in a mathematically tractable way. Intuitively it is clear which of the assets we would call liquid, yet there is still no widely accepted way of defining illiquidity of an asset as a measurable parameter. The mathematically correct definition, being a problem itself, is not the biggest challenge in this area. The exact formulation of the goals of the portfolio optimization is even more tedious, since illiquidity is usually connected with different sale mechanisms and with an essential liquidation lag-time. The stochastic processes that describe such effects are not studied profoundly in financial mathematics. Now let us give a brief overview of the models that are relevant to this paper.

In 1993 *Duffie and Zariphopoulou* in [10] develop the framework of the optimal consumption for the continuous time model, proposed by *Merton*, [19]. They considered an infinite time horizon and proved the existence and uniqueness of the viscosity solution of the associated HJB equation for the class of concave utility functions $U(c)$ satisfying the following conditions: U in c is strictly concave; $C^2(0,+\infty)$, $U(c) \leq M(1+c)^\gamma$, with

[*]This research is supported by the European Union in the FP7-PEOPLE-2012-ITN Program under Grant Agreement Number 304617 (FP7 Marie Curie Action, Project Multi-ITN STRIKE - Novel Methods in Computational Finance). Short reference for contract: PITN-GA-2012-304617 STRIKE.

$0 < \gamma < 1, M > 0; U(0) \geq 0, \lim_{c\to 0} U'(c) = +\infty, \lim_{c\to\infty} U'(c) = 0.$

Later, in 1997, in [9] an extended problem of hedging in incomplete markets with hyperbolic absolute risk aversion (so called HARA) utility function was studied. Here the stochastic income cannot be replicated by trading available securities. An investor receives stochastic income in moment t at a rate Y_t, where $dY_t = \mu Y_t dt + \eta Y_t dW_t^1$, $t \geq 0, Y_0 = y$, $y \geq 0$ and $\mu, \eta > 0 - const$ here W^1 is a standard Brownian motion. The riskless bank account has a constant continuously compound interest rate r. A traded security has a price S given by $dS_t = \alpha S_t dt + \sigma S_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2)$, α, $\sigma > 0 - const$ and W^2 is an independent standard Brownian motion, $\rho \in (-1,1)$ is a correlation between price processes S_t and Y_t. The investor utility function for consumption process c_t is given by $\mathscr{U}(c(t)) = E\left[\int_0^\infty e^{-\kappa t} U(c(t))dt\right]$, $U(c(t)) = c(t)^\gamma$, where $\gamma \in (0,1)$ and κ is a discount factor $\kappa > r$. The investors wealth process L evolves $dL_t = [rL_t + (\alpha + \delta - r)\pi_t - c_t + Y_t]dt + \sigma\pi_t(\rho dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2)$, $t \geq 0, L_0 = l$, where δ could be regarded as the dividends payed constantly from an illiquid asset or as the possession costs, l is an initial wealth endowment and π_t represents an investment in the risky asset S, with the remaining wealth held in riskless borrowing or lending. The goal is to characterize an investor value function $V(l,y) = \sup_{(\pi,c)\in\mathscr{A}(l,y)} \mathscr{U}(C)$. The set $\mathscr{A}(l,y)$ is a set of admissible controls (π, c) such that $L_t \geq 0$.

Remark. The notation of the strategy (π, c) is standard for the problems of such kind. Throughout this paper we will denote the amount of the investment in a liquid risky asset as π and investor's consumption as c. Both controls do depend on time, so to emphasize it to the reader we might also use $(\pi(t), c(t))$ or even (π_t, c_t) from time to time.

The authors in [9] proved the smoothness of the viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation in the case of the HARA utility function and the infinite time horizon. This proof heavily relies on a reduction of the initial HJB equation to an ODE. After this reduction the main result follows from the uniform convergence of the classical solution of a uniformly elliptic equation to the viscosity solution, which is unique.

It is important to mention that the authors use the discount factor $e^{-\kappa t}$ in (1) as a technical factor which is not related to stochastic income. The economical setting does not imply any liquidation of an illiquid asset which provides stochastic income Y_t.

Schwarz and Tebaldi in [21] broadened a model of random income proposed before. They assumed that the non-traded illiquid asset generates a flow of random income in the form of dividends, until it is sold at a fixed moment of time. The authors define illiquid asset as an asset that can not be sold neither piece by piece nor at once before the investment's horizon, denoted as T, which is a fixed deterministic value at which the asset generates a random cash-flow equal to its' paper-value at this moment T (the cash-flow is denoted as H_T). With this economical reasoning behind it this model of illiquidity looks rather promising yet needs a more exact qualitative and quantitive description. In this particular paper we will talk about a further improvement of this framework, especially, weakening the trading conditions for an illiquid asset that can move a model closer to the practical needs.

One of the possible extensions of this problem was done by *Ang, Papanikolaou and Westerfeld* in [3]. They considered exactly the same model as in [21]. However, they assumed that an illiquid asset can be traded but only at infrequent, stochastic moments of time and thus the whole three-asset portfolio could be rebalanced. With a series of numerical calculations they provide an intuition of the influence of illiquidity on the marginal utility of the investor. The authors numerically study the cases when amount of the illiquid wealth is significantly bigger than the amount of the liquid capital and comparing it with the opposite case (insignificantly small amount of illiquid wealth) they show that the effects of the asset being illiquid may cause unbounded deviations from the Merton solution.

In 2008 *He* [13] proposed a model with the same set-up but different constraints on illiquid asset. While the investor can instantaneously transfer funds from the liquid to the illiquid asset, the vice versa transaction is allowed only in exponentially distributed moments of time. The author finds an approximate numerical solution of the problem for the constant risk-aversion (CARA) utility function.

Industry is highly interested in feasible illiquidity models. The practitioners constantly state that portfolios that include illiquid assets have a heavily time-dependent behavior (see, for example, [7]). There were several attempts made in this direction. In [16], for example, the authors use endogenous random time horizon and demonstrate that a standard optimization problem with an endogenous stopping time differs from classical

Metron case. However, in this paper we focus on the time-horizon is an exogenous random variable. We would like to note is that the set-up with exogenous time is actually economically motivated. For example, standard inheritance procedures in several EU countries assume that the illiquid assets are sold and the cash is then divided between the heirs. Naturally the sale occurs in a random moment of time and the inheritance manager splits the cash between the heirs immediately after the sale. Another example of an exogenous liquidation time that justifies our model are shares-for-the-loan auctions. This phenomenon is typical for the the emerging markets where governmentally owned businesses are at some point privatized fully or partially. For example, it was very typical for a post-soviet markets in their transition period and is still relevant for a number of states in the Eastern Europe.

We develop a connection between the model of illiquidy [21] and the optimal consumption problem with an undiversifiable future income. We substitute the somewhat artificial constraint that the liquidation time T is fixed from very beginning with the assumption that it is stochastically distributed exogenous time. We formulate the problem for a general case with an arbitrary liquidation time distribution and some utility functions in the next Section and in the Section 5 we find the bounds for the value function. In the Section 6 and Section 7 we investigate two special cases with logarithmic utility function and two liquidation time distributions: exponential and the Weibull distribution. One of the important contributions of this paper is that using the technique of the viscosity solutions we show the existence and uniqueness of the solution of the HJB equation that corresponds to the case of Weibull-distribution which probability density function under certain conditions has a local maximum. This situation corresponds to a measurable time-lag between the sell-offer and an actual deal. Strong stability of the viscosity solutions allows to get numerical approximations with a range of monotone and consistent schemes, for example, as it was done by *Munk* in [20].

Remark. The idea to work with a non-exponential discounting is not new, for example *Ivar Ekelnad* in [11] has show the possibility to work with different discounting factors, however to our knowledge the idea of a different discounting in a framework of illiquidity was never proposed before.

2. Economical setting

We assume that the investor's portfolio includes a riskless bond, a risky asset and a non-traded asset that generates stochastic income i.e. dividends. However, in contrast with the previous works we replace the liquidation time that was deterministic before with a stochastically distributed time T. A risk-free bank account B_t with the interest rate r and a stock price S_t follow

$$dB_t = rB_t dt, \;\; dS_t = S_t(\alpha dt + \sigma dW_t^1), t \leq T \tag{1}$$

where r is assumed to be constant, the continuously compounded rate of return $\alpha > r$ and the standard deviation $\sigma, r, \alpha, \sigma - const$. An illiquid asset H_t that can not be traded up to the time T and which paper value is correlated with the stock price and follows

$$\frac{dH_t}{H_t} = (\mu - \delta)dt + \eta(\rho dW^1 + \sqrt{1-\rho^2}dW^2), t \leq T. \tag{2}$$

where μ is the expected rate of return of the risky illiquid asset, (W^1, W^2) are two independent standard Brownian motions, δ is the rate of dividend paid by the illiquid asset, η is the continuous standard deviation of the rate of return, and $\rho \in (-1;1)$ is the correlation coefficient between the stock index and the illiquid risky asset. The parameters μ, δ, η, ρ are all assumed to be constant. The liquidation time T is a random-distributed continuous variable which does not depend on the Brownian motions (W^1, W^2). The probability density function of T distribution is denoted by $\phi(t)$ whereas $\Phi(t)$ denotes the cumulative distribution function, and $\overline{\Phi}(t)$ the survival function also known as *a reliability function* $\overline{\Phi}(t) = 1 - \Phi(t)$. We omit here the explicit notion of the possible parameters of distribution in order to make the formulae shorter.

Given the filtration $\{\mathscr{F}_t\}$ generated by the Brownian motion $W = (W^1, W^2)$ we assume that the consumption process is an element of the space $\mathscr{L}_+$ of non-negative $\{\mathscr{F}_t\}$-progressively measurable processes c_t such that

$$E\left(\int_0^s c(t)dt\right) < \infty, \;\; s \in [0,T]. \tag{3}$$

The investor wants to maximize the average utility consumed up to the time of liquidation, given by

$$\mathscr{U}(c) := \mathscr{E}\left[\int_0^T U(c(t))dt\right]. \tag{4}$$

Here we used $\mathscr{E}$ to indicate that we are averaging over all random variables. The wealth process L_t is the sum of cash holdings in bonds, stocks and *random* dividends from the non-traded asset minus the consumption stream. Thus, we can write

$$dL_t = (rL_t + \delta H_t + \pi_t(\alpha - r) - c_t)dt + \pi_t \sigma dW_t^1. \tag{5}$$

The set of admissible policies is standard and consists of investment strategies (π_t, c_t) such that

1. c_t belongs to $\mathscr{L}_+$,
2. π_t is $\{\mathscr{F}_t\}$-progressively measurable and $\int_t^s (\pi_\tau)^2 d\tau < \infty$ a.s. for any $t \le \tau \le T$,
3. L_t, defined by the stochastic differential equation (5) and initial conditions $L_t = l > 0, H_t = h > 0$ a.e. $(t \le T)$.

We claim that one can explicitly average (4) over T and with the certain conditions posed on $\overline{\Phi}$ and $U(c)$ the problem (4) is equivalent to the maximization of

$$\mathscr{U}(c) := E\left[\int_0^\infty \overline{\Phi}(t)U(c(t))dt\right], \tag{6}$$

where E is an expectation over space coordinates excluding T.

Remark. It is important to note, that if T is exponentially distributed we get *precisely* the problem of optimal consumption with random income that was studied in [9] and already discussed in introduction.

We demonstrate here a formal derivation of the equivalence between two optimal problems briefly mentioned by Merton in [19].

Proposition 2.1. *The problems (4) and (6) are equivalent provided*

$$\lim_{t\to\infty} \overline{\Phi}(t) E\left[U(c(t)\right] = 0. \tag{7}$$

Proof. We have

$$\mathscr{E}\left[\int_0^T U(c(t))dt\right] = \int_0^\infty \phi(T) E\left[\int_0^T U(c(t))dt\right] dT = \int_0^\infty \int_0^T \phi(T) g(t) dT dt, \tag{8}$$

where $g(t) = E[U(c(t)]$. Because of the absolute convergence $\mathscr{E}\left[\int_0^T U(c(t))dt\right] = \int_0^T g(t)dt$ and integrating (8) by parts we get

$$\int_0^\infty \int_0^T \phi(T) g(t) dT dt = \overline{\Phi}(T) \int_0^T g(t)dt\Big|_0^\infty + \int_0^\infty \overline{\Phi}(t) g(t) dt = E\left[\int_0^\infty \overline{\Phi}(t) U(c(t)) dt\right],$$

where we used the condition (7) to eliminate the first term, and the absolute convergence of the integral to move the expectation out. • □

Remark. In the majority of the models consumption $c(t)$ is bounded as time goes to infinity. For all these models condition (7) is satisfied automatically. Yet if one regards absolute values of consumption and it grows as time goes to infinity this constraint is needed.

From now on in this paper we will work with the problem (4) with random liquidation time T that has a distribution satisfying the condition (7) in Proposition 2.1 and, therefore, corresponds to the *value function* $V(t,l,h)$ which is defined as

$$V(t,l,h) = \max_{(\pi,c)} E\left[\int_t^\infty \overline{\Phi}(\tau) U(c(\tau)) d\tau | L(t) = l, H(t) = h\right]. \tag{9}$$

For the value function we can derive a HJB equation on which we focus in this paper

$$\begin{aligned} V_t(t,l,h) &+ \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ &+ \max_{\pi} G[\pi] + \max_{c\geq 0} H[c] = 0, \end{aligned} \tag{10}$$

$$G[\pi] = \frac{1}{2}V_{ll}(t,l,h)\pi^2\sigma^2 + V_{lh}(t,l,h)\eta\rho\pi\sigma h + \pi(\alpha-r)V_l(t,l,h), \tag{11}$$

$$H[c] = -cV_l(t,l,h) + \overline{\Phi}(t)U(c), \tag{12}$$

with the boundary condition

$$V(t,l,h) \to 0, \text{ as } t \to \infty.$$

3. Viscosity solution of the problem. Comparison Principle

Definition 3.1. A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity solution of (10) if both conditions are satisfied

- u is a *viscosity subsolution*, i.e. for any $\phi \in C^2(\bar{\Omega})$ and any local maximum point $z_0 \in \bar{\Omega}$ of $u-\phi$ holds,

$$F(z_0, u(z_0), D\phi(z_0), D^2\phi(z_0)) \leq 0$$

- u is a *viscosity supersolution*, i.e. for any $\phi \in C^2(\bar{\Omega})$ and any local minimum point $z_0 \in \bar{\Omega}$ of $u-\phi$ holds,

$$F(z_0, u(z_0), D\phi(z_0), D^2\phi(z_0)) \geq 0,$$

where F is a left-hand side of (10).

The fact that the value function for a problem of such kind is a viscosity solution is well known (see e.g. [8]) and generally holds if the control and state variables are uniformly bounded. However, this is not the case for the optimal consumption problem and thus a more sophisticated proof is needed. This problem was previously studied in [10], [9], [22]. The main difficulties in our case come from the non-exponential time discounting we are using in the utility functional (9). As we mentioned before, this leads to the HJB equation (10) being three dimensional. This demands additional work. We will concentrate on the new results and will omit the details of the arguments that work in our problem and could be found in [10].

Theorem 3.1. *There exists a unique viscosity solution of the corresponding HJB equation (9) if*

1. *$U(c)$ is strictly increasing, concave and twice differentiable in c,*
2. *$\lim_{t\to\infty}\overline{\Phi}(t)E[U(c(t))] = 0$, $\overline{\Phi}(t) \sim e^{-\kappa t}$ or faster as $t \to \infty$,*
3. *$U(c) \leq M(1+c)^{\gamma}$ with $0 < \gamma < 1$ and $M > 0$,*
4. *$\lim_{c\to 0}U'(c) = +\infty$, $lim_{c\to+\infty}U'(c) = 0$.*

The proof of this statement is to be done in three steps. At first we need to establish certain properties of the value-function $V(t,l,h)$ that corresponds to our problem. This properties are formulated and proved in Lemma 3.2 that follows. Then we show that the value function with such properties is a viscosity solution of the problem, this is done in Lemma 3.3. The uniqueness of this solution follows from the *comparison principle* that is actually a very useful tool by itself and is formulated and proved in Theorem 3.4.

Lemma 3.2. *Under the conditions $(1)-(4)$ from Theorem 3.1 the value function $V(t,l,h)$ (9) has the following properties:*

(i) *$V(t,l,h)$ is concave and non-decreasing in l and in h,*

(ii) *$V(t,l,h)$ is strictly increasing in l,*

(iii) *$V(t,l,h)$ is strictly decreasing in t starting from some point,*

(iv) $0 \leq V(t,l,h) \leq O(|l|^\gamma + |h|^\gamma)$ *uniformly in* t.

1. *Proof.* Let us look on the points (l_1,h_1) and (l_2,h_2) with corresponding $(\pi_1^\varepsilon, c_1^\varepsilon)$ and $(\pi_2^\varepsilon, c_2^\varepsilon)$ which are ε-optimal controls in each of this points respectively or in another words:

$$V(t,l,h) \leq E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c^\varepsilon)d\tau\right] + \varepsilon,$$

where $l = l_1, l_2$, $h = h_1, h_2$ and $c = c_1, c_2$ correspondingly. We choose the point $(\alpha c_1^\varepsilon + (1-\alpha)c_2^\varepsilon)$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$. The policy $(\alpha l_1 + (1-\alpha)l_2, \alpha h_1 + (1-\alpha)h_2)$ is admissible for this point

$$V(t, \alpha l_1 + (1-\alpha)l_2, \alpha h_1 + (1-\alpha)h_2) \geqslant E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(\alpha c_1^\varepsilon + (1-\alpha)c_2^\varepsilon)d\tau\right]. \tag{13}$$

The utility function is concave (see condition 1. from Theorem 3.1), so we can write

$$\begin{aligned} & E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(\alpha c_1^\varepsilon + (1-\alpha)c_2^\varepsilon)d\tau\right] \geqslant \alpha E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c_1^\varepsilon)d\tau\right] \qquad (14) \\ + \quad & (1-\alpha)E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c_2^\varepsilon)d\tau\right] \geqslant \alpha V(t,l_1,h_1) + (1-\alpha)V(t,l_2,h_2) + 2\varepsilon. \end{aligned}$$

Now that we have proved the concavity of $V(t,l,h)$ in l and h. We can show that it is not decreasing. Without any loss of generality we can assume that $l_1 \leqslant l_2$ and $h_1 \leqslant h_2$. Note that if $(\pi_1^\varepsilon, c_1^\varepsilon)$ is ε-optimal for (l_1,h_1) it is admissible for (l_2,h_2) which means that

$$V(t,l_1,h_1) \leqslant V(t,l_2,h_2) + \varepsilon,$$

setting $\varepsilon \to 0$ we get that $V(t,l,h)$ is non-decreasing in first two variables. • □

2. *Proof.* To show that $V(t,l,h)$ is strictly increasing in l we can assume the contrary. Let us look at $l_1 < l_2$ such that $V(t,l_1,h) = V(t,l_2,h)$. Since we already know that $V(t,l,h)$ is non-decreasing in l the function V should be constant on the interval $[l_1,l_2]$, moreover, since V is concave in l this interval has to be infinite. This means that there is such l_0 that $V(t,l,h) = V(t,l_0,h)$ for any $l \geqslant l_0$. Let $(\pi^\varepsilon, c^\varepsilon)$ be ε-optimal for (t,l_0,h)

$$V(t,l_0,h) \leq E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c^\varepsilon)d\tau\right] + \varepsilon. \tag{15}$$

We denote $\int_t^{+\infty} \overline{\Phi}(\tau)d\tau$ as $K(t)$ and look on the inequality

$$l_1 > \max(l_0, U^{-1}[1/K(t)(E[\int_t^{+\infty} \overline{\Phi}(\tau)U(c^\varepsilon)d\tau] + \varepsilon)]/r),$$

where U^{-1} denotes an inverse utility function. The strategy $\pi = 0$ and $c = rl_1$ does not depend on time but is admissible for (t,l_1,h). Indeed, due to the fact that the strategy $(0,rl_1)$ does not depend on time one can write

$$K(t)U(rl_1) = E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(rl_1)d\tau\right] \leqslant V(t,l_1,h).$$

But if we look at $K(t)U(rl_1)$ and use the formula for l_1 given above we get

$$K(t)U(rl_1) > E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c^\varepsilon)d\tau\right] + \varepsilon,$$

which is greater or equal to $V(t,l_0,h)$ according to the Equation (15). That gives us $V(t,l_0,h) < V(t,l_1,h)$ which is a contradiction keeping in mind that $l_1 > l_0$. So, V is strictly increasing in l. • □

3. *Proof.* According to condition 2 from Theorem 3.1 the product of $\overline{\Phi}(t)$ and $U(c(t))$ as well as $\overline{\Phi}(t)$ itself should be both decreasing for $t > \tau$ starting from a large enough τ. So we choose two moments of time t_1 and t_2 such that $\tau < t_1 < t_2$, $\Delta t = t_2 - t_1$ and look at $V(t_2, l, h)$ then

$$V(t_2,l,h) = \int_{t_2}^{\infty} \overline{\Phi}(t)U(c_t)dt \overset{\tau=t-\Delta t}{=} \int_{t_1}^{\infty} \overline{\Phi}(\tau+\Delta t)U(c_{\tau+\Delta t})d\tau,$$

since $\overline{\Phi}(t)$ is decreasing for every $t > t_1$ and the process $c_{\tau+\Delta t}$ for $\tau \geq t_1$ with $L(t_2) = l, H(t_2) = h$ has exactly the same realisations as c_τ for $\tau \geq t_1$ with $L(t_1) = l, H(t_1) = h$ one can write

$$\int_{t_1}^{\infty} \overline{\Phi}(\tau+\Delta t)U(c_{\tau+\Delta t})d\tau < \int_{t_1}^{\infty} \overline{\Phi}(\tau)U(c_{\tau+\Delta t})d\tau \leqslant V(t_1,l,h).$$

So for any t_1 and t_2 such that $\tau < t_1 < t_2$ we get $V(t_1,l,h) > V(t_2,l,h)$. • □

4. *Proof.* Instead of the original problem with the non-traded income generated by $H_t, H_0 = h$ one can consider a fiction consumption-investment problem with a special asset on the market, such that has a sufficient initial endowment (meaning that one can generate exactly the same income flow as H_t would by investing in the market). Suppose the synthetic asset follows geometrical Brownian motion

$$dS'_t = \alpha' S'_t + \sigma' S'_t dW_t, \quad t \geq 0 \;\; S'_0 = s', \quad s' > 0, \tag{16}$$

with constants α' and σ' to be defined later. Next, the *initial wealth equivalent* of the stochastic income is defined by

$$\begin{aligned} f(h) &= \delta E_h \left[\int_0^{\infty} e^{-\kappa t} \xi_t H_t dt \right], \\ \xi_t &= \exp\left(-\frac{1}{2}(\theta_1^2 + \theta_2^2) + \theta_1 W_t^1 + \theta_2 W_t^2 \right), \end{aligned}$$

where $\theta_1 = (\alpha - r)/\sigma$ and $\theta_2 = (\alpha' - r)/\sigma'$.

It turns out that with the properly chosen α' and σ' we achieve that $f(h) < C_1 h$. Moreover, the stochastic income rate H_t can be replicated by a self-financing strategy on the complete market (B_t, S_t, S'_t) with the additional initial endowment $f(h)$. This fact is well known from the martingale-based studies of the consumption-investment problem, primarily carried out in [14] and [17].

To finish the proof, we notice that since the stochastic income can be replicated, any admissible strategy for the original problem with initial conditions (l,h) is dominated by a strategy on the synthetic market with initial endowment $l + f(h) < l + C_1 h$. On the other hand, we have the growth conditions for $\overline{\Phi}(t)$ and $U(c)$. So, the maximal utility is bounded from above by the solution of the classic investment-consumption problem with initial wealth $l + C_1 h$, HARA utility and exponential discounting. Due to Merton we have a closed form solution for this case. Putting everything together, we obtain the desired bound (all the further details can be found in [14] and [10]). • □

Now we can prove the existence of the viscosity solution of the problem (10).

Lemma 3.3. *Under the conditions of Lemma 3.2 the function $V(t,l,h)$ is a viscosity solution of (10) on the domain $D = (0,\infty) \times (0,\infty) \times (0,\infty)$.*

Proof. We again use the reasoning from the proof of Theorem 4.1 in [10] but modify it for our case. To show that V is a viscosity solution one need to show that it is a viscosity supersolution and a viscosity subsolution of the problem.

Let us show at first that $V(t,l,h)$ is a viscosity supersolution for (10). Let us look at $\phi \in C^2(D)$ and assume that $(t_0,l_0,h_0) \in D$ is a point where a minimum of $V - \phi$ is achieved. We can assume that $V(t_0,l_0,h_0) = \phi(t_0,l_0,h_0)$ and $V > \phi$ in D without any loss of generality. To show that V is a supersolution we need to check that $\mathscr{F}[\phi](t_0,l_0,h_0,\pi,c) \leq 0$, where

$$\begin{aligned} \mathscr{F}[\phi](t_0,l_0,h_0,\pi,c) &= \phi_t(t_0,l_0,h_0) + \frac{1}{2}\eta^2 h_0^2 \phi_{hh}(t_0,l_0,h_0) + (rl_0 + \delta h_0)\phi_l(t_0,l_0,h_0) \\ + \quad & (\mu - \delta)h_0 \phi_h(t_0,l_0,h_0) + \max_{\pi} G[t_0,l_0,h_0,\pi] + \max_{c} H[t_0,l_0,h_0,c], \end{aligned}$$

with $G[\pi]$ and $H[c]$ defined in (10).

We consider a locally constant strategy (π_0, c_0) for the period of time θ tending to zero. One can take $\theta = \min\{1/n, \tau\}$ where $\tau = \inf\{t \geq t_0 : W_t = 0\}$ to ensure feasibility of this strategy. Since this strategy is suboptimal we can write (using the dynamic programming principle, [12])

$$V(t_0,l_0,h_0) \geq E\left[\int_{t_0}^{t_0+\theta} \overline{\Phi}(t)U(c_0)dt + V(L_\theta, H_\theta, \theta)\right] \geq E\left[\int_{t_0}^{t_0+\theta} \overline{\Phi}(t)U(c_0)dt + \phi(L_\theta, H_\theta, \theta)\right]. \quad (17)$$

On the other hand, applying Itô calculus to the smooth function ϕ we can expand

$$E[\phi(\theta, L_\theta, H_\theta)] = \phi(t_0,l_0,h_0) + E\left[\int_{t_0}^{t_0+\theta} D\phi(s, L_s, H_s)ds\right].$$

Substituting into (17) and using standard estimates to approximate the terms with $\phi(s,l_s,h_s)$, $\phi_l(s,l_s,h_s)$, $\phi_h(s,l_s,h_s)$, etc. via $\phi(t_0,l_0,h_0)+O(s)$, $\phi_l(t_0,l_0,h_0)+O(s)$, $\phi_h(t_0,l_0,h_0)+O(s)$ respectively, we obtain the bound

$$E\left[\int_{t_0}^{t_0+\theta} \mathscr{F}[\phi](t_0,l_0,h_0,c_0,\pi_0)\right] + E\left[\int_{t_0}^{t_0+\theta} h(s)ds\right] \leq 0,$$

with $h(s) = O(s)$. Dividing by $E[t_0+\theta]$ and taking the limit $n \to \infty$ (so $\theta \to 0$ and $E\left[\int_{t_0}^{t_0+\theta} h(s)ds\right] \to 0$) we get (17) as (π_0, c_0) can be arbitrary admissible pair.

The second part of the proof is to show that $V(t,l,h)$ is a subsolution as well. However, the proof of the second part of Theorem 4.1 in [10] can be applied verbatim here so we omit further details. •

□

The third result that is needed to finalize the proof of Theorem 3.1 is a *comparison principle* formulated below as Theorem 3.4. Results of this type are well-known in general for bounded controls, but due to the unboundness of the controls, classical proofs require adaptations for our case.

Theorem 3.4. ***(Comparison Principle)*** *Let $u(t,l,h)$ be an upper-semicontinuous concave viscosity subsolution of (10) on D and $V(t,l,h)$ is a supersolution of (10) on D which is bounded from below, uniformly continuous on D, and locally Lipschitz in D, such that $u(t,l,h) \to 0$, $V(t,l,h) \to 0$ as $t \to \infty$ and $|u(t,l,h)| + |V(t,l,h)| \leq O(|l|^\gamma + |h|^\gamma)$ for large l,h, where $0 < \gamma < 1$, uniformly in t. Then $u \leq v$ on $\overline{D}$.*

Proof. Let us introduce $x := (l,h)$, $x \in \mathbb{R}^+ \times \mathbb{R}^+$ to make formulae shorter. Assume for contradiction that $\sup_{(t,x)\in\overline{D}}[u(t,x) - v(t,x)] > 0$. Let $T_n \to \infty$ be an increasing sequence of time moments, $m > 0$ be a parameter and

$$\Psi^{m,n}(t,x) = u(t,x) - v(t,x) - m(T_n - t).$$

Since $u, v \to 0$ as $t \to \infty$, for sufficiently large n and sufficiently small m the maximum of $\Psi^{m,n}$ must occur in an internal point of D. So let us assume that $\overline{m} > 0$ and T_n are such that $\sup_{(x,t)\in\overline{D}} \Psi^{\overline{m},n}(x,t)$ occurs in some point (t_0,x_0) with $t_0 < T_n$. Let us define two functions $\tilde{u}$ and ϕ

$$\begin{aligned} \tilde{u}(t,x) &= u(t,x) - \overline{m}(T_n - t), \\ \phi(t,x,y) &= \left|\frac{y-x}{\xi} - 4\varpi\right|^4 + \theta(l_x + h_x)^\lambda + \overline{m}(T_n - t) \end{aligned}$$

where $x = (l_x, h_x)$, $y = (l_y, h_y)$ and $\lambda \in (\gamma, 1)$, $\theta, \xi > 0$, $\varpi \in \mathbb{R}^2_+$ being parameters to be varied later. Finally, we look at the point $(\overline{x}, \overline{y}, \overline{t})$ where the following function achieves a maximum

$$\psi(t,x,y) = \tilde{u}(t,x) - v(t,y) - \phi(t,x,y).$$

Since $\overline{t}$ is an interior point we can write

$$2\overline{m} = u_t(\overline{t}, \overline{x}) - v_t(\overline{t}, \overline{y}). \quad (18)$$

On the other hand, one can bound $u_t(\overline{t}, \overline{x}, \overline{t}) - v_t(\overline{y})$ merely by ϕ and its derivatives which can be written down explicitly. It appears then, that as $\theta, \xi, \|\varpi\| \to 0$ the distance $\|\overline{x} - \overline{y}\|$ tends to zero and both $(\overline{t}, \overline{x})$, $(\overline{t}, \overline{y})$ are close to (t_0, x_0), so in the limit in terms of $\|\overline{x} - \overline{y}\| \to 0$ (18) leads to $\overline{m} \leq 0$ and we get a contradiction. • □

4. Homotheticity reduction for utility functions of the HARA type

Though the HJB equation (10) generally fails to have a reduction with respect to the time variable, it is possible to reduce its dimension if the utility function is of the HARA type. In this paper we work just with the logarithmic utility function. First of all, the logarithmic case allows one to consider time distributions with subexponential tails, while enjoying the homotheticity reduction available for utility functions of the general HARA type. Secondly, the logarithmic case could in some sense be regarded as a limiting of the HARA case with γ tending to zero. This allows to translate all the obtained results to the general power case of HARA utility with only straightforward modifications.

Rewriting the HJB equation (10) for the logarithmic utility function $U(c(t)) = \log c(t)$ we get

$$\begin{aligned} V_t(t,l,h) &+ \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ &+ \max_{\pi} G[\pi] + \max_{c\geq 0} H[c] = 0 \end{aligned} \tag{19}$$

$$G[\pi] = \frac{1}{2}V_{ll}(t,l,h)\pi^2\sigma^2 + V_{lh}(t,l,h)\eta\rho\pi\sigma h + \pi(\alpha-r)V_l(t,l,h), \tag{20}$$

$$H[c] = -cV_l(t,l,h) + \overline{\Phi}(t)\log(c). \tag{21}$$

Using the homotheticity of the logarithm and homogeneity of the differential operator applied to the value function in (19) we rewrite $V(t,l,h)$ in the following way

$$V(t,l,h) = W(t,z) - \Psi_1(t)\log h + \Psi_2(t), \tag{22}$$

having $z = l/h$ and $\Psi_1(t) = \int_t^\infty \overline{\Phi}(s)ds$ and $\Psi_2(t)$ to be chosen later.

Remark. The form of the substitution can be defined via Lie group analysis of the given equation. In [4] such analysis was carried out for a model of illiquidity with frictions. The analysis for the current model with logarithmic and general HARA-type utility is done in [5].

The Hamiltonian terms $\max_\pi G[\pi]$ and $\max_c H[c]$ in (19) now become

$$\max_{\pi} G[\pi] = \max_{\pi'=\pi/h}\left[\frac{1}{2}W_{zz}\sigma^2\pi'^2 + \pi'(-\eta\rho\sigma(W_z + zW_{zz}) + (\alpha-r)W_z)\right], \tag{23}$$

$$\max_{c} H[c] = \max_{c'=c/h}\left[-c'W_z + \overline{\Phi}(t)\log(c')\right] + \overline{\Phi}(t)\log(h), \tag{24}$$

and the optimal policies after formal maximization are

$$\begin{aligned} \pi_\star(l,h) &= h\sigma^{-2}\left(\eta\rho\sigma z - ((\alpha-r) - \eta\rho\sigma)\frac{W_z}{W_{zz}}\right), \\ c_\star(l,h) &= h\frac{\overline{\Phi}(t)}{W_z}, \end{aligned} \tag{25}$$

We rewrite (19) using formulae (23) and (24)

$$\begin{aligned} &W_t + \Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu-\delta)\right)\Psi_1(t) + \frac{\eta^2}{2}z^2W_{zz} + (\eta^2 + r - (\mu-\delta))zW_z + \delta W_z \\ &+ \max_{\pi'}\left[\frac{1}{2}W_{zz}\sigma^2\pi'^2 + \pi'(-\eta\rho\sigma(W_z + zW_{zz}) + (\alpha-r)W_z)\right] + \max_{c'\geq 0}\left[-c'W_z + \overline{\Phi}(t)\log(c')\right] = 0. \end{aligned}$$

We provide the formal maximization of $H[\pi]$ and $G[c]$ and obtain

$$\begin{aligned} \max_{\pi} H[\pi] &= -\frac{1}{2}((\eta\rho - (\alpha-r)/\sigma)^2\frac{W_z^2}{W_{zz}} + 2\eta\rho(\eta\rho - (\alpha-r)/\sigma)zW_z + \eta\rho^2z^2W_{zz}) \\ \max_{c} G[c] &= \overline{\Phi}(t)\left(\log\overline{\Phi}(t) - 1\right) - \overline{\Phi}(t)\log W_z. \end{aligned}$$

so (19) becomes

$$\begin{aligned} &W_t + \Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu-\delta)\right)\Psi_1(t) + \overline{\Phi}(t)(\log\overline{\Phi}(t) - 1) + d_2z^2W_{zz} - \frac{d_1^2}{2}\frac{(W_z)^2}{W_{zz}} \\ &+ d_3zW_z + \delta W_z - \overline{\Phi}(t)\log W_z = 0, \end{aligned} \tag{26}$$

where

$$\begin{aligned} d_1 &= \frac{\alpha - r - \eta\rho\sigma}{\sigma^2},\ d_2 = \frac{1}{2}\eta^2(1-\rho^2), \\ d_3 &= 2d_2 + \frac{\rho\eta}{\sigma}(\alpha - r) + r - (\mu - \delta). \end{aligned} \tag{27}$$

Now by choosing $\Psi_2(t)$ as a solution of the equation

$$\Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu - \delta)\right)\Psi_1(t) + \overline{\Phi}(t)(\log\overline{\Phi}(t) - 1) = 0,\ \ \Psi_2(t) \to 0, t \to \infty,$$

we can cancel out the terms dependent only on t in the equation (26). We arrive at

$$W_t - \frac{d_1^2}{2}\frac{(W_z)^2}{W_{zz}} + d_2 z^2 W_{zz} + d_3 z W_z + \delta W_z - \overline{\Phi}(t)\log W_z = 0. \tag{28}$$

5. Bounds for the value function

The main tool we are going to use to obtain the bounds is the comparison principle given by Theorem 3.4. Since (28) is a two-dimensional PDE and by itself is not a HJB equation, we argue as follows. Any formal sub- or super- solution of (28) can be transformed to a sub- or super- solution of (19) with a substitution described by (22). On the other hand, for the HJB equation (19) Theorem 3.1 and Theorem 3.4 hold and we can obtain a lower and upper bound. In order to comply with the Definition 3.1 we have to take the equation (28) with the minus sign.

Determining an upper bound demands specific information on the cumulative distribution function $\Phi(t)$ of the liquidation time. In the next Section this issue is addressed specifically for two practically applicable cases of exponentially and Weibull distributed liquidation time T.

A lower bound, however, could be found without any specific information on $\Phi(t)$. Let us look on an optimal consumption problem without random income. This is a classical two dimensional Merton's problem for which we can write the HJB equation on the value function $u(t,z)$. This problem corresponds to (19) but without any terms, containing the derivatives with respect to h and with a notation $V \to u$, $l \to z$

$$u_t + rlu_z + \max_{\pi} G[\pi] + \max_{c \geq 0} H[c] = 0, \tag{29}$$

$$G[\pi] = \frac{1}{2}u_{zz}(t,z)\pi^2\sigma^2 + \pi(\alpha - r)u_z(t,z),$$

$$H[c] = -cu_z(t,z) + \overline{\Phi}(t)\log(c). \tag{30}$$

After the formal maximization, one gets

$$u_t + rlu_z - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \frac{u_z^2}{u_{zz}} + \overline{\Phi}(t)\left(\log\overline{\Phi}(t) - \overline{\Phi}(t)\right) - \overline{\Phi}(t)\log u_z = 0.$$

We look for a solution in the form $u(t,z) = \Psi_1(t)\log z + \Theta_1(t)$, where again $\Psi_1(t) = \int_t^\infty \overline{\Phi}(s)ds$ and $\Theta_1(t)$ is a solution of

$$\Theta_1' + \Psi_1\left(r + \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2}\right) - \overline{\Phi}(\overline{\Phi} - \log\overline{\Phi} + \log\Psi_1) = 0. \tag{31}$$

One can easily check that such u tends to zero uniformly as $t \to \infty$ and since the solution of (29) is a lower bound for our three-dimensional problem we obtain the following inequality for the lower bound

$$\Psi_1(t)\log z + \Theta_1(t) \leq W(z,t) = V(t,l,h) - \Psi_1\log h + \Psi_2(t),$$

or

$$\Psi_1(t)\log l + \Theta_1(t) - \Psi_2(t) \leq V(t,l,h).$$

In the next Sections we consider specific liquidation time distributions. First we take the most simple one - an exponential distribution. We get asymptotically tight bounds for the value function and derivatives, which lead to asymptotic formulae for the optimal policies. Not surprisingly, in the limit case when the random income vanishes the value function and optimal policies coincide with the classical Merton solution for the logarithmic case. Another somewhat more complicated case is the Weibull distribution, where the bounds have no elementary representation, but their asymptotic can be derived using incomplete gamma functions.

6. The case of exponential distributed liquidation time and logarithmic utility function

Now we examine the optimal consumption problem introduced before in the case of the logarithmic utility. Despite that we know from [10] that the optimal strategy does exist and the value function is the viscosity solution of the HJB equation, it is desirable to have the optimal policy in the feedback form (25). In a general situation the feedback optimal policy is hard to establish since the value function is not a priori smooth. On the other hand, smoothness of the value function simplifies the problem so it becomes amenable to standard verification theorems of optimization theory, see e.g. [12]. Here we prove that in the case at hand the value function is twice differentiable. As far as we know this fact was not explicitly addressed before, though the structure of our proof is similar to the paper [9] where the smoothness was proved for the HARA utility case. Since the case without stochastic income is known to have a closed form solution and was derived by Merton [19], it is plausible to consider it as a zero-term approximation. Keeping that in mind, we will rigorously prove that value function tends to the Merton closed form solution in the limit of vanishing random income. Recall the definition of the value function

$$V(t,l,h)=\max_{(\pi,c)}E\left[\int_t^\infty e^{-\kappa t}\log(c)dt|L(t)=l,H(t)=h\right],\ \ \kappa>0. \tag{32}$$

At first let us note that in the exponential liquidation time distribution case the problem is homogenous in time. We introduce $\tilde{V}(l,h)$

$$\tilde{V}(l,h)=\max_{(\pi,c)}E\left[\int_t^\infty e^{-\kappa(s-t)}\log(c)ds\right]=\max_{(\pi,c)}E\left[\int_0^\infty e^{-\kappa v}\log(c)dv\right],$$

which is independent on time. Substituting

$$V(t,l,h)=e^{-\kappa t}\tilde{V}(l,h)$$

into the HJB equation (10) we arrive at a time-independent PDE on $\tilde{V}(l,h)$. With a slight abuse of notation, hereafter we will use the same letter V for $\tilde{V}$. The reduced equation takes the form

$$\frac{1}{2}\eta^2h^2V_{hh}(l,h)+(rl+\delta h)V_l(l,h)+(\mu-\delta)hV_h(l,h)+\max_\pi G[\pi]+\max_{c\geq 0}H[c]=\kappa V(l,h),$$

$$G[\pi]=\frac{1}{2}V_{ll}(l,h)\pi^2\sigma^2+V_{lh}(l,h)\eta\rho\pi\sigma h+\pi(\alpha-r)V_l(l,h), \tag{33}$$

$$H[c]=-cV_l(l,h)+\log(c). \tag{34}$$

Now using substitution (22) with $\Psi_1=\frac{1}{\kappa}$ and $\Psi_2=\frac{1}{\kappa^2}(\mu-\delta-\frac{\eta^2}{2})$ we can argue exactly as in the general case and represent $V(l,h)$ in the form

$$V(l,h)=v(z)+\frac{\log h}{\kappa}+\frac{1}{\kappa^2}\left(\mu-\delta-\frac{\eta^2}{2}\right),\ \ z=l/h, \tag{35}$$

so $v(z)$ satisfies the ordinary differential equation of second order

$$\frac{\eta^2}{2}z^2v''+\max_\pi\left[\frac{1}{2}\pi^2\sigma^2v'-\pi\left((v'+zv'')\eta\rho\sigma+(\alpha-r)v'\right)\right]+\max_{c\geq-\delta}\left[-cv_z+\log(c+\delta)\right]=\kappa v, \tag{36}$$

where $v'=v_z$ and the dimension of the problem is reduced to one. It is important to note that such reduction was possible due to the exponential decay, the homotethicity of the logarithmic function and the linearity of the control equations, which make the reduction (35) sound.

Assuming that v is smooth and strictly concave, we perform a formal maximization of the quadratic part (33) which leads to

$$\kappa vv''=-\frac{d_1^2}{2}(v')^2+d_2z^2(v'')^2+d_3zv'v''-v''\left[1+\log(v')\right], \tag{37}$$

where again d_1, d_2 and d_3 are defined in (27).

Coming back to the original variables we obtain the optimal policies in the form

$$c_\star(l,h) = \frac{h}{v'(l/h)}, \;\; \pi_\star(l,h) = -\frac{\eta\rho}{\sigma} l - h\frac{d_1}{\sigma}\frac{v'(l/h)}{v''(l/h)}. \tag{38}$$

Summing up, we announce the main result of this Section.

Theorem 6.1. *Suppose $r-(\mu-\delta)>0$ and $d_1 \neq 0$.*

(i) *There is the unique $C^2(0,+\infty)$ solution $v(z)$ of (37) in a class of concave functions.*

(ii) *For $l,h>0$ the value function is given by (35). For $h=0, l>0$ the value function $V(l,0)$ coincides with the classical Merton solution*

$$V(l,0) = \frac{1}{\kappa^2}\left[r + \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2} - \kappa\right] + \frac{\log(\kappa l)}{\kappa} = M + \frac{\log(\kappa l)}{\kappa}. \tag{39}$$

(iii) *If the ratio between the stochastic income and the total wealth tends to zero, the policies $(\pi^\star, c^\star)$ given by (38) tend to the classical Merton's policies*

$$c_\star(l,0) \sim \kappa l, \;\; \pi_\star(l,0) \sim -\frac{(\alpha-r)l}{\sigma^2}\frac{V_l^2}{V_{ll}}. \tag{40}$$

(iv) *Policies (38) are optimal.*

We have shown that the solution exists and tends to Merton case when $h=0$. In the next step we will show the smoothness of the solution.

6.1 The dual optimization problem and smoothness of the viscosity solution

In this Section we introduce the dual optimization problem with a synthetic asset such that the optimization equation formally coincides with (37). The regularity of the dual problem proves the regularity of the original one due to the uniqueness of the viscosity solution.

Consider the investment-consumption problem with the wealth process Z_t defined by

$$\begin{aligned} Z_t &= (d_3 Z_t + d_1\sigma\pi_t - c_t)dt + \sigma\pi_t dW_t^1 + \eta Z_t\sqrt{1-\rho^2}dW_t^2, \\ Z_0 &= z \geq 0, \end{aligned} \tag{41}$$

where d_1 and d_3 are defined in (27). We define the set of admissible controls $\hat{\mathscr{A}}(z)$ as the set of pairs (π, c) such that there exists an a.s. positive solution Z_t of the stochastic differential equation (41), $c_t \geq -\delta$ and c and π satisfy the integrability conditions (3).

The investor wants to maximize the average utility given by

$$\hat{\mathscr{U}}(c) = E\left[\int_0^\infty e^{-\kappa\tau}\log(\delta + c(\tau))d\tau\right]$$

and the value function w is defined as $w(z) = \sup_{(\pi,c)\in\hat{\mathscr{A}}(z)} \hat{\mathscr{U}}(c)$.

The associated HJB equation is reduced to the ODE

$$\kappa w = d_2 z^2 w'' + \max_\pi\left[\frac{1}{2}\sigma^2\pi^2 w'' + d_1\sigma\pi w'\right] + d_3 z w' + \max_{c\geq-\delta}\left[-cw' + \log(c+\delta)\right], \tag{42}$$

Next, keeping in mind $w'>0, w''<0$, we can rewrite (42) as

$$-\frac{d_1^2}{2}\frac{(w')^2}{w''} + d_2 z^2 w'' + d_3 z w' + \delta w' - 1 - \log w' - \kappa w = 0. \tag{43}$$

Now, it is easy to see that (42) reduces to (37) assuming that w is smooth. Thus, if we prove that w is smooth and concave, we will get the desired result for v in (36) as well. The possibility to switch back and forth from

V in (34) to v in (36) and w in (42) is guaranteed by the existence and uniqueness of the viscosity solutions given by Theorem 3.1. On the other hand, if a function is the value function for the corresponding optimization problem, and the HJB equations formally coincide, the value functions must coincide as well due to uniqueness. Therefore, it is sufficient to prove that w is smooth.

From the previous Section we already know that if $D=(0,\infty)$ and $\overline{D}=[0,\infty]$ the following theorem hold.

Theorem 6.2. *The function w is the unique viscosity solution of (42) in D. And the value function $V(l,h)$ is the unique viscosity solution of (35) in $D\times D$.*

Let us now prove the smoothness of the solution and of its' first derivative.

Theorem 6.3. *The function w in (43) is the unique concave $C^2(D)$ solution of (42).*

To start with the proof of the theorem we need some explicit bounds for w.

Lemma 6.4. *The following bounds hold for $w(z)$*

$$C_1\log(z+C_2)<w(z)<(z+C_3)^\gamma,\quad z\in\Omega \tag{44}$$

for some constants $C_1,C_2,C_3>0$ and $0<\gamma<1$.

Proof. The function

$$W^-(z)=C_1\log(z+C_2),\quad z\in\Omega$$

is a subsolution for (43) as the coefficient of the leading logarithmic term is negative provided $C_1,C_2>0$ are appropriately chosen. On the other hand, the function

$$W^+(z)=(z+C_3)^\gamma,\quad z\in\Omega$$

is a supersolution provided $0<\gamma<1$ is sufficiently close to 1. Indeed, the leading term is z^γ with the coefficient $-(d_1^2(w')^2)/(2w'')$, which in turn grows as $-\gamma/(\gamma-1)$ and becomes arbitrarily large as γ tends to 1.

Thus, the desired bound (44) is a consequence of the comparison principle formulated in Theorem 3.4. • □

Now we can prove Theorem 6.3.

Proof. It is known that uniformly elliptic equations enjoy regularity, but as before the main obstacle is the lack of uniform bounds. The main idea of the proof is to approximate the original problem with a convergent family of optimization problems such that the approximating equations is uniformly elliptic and thus smooth. Then the smoothness follows from the stability of viscosity solutions and uniqueness.

Step 1. Consider the value function $w_L(z)=\sup_{(\pi,c)\in\hat{\mathscr{A}}(z)}\hat{\mathscr{U}}(c)$ for the problem with the additional strategy constraint $-L\le\pi_t\le L$ for almost every t. Arguing as in Section 2 we conclude that that w_L is an increasing continuous function, which is the unique viscosity solution to

$$\kappa w_L=d_2z^2w_L''+\max_{-L\le\pi\le L}\left[\frac{1}{2}\sigma^2\pi^2w_L''+d_1\pi w_L'\right]+d_3zw_L'+\max_{c\ge-\delta}\left[-cw_L'+\log(c+\delta)\right]. \tag{45}$$

Moreover, the bounds of Lemma 6.4 hold so $C_1\log(z+C_2)<w^L(z)<(z+C_3)^\gamma$.

Thus, there exists a concave function $\hat{w}$ such that $w_L\to\hat{w},L\to\infty$ locally uniformly. Then due to the stability property and uniqueness of the viscosity solution the function $\hat{w}$ is a viscosity solution of (42) and thus coincides with w. Therefore $w_L\to w,L\to\infty$ locally uniformly.

Step 2. We claim that w_L is a smooth function on an arbitrary interval $[z_1,z_2]$ such that $z_1>0$. Due to concavity we may assume that derivatives $w_L'(z_1),w_L'(z_2)$ exist. On the one hand the function w_L is the unique solution of the boundary problem

$$\begin{aligned}\kappa u &= d_2z^2u''+\max_{-L\le\pi\le L}\left[\frac{1}{2}\sigma^2\pi^2u''+d_1\sigma\pi u'\right]+d_3zu'+\max_{c\ge-\delta}\left[-cu'+\log(c+\delta)\right], \\ u(z_1) &= w_L(z_1),\quad u(z_2)=w_L(z_2),\quad z\in[z_1,z_2].\end{aligned} \tag{46}$$

On the other hand, according to the general theory of fully nonlinear elliptic equations of second order of Bellman type in a compact region, (see *Krylov* [18]), (45) has a unique C^2 solution in $[z_1,z_2]$ that coincides with w_L and w_L is smooth on $[z_1,z_2]$.

Step 3. We show that the constraint $-L \le \pi_t \le L$ is superfluous for sufficiently large L and can be eliminated. First it is clear that due to concavity and monotonicity of w_L, the condition $-L \le \pi_t \le L$ in (45) can be substituted with $\pi_t \le L$. Now we prove that

$$\sup_{z\in(z_1,z_2)} \left[-\frac{d_1^2 (w_L')^2}{2w_L''} \right] < L$$

for sufficiently large L. Assume the contrary for contradiction. Then there is a sequence $z_n \in (z_1,z_2)$, $L_n \to \infty$ such that

$$\begin{aligned} - \quad & \frac{d_1^2 (w_L'(z_n))^2}{2w_L''(z_n)} > L_n, \qquad (47) \\ \kappa w_L \quad \ge \quad & d_2 z^2 w_L'' - L_n + d_3 z w_L' + \left[\delta w_L' - 1 - \log w_L' \right]. \end{aligned}$$

Since $w_L \to w$ and both function are monotone and concave there exist constants C_1, C_2 such that $C_1 < w_L'(z) < C_2, \quad z \in [z_1,z_2]$ for all sufficiently large L, and also $w_L'' \to 0$ as $n \to \infty$. But this contradicts (47) as z_n takes values in a bounded interval so $w_L(z_n)$ is bounded as well.

Step 4. We are going to show that there is a constant $K < 0$ which does not depend on L such that $w_L''(z) < K, \quad z \in [z_1,z_2]$. Arguing again by contradiction suppose there is a sequence $z_n \in [z_1,z_2]$ such that $w_L''(z_n) \to \infty$. Then analogously to Step 3, the right hand side of (46) grows to infinity since $w_L'(z)$ on the interval that is bounded. At the same time the left hand side stays bounded as a value of a continuous function on a bounded interval.

Step 5. Putting it all together, we have the following chain of implications. The functions w_L are unique smooth solutions in the class of concave functions to the boundary problem (46) for some sufficiently large $M > 0$. Since $w_L \to w$, it follows that w is the unique viscosity solution of (46) in the class of concave functions. On the other hand, the equation (46) possesses the unique smooth solution, see [18], which must coincide with the viscosity solution. Thus w is a C^2-smooth function on $[z_1,z_2]$ and the claim of the theorem follows since the interval is arbitrary. • □

6.2 Asymptotic behavior of the value function

In this Section we examine the asymptotic behavior of the value function $V(t,l,h)$ in (34) and show that as $l/h \to \infty$ it becomes the classical Merton solution.

Theorem 6.5. *There is a positive constant C_1 such that*

$$\begin{aligned} M \quad + \quad & \frac{\log(\kappa l)}{\kappa} \le V(l,h) \le M + \frac{\log(\kappa(l + C_1 \delta h))}{\kappa}, \\ M \quad = \quad & \frac{1}{\kappa^2}\left[r + \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2} - \kappa \right] \end{aligned}$$

where M is a constant from the Merton's formula (39).

Proof. The proof is based on the idea mentioned in Lemma 3.2, but in the specific exponentially distributed liquidation time case the bounds could be found explicitly. The left-hand inequality is obvious since any strategy (π, c) for the classical problem with $L_0 = l, H_0 = 0$ is admissible for the problem with any non-zero initial endowment as well. For the right-hand side, let us consider a fictitious investment-consumption problem without any stochastic income but with an additional synthetic asset with the price process S': $dS_t' = \alpha' S_t' + \sigma_1' S_t' dW_t, \quad t \ge 0$, $S_0' = s', \quad s' > 0$ with appropriate constants α' and σ'. Next, we define the *initial wealth equivalent* of the stochastic income defined by

$$\begin{aligned} V_\delta(l,h) \quad = \quad & \delta E_h \left[\int_0^\infty e^{-rt} \xi_t H_t dt \right], \\ \xi_t \quad = \quad & \exp\left(-\frac{1}{2}(\theta_1^2 + \theta_2^2) + \theta_1 W_t^{(1)} + \theta_2 W_t \right), \end{aligned}$$

where $\theta_1 = (\alpha - r)/\sigma_1$ and $\theta_2 = (\alpha' - r)/\sigma_1'$.

As we mention in the proof of Lemma 3.2 (see page 5), by a careful choice of the constants α', σ' the stochastic income rate H_t can be replicated by a self-financing strategy on the complete market (B_t, S_t, S_t') with

the additional initial endowment $f(h) < C_1\delta h$, see [17], [14] and [10]. Thus, any average utility generated by the strategy $(\pi, c) \in \mathscr{A}(l,h)$ can be attained in the settings of a classical Merton's problem with the initial wealth $l + f(h) < l + C_1\delta h$. This actually gives the right-hand bound in Theorem 3.1. • □

From this theorem we immediately get that $V(l,h)$ behaves as the classical Merton solution (39) as $\delta \to 0$ or $l/h \to \infty$.

Corollary 6.6. *$V_\delta(l,h)$ converges locally uniformly to $M + \log(\kappa l)/\kappa$ as $\delta \to 0$.*

Corollary 6.7. *$V(l,h) = M + \log(\kappa l)/\kappa + O(1/z)$ as $z = l/h \to \infty$. Also for the function $w(z)$ we obtain $w(z) = (M-K) + \frac{\log(\kappa z)}{\kappa} + O(1/z)$.*

Proof. Indeed,

$$\left| V(l,h) - M - \frac{\log(\kappa l)}{\kappa} \right| < \left| \frac{1}{\kappa} \left(\log(\kappa(l + \delta C_1 h)) - \log(\kappa l) \right) \right| = O\left(\frac{1}{z}\right). \tag{48}$$

The formula immediately follows from the form of $V(l,h)$.• □

Finally, we verify that the optimal policies given by (38) asymptotically give the Merton strategy (40).

Lemma 6.8. *For the value function $w(z)$ holds*

$$w'(z) = \frac{1}{\kappa z} + o\left(\frac{1}{z}\right), \quad z \to \infty. \tag{49}$$

Proof. Consider the function w_λ defined as

$$w_\lambda(z) = w(\lambda z) - \frac{\log(\lambda)}{\kappa},$$

so that w_λ solves (42) but with the term

$$F(w_z) = \max_{c \geq -\delta} [-c w_z + \log(c + \delta)]$$

replaced by

$$F_\lambda(w_z) = \max_{c \geq -\delta/\lambda} \left[-c w_z + \log(c + \frac{\delta}{\lambda}) \right].$$

Then, by Corollary 6.6 w_λ converges locally uniformly to the Merton's value function

$$v(z) = (M-K) + \frac{\log(\kappa z)}{\kappa}.$$

We note that v solves (42) with $\delta = 0$ that is delivered by $F_\infty(\cdot) = \lim_{\lambda\to\infty} F_\lambda(\cdot)$. Thus, since w_λ is concave, the uniform convergence of w_λ to v implies the convergence of derivatives, so $\lim_{\lambda\to\infty} w'_\lambda(z) = v'(z) = \frac{1}{\kappa z}$. Hence,

$$\lim_{\lambda\to\infty} w'_\lambda(1) = \lim_{\lambda\to\infty} \lambda v'(\lambda) = \frac{1}{\kappa},$$

which proves the lemma. • □

Theorem 6.9. *The following asymptotic formulae hold for the optimal policies (38) as $z = l/h \to \infty$.*

$$\frac{c^\star}{l} \sim \frac{1}{\kappa}, \quad \frac{\pi^\star}{l} \sim \frac{\alpha - r}{\sigma^2}. \tag{50}$$

Proof. The formula for $c^\star$ in (50) immediately follows from Lemma 6.8. For the second part, we rewrite (38) in a form

$$\frac{\pi^\star}{l} = \frac{\eta\rho}{\sigma} - \frac{k_1}{\sigma^2}\frac{zv'(z)}{z^2v''(z)}.$$

To calculate the limit value of $z^2v''(z)$ we rewrite (37) as a quadratic equation with respect to v_{zz}. Since $v_{zz} < 0$ we choose the negative root and obtain

$$v_{zz}(z) = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

where

$$\begin{aligned} A &= \frac{1}{2}\eta^2(1-\rho)^2z^2, \\ B &= k(zv_z) - 1 - (M-C)\kappa + o(1), \\ C &= -\frac{k_1^2}{2\sigma^2}(v_z)^2. \end{aligned}$$

Expanding all constants and using $zw_z = 1/\kappa + o(1)$ we finally get

$$z^2v_{zz}(z) = \frac{(\alpha - r)l}{\sigma^2} + o(1).\bullet$$

□

The facts that the solution exists, is unique and smooth give an opportunity for numerical calculations. For example, basing on a script, developed by *Andersson, Svensson, Karlsson and Elias*, see [2], with some modifications and corrections of minor mistakes we can obtain the solution for the exponential case and compare it with a two-dimensional Merton solution as shown on the Figure 1.

7. The case of a Weibull distributed liquidation time and a logarithmic utility function

One of the most natural ways to extend the framework of a randomly distributed liquidation time described in the Section 2 is to introduce a distribution with a probability density function that has a local maximum unlike exponential distribution. It is very natural to expect that the assets of a certain type might have a time-lag between the moment when the sell offer is opened and a time when someone reacts on it. From the practitioner's point of view an empirical estimation of such time-lag is a natural measure of illiquidity that can give an insight into the strategy of a portfolio management. In this Section we look closely on a Weibull distribution that has a local maximum. The Weibull distribution is commonly used in survival analysis, in reliability engineering and failure analysis, and in industrial engineering to describe manufacturing and delivery times. It seems to be quite adequate for the studied case. We demonstrate that the proposed framework is applicable for this case, show the existence and uniqueness of the solution and using a numerical algorithm generate an insight into how this case differs from the exponential illiquid and Merton's absolutely liquid cases.

In this Section we will discuss the case when the liquidation time T is a random Weibull-distributed variable independent of the Brownian motions (W^1, W^2).
The probability density function of the Weibull distribution is

$$\phi(x,\lambda,k) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k},$$

if $t \geq 0$ and $= 0$, if $t < 0$.

Let us also introduce as before the cumulative distribution function $\Phi(x,\lambda,k) = 1 - e^{-(t/\lambda)^k}$, if $t \geq 0$ and $= 0$, if $t < 0$ and the survival function $\overline{\Phi}(t) = 1 - \Phi(t)$. We will often omit the constant parameters λ and k in notations for shortness.

It is important to notice that when $k = 1$ the Weibull-distribution turns into exponential one, that we have already discussed before and for $k > 1$ its probability density has a local maximum. This situation corresponds to our economical motivation.

The equation (10) is the same as before but the term that corresponds to $\overline{\Phi}$ is naturally replaced by Weibull survival function

$$\begin{aligned} V_t(t,l,h) &+ \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \qquad (51)\\ &+ \max_{\pi} G[\pi] + \max_{c\geq 0} H[c] = 0, \end{aligned}$$

$$G[\pi] = \frac{1}{2}V_{ll}(t,l,h)\pi^2\sigma^2 + V_{lh}(t,l,h)\eta\rho\pi\sigma h + \pi(\alpha-r)V_l(t,l,h), \qquad (52)$$

$$H[c] = -cV_l(t,l,h) + e^{-(t/\lambda)^k}U(c), \qquad (53)$$

Proposition 7.1. *All the conditions of the Theorem 3.1 hold for the case of the Weibull distribution with $k>1$ and, therefore, there exists a unique solution for the problem (51).*

Indeed the conditions 1., 3. and 4. are not altered since we work with the same logarithmic utility and one can easily see that the Weibull cumulative function satisfies the condition 2. for the case $k>1$.

Analogously to the equation (28) one can obtain a two dimensional equation using a known reduction $z=l/h$. We study all the symmetry reductions of this model for the exponential and Weibull case in [5]. Yet here let us just list a two dimensional equation that corresponds to the Weibull case

$$W_t - \frac{d_1^2}{2}\frac{(W_z)^2}{W_{zz}} + d_2 z^2 W_{zz} + d_3 z W_z + \delta W_z - e^{-(t/\lambda)^k}\log W_z = 0, \qquad (54)$$

where d_1, d_2 and d_3 correspond to the constants for the general case (27).

The function $\Psi_1(t) = \int_t^\infty \overline{\Phi}(s)ds$ can be defined explicitly as $\Psi_1(t) = \frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right)$, where $\Gamma(\alpha,x)$ is an incomplete gamma function. For this function we can use the series representation by Laguerre polynomials and asymptotic representation [1], [15].

The lower bound for $W(z,t)$ can be found exactly as in Section 5

$$W(z,t) = V(t,l,h) - \Psi_1 \log h - \Psi_2(t) \geq \Psi_1(t)\log z + (\Theta(t) - \Psi_2(t)),$$

where the behavior of the functions Ψ_1, Ψ_2 and Θ by $t\to\infty$ can be now well defined.

The equation (**??**) for the auxiliary function $\Psi_2'(t)$ takes the form

$$\begin{aligned} &\Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu-\delta)\right)\frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) - e^{-(t/\lambda)^k}((t/\lambda)^k + 1) = 0, \qquad (55)\\ &\Psi_2(t) \to 0, t\to\infty, \end{aligned}$$

The solution for this equation can be found explicitly

$$\Psi_2(t) = -\left(-\frac{\eta^2}{2} + (\mu-\delta)\right)\frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) + e^{-\left(\frac{t}{\lambda}\right)^k}\left(\left(\frac{t}{\lambda}\right)^k + 1\right) \qquad (56)$$

Equation (31) for Θ is now

$$\begin{aligned} &\Theta' + \left(r + \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2}\right)\frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) - e^{-(t/\lambda)^k}(e^{-(t/\lambda)^k} + (t/\lambda)^k + \log\lambda - \log k\\ &+ \log\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right)) = 0. \qquad (57) \end{aligned}$$

And one can find an explicit solution for it as well

$$\Theta(t) = -\left(r + \frac{(\alpha-r)^2}{2\sigma^2}\right)\frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) + e^{-\left(\frac{t}{\lambda}\right)^k}\left(e^{-\left(\frac{t}{\lambda}\right)^k} + \left(\frac{t}{k}\right)^k + \ln\left(\frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right)\right)\right).$$

Since $\frac{1}{k} > 0$ we can show that asymptotically as $t\to\infty$

$$\begin{aligned} \Psi_1(t) &\to \frac{\lambda^k}{k}(t)^{1-k}e^{-(t/\lambda)^k}\left(1 + O\left(t^{-k}\right)\right),\\ \Psi_2(t) &\to -\frac{1}{k}te^{-(t/\lambda)^k}\left(1 + O\left(t^{-k}\right)\right),\ k>1,\\ \Theta(t) &\to \frac{\lambda-k}{\lambda k}te^{-(t/\lambda)^k}\left(1 + \frac{(k-1)k\lambda}{\lambda-k}t^{-k}\ln t + O\left(t^{-k}\right)\right). \end{aligned}$$

It follows from the asymptotic behavior that the value function in (51) tends to zero faster than $e^{-\kappa t}$ and consequently Theorem 3.1 is applicable for the Weibull-distributed liquidation time.

On the Figure 1 one can see the results of the numerical simulation for consumption and investment strategies that we run for a Weibull and exponential case. As the parameter k that is responsible for the form of Weibull distribution increases the optimal policies differ significantly from the exponential liquidation time case. As z increases, i.e. the illiquid part of the portfolio becomes insufficiently small, we can see that all the policies tend to one solution which is, in fact, a Merton solution for a two-asset problem derived in [19].

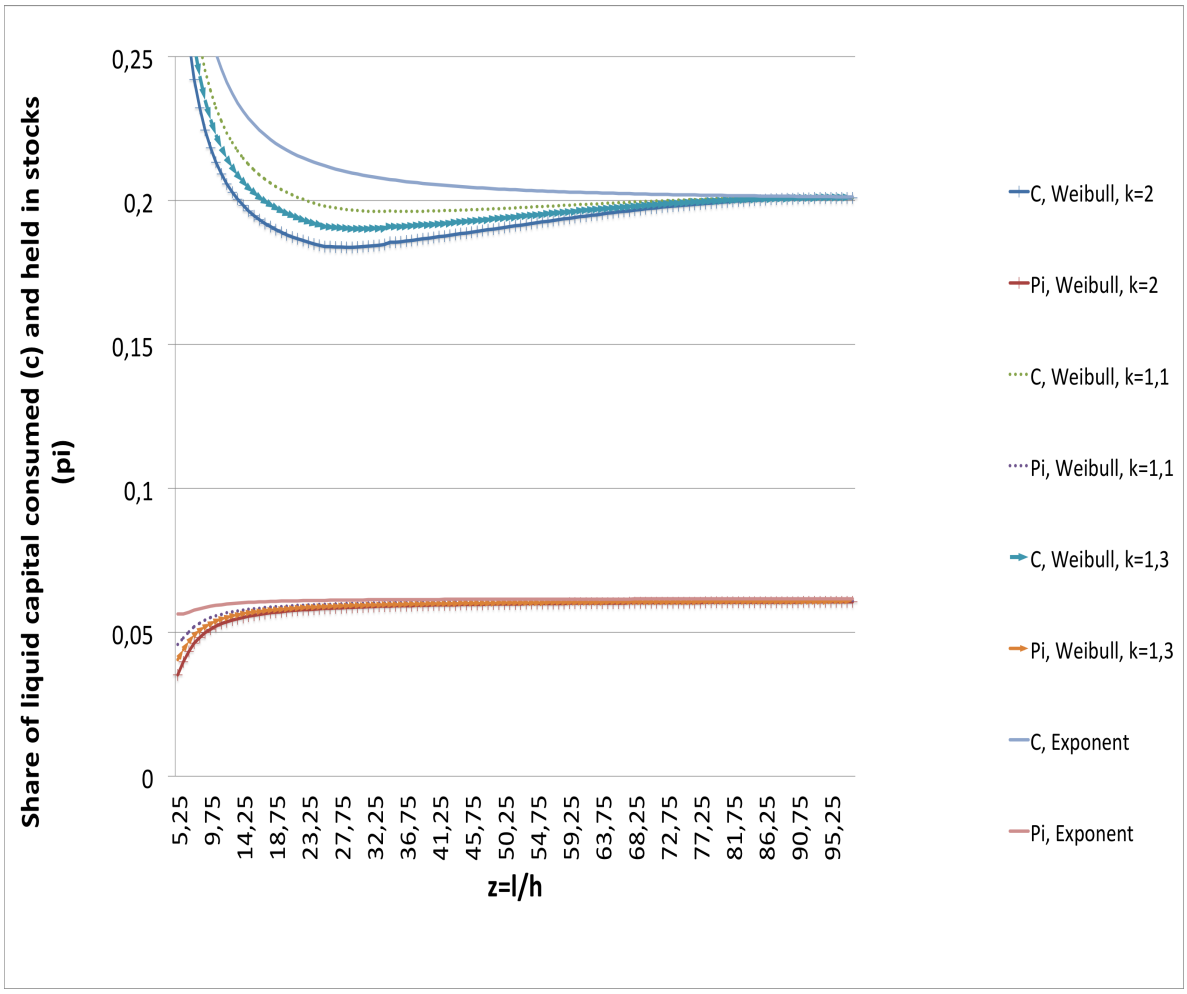

Figure 1. *Consumption stream c and the share of liquid capital π stored in a risky asset depending on the ratio between the liquid and illiquid asset. As illiquid asset value becomes infinitely small the policies tend to Merton policies for a two-asset problem. We used the following parameters for assets* $r = 0.01, \sigma = 0.5, \delta = 0.02, \rho = 0.4, \mu = 0.05, \eta = 0.3$ *and* $\lambda = 2$

It is especially important to note that the optimal policies significantly differ from Merton solution when illiquidity becomes higher. Already when an amount of illiquid asset is more than 5% of the portfolio value the percentage of capital that is not invested in a risky stock is higher than in Merton model.

8. Conclusion

We have proposed a framework with which one can obtain a management strategy for a portfolio that consists of a liquid riskless and liquid risky assets and of an illiquid asset. We suppose that illiquid asset is liquidated in a random moment of time that has a prescribed distribution. In Theorem 3.1 we have proved the existence and uniqueness of the solution for a variety of the portfolio optimization problems. We have applied the obtained

theorem to two different cases of exponential and Weibull liquidation time distributions. For the exponentially distributed random liquidation time but for existence and uniqueness we have proved the smoothness of the solution and found a lower and upper bound. For the Weibull distributed liquidation time with parameter $k > 1$ we have demonstrated the applicability of a general Theorem 3.1 that proves the existence and uniqueness of a viscosity solution and also found a lower and upper bound for it. We have also demonstrated numerically that the resulting strategies for such portfolio differ from the Merton case yet tend to it when illiquidity becomes infinitely small.

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