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# Short time heat diffusion in compact domains with discontinuous transmission boundary conditions

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September 7, 2015

## Abstract

We consider a heat problem with discontinuous diffusion coefficients and discontinuous transmission boundary conditions with a resistance coefficient. For all compact  $(\epsilon, \delta)$ -domains  $\Omega \subset \mathbb{R}^n$  with a  $d$ -set boundary (for instance, a self-similar fractal), we find the first term of the small-time asymptotic expansion of the heat content in the complement of  $\Omega$ , and also the second-order term in the case of a regular boundary. The asymptotic expansion is different for the cases of finite and infinite resistance of the boundary. The derived formulas relate the heat content to the volume of the interior Minkowski sausage and present a mathematical justification to the de Gennes' approach. The accuracy of the analytical results is illustrated by solving the heat problem on prefractal domains by a finite elements method.

**Keywords:** heat content; discontinuous transmission condition; Minkowski sausage.

## 1 Introduction

We consider a compact domain  $\Omega \subset \mathbb{R}^n$  with boundary  $\partial\Omega$  that splits  $\mathbb{R}^n$  into “hot” and “cold” media,  $\Omega_+ = \Omega$  and  $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$ , characterized by (distinct) heat diffusion coefficients  $D_+$  and  $D_-$  (Fig. 1). On the boundary  $\partial\Omega$  is also defined a function  $0 \leq \lambda(x) \leq \infty$  which describes the resistivity to heat exchange through the boundary.

We are interested in propagation of the heat content associated with the following

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problem:

$$\partial_t u_{\pm} - D_{\pm} \Delta u_{\pm} = 0 \quad x \in \Omega_{\pm}, \quad t > 0, \quad (1)$$

$$u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \quad (2)$$

$$D_- \frac{\partial u_-}{\partial n} |_{\partial\Omega} = \lambda(x)(u_- - u_+) |_{\partial\Omega}, \quad (3)$$

$$D_+ \frac{\partial u_+}{\partial n} |_{\partial\Omega} = D_- \frac{\partial u_-}{\partial n} |_{\partial\Omega}, \quad (4)$$

where  $\partial/\partial n$  is the normal derivative directed outside the domain  $\Omega$ .

A rigorous analysis of the problem (1)–(4) for irregular boundaries requires its variational formulation in appropriate functional spaces (see Section 2). The variational problem is shown to have a unique weak solution with the desired trace properties on the boundary  $\partial\Omega$  (see Section 2). The variational problem is equivalent to the problem (1)–(4) for a piecewise Lipschitz  $\partial\Omega$  according to the classical trace theorem. In turn, extensions of the trace theorem have to be used for fractal boundaries or, more precisely,  $d$ -sets (see Subsection 2.2).

Once a unique solution  $u_{\pm}$  of the problem (1)–(4) is established, we study the asymptotic expansion of the heat content as  $t \rightarrow 0$

$$N(t) = \int_{\mathbb{R}^n \setminus \Omega} u_-(x, t) dx = \text{Vol}(\Omega) - \int_{\Omega} u_+(x, t) dx. \quad (5)$$

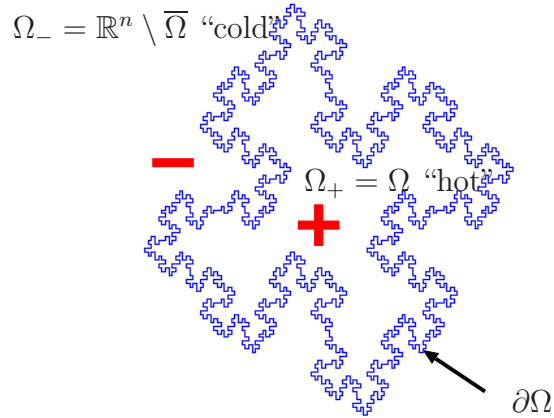


Figure 1: Illustration of the heat content problem for a planar domain  $\Omega$  with prefractal boundary  $\partial\Omega$  presented by the third generation of the Minkowski fractal (of fractal dimension  $3/2$ ). This boundary splits the plane into two complementary regions. At time  $t = 0$ , the inner region  $\Omega_+ = \Omega$  is “hot” (functions on  $\Omega_+$  are denoted with subscript  $+$ ), while the outer region  $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$  is “cold” (functions on  $\Omega_-$  are denoted with subscript  $-$ ).

Eqs. (1)–(4) describe heat exchange between two media prepared initially at different temperatures and separated by a partially isolating boundary[1, 2]. In fact,  $u(x, t)$

can describe how the distribution of (normalized) temperature evolves with time. The transmission boundary conditions (3), (4) impose the continuity of the temperature flux across the boundary, and relate this flux to the temperature drop at the boundary due to thermal isolation. The growth rate of the heat content with time characterizes the efficiency of thermal isolation. Understanding this problem is relevant to improve heat exchangers, e.g., cooling of metallic radiators or thermal isolation of pipes and buildings. Depending on application, cooling rate has to be either enhanced (e.g., in the case of microprocessors or nuclear reactors), or slowed down (e.g., in the case of pipes and buildings). For these purposes, one can either modify the thermal isolation (i.e., the resistivity  $\lambda$ ), or the shape of the exchange boundary. It is therefore crucial to understand how the shape of the boundary influences heat exchange. In particular, would an irregular (e.g., fractal) boundary with a very large exchange area significantly speed up cooling?

Similar equations can describe molecular diffusion between two media across semi-permeable membranes [3, 4]. In that case,  $u(x, t)$  represents the (normalized) concentration of molecules, while Eqs. (1)–(4) can model the leakage of molecules from a cell ( $\Omega_+$ ) to the extracellular space ( $\Omega_-$ ) or, more generally, the diffusive exchange between two compartments (e.g., oxygen or carbon dioxide exchange between air and blood across the alveolar membrane in the lungs). The resistance  $\lambda$  is related to the cellular membrane permeability. As for heat exchange, one may need to enhance or to slow down the molecular leakage, and the shape of the boundary may play an important role.

The discontinuity of the initial condition, of the diffusion coefficient, and of the solution  $u(x, t)$  across the boundary between two domains constitutes one of the mathematical difficulties to be treated. From a physical point of view, such discontinuities might appear unrealistic. For instance, the diffusive flux at the boundary at time  $t = 0$  is infinite. For any physical setting of heat or molecular diffusion, there would be an intermediate layer between two media in which the material properties would change rapidly but continuously. When the thickness of this intermediate layer is much smaller than the size of the domain, the physical problem with continuously varying parameters can be approximated by the heat problem (1)–(4). Such an approximation is applicable starting from a small cut-off time while understanding the heat exchange at smaller time scales would need either restituting an intermediate layer, or introducing nonlinear terms into the heat equation. Throughout this paper, we focus on the mathematical problem (1)–(4).

The physical properties of the two media  $\Omega_+$  and  $\Omega_-$  are supposed to be different:  $D_+ \neq D_-$ . This implies the discontinuity of the metric on  $\partial\Omega$ . The case of continuous metric ( $g_-|_{\partial\Omega} = g_+|_{\partial\Omega}$ ) on smooth compact  $n$ -dimensional Riemannian manifolds with a smooth boundary  $\partial\Omega$  was considered in Ref. [5]. The case of continuous transmission boundary conditions for the expansion of the heat kernel on the diagonal was treated in Ref. [6] (see also Ref. [7] for a survey of results on asymptotic expansion of the heat kernel for different boundary conditions). The heat content asymptotic expansion with Dirichlet boundary condition was found

- up to the third-order term for a compact connected domain  $\Omega \subset \mathbb{R}^n$  with a regular boundary  $\partial\Omega \in C^3$  (Refs. [8, 9]);
- up to an exponentially small error for a compact connected domain  $\Omega \subset \mathbb{R}^2$  with a polygonal  $\partial\Omega$  (Ref. [10]) and for  $\Omega \subset \mathbb{R}^2$  with  $\partial\Omega$  given by the triadic Von Koch

snowflake (Ref. [11]);

- up to the second-order term for the general case of self-similar fractal compact connected domains in  $\mathbb{R}^n$  (Ref. [12]).

In general, the boundary between two media can have some resistance to heat exchange, described by the function  $\lambda(x) \geq 0$  ( $x \in \partial\Omega$ ) that may account for partial thermal isolation. We outline three cases of boundary conditions according to  $\lambda$ :

1. If  $0 < \lambda(x) < \infty$  for all  $x \in \partial\Omega$ ,  $u$  is discontinuous on  $\partial\Omega$  and we have:

$$\left( \lambda(x)u_- - D_- \frac{\partial u_-}{\partial n} \right) \Big|_{\partial\Omega} = \lambda(x)u_+ \Big|_{\partial\Omega}, \quad D_+ \frac{\partial u_+}{\partial n} \Big|_{\partial\Omega} = D_- \frac{\partial u_-}{\partial n} \Big|_{\partial\Omega}.$$

2. If  $\lambda = +\infty$  for all  $x \in \partial\Omega$ ,  $u$  is continuous on  $\partial\Omega$  due to the transmission condition and in this case

$$u_+ \Big|_{\partial\Omega} = u_- \Big|_{\partial\Omega}, \quad D_+ \frac{\partial u_+}{\partial n} \Big|_{\partial\Omega} = D_- \frac{\partial u_-}{\partial n} \Big|_{\partial\Omega}.$$

3. If  $\lambda = 0$  for all  $x \in \partial\Omega$ , we have the Neumann boundary condition

$$\frac{\partial u_-}{\partial n} \Big|_{\partial\Omega} = \frac{\partial u_+}{\partial n} \Big|_{\partial\Omega} = 0$$

that models the complete thermal isolation of  $\partial\Omega$  and implies the trivial solution given by  $u_-(x, t) = 0$  and  $u_+(x, t) = 1$  for all time  $t \geq 0$ .

The main goal of the article is to develop the preliminary study[13] and especially to formalize the seminal approach by de Gennes[14]. In the case  $\lambda = +\infty$ , de Gennes argued that as  $t \rightarrow +0$ ,  $N(t)$  is proportional to the volume  $\mu(\partial\Omega, \sqrt{D_+ t})$  of the interior Minkowski sausage of  $\partial\Omega$  of the width equal to the diffusion length  $\sqrt{D_+ t}$ :

$$\mu(\partial\Omega, \ell) = \text{Vol}(\{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \ell\})$$

(see also Ref. [12]). In particular,

- for a regular boundary  $\partial\Omega$ ,  $N(t)$  is proportional to  $\text{Vol}(\partial\Omega)\sqrt{D_+ t}$ ;
- for a fractal boundary  $\partial\Omega$  of the Hausdorff dimension  $d$ ,  $N(t)$  is proportional to  $(D_+ t)^{\frac{n-d}{2}}$ .

The de Gennes scaling argument was further investigated in Ref. [13], both experimentally and numerically. It was shown that irregularly shaped passive coolers rapidly dissipate at short times, but their efficiency decreases with time. The de Gennes scaling argument was shown to be only a large scale approximation, which is not sufficient to describe adequately the temperature distribution close to the irregular frontier.

In the present paper, we provide a mathematical foundation and further understanding for the de Gennes approach. We obtain three results valid for all compact  $(\epsilon, \delta)$ -domains  $\Omega$  in  $\mathbb{R}^n$  with connected boundary  $\partial\Omega$ , presented by a closed  $d$ -set (see Section 2.2 for

the definitions of  $(\epsilon, \delta)$ -domains and  $d$ -sets): the well-posedness of the problem (1)–(4), the continuity of the solution on  $\lambda$  (see Section 2), and the asymptotic expansion of the heat content (5). In particular, these results hold for domains with a self-similar fractal boundary.

We show in Theorem 5 that the heat content  $N(t)$  is approximated by the volume of the interior Minkowski sausage of  $\partial\Omega$  of the radius  $\sqrt{4D_+t}$ :

$$N(t) = \tau_\lambda \left[ C_\lambda(\partial\Omega) \mu \left( \partial\Omega, \sqrt{4D_+t} \right) + O \left( \mu^2 \left( \partial\Omega, \sqrt{4D_+t} \right) \right) \right], \quad (6)$$

where  $\tau_\lambda$  is equal to 1 if  $\lambda = \infty$  and  $\sqrt{t}$  if  $\lambda > 0$  is finite. Here  $C_\lambda(\partial\Omega)$  is a constant depending only on the shape of  $\partial\Omega$  and finiteness of  $\lambda$  (see Theorem 5 for the exact formulas). Formula (6) is the first approximation of Eqs (80), (82) given in Theorem 5, which allows to find  $N(t)$  up to terms of the order  $\tau_\lambda O(\sqrt{t} \mu(\partial\Omega, \sqrt{4D_+t}))$ .

Moreover, the asymptotic relation (6) remains valid even for mixed boundary conditions for three disjoint boundary parts, i.e. when  $\lambda = \infty$  on one part of the boundary,  $\lambda = 0$  on another part, and  $0 < \lambda < \infty$  on the remaining boundary (see Theorem 3). However, changes of the type of the boundary condition should be continuous (see Theorem 2) such that  $u$  remains a continuous function of  $\lambda$ . In this more general case, the coefficient  $C_\lambda(\partial\Omega)$  in Eq. (6) is given either by Eq. (83) for  $0 < \lambda < \infty$ , or by Eq. (84) for  $\lambda = \infty$ , or is equal to 0 for  $\lambda = 0$  (the boundary with  $\lambda = 0$  does not contribute to the short-time asymptotics of the heat content). Finding the asymptotics for mixed boundary conditions with a discontinuous jump from a finite  $\lambda$  to  $\lambda = \infty$  is still an open problem.

As expected, the resistivity of the boundary to heat transfer makes heat diffusion *slower* due to the presence of the coefficient  $\tau_\lambda = \sqrt{t}$ .

For a fractal boundary we replace  $\mu(\partial\Omega, \sqrt{4D_+t})$  by the volume of the interior Minkowski sausage which scales as  $(4D_+t)^{(n-d)/2}$ , where  $d$  is the fractal dimension[12]. In the fractal case the integral over  $\partial\Omega$  should be understood by using the Hausdorff measure (see Ref. [15, 16, 17]).

The comparison between the asymptotic formula (6) and a numerical solution of the problem (1)–(4) for the unit square and a prefractal domain is shown in Fig. 2 for a finite  $\lambda$  and in Fig. 3 for  $\lambda = +\infty$ . The numerical solution was obtained in FreeFem++ by a finite elements method with the implicit  $\theta$ -schema, also known as Crank-Nicolson schema, for the time discretization with  $\theta = \frac{1}{2}$  and  $\Delta t = 10^{-6}$ . The domain  $\Omega$  was centered in a ball  $B$  of diameter (at least) twice bigger than the diameter of  $\Omega$ . The Neumann boundary condition was imposed on the boundary of the ball. According to the principle “not feeling the boundary”[11] (see also Section 3), the heat content propagation in  $\mathbb{R}^2$  with a prescribed boundary  $\partial\Omega$  can be very accurately approximated at small times by the heat content propagation computed in  $B$ . The accuracy of this approximation can also be checked by changing the diameter of the ball. In the case of the square domain  $\Omega$ , the ball was replaced by a square with four times bigger edge. Each prefractal edge was discretized with 27 space points while 57 points were used in the external boundary of the ball. The mesh size was varied to check the accuracy of the presented numerical solutions. For the case of the discontinuous solution on the boundary (when  $0 < \lambda < \infty$ ) we apply the domain decomposition method and match the boundary

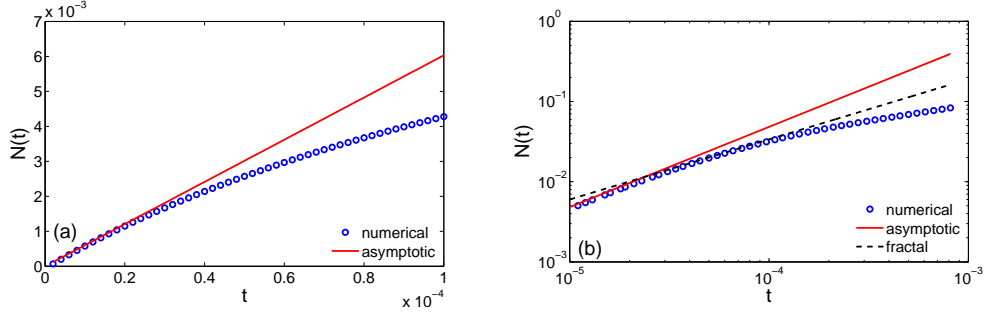


Figure 2: Comparison between the asymptotic formula (6) (solid line) and a FreeFem++ numerical solution of the problem (1)–(4) (circles) for two domains: (a) the unit square ( $\text{Vol}(\partial\Omega) = 4$ ) and (b) the third generation of the Minkowski fractal ( $\text{Vol}(\partial\Omega) = 2^3 \cdot 4$ ), with  $D_+ = 1/100$ ,  $D_- = 1$ , and  $\lambda = 17$ . Since the Hausdorff dimension of the boundaries of these domains is 1 (even for the *prefractal* case), Eq. (6) for a constant  $\lambda$  is reduced, according to Theorem 5, to  $N(t) = 2\sqrt{t}C_0\lambda\mu(\partial\Omega, \sqrt{4D_+t}) + O(t^{\frac{3}{2}})$  with  $\mu(\partial\Omega, \sqrt{4D_+t}) \simeq \sqrt{4D_+t}\text{Vol}(\partial\Omega)$  and  $C_0$  given by Eq. (97). For plot (b), dashed line shows the fractal asymptotic (that would be exact for the infinite generation of the fractal) with de Gennes approximation of  $\mu(\partial\Omega, \sqrt{4D_+t})$  in Eq. (6) by  $(4D_+t)^{\frac{1}{4}}$ . This approximation is valid for intermediate times.

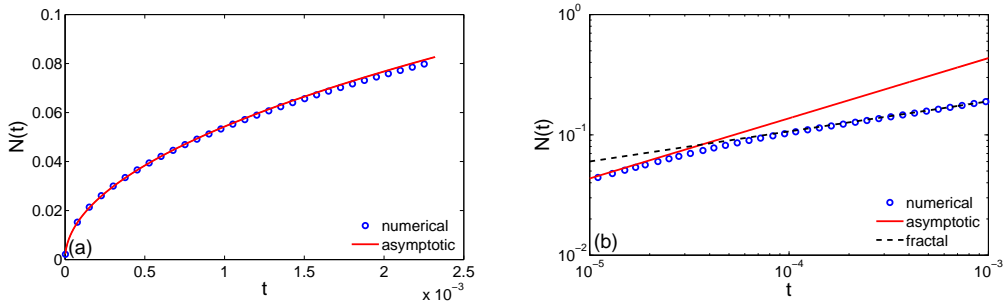


Figure 3: Comparison between the asymptotic formula (6) (solid line) and a FreeFem++ numerical solution of the problem (1)–(4) (circles) for two domains: (a) the unit square ( $\text{Vol}(\partial\Omega) = 4$ ), and (b) the third generation of the Minkowski fractal ( $\text{Vol}(\partial\Omega) = 2^3 \cdot 4$ ), with  $D_+ = 0.4$ ,  $D_- = 1$ , and  $\lambda = \infty$ . Since the *prefractal* boundary  $\partial\Omega$  has the Hausdorff dimension 1, Eq. (6) is reduced to Eq. (95), i.e.,  $N(t) \propto \sqrt{t}$ . In turn, dashed line shows the fractal asymptotic (that would be exact for the infinite generation of the fractal) with de Gennes approximation of  $\mu(\partial\Omega, \sqrt{4D_+t})$  in Eq. (6) by  $2.5(4D_+t)^{\frac{1}{4}}$ . This approximation is valid for intermediate times.

values of the respective solutions on  $\partial\Omega$  by a Picard fixed point method. We consider therefore the numerical solution of heat propagation for small times as a reference, to which asymptotic formulas are compared with. In particular, deviations between the numerical solution and the asymptotic formulas observed at longer times illustrate the range of validity of the short-time expansion.

For the regular case  $\partial\Omega \in C^3$ , we obtain the heat content approximation up to the third-order term. The formulas are given in Theorem 6. For the case  $\lambda < \infty$ , the coefficient in front of the second-order term ( $t^{\frac{3}{2}}$ ) in the asymptotic expansion depends on the mean curvature. In turn, for  $\lambda = \infty$ , the second-order term (here,  $t$ ) in the asymptotic expansion vanishes:

$$N(t) = 2 \frac{1 - e^{-4}}{\sqrt{\pi}} \frac{\sqrt{D_- D_+}}{\sqrt{D_+} + \sqrt{D_-}} \text{Vol}(\partial\Omega) \sqrt{t} + O(t^{\frac{3}{2}}). \quad (7)$$

The rest of the paper is organized as follows. In Section 2, we describe the class of irregular boundaries and prove the well-posedness of the model relying on the variational formulation of the problem. The boundary conditions are treated in the weak sense by generalizing the trace operator and the Green formula to fractals using fractal Besov spaces,  $B_{\beta}^{2,2}(\partial\Omega)$  and  $B_{-\beta}^{2,2}(\partial\Omega)$  ( $\beta = 1 - \frac{n-d}{2} > 0$  for a  $d$ -dimensional  $\partial\Omega$ ) defined in A. In Section 2 we also establish the continuity of  $u$  as a function of  $\lambda$ . In Section 3 we prove that the problem to find  $N(t)$  can be replaced by a heat problem localized in  $O(\sqrt{t})$ -interior Minkowski sausage of the boundary by a variant of the principle “not feeling the boundary”[11] in the general case in  $\mathbb{R}^n$ . This allows, due to the continuity of  $u$  on  $\lambda$ , to establish Theorem 3 for a mixed boundary condition including zero, finite, or infinite values of  $\lambda$ . Considering a regular  $\partial\Omega$  (at least in  $C^3$ ) and using the localization properties from Section 3, we rewrite in Section 4 the formula for  $N(t)$  in the terms of the local coordinates. Section 5 gives the approximation of the heat problem solution through the solution of one-dimensional constant coefficient problem. The heat content is calculated in terms of the volume of the interior Minkowski sausage of the boundary in Section 6. Firstly, to illustrate the technique of the proof on a simple case, we give the proof for the case of continuous diffusion coefficients  $D_+ = D_-$ , just with discontinuity of the initial condition. In this case, all formulas given in Section 6 are valid for all types of the boundary introduced in Subsection 2.2. The calculation relies on the Green function of the problem with constant coefficients for  $\Omega$  being a half-space (see B). We also obtain the Green function used in Section 7 for the proof of the asymptotic heat expansion up to the third-order term for a regular  $\partial\Omega \in C^3$ .

## 2 Well-posedness of the model

Let  $\Omega$  be an open connected bounded subset of  $\mathbb{R}^n$  such that  $\partial\Omega$  is closed with  $\text{Vol}(\Omega) < \infty$ . We denote by  $\Omega_+ = \Omega$  and  $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$  (Fig. 1).

We are looking for the solution of the problem (1)–(4), where  $D_+ \neq D_-$ ,  $D_+ > 0$  and  $D_- > 0$ ,  $\lambda(x) \geq 0$  for all  $x \in \partial\Omega$ . The boundary  $\partial\Omega$  is divided into two disjoint parts:  $\Gamma_{\infty} = \{x \in \partial\Omega \mid \lambda(x) = +\infty\}$  and  $\partial\Omega \setminus \Gamma_{\infty} = \{x \in \partial\Omega \mid 0 \leq \lambda(x) < +\infty\}$ . Each of the parts can be the empty set. We thus assume that  $\lambda \in L^{\infty}(\partial\Omega \setminus \Gamma_{\infty})$ .

### 2.1 Regular boundary: at least piecewise Lipschitz

Firstly, we consider the case when  $\partial\Omega$  is regular (at least piecewise Lipschitz) and  $\Gamma_{\infty}$  is the empty set.



To prove the existence, the uniqueness, and the stability of a solution of the problem (1)–(4), we proceed with its variational formulation.

We introduce the space  $H = L^2(\mathbb{R}^n)$  and the space

$$V = \{f \in H \mid f_+ = f|_{\Omega_+} \in H^1(\Omega_+), \text{ and } f_- = f|_{\Omega_-} \in H^1(\Omega_-)\}$$

of functions  $f = f_+ \mathbf{1}_{\Omega_+} + f_- \mathbf{1}_{\Omega_-}$  defined on  $\Omega_+ \cup \Omega_-$  such that their restrictions  $f_+ = f|_{\Omega_+}$  and  $f_- = f|_{\Omega_-}$  belong to  $H^1$ . We equip  $V$  with the norm:

$$\|u\|_V^2 = D_+ \int_{\Omega_+} |\nabla u_+|^2 dx + D_- \int_{\Omega_-} |\nabla u_-|^2 dx + \int_{\Omega_+ \cup \Omega_-} |u|^2 dx.$$

We notice that  $V$  is a Hilbert space,  $V \subset L^2(\Omega)$ , and  $V$  is dense in  $L^2(\Omega)$ . In addition,  $V \subset L^2(\mathbb{R}^n) \subset V'$ , where  $V'$  is the dual space to  $V$ . Finally, since  $\partial\Omega$  is regular, the inclusion  $V \subset L^2(\mathbb{R}^n)$  is compact.

Applying the usual trace theorem under the assumptions that  $\Omega$  is bounded and  $\partial\Omega$  is at least piecewise Lipschitz, the bilinear form

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_- + \int_{\partial\Omega_+} \lambda(x)(u_+ - u_-)(v_+ - v_-) d\sigma \quad (8)$$

is continuous,

$$|a(u, v)| \leq C(\|\lambda\|_{L^\infty(\partial\Omega)}, D_+, D_-, \Omega_+) \|u\|_V \|v\|_V \quad (\text{for a constant } C > 0),$$

and coercive on  $V \times V$ , i.e.,

$$\begin{aligned} a(u, u) &= D_+ \int_{\Omega_+} |\nabla u_+|^2 dx + D_- \int_{\Omega_-} |\nabla u_-|^2 dx + \int_{\partial\Omega_+} \lambda(x) |u_+ - u_-|^2 d\sigma \\ &\geq \|u\|_V^2 - \|u\|_{L^2(\mathbb{R}^n)}^2 > 0. \end{aligned}$$

Thus we conclude[18] that the bilinear form  $a(u, v)$  defines an operator  $A : V \rightarrow V'$  by  $a(u, v) = \langle Au, v \rangle$ . Moreover,  $-A|_{L^2(\mathbb{R}^n)}$  with  $D(A) = \{u \in V \mid Au \in L^2(\mathbb{R}^n)\}$  generates an analytical semigroup.

**Remark 1** When  $\Gamma_\infty$  is not empty, the variational form (8) is well adaptable to the case where  $u$  is continuous across the part  $\Gamma_\infty \subset \partial\Omega$  of the interface. By convention we put on this part  $\lambda(x) = \infty$  which implies  $u_+ = u_-$  on  $\Gamma_\infty$  (see also Theorem 2).

For  $\Gamma_\infty \neq \emptyset$ , we introduce  $V$  as the space of functions  $u \in L^2(\mathbb{R}^n)$  such that

$$u_+ = u|_{\Omega_+} \in H^1(\Omega_+), \quad u_- = u|_{\Omega_-} \in H^1(\Omega_-), \quad u_+|_{\Gamma_\infty} = u_-|_{\Gamma_\infty},$$

and, therefore, we consider the bilinear continuous and coercive form on  $V \times V$

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_- + \int_{\partial\Omega \setminus \Gamma_\infty} \lambda(x)(u_+ - u_-)(v_+ - v_-) d\sigma. \quad (9)$$

In particular, for  $\Gamma_\infty = \partial\Omega$ , we get  $V = H^1(\Omega_+ \cup \Omega_-)$  and

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_-.$$

## 2.2 Extension to $d$ -sets (fractal case for $d > n - 1$ )

Let us define a class of fractal domains to be considered. We will see that the existence and uniqueness results of a weak solution of the problem (1)–(4) hold for a class of bounded  $(\epsilon, \delta)$ -domains [20, 21, 22]  $\Omega_+$  such that  $\partial\Omega$  is a  $d$ -set [21]:

**Definition 1** ( *$d$ -set* [21, 22, 23]) *Let  $\Gamma$  be a closed subset of  $\mathbb{R}^n$  and  $0 < d \leq n$ . A positive Borel measure  $m_d$  with support  $\Gamma$  is called a  $d$ -measure of  $\Gamma$  if, for some positive constants  $c_1, c_2 > 0$ ,*

$$c_1 r^d \leq m_d(\Gamma \cap U_r(x)) \leq c_2 r^d, \quad \text{for } \forall x \in \Gamma, 0 < r \leq 1,$$

where  $U_r(x) \subset \mathbb{R}^n$  denotes the Euclidean ball centered at  $x$  and of radius  $r$ .

The set  $\Gamma$  is a  $d$ -set if there exists a  $d$ -measure on  $\Gamma$ .

As it is known from Ref. [[24], p.30], any two  $d$ -measures on  $\Gamma$  are equivalent.

**Definition 2** ( *$(\epsilon, \delta)$ -domain* [20, 21, 22]) *An open connected subset  $\Omega$  of  $\mathbb{R}^n$  is an  $(\epsilon, \delta)$ -domain,  $\epsilon > 0$ ,  $0 < \delta \leq \infty$ , if whenever  $x, y \in \Omega$  and  $|x - y| < \delta$ , there is a rectifiable arc  $\gamma \subset \Omega$  with length  $\ell(\gamma)$  joining  $x$  to  $y$  and satisfying*

1.  $\ell(\gamma) \leq \frac{|x-y|}{\epsilon}$  and
2.  $d(z, \partial\Omega) \geq \epsilon|x - z| \frac{|y-z|}{|x-y|}$  for  $z \in \gamma$ .

In particular, a Lipschitz domain  $\Omega$  is an  $(\epsilon, \delta)$ -domain and also a  $n$ -set [22] (i.e., a  $d$ -set with  $d = n$ ). Self-similar fractals (e.g., von Koch's snowflake domain) are examples of  $(\epsilon, \delta)$ -domains with the  $d$ -set boundary [19, 22],  $d > n - 1$ .

In order to describe irregular boundaries of fractal dimension  $d > n - 1$ , we define sets preserving Markov's inequality (Ref. [21] Ch. II):

**Definition 3** *A closed subset  $V$  in  $\mathbb{R}^n$  preserves Markov's inequality if for every fixed positive integer  $k$ , there exists a constant  $c = c(V, n, k) > 0$ , such that*

$$\max_{V \cap U_r(x)} |\nabla P| \leq \frac{c}{r} \max_{V \cap U_r(x)} |P|$$

for all polynomials  $P \in \mathcal{P}_k$  and all closed balls  $U_r(x)$ ,  $x \in V$  and  $0 < r \leq 1$ .

Examples of sets that preserves Markov's inequality are  $d$ -sets in  $\mathbb{R}^n$ , where  $d > n - 1$ , and self-similar sets that are not a subset of any  $(n - 1)$ -dimensional subspace of  $\mathbb{R}^n$  (see Refs. [22, 25]).

To extend the variational formulation introduced in Subsection 2.1 to fractal boundaries of the type of  $d$ -sets, we use the existence of the  $d$ -dimensional Hausdorff measure  $m_d$  on  $\partial\Omega$  (the  $d$ -measure from Definition 1) and the theorem which generalizes the usual trace theorem and the Green formula.

For example, for  $d = n - 1$  and a Lipschitz  $\partial\Omega$ , we know [18, 26] that the trace operator is linear continuous and surjective from  $H^1(\Omega)$  onto  $H^{\frac{1}{2}}(\partial\Omega)$ , and the formula

$$\int_{\Omega} v \Delta u dx = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \right\rangle_{(H^{\frac{1}{2}}(\partial\Omega))', H^{\frac{1}{2}}(\partial\Omega)} - \int_{\Omega} \nabla v \nabla u dx,$$

holds whatever  $u \in H^1(\Omega)$  such that  $\Delta u \in L^2(\Omega)$  and  $v \in H^1(\Omega)$ .

To generalize the trace operator and the Green formula to fractal boundaries, one introduces the Besov space  $B_{\beta}^{2,2}(\partial\Omega)$  with  $\beta = 1 - \frac{n-d}{2} > 0$  (see A). Note that for  $d = n - 1$ , one has  $\beta = \frac{1}{2}$  and

$$B_{\frac{1}{2}}^{2,2}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega),$$

i.e., one recovers the above relations. In general,

1. For an arbitrary open set  $\Omega$  of  $\mathbb{R}^n$ , the trace operator  $\text{Tr}$  is defined[21, 25, 27] for  $u \in L_{loc}^1(\Omega)$  by

$$\text{Tr}u(x) = \lim_{r \rightarrow 0} \frac{1}{m(\Omega \cap U_r(x))} \int_{\Omega \cap U_r(x)} u(y) dy, \quad (10)$$

where  $m$  denotes the Lebesgue measure. The trace operator  $\text{Tr}$  is considered for all  $x \in \bar{\Omega}$  for which the limit exists.

2. If  $\Omega$  is a bounded  $(\epsilon, \delta)$ -domain in  $\mathbb{R}^n$  such that its boundary  $\partial\Omega$  is a closed  $d$ -set preserving Markov's inequality, then[21, 22]
  - (a) the trace operator  $\text{Tr} : H^1(\Omega) \rightarrow B_{\beta}^{2,2}(\partial\Omega)$  is linear continuous and surjective;
  - (b) the Green formula holds (see also Refs. [27, 28] for the von Koch case in  $\mathbb{R}^2$ ):

$$\int_{\Omega} v \Delta u dx = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr}v \right\rangle_{(B_{\beta}^{2,2}(\partial\Omega))', B_{\beta}^{2,2}(\partial\Omega)} - \int_{\Omega} \nabla v \nabla u dx, \quad (11)$$

where the dual Besov space  $(B_{\beta}^{2,2}(\partial\Omega))' = B_{-\beta}^{2,2}(\partial\Omega)$  is introduced in Ref. [23] (see A).

Let us also notice that the Green's formula (11) still holds whatever  $u \in H^1(\Omega)$  such that  $\Delta u \in L^2(\Omega)$  and  $v \in H^1(\Omega)$ .

### 2.3 Well-posedness

The above preliminaries allow us to prove the following Proposition:

**Proposition 1** *1. Let  $\Omega = \Omega_+$  be a bounded domain in  $\mathbb{R}^n$  with a closed piecewise Lipschitz boundary  $\partial\Omega$  and  $0 < \lambda(x) \leq +\infty$  be a given function defined on  $\partial\Omega$ . By  $\Gamma_{\infty}$  is denoted the part of  $\partial\Omega$  such that*

$$\forall x \in \Gamma_{\infty} \quad \lambda(x) = +\infty,$$

*in the such way that  $\lambda \in L^{\infty}(\partial\Omega \setminus \Gamma_{\infty})$ . Then the bilinear form*

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_- + \int_{\partial\Omega \setminus \Gamma_{\infty}} \lambda(x) (u_+ - u_-) (v_+ - v_-) d\sigma$$

*is continuous and coercive on  $V \times V$  with*

$$V = \{u \in L^2(\mathbb{R}^n) \mid u_+ = u|_{\Omega_+} \in H^1(\Omega_+), u_- = u|_{\Omega_-} \in H^1(\Omega_-), \\ u_+ = u_- \text{ on } \Gamma_{\infty}\}. \quad (12)$$

2. Let  $\Omega = \Omega_+$  be a bounded  $(\epsilon, \delta)$ -domain in  $\mathbb{R}^n$  with a closed  $d$ -set boundary  $\partial\Omega$  and  $\lambda \in C(\partial\Omega)$  be a positive continuous function defined on  $\partial\Omega$ . By  $m_d$  is denoted the  $d$ -measure on  $\partial\Omega$  (see Definition 1). Then the bilinear form

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_- + \int_{\partial\Omega} \lambda(x) \text{Tr}(u_+ - u_-) \text{Tr}(v_+ - v_-) dm_d$$

is continuous and coercive on  $V \times V$  ( $V$  is defined in Eq. (12)).

3. Let  $\Omega = \Omega_+$  be a bounded  $(\epsilon, \delta)$ -domain in  $\mathbb{R}^n$  with a closed  $d$ -set boundary  $\partial\Omega$  and  $\lambda(x) = +\infty$  for all  $x \in \partial\Omega$ . Then the bilinear form

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_-$$

is continuous and coercive on  $V \times V$  with  $V = H^1(\mathbb{R}^n)$ .

Consequently, we obtain the following theorem:

**Theorem 1 (Well-posedness)** *In all cases from Proposition 1 for all  $u_0 \in H = L^2(\mathbb{R}^n)$  there exists a unique solution  $u \in C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, V)$  of the variational problem*

$$\forall v \in V \quad \frac{d}{dt} \langle u, v \rangle_H + a(u, v) = 0, \quad u(x, 0) = u_0 \in L^2(\mathbb{R}^n), \quad (13)$$

where by  $\langle \cdot, \cdot \rangle_H$  is denoted the inner product in  $H$ . In addition, this solution verifies the energy equality:

$$\frac{1}{2} \int_{\mathbb{R}^n} |u(t)|^2 dx + \int_0^t a(u, u) ds = \frac{1}{2} \int_{\mathbb{R}^n} |u_0(x)|^2 dx. \quad (14)$$

**Remark 2** *On one hand, any “smooth enough” solution of the problem (1)–(4) gives the solution of Theorem 1. On the other hand, any solution from Theorem 1 satisfies the relations (1)–(2) and, in a weak sense (in the sense of the duality presented above), satisfies the relations (3)–(4).*

Finally, we prove

**Theorem 2 (Continuity of  $u_\lambda$  on  $\lambda$  and the case  $\lambda = \infty$ )** *Let  $(\lambda_k)_{k \in \mathbb{N}}$  be a positive sequence converging to  $\lambda^*$  in  $L^\infty(\partial\Omega)$ . Then the corresponding sequence of the solutions  $(u_{\lambda_k})_{k \in \mathbb{N}}$  of the system (1)–(4) converges strongly to  $u_\lambda^*$  in  $C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, V)$ , i.e.,  $u_\lambda$  is continuous as a function of  $\lambda$ .*

*If  $\lambda_k \rightarrow \infty$  in  $L^\infty(\partial\Omega)$ , then  $u_{\lambda_k} \rightarrow u_\infty$  in  $C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, V)$  with  $(u_\infty)_+ = (u_\infty)_-$  on  $\partial\Omega$ . In this case,  $u_\infty \in C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, H^1(\mathbb{R}^n))$  solves*

$$\forall v \in H^1(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} \partial_t u_\infty v dx + \int_{\mathbb{R}^n} D(x) \nabla u_\infty \nabla v dx = 0, \quad u_\infty(x, 0) = u_0 \in L^2(\mathbb{R}^n), \quad (15)$$

with  $D(x) = \mathbf{1}_{\Omega_+} D_+ + \mathbf{1}_{\Omega_-} D_-$ .

**Proof.** Firstly we suppose that  $\lambda^*$  is a finite bounded function on  $\partial\Omega$  ( $\|\lambda^*\|_{L^\infty(\partial\Omega)} < \infty$ ). Since  $u(0) = u_0$  does not depend on  $\lambda$ , the equality (14) implies that the sequence  $(u_{\lambda_k})$  is bounded in  $C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, V)$ .

Therefore, due to the unicity of the solution for  $\lambda^*$  and the unicity of the weak limit, the convergence  $\lambda_k \rightarrow \lambda^*$  in  $L^\infty(\partial\Omega)$  implies  $u_{\lambda_k} \rightharpoonup u_{\lambda^*}$ . Since  $u_k|_{\partial\Omega} \in B_\beta^{2,2}(\partial\Omega)$  with  $\beta = 1 - \frac{n-d}{2} > 0$ , with the help of (13) and the coercive behavior of  $a(u, u)$ ,

$$a(u, u) > \alpha \|u\|_V^2, \quad \text{for } \alpha > 0,$$

we conclude that  $u_{\lambda_k} \rightarrow u_{\lambda^*}$  in  $C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, V)$ .

In the case  $\|\lambda_k\|_{L^\infty(\partial\Omega)} \rightarrow +\infty$ , we find from (14) that

$$\begin{aligned} \forall k \in \mathbb{N} \quad \int_{\partial\Omega} \lambda_k(x) \text{Tr}((u_k)_+ - (u_k)_-)^2 dm_d < \infty, \\ \left( \int_{\partial\Omega} \text{Tr}((u_k)_+ - (u_k)_-)^2 dm_d \right)^{\frac{1}{2}} \leq \frac{1}{2\sqrt{\|\lambda_k\|_{L^\infty(\partial\Omega)}}} \int_{\mathbb{R}^n} |u_0(x)|^2 dx. \end{aligned}$$

Therefore, we obtain in this case that  $(u_\infty)_+ = (u_\infty)_-$  on  $\partial\Omega$ , where by  $u_\infty$  we denote the limit of  $u_k$  as  $\lambda \rightarrow +\infty$ . In addition,  $u_\infty(t, \cdot) \in H^1(\mathbb{R}^n)$  and it is the solution of (15).

□

### 3 Heat content localization to a small neighborhood of the boundary

s the initial condition is zero in  $\mathbb{R}^n \setminus \Omega$ , we have

$$N(t) = \int_{\Omega} (1 - u(x, t)) dx = \text{Vol}(\Omega) - \int_{\Omega} u(x, t) dx, \quad (16)$$

or equivalently, in terms of the Green function of the problem (1)–(4),

$$N(t) = \text{Vol}(\Omega) - \int_{\Omega} \int_{\Omega} G(x, y, t) dy dx.$$

Let us show that it is sufficient to integrate only on a small neighborhood of the boundary  $\partial\Omega$  to obtain the desired heat content with an exponentially small error:

**Lemma 1** *Let  $F \subset \Omega$  be a non-empty open bounded set in  $\mathbb{R}^n$ , such that  $\text{dist}(F, \partial\Omega) = \epsilon > 0$ . Then for  $t \rightarrow +0$  and  $u = u_+ \mathbf{1}_\Omega + u_- \mathbf{1}_{\mathbb{R}^n \setminus \Omega}$  the solution of (1)–(4), associated with the Green function  $G(x, y, t)$ ,*

1. *it holds*

$$\begin{aligned} \int_F (1 - u_+(x, t)) dx &= \int_F \left( 1 - \int_{\Omega} G(x, y, t) dy \right) dx \\ &= O \left( \left( \frac{\epsilon}{\sqrt{4D_+ t}} \right)^{n-2} e^{-\epsilon^2/(4D_+ t)} \right). \end{aligned} \quad (17)$$

2. for  $\epsilon > 2\sqrt{D_+t}$  such that  $\epsilon = O(\sqrt{t})$ , there exists  $\delta > 0$  (a constant independent on time) such that the heat content  $N(t)$  can be expressed as

$$\begin{aligned} N(t) &= \int_{\mathbb{R}^n \setminus \Omega} u_-(x, t) dx \\ &= \int_{\Omega_\epsilon} \left( 1 - \int_{\Omega_\epsilon} G(x, y, t) dy \right) dx + O\left(e^{-\frac{1}{t^\delta}}\right), \end{aligned} \quad (18)$$

where  $\Omega_\epsilon$  is the  $\epsilon$ -neighborhood of  $\partial\Omega$ .

**Proof.** As it was shown, the problem (1)–(4) has a unique solution  $u = u_+ \mathbf{1}_\Omega + u_- \mathbf{1}_{\mathbb{R}^n \setminus \Omega}$ . Let  $G(x, y, t)$  be the Green function so that

$$u(x, t) = \int_{\Omega} G(x, y, t) dy.$$

Thus, using the properties of  $G$  such as  $G \geq 0$  for all  $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$  and  $\int_{\mathbb{R}^n} G(x, y, t) dy = 1$ , we easily see that

$$0 \leq \int_{\Omega} G(x, y, t) dy = u(x, t) \leq \int_{\mathbb{R}^n} G(x, y, t) dy = 1.$$

We notice that, by the assumption,  $\lambda(x) > 0$  is a regular function on  $\partial\Omega$  and all other coefficients are constant. By definition  $u_+$  is the solution of the system

$$\begin{aligned} (\partial_t - D_+ \Delta) u_+ &= 0, \quad x \in \Omega \subset \mathbb{R}^n, \\ u_+|_{t=0} &= 1, \\ u_+|_{\partial\Omega} &= \left( u_- - \frac{D_-}{\lambda} \frac{\partial u_-}{\partial n} \right) |_{\partial\Omega}, \quad \lambda > 0, \end{aligned}$$

which can be reformulated for  $\hat{v} = 1 - u_+$

$$\begin{aligned} (\partial_t - D_+ \Delta) \hat{v} &= 0, \quad x \in \Omega \subset \mathbb{R}^n, \\ \hat{v}|_{t=0} &= 0, \\ (1 - \hat{v})|_{\partial\Omega} &= \left( u_- - \frac{D_-}{\lambda} \frac{\partial u_-}{\partial n} \right) |_{\partial\Omega}, \end{aligned}$$

where  $0 \leq u_- \leq 1$  for all  $t$ . Moreover, as  $0 \leq \hat{v} \leq 1$ , it follows that

$$0 \leq \left( u_- - \frac{D_-}{\lambda} \frac{\partial u_-}{\partial n} \right) |_{\partial\Omega} \leq 1$$

and, as  $u_-$  is increasing in time on  $\partial\Omega$ , then  $\hat{v}$  is decreasing in time on  $\partial\Omega$ . Therefore,  $\hat{v} \leq v$ , where  $v$  is the solution of the following problem:

$$\begin{aligned} (\partial_t - D_+ \Delta) v &= 0, \quad x \in \Omega \subset \mathbb{R}^n, \\ v|_{t=0} &= 0, \\ v|_{\partial\Omega} &= 1, \end{aligned}$$

Thus, as in Ref. [29] (p.231 Lemma 12.7) for  $n = 2$ , but now in  $\mathbb{R}^n$  ( $n \geq 2$ ), we find that for the ball  $\Omega = U_r(z)$  centered at  $z$  and of radius  $r$ , the solution satisfies as  $t \rightarrow +0$

$$v(z, t) \leq C \left( \frac{r}{\sqrt{4D_+t}} \right)^{n-2} \exp \left( -\frac{r^2}{4D_+t} \right),$$

with a constant  $C > 0$  depending only on  $n$  ( $C$  can be explicitly obtained by the integration by parts in the generalized spherical coordinates in  $\mathbb{R}^n$ , where the coefficient  $\left( \frac{r}{\sqrt{4D_+t}} \right)^{n-2}$  corresponding to the leading term as  $t \rightarrow +0$ , appears from the integral  $\int_{\frac{r}{\sqrt{4D_+t}}}^{+\infty} e^{-w^2} w^{n-1} dw$ ). Consequently (see Ref. [29] Corollary 12.8 p.232), for  $z \in \text{int}\{\Omega\}$  and  $t \rightarrow +0$  we find

$$v(z, t) \leq C \left( \frac{\text{dist}(z, \partial\Omega)}{\sqrt{4D_+t}} \right)^{n-2} \exp \left( -\frac{\text{dist}(z, \partial\Omega)^2}{4D_+t} \right).$$

Then we immediately obtain Eq. (17) by integration.

For  $n = 2$  we obtain directly the exponential decay in Eq. (17) for all  $\epsilon > 0$ . If  $n > 2$ , we still have the exponential decay for a small constant  $\alpha > 0$  depending only on  $\epsilon$ :

$$O \left( \left( \frac{\epsilon}{\sqrt{4D_+t}} \right)^{n-2} e^{-\epsilon^2/(4D_+t)} \right) = O(e^{-\alpha/t}).$$

Note that  $O \left( e^{-\epsilon^2/(4D_+t)} \right)$  gives an exponentially small remaining term iff  $\epsilon = 2\sqrt{D_+} t^{\frac{1}{2}-\delta_0}$  for a constant  $\delta_0 > 0$ . For small enough  $\delta_0$  we have  $\epsilon = O(\sqrt{4D_+t})$ , also knowing that  $\epsilon > \sqrt{4D_+t}$ .

So, for this  $\epsilon$ , we split  $\Omega$  in two parts:  $\Omega_\epsilon$ , the neighborhood of  $\partial\Omega$  such that  $\text{dist}(x, \partial\Omega) \leq \epsilon$ , and  $\Omega \setminus \Omega_\epsilon$ . For all  $F \subseteq \Omega \setminus \Omega_\epsilon$ ,  $\text{dist}(F, \partial\Omega) > \epsilon > 2\sqrt{D_+t}$ , we have

$$\int_F \left( 1 - \int_\Omega G(x, y, t) dy \right) dx = O \left( e^{-c(F)/t^{\delta(F)}} \right),$$

where  $c(F)$  and  $\delta(F)$  are positive constants depending only on the distance between  $F$  and  $\partial\Omega$  and the dimension  $n$ .

To complete the proof of the second statement, we first find that

$$\begin{aligned} N(t) &= \int_{\mathbb{R}^n \setminus \Omega} u_-(x, t) dx \\ &= \int_{\mathbb{R}^n \setminus \Omega} \int_\Omega G(x, y, t) dy dx = \int_{\mathbb{R}^n} \int_\Omega G(x, y, t) dy dx - \int_\Omega \int_\Omega G(x, y, t) dy dx \\ &= \text{Vol}(\Omega) - \int_\Omega \int_\Omega G(x, y, t) dy dx. \end{aligned}$$

For  $\Omega = \Omega_\epsilon \cup (\Omega \setminus \Omega_\epsilon)$  we can write

$$\begin{aligned}
\int_{\Omega} \int_{\Omega} G(x, y, t) dy dx &= \left( \int_{\Omega_\epsilon} \int_{\Omega} + \int_{\Omega \setminus \Omega_\epsilon} \int_{\Omega} \right) G(x, y, t) dy dx \\
&= \text{Vol}(\Omega \setminus \Omega_\epsilon) + \int_{\Omega_\epsilon} \int_{\Omega} G(x, y, t) dy dx + O\left(e^{-\frac{1}{t^\delta}}\right) \\
&= \text{Vol}(\Omega) - \text{Vol}(\Omega_\epsilon) + \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} G(x, y, t) dy dx \\
&+ \int_{\Omega_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx + O\left(e^{-\frac{1}{t^\delta}}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\Omega_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx &= \int_{\Omega} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx - \int_{\Omega \setminus \Omega_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx \\
&= \text{Vol}(\Omega \setminus \Omega_\epsilon) + O\left(e^{-\frac{1}{t^\delta}}\right) - \int_{\Omega \setminus \Omega_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx
\end{aligned}$$

and since

$$\begin{aligned}
&\int_{\Omega \setminus \Omega_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx - \text{Vol}(\Omega \setminus \Omega_\epsilon) \\
&\leq \int_{\Omega \setminus \Omega_\epsilon} \int_{\Omega} G(x, y, t) dy dx - \text{Vol}(\Omega \setminus \Omega_\epsilon) = O\left(e^{-\frac{1}{t^\delta}}\right),
\end{aligned}$$

we conclude that

$$-\int_{\Omega_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} G(x, y, t) dy dx = O\left(e^{-\frac{1}{t^\delta}}\right)$$

and finally

$$N(t) = \text{Vol}(\Omega_\epsilon) - \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} G(x, y, t) dy dx + O\left(e^{-\frac{1}{t^\delta}}\right),$$

that completes the proof.  $\square$

A variant of Lemma 1 for  $n = 2$  can be found in Ref. [11], where the heat localization near the boundary is also called by the principle of “not feeling the boundary”. In addition, we can consider the case of the distinct parts of the boundary:

**Corollary 1** *Let  $X$  and  $Y$  be different closed parts of  $\partial\Omega$  such that  $\text{dist}(X, Y) > 2\epsilon$ , where  $\epsilon = O(\sqrt{t}) > 2\sqrt{D_+} t$ . Let  $U_r(X) = \{x \in \mathbb{R}^n \mid d(x, X) < r\}$  be the open neighborhood of  $X$  of size  $r > 0$ . Consider  $u_+$  and  $\hat{u}_+$  as the respective solutions of the following systems:*

$$\begin{aligned}
\partial_t u_+ - D_+ \Delta u_+ &= 0, \quad x \in \Omega \subset \mathbb{R}^n, \\
u_+|_{t=0} &= 1, \\
u_+|_{\partial\Omega} &= \left( u_- - \frac{D_-}{\lambda} \frac{\partial u_-}{\partial n} \right) |_{\partial\Omega}, \quad \lambda > 0,
\end{aligned}$$



$$\begin{aligned}
\partial_t \hat{u}_+ - D_+ \Delta \hat{u}_+ &= 0, \quad x \in \Omega \subset \mathbb{R}^n, \\
\hat{u}_+|_{t=0} &= 1, \\
\hat{u}_+|_{\partial\Omega \cap \overline{U(X)}} &= \left( u_- - \frac{D_- \partial u_-}{\lambda \partial n} \right) |_{\partial\Omega \cap \overline{U(X)}}, \quad \lambda > 0 \\
\hat{u}_+|_{\partial\Omega \setminus (\partial\Omega \cap \overline{U(X)})} &= 1,
\end{aligned}$$

where  $U(X)$  is an open neighborhood of  $X$  of a radius strictly greater than  $2\epsilon$ :  $U_{2\epsilon}(X) \subset U(X)$ .

Then there exists  $\delta > 0$  such that

$$\int_{U_\epsilon(X)} |u_+ - \hat{u}_+| dx = O\left(e^{-\frac{1}{t^\delta}}\right).$$

Moreover, if  $\tilde{u}_+$  is the solution of the system:

$$\begin{aligned}
\partial_t \tilde{u}_+ - D_+ \Delta \tilde{u}_+ &= 0, \quad x \in \Omega \subset \mathbb{R}^n, \\
\tilde{u}_+|_{t=0} &= 1, \\
\tilde{u}_+|_Y &= \left( u_- - \frac{D_- \partial u_-}{\lambda \partial n} \right) |_Y, \quad \lambda > 0 \\
\tilde{u}_+|_{\partial\Omega \setminus Y} &= 1,
\end{aligned}$$

then

$$\int_{U_\epsilon(X)} (1 - \tilde{u}_+) dx = \int_{\Omega \setminus U_\epsilon(Y)} (1 - \tilde{u}_+) dx = O\left(e^{-\frac{1}{t^\delta}}\right).$$

The proof of Corollary 1 follows from the proof of the first statement of Lemma 1.

Note that the continuity of  $u$  on  $\lambda$  (see Theorem 2) and the localization of the heat content near the boundary allow one to consider mixed boundary conditions:

**Theorem 3** *Let  $\Omega$  be a bounded  $(\epsilon, \delta)$ -domain (see Section 2) with a closed connected  $d$ -set boundary  $\partial\Omega = \Gamma_0 \sqcup \Gamma_\lambda \sqcup \Gamma_\infty$ . Let  $\lambda \in C(\Gamma_0 \sqcup \Gamma_\lambda)$  such that*

$$\lambda(x) = \begin{cases} 0, & x \in \Gamma_0, \\ 0 < f(x) < \infty, & x \in \Gamma_\lambda, \\ +\infty, & x \in \Gamma_\infty \end{cases}$$

and  $\epsilon = O(\sqrt{t}) > \sqrt{4D_+ t}$ . We assume that the connection between different types of boundary is performed in the continuous way (see Theorem 2) such that the solution  $u$  remains continuous as a function of  $\lambda$ .

We split the  $\epsilon$ -interior Minkowski sausage of  $\partial\Omega$  into disjoint subsets

$$\Omega_\epsilon = \Omega_\epsilon^{\Gamma_0} \sqcup \Omega_\epsilon^{\Gamma_\lambda} \sqcup \Omega_\epsilon^{\Gamma_\infty}$$

such that each subset  $\Omega_\epsilon^\Gamma$  is contained in the  $\epsilon$ -interior Minkowski sausage of  $\Gamma$  ( $\Gamma \subset \partial\Omega$ ). Then, for  $\delta > 0$  from Lemma 1, the heat content of the problem (1)–(4),

$$N(t) = \int_\Omega (1 - u(x, t)) dx = \int_{\Omega_\epsilon} (1 - u(x, t)) dx + O\left(e^{-\frac{1}{t^\delta}}\right),$$

can be found as a sum of two heat contents:

$$N(t) = \int_{\Omega_\epsilon^{\Gamma_\lambda}} (1 - u(x, t)) dx + \int_{\Omega_\epsilon^{\Gamma_\infty}} (1 - u(x, t)) dx + O(e^{-\frac{1}{t^\delta}}).$$

In order to locally approximate the solution of the problem (1)–(4) by considering the problem with coefficients frozen on a fixed boundary point, according to Corollary 1, we also obtain the following proposition:

**Proposition 2** *Let  $\sigma$  be a fixed point of the boundary  $\partial\Omega$  and let define*

$$B_{l\epsilon, \epsilon} = U_{l\epsilon}(\sigma) \cap (\Omega_\epsilon \cup \Omega_{-\epsilon}) \quad \text{for } l \in \mathbb{N}, \quad (19)$$

where  $U_{l\epsilon}(\sigma) \subset \mathbb{R}^n$  is a ball of radius  $l\epsilon$  centered at  $\sigma$ ,  $\epsilon$  is defined in Corollary 1. Let  $\phi_\sigma \in C^\infty(B_{4\epsilon, \epsilon}(\sigma))$  be a smooth cut-off function with a compact support on  $B_{4\epsilon, \epsilon}(\sigma)$ :

$$\phi_\sigma(x) = \begin{cases} 1 & x \in B_{3\epsilon, \epsilon}(\sigma), \\ a \text{ smooth function } 0 \leq \eta < 1 & x \in B_{4\epsilon, \epsilon}(\sigma) \setminus B_{3\epsilon, \epsilon}(\sigma), \\ 0 & x \in \Omega \setminus B_{4\epsilon, \epsilon}(\sigma) \end{cases} \quad (20)$$

If  $u$  is the solution of the problem (1)–(4), then  $\phi_\sigma u$  is the solution of the following problem:

$$\partial_t(\phi_\sigma u_\pm) - D_\pm \Delta(\phi_\sigma u_\pm) = \begin{cases} -(1 - u_\pm) D_\pm \Delta \phi_\sigma & x \in B_{4\epsilon, \epsilon}(\sigma) \setminus B_{3\epsilon, \epsilon}(\sigma), \\ 0 & \text{elsewhere in } \Omega, \end{cases} \quad (21)$$

$$(\phi_\sigma u_\pm)|_{t=0} = \mathbf{1}_\Omega(x) \phi_\sigma(x), \quad (22)$$

$$D_- \frac{\partial(\phi_\sigma u_-)}{\partial n} \Big|_{\partial\Omega} = \lambda(x) \phi_\sigma(x) (u_- - u_+) \Big|_{\partial\Omega}, \quad (23)$$

$$D_+ \frac{\partial(\phi_\sigma u_+)}{\partial n} \Big|_{\partial\Omega} = D_- \frac{\partial(\phi_\sigma u_-)}{\partial n} \Big|_{\partial\Omega}. \quad (24)$$

Therefore, there exists  $\delta > 0$  such that

$$\int_{B_{2\epsilon, \epsilon}(\sigma)} |u - \phi_\sigma u| dx = O\left(e^{-\frac{1}{t^\delta}}\right),$$

and if  $\phi_\sigma u^\sigma$  is the solution of the problem (21)–(24) with frozen coefficients in the boundary point  $\sigma$ , then

$$\int_{\Omega \setminus B_{\epsilon, \epsilon}(\sigma)} \phi_\sigma (1 - u^\sigma) dx = O\left(e^{-\frac{1}{t^\delta}}\right). \quad (25)$$

## 4 Local coordinates for a regular $\partial\Omega \in C^3$

In order to prove Eq. (6) for a large class of  $(\epsilon, \delta)$ -compact connected domains  $\Omega$  in  $\mathbb{R}^n$ , we first prove it for the case of domains with regular boundary  $\partial\Omega \in C^\infty$  or at least in  $C^3$ . As  $\Omega$  is compact, for all types of connected  $\partial\Omega$ , the volume of  $\Omega$  is finite and, therefore, the volume of the  $\epsilon$ -neighborhood of  $\partial\Omega$  in  $\Omega$  is also finite and can be

approximated by a sequence of volumes of Minkowski sausages with regular boundaries (the same argument was used in Ref. [11] p.378).

Let us consider the regular boundary  $\partial\Omega \in C^3$ .

Given a positive  $\epsilon > 0$  provided in Lemma 1, we denote by  $\Omega_\epsilon$  and  $\Omega_{-\epsilon}$  the open  $\epsilon$ -neighborhoods of  $\partial\Omega$  in  $\Omega$  and in  $\mathbb{R}^n \setminus \Omega$ , respectively.

According to Eq. (18) and the regularity of the boundary  $\partial\Omega$ , we can decompose  $\Omega_\epsilon \cup \partial\Omega \cup \Omega_{-\epsilon} = \bigsqcup_{i=1}^I B_{i,\epsilon}$  ( $I$  is a finite integer because  $\overline{\Omega}_+ \cup \Omega_{-\epsilon}$  is a compact domain) in such way that on each  $B_{i,\epsilon}$  it is possible to introduce the local coordinates. In addition, we assume that for all  $i = 1, \dots, I$  there exists  $\sigma_i \in \partial\Omega \cap B_{i,\epsilon}$  such that  $B_{i,\epsilon} \subset B_{2\epsilon,\epsilon}(\sigma_i)$  (see Eq. (19) for the definition). Due to Proposition 2, the last assumption ensures that

$$\int_{B_{i,\epsilon}} (1-u)dx = \int_{B_{i,\epsilon}} \phi_{\sigma_i}(1-u)dx + O\left(e^{-\frac{1}{t^{\delta}}}\right).$$

For all  $i$  we perform the change of the space variables  $(x_1, \dots, x_n) \in B_{i,\epsilon}$  to the local coordinates  $(\theta_1, \dots, \theta_{n-1}, s)$  by the formula

$$x = \hat{x}(\theta_1, \dots, \theta_{n-1}) - sn(\theta_1, \dots, \theta_{n-1}) \quad \begin{cases} 0 < s < \epsilon \text{ for } x \in B_{i,\epsilon} \cap \Omega_\epsilon \\ -\epsilon < s < 0 \text{ for } x \in B_{i,\epsilon} \cap \Omega_{-\epsilon} \end{cases}, \quad (26)$$

where  $\hat{x}(\theta_1, \dots, \theta_{n-1}) \in \partial\Omega$  and  $x$ ,  $\hat{x}$  and  $n$  are the vectors in  $\mathbb{R}^n$  such that

$$\left\{ \frac{\partial \hat{x}}{\partial \theta_1}, \dots, \frac{\partial \hat{x}}{\partial \theta_{n-1}}, n \right\}$$

is an orthonormal basis in  $\mathbb{R}^n$ .

In what follows we denote  $B_{i,\epsilon} \cap \Omega_\epsilon$  by  $\Omega_{i,+ \epsilon}$  and  $B_{i,\epsilon} \cap \Omega_{-\epsilon}$  by  $\Omega_{i,- \epsilon}$  respectively. In each of two regions,  $\Omega_{i,+ \epsilon}$  and  $\Omega_{i,- \epsilon}$ , the change of variables  $(x_1, \dots, x_n) \mapsto (\theta_1, \dots, \theta_{n-1}, s)$  is a local  $C^1$ -diffeomorphism.

In local coordinates  $\partial\Omega$  is described by  $s = 0$ .

Thus, Eq. (18) becomes

$$N(t) = \sum_{i=1}^I \int_{\Omega_{i,+ \epsilon}} (1-u(x,t))dx + O(e^{-\frac{1}{t^{\delta}}}). \quad (27)$$

Denoting  $\theta = (\theta_1, \dots, \theta_{n-1})$ , the integration domain  $\Omega_{i,+ \epsilon}$  in (27) becomes

$$\Omega_{i,+ \epsilon} = \{0 < s < \epsilon, \quad \theta \in \partial\Omega \cap \overline{\Omega}_{i,+ \epsilon}\},$$

which is actually a parallelepiped neighborhood  $(\partial\Omega \cap \overline{\Omega}_{i,+ \epsilon}) \times ]0, \epsilon[$ .

For this change of variables we have

$$|\nabla_x s|^2 = 1, \quad \nabla_x s \nabla_x \theta_i = 0, \quad \nabla_x \theta_j \nabla_x \theta_i = \frac{\delta_{ij}}{(1 - sk_i)^2} \quad \text{for } i, j = 1, \dots, n-1,$$

$$\nabla u_\pm \nabla \phi_\pm = \frac{\partial u_\pm}{\partial s} \frac{\partial \phi_\pm}{\partial s} + \sum_i^{n-1} \frac{\partial u_\pm}{\partial \theta_i} \frac{\partial \phi_\pm}{\partial \theta_i} \frac{1}{(1 - sk_i)^2},$$

and therefore, using twice the integration by parts and the notations

$$|J(s, \theta)| = \prod_{i=1}^{n-1} (1 - sk_i) \quad (28)$$

for the Jacobian and  $k_i = k_i(\theta_1, \dots, \theta_{n-1})$  of the principal curvatures for  $\partial\Omega$  curving away the outward normal  $n$  to  $\partial\Omega$  like in the case of the sphere, we find that for all test functions  $\phi = (\phi_+, \phi_-) \in V|_{B_{i,\epsilon}}$

$$\begin{aligned} & \int_{B_{i,\epsilon}} \partial_t u |J(s, \theta)| \phi \, ds d\theta_1 \cdots d\theta_{n-1} \\ & - \int_{\Omega_{i,+ \epsilon}} \left[ \frac{\partial}{\partial s} \left( D_+ |J(s, \theta)| \frac{\partial u_+}{\partial s} \right) + \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_i} \left( \frac{D_+ |J(s, \theta)|}{(1 - sk_i)^2} \frac{\partial u_+}{\partial \theta_i} \right) \right] \phi_+ \, ds d\theta_1 \cdots d\theta_{n-1} \\ & - \int_{\Omega_{i,- \epsilon}} \left[ \frac{\partial}{\partial s} \left( D_- |J(s, \theta)| \frac{\partial u_-}{\partial s} \right) + \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_i} \left( \frac{D_- |J(s, \theta)|}{(1 - sk_i)^2} \frac{\partial u_-}{\partial \theta_i} \right) \right] \phi_- \, ds d\theta_1 \cdots d\theta_{n-1} \\ & + \int_{s=0} \lambda(\theta) (u_+ - u_-) (\phi_+ - \phi_-) \, d\theta = 0. \end{aligned}$$

The regularity of the boundary ensures that the principal curvatures  $k_i(\theta)$  are at least in  $C^1(\partial\Omega \cap \partial B_{i,\epsilon})$ .

Therefore, the problem (1)–(4) locally becomes

$$\begin{aligned} & \frac{\partial}{\partial t} u_+ - D_+ \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_+ = D_+ \sum_{i=1}^{n-1} \frac{sk_i(\theta)}{1 - sk_i(\theta)} \left( 1 + \frac{1}{1 - sk_i(\theta)} \right) \frac{\partial^2 u_+}{\partial \theta^2} \\ & - D_+ \left( \sum_{i=1}^{n-1} k_i(\theta) + s \sum_{i=1}^{n-1} \frac{k_i^2(\theta)}{1 - sk_i(\theta)} \right) \frac{\partial u_+}{\partial s} \\ & + \frac{D_+}{|J(s, \theta)|} \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_i} \left( \frac{|J(s, \theta)|}{(1 - sk_i(\theta))^2} \right) \frac{\partial u_+}{\partial \theta_i}, \quad 0 < s < \epsilon, \quad \theta \in (\partial\Omega \cap \bar{\Omega}_{i,+ \epsilon}) \quad (29) \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial t} u_- - D_- \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_- = D_- \sum_{i=1}^{n-1} \frac{sk_i(\theta)}{1 - sk_i(\theta)} \left( 1 + \frac{1}{1 - sk_i(\theta)} \right) \frac{\partial^2 u_-}{\partial \theta^2} \\ & - D_- \left( \sum_{i=1}^{n-1} k_i(\theta) + s \sum_{i=1}^{n-1} \frac{k_i^2(\theta)}{1 - sk_i(\theta)} \right) \frac{\partial u_-}{\partial s} \\ & + \frac{D_-}{|J(s, \theta)|} \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_i} \left( \frac{|J(s, \theta)|}{(1 - sk_i(\theta))^2} \right) \frac{\partial u_-}{\partial \theta_i}, \quad -\epsilon < s < 0, \quad \theta \in (\partial\Omega \cap \bar{\Omega}_{i,+ \epsilon}), \quad (30) \end{aligned}$$

$$u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \quad (31)$$

$$D_- \frac{\partial u_-}{\partial s} |_{s=-0} = \lambda(\theta) (u_- - u_+) |_{s=0}, \quad (32)$$

$$D_+ \frac{\partial u_+}{\partial s} |_{s=+0} = D_- \frac{\partial u_-}{\partial s} |_{s=-0}. \quad (33)$$

We emphasize that the problem (29)–(33) should be considered as the trace of Eqs. (1)–(4) on  $B_{i,\epsilon}$  in the sense of the problem (21)–(24) with  $\phi_{\theta_i} \equiv 1$  on  $B_{i,\epsilon}$ .

Therefore, we can rewrite (27) in new coordinates and use the parallelepiped property of  $\Omega_{i,+ \epsilon}$  in the space of variables  $(s, \theta)$ :

$$\begin{aligned} N(t) &= \sum_{i=1}^I \int_{\Omega_{i,+ \epsilon}} (1 - u(s, \theta, t)) |J(s, \theta)| ds d\theta + O(e^{-\frac{1}{i^\delta}}) \\ &= \sum_{i=1}^I \int_{\partial\Omega \cap \bar{\Omega}_{i,+ \epsilon}} d\theta \int_{[0, \epsilon]} ds (1 - u(s, \theta, t)) |J(s, \theta)| + O(e^{-\frac{1}{i^\delta}}). \end{aligned}$$

Since this local representation holds for all  $i$  (the form of the problem (29)–(33) is the same for all  $i$ ) and  $\sum_{i=1}^I \int_{\partial\Omega \cap \bar{\Omega}_{i,+ \epsilon}} d\theta = \int_{\partial\Omega} d\theta$ , we can formally write

$$N(t) = \int_{\partial\Omega} d\theta \int_{[0, \epsilon]} ds (1 - u(s, \theta, t)) |J(s, \theta)| + O(e^{-\frac{1}{i^\delta}}), \quad (34)$$

where  $u$  is the solution of (29)–(33) in  $] - \epsilon, \epsilon[ \times \partial\Omega$  in the local sense, as explained previously.

## 5 Approximation of the heat content by solutions of one dimensional problems (for a regular boundary)

We denote by  $\tilde{G}(s_1, \theta_1, s_2, \theta_2, t)$  the Green function of the problem (29)–(33) in  $\partial\Omega \times ] - \epsilon, \epsilon[$ . Let us fix a boundary point  $(0, \theta_0)$ .

We denote by  $G^{\theta_0}$  the Green function corresponding to the following constant coefficient problem, considered as a local trace problem, i.e. in the sense of the problem (21)–(24) with  $\phi_{\theta_i} \equiv 1$  on  $B_{i,\epsilon}$ :

$$\frac{\partial}{\partial t} u_+ - D_+ \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_+ = 0, \quad 0 < s < \epsilon \quad (35)$$

$$\frac{\partial}{\partial t} u_- - D_- \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_- = 0, \quad -\epsilon < s < 0 \quad (36)$$

$$u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \quad (37)$$

$$\begin{aligned} D_- \frac{\partial u_-}{\partial s} |_{s=-0} &= \lambda(\theta_0) (u_- - u_+) |_{s=0}, \\ D_+ \frac{\partial u_+}{\partial s} |_{s=+0} &= D_- \frac{\partial u_-}{\partial s} |_{s=-0}. \end{aligned} \quad (38)$$

Next, let

$$G_{\mathbb{R}^n}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t) = \mathbb{1}_{\{s_1 > 0\}} G_{++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t) + \mathbb{1}_{\{s_1 < 0\}} G_{-+}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t)$$

be the Green function of the constant coefficient problem in the half space, explicitly obtained in B. Then, according to Ref. [30] p.48–49, due to Varadhan’s bound property of Green functions, in  $U_\epsilon(0, \theta_0)$  the difference between the Green function  $\phi_{\theta_0} G^{\theta_0}$  of the problem (35)–(38) and the analogous Green function in  $\mathbb{R}^n$ ,  $G_{\mathbb{R}^n}^{\theta_0}$ , is exponentially small:

$$|(\phi_{\theta_0} G^{\theta_0} - G_{\mathbb{R}^n}^{\theta_0})|_{U_\epsilon(0, \theta_0) \times U_\epsilon(0, \theta_0)} = O\left(e^{-\frac{1}{t^\delta}}\right).$$

Therefore, following the ideas of McKean and Singer[30] (p.49), we approximate  $\tilde{G}$  by the Green function  $G^{\theta_0}$  with the frozen coefficients on  $(0, \theta_0)$ , whose replacement by  $G_{\mathbb{R}^n}^{\theta_0}$  yields only an exponentially small error.

For an abstract operator Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u - Au &= \mathcal{R}u, \\ u|_{t=0} &= u_0 \end{aligned} \quad (39)$$

the solution  $u$  can be found by the Duhamel formula

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-\tau)A} \mathcal{R}u(\tau) d\tau. \quad (40)$$

Therefore, by the Duhamel formula, locally, we have the following infinite expansion

$$\begin{aligned} u_+(s, \theta_0, t) &= \int_{\Omega_\epsilon} d\theta_1 ds_1 G_{++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t) \\ &+ \int_0^t d\tau \int_{\Omega_\epsilon} d\theta_1 ds_1 G_{++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t - \tau) \cdot \\ &\cdot \int_{\Omega_\epsilon} d\theta_2 ds_2 \mathcal{R}G_{++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, \tau) \\ &+ \int_0^t d\tau \int_{\Omega_\epsilon} d\theta_1 ds_1 G_{++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t - \tau) \cdot \\ &\cdot \int_0^\tau d\tau_1 \int_{\Omega_\epsilon} d\theta_2 ds_2 \mathcal{R}G_{++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, \tau - \tau_1) \cdot \\ &\cdot \int_{\Omega_\epsilon} d\theta_3 ds_3 \mathcal{R}G_{++}^{\theta_0}(s_2, \theta_2, s_3, \theta_3, \tau_1) + \dots + O\left(e^{-\frac{1}{t^\delta}}\right) \end{aligned} \quad (41)$$

where the operator  $\mathcal{R}$  is defined by

$$\mathcal{R} = \mathcal{R}_{s_1}(s_1, \theta_1) + \mathcal{R}_{\theta_1}(s_1, \theta_1), \quad (42)$$

$$\mathcal{R}_s(s, \theta) = R(s, \theta) \frac{\partial}{\partial s} = -D_+ \left( \sum_{i=1}^{n-1} k_i(\theta) + s \sum_{i=1}^{n-1} \frac{k_i^2(\theta)}{1 - sk_i(\theta)} \right) \frac{\partial}{\partial s}, \quad (43)$$

$$\begin{aligned} \mathcal{R}_\theta(s, \theta) &= \sum_{i=1}^{n-1} \frac{D_+ sk_i(\theta)}{1 - sk_i(\theta)} \left( 1 + \frac{1}{1 - sk_i(\theta)} \right) \frac{\partial^2}{\partial \theta_i^2} \\ &+ \frac{D_+}{|J(s, \theta)|} \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_i} \left( \frac{|J(s, \theta)|}{(1 - sk_i(\theta))^2} \right) \frac{\partial}{\partial \theta_i}. \end{aligned} \quad (44)$$

We substitute Eq. (41) into Eq. (34) with  $\theta = \theta_0$  and prove the following theorem:

**Theorem 4** *Let*

$$\hat{u} = \begin{cases} \hat{u}_+, & 0 < s < \epsilon \\ \hat{u}_-, & -\epsilon < s < 0 \end{cases}$$

*be the solution of the one-dimensional problem*

$$\frac{\partial}{\partial t} \hat{u} - D_{\pm} \frac{\partial^2}{\partial s^2} \hat{u} = \mathcal{R}_s(s, \theta_0) \hat{u} \quad -\epsilon < s < \epsilon, \quad \theta \equiv \theta_0, \quad (45)$$

$$\hat{u}|_{t=0} = \mathbf{1}_{0 < s < \epsilon}(s),$$

$$D_- \frac{\partial \hat{u}_-}{\partial s} \Big|_{s=-0} = \lambda(\theta_0) (\hat{u}_- - \hat{u}_+) \Big|_{s=0}, \quad (46)$$

$$D_+ \frac{\partial \hat{u}_+}{\partial s} \Big|_{s=+0} = D_- \frac{\partial \hat{u}_-}{\partial s} \Big|_{s=-0}, \quad (47)$$

*obtained from (29)–(33) setting  $\theta \equiv \theta_0$  ( $\mathcal{R}_s(s, \theta_0)$  is given by (43)). Then the heat content  $N(t)$ , defined in (34), satisfies*

$$N(t) - \int_{\partial\Omega} d\theta_0 \int_{[0, \epsilon]} ds (1 - \hat{u}(s, \theta_0, t)) |J(s, \theta_0)| = \begin{cases} O(t^{\frac{5}{2}}), & 0 < \lambda < \infty \\ O(t^2), & \lambda = \infty \end{cases}. \quad (48)$$

*If all principal curvatures of  $\partial\Omega$  are constant, then*

$$N(t) = \int_{\partial\Omega} d\theta_0 \int_{[0, \epsilon]} ds (1 - \hat{u}(s, \theta_0, t)) |J(s, \theta_0)| + O(e^{-\frac{1}{t^{\frac{1}{\lambda}}}}).$$

*Moreover, if  $\hat{u}^{hom}$  is the solution of the homogeneous constant coefficients problem*

$$\partial_t \hat{u} - D_{\pm} \frac{\partial^2}{\partial s^2} \hat{u} = 0, \quad -\epsilon < s < \epsilon, \quad \theta \equiv \theta_0, \quad (49)$$

$$\hat{u}|_{t=0} = \mathbf{1}_{0 < s < \epsilon}(s),$$

$$D_- \frac{\partial \hat{u}_-}{\partial s} \Big|_{s=-0} = \lambda(\theta_0) (\hat{u}_- - \hat{u}_+) \Big|_{s=0}, \quad (50)$$

$$D_+ \frac{\partial \hat{u}_+}{\partial s} \Big|_{s=+0} = D_- \frac{\partial \hat{u}_-}{\partial s} \Big|_{s=-0}, \quad (51)$$

*then*

$$N(t) - \int_{\partial\Omega} d\theta_0 \int_{[0, \epsilon]} ds (1 - \hat{u}^{hom}(s, \theta_0, t)) |J(s, \theta_0)| = \begin{cases} O(t^{\frac{3}{2}}), & 0 < \lambda < \infty \\ O(t), & \lambda = \infty \end{cases} \quad (52)$$

From B we get

$$\begin{aligned} G^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t) &= \mathbf{1}_{\{s_1 > 0, s_2 > 0\}} G_{++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t) \\ &\quad + \mathbf{1}_{\{s_1 < 0, s_2 > 0\}} G_{-+}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t). \end{aligned}$$

Due to Eq. (34), we need to know only  $G_{++}^{\theta_0}$

$$G_{++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, t) = (h_+^{\theta_0}(s_1, s_2, t) - f_+^{\theta_0}(s_1, s_2, t)) K(\theta_1, \theta_2, D_+ t),$$

with notations

$$h_+^{\theta_0}(s_1, s_2, t) = \frac{1}{\sqrt{4\pi D_+ t}} \left( \exp\left(-\frac{(s_1 - s_2)^2}{4D_+ t}\right) + a(\lambda, 0, \theta_0) \exp\left(-\frac{(s_1 + s_2)^2}{4D_+ t}\right) \right), \quad (53)$$

$$f_+^{\theta_0}(s_1, s_2, t) = b(\lambda, 0, \theta_0) \frac{\lambda(\theta_0)}{D_+} \exp\left(\frac{\lambda(\theta_0)\alpha}{\sqrt{D_+}}(s_1 + s_2) + \lambda(\theta_0)^2 \alpha^2 t\right) \cdot \operatorname{Erfc}\left(\frac{s_1 + s_2}{2\sqrt{D_+ t}} + \lambda(\theta_0)\alpha\sqrt{t}\right), \quad (54)$$

where

$$a(\lambda, 0, \theta_0) = \begin{cases} 1, & \lambda(\theta_0) < \infty, \\ \frac{\sqrt{D_+} - \sqrt{D_-}}{\sqrt{D_+} + \sqrt{D_-}}, & \lambda(\theta_0) = \infty, \end{cases}$$

$$b(\lambda, 0, \theta_0) = \begin{cases} 1, & \lambda(\theta_0) < \infty, \\ 0, & \lambda(\theta_0) = \infty, \end{cases}$$

and  $K(\theta_1, \theta_2, D_{\pm}t)$  is the heat kernel in  $\mathbb{R}^{n-1}$ :

$$K(\theta_1, \theta_2, D_{\pm}t) = \frac{1}{(4\pi D_{\pm}t)^{\frac{n-1}{2}}} \exp\left(-\frac{|\theta_1 - \theta_2|^2}{4D_{\pm}t}\right). \quad (55)$$

Since

$$N(t) = \int_{\Omega_\epsilon} (1 - u_\epsilon(s, \theta, t)) |J(s, \theta)| ds d\theta + O(e^{-\frac{1}{t^\delta}})$$

$$= \operatorname{Vol}(\Omega_\epsilon) - \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} G(s, \theta, s_1, \theta_1, t) |J(s, \theta)| ds d\theta ds_1 d\theta_1 + O(e^{-\frac{1}{t^\delta}}), \quad (56)$$

in what follows we use  $P(t)$  for the notation of the principal part of  $N(t)$ :

$$P(t) = \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} G(s, \theta, s_1, \theta_1, t) |J(s, \theta)| ds d\theta ds_1 d\theta_1. \quad (57)$$

To prove Theorem 4 we need the following Lemma:

**Lemma 2** *The principal part  $P(t)$  of the heat content for the solution of the system (29)–(33), defined in Eq. (57), is given by*

$$P(t) = \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} G_{++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t) |J(s, \theta_0)| ds d\theta_0 d\theta_1 ds_1 + G_{++}^{\theta_0} \# (\mathcal{R}_s^{\theta_0} + \mathcal{R}_\theta^{\theta_0}) G_{++}^{\theta_0} + G_{++}^{\theta_0} \# (\mathcal{R}_s^{\theta_0} + \mathcal{R}_\theta^{\theta_0}) G_{++}^{\theta_0} \# (\mathcal{R}_s^{\theta_0} + \mathcal{R}_\theta^{\theta_0}) u_+ + O(e^{-\frac{1}{t^\delta}}), \quad (58)$$

with notation

$$G_{++}^{\theta_0} \# (\mathcal{R}_s^{\theta_0} + \mathcal{R}_\theta^{\theta_0}) G_{++}^{\theta_0} = \int_0^t d\tau \int_{\Omega_\epsilon} ds d\theta_0 |J(s, \theta_0)| \int_{\Omega_\epsilon} G_{++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t - \tau) \cdot \int_{\Omega_\epsilon} (\mathcal{R}_{s_1}^{\theta_0}(s_1, \theta_1) + \mathcal{R}_{\theta_1}^{\theta_0}(s_1, \theta_1)) G_{++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, \tau) d\theta_2 ds_2 d\theta_1 ds_1. \quad (59)$$



Moreover, the following identities hold

$$\begin{aligned} M(t) &= \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} G_{++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t) |J(s, \theta_0)| ds d\theta_0 d\theta_1 ds_1 \\ &= \int_{\partial\Omega} d\theta_0 \int_{[0, \epsilon]^2} ds_1 ds (h_+^{\theta_0}(s, s_1, t) - f_+^{\theta_0}(s, s_1, t)) |J(s, \theta_0)|, \end{aligned} \quad (60)$$

$$G_{+++}^{\theta_0} \# \mathcal{R}_\theta^{\theta_0} G_{+++}^{\theta_0} = G_{+++}^{\theta_0} \# \mathcal{R}_\theta^{\theta_0} G_{+++}^{\theta_0} \# \mathcal{R}_\theta^{\theta_0} G_{+++}^{\theta_0} = \dots = 0, \quad (61)$$

$$\begin{aligned} &\int_{\partial\Omega} d\theta_1 K(\theta_0, \theta_1, D_+ t) \\ &= \int_{\partial\Omega} d\theta_1 \int_{\partial\Omega} d\theta_2 K(\theta_0, \theta_1, D_+(t - \tau)) K(\theta_1, \theta_2, D_+ \tau) \\ &= \dots = \mathbb{1}_{\partial\Omega}(\theta_0). \end{aligned} \quad (62)$$

**Proof.** Formula (58) is the direct corollary of the Duhamel formula (see (40) and (41)). Let us start to prove (60).

Indeed, we find that

$$\begin{aligned} M(t) &= \int_{\Omega_\epsilon} \int_{\Omega_\epsilon} (h_+^{\theta_0}(s_1, s_2, t) - f_+^{\theta_0}(s, s_1, t)) K(\theta_1, \theta_2, D_+(\theta_0)t) |J(s, \theta_0)| ds_1 ds d\theta_1 d\theta_0 \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} d\theta_0 d\theta_1 \frac{1}{(4\pi D_+ t)^{\frac{n-1}{2}}} \exp\left(-\frac{|\theta_0 - \theta_1|^2}{4D_+ t}\right) \mathbb{1}_{\partial\Omega}(\theta_0) \mathbb{1}_{\partial\Omega}(\theta_1) \Phi(\theta_0, t), \end{aligned}$$

where

$$\Phi(\theta_0, t) = \int_{[0, \epsilon]^2} ds ds_1 (h_+^{\theta_0}(s, s_1, t) - f_+^{\theta_0}(s, s_1, t)) |J(s, \theta_0)|. \quad (63)$$

With the change of variables  $\theta^1 \mapsto v = \frac{\theta_0 - \theta_1}{\sqrt{4D_+ t}}$ ,  $M(t)$  becomes

$$M(t) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{e^{-|v|^2}}{\pi^{\frac{n-1}{2}}} \mathbb{1}_{\partial\Omega}(\theta_0) \mathbb{1}_{\partial\Omega + \sqrt{4D_+ t}v}(\theta_0) \Phi(\theta_0, t) dv d\theta_0.$$

By our construction,

$$\theta_0 \in \partial\Omega \quad \text{and} \quad \theta_1 = \theta_0 - \sqrt{4D_+ t}v \in \partial\Omega,$$

that implies

$$\mathbb{1}_{\partial\Omega}(\theta_0) - \mathbb{1}_{\partial\Omega}(\theta_0) \mathbb{1}_{\partial\Omega + \sqrt{4D_+ t}v}(\theta_0) \equiv 0.$$

It can be interpreted in the following way: if we take a point on the boundary and move it along the boundary, we obtain another point which is still a boundary point.

Consequently, we find (60)

$$M(t) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{e^{-v^2}}{\pi^{\frac{n-1}{2}}} \mathbb{1}_{\partial\Omega}(\theta_0) \Phi(\theta_0, t) dv d\theta_0 = \int_{\partial\Omega} \Phi(\theta_0, t) d\theta_0,$$

which also implies the first part of (62):

$$\int_{\partial\Omega} d\theta_1 K(\theta_0, \theta_1, D_+t) = \mathbb{1}_{\partial\Omega}(\theta_0).$$

Let us now prove that in the computation of  $P(t)$  all terms containing the derivatives over the transversal variable  $\theta$  vanish.

For all terms in (29) containing a derivative over  $\theta_1$ , we calculate (see (44))

$$\begin{aligned} & \mathcal{R}_{\theta_1} K(\theta_1, \theta_2, D_+t) \\ &= \sum_{i=1}^{n-1} \frac{D_+ s_1 k_i(\theta_1)}{1 - s_1 k_i(\theta_1)} \left( 1 + \frac{1}{1 - s_1 k_i(\theta_1)} \right) \\ & \cdot \frac{1}{2D_+t} \left( \frac{(\theta_1^i - \theta_2^i)^2}{2D_+t} - 1 \right) K(\theta_1, \theta_2, D_+t) \\ & - \frac{1}{|J(s_1, \theta_1)|} \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_1^i} \left( \frac{D_+ |J(s_1, \theta_1)|}{(1 - s_1 k_i(\theta_1))^2} \right) \frac{(\theta_1^i - \theta_2^i)}{2D_+t} K(\theta_1, \theta_2, D_+t). \end{aligned}$$

Let us prove Eq. (61), noting that

$$\begin{aligned} G_{+++}^{\theta_0} \# \mathcal{R}_{\theta_1} G_{+++}^{\theta_0} &= \int_0^t d\tau \int_{\Omega_\epsilon} ds d\theta_0 |J(s, \theta_0)| \int_{\Omega_\epsilon} G_{+++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t - \tau) \cdot \\ & \cdot \int_{\Omega_\epsilon} \mathcal{R}_{\theta_1} G_{+++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, \tau) d\theta_2 ds_2 d\theta_1 ds_1. \end{aligned}$$

We can schematically rewrite  $G_{+++}^{\theta_0} \# \mathcal{R}_{\theta_1} G_{+++}^{\theta_0}$  in the following form:

$$\begin{aligned} & G_{+++}^{\theta_0} \# \mathcal{R}_{\theta_1} G_{+++}^{\theta_0} = \\ &= \int_0^t d\tau \int_{\mathbb{R}^{n-1}} d\theta_0 \int_{\mathbb{R}^{n-1}} d\theta_1 \int_{\mathbb{R}^{n-1}} d\theta_2 K(\theta_0, \theta_1, D_+(t - \tau)) \mathcal{R}_{\theta_1} K(\theta_1, \theta_2, D_+\tau) \\ & \cdot \mathbb{1}_{\partial\Omega}(\theta_0) \mathbb{1}_{\partial\Omega}(\theta_1) \mathbb{1}_{\partial\Omega}(\theta_2) \int_{[0, \epsilon]^3} ds ds_1 ds_2 \phi(s, s_1, s_2, t, \tau, \theta_0). \end{aligned}$$

With the change of variables involving  $\theta_0$ :

$$\tilde{\theta}_1 = \frac{\theta_0 - \theta_0^1}{2\sqrt{D_+(t - \tau)}}, \quad \theta_1 = \theta_0 - 2\sqrt{D_+(t - \tau)}\tilde{\theta}_1, \quad (64)$$

$$\tilde{\theta}_2 = \frac{\theta_0^1 - \theta_0^2}{2\sqrt{D_+\tau}}, \quad \theta_2 = \theta_1 - 2\sqrt{D_+\tau}\tilde{\theta}_2, \quad \text{and so}$$

$$\theta_2 = \theta_0 - 2\sqrt{D_+(t - \tau)}\tilde{\theta}_1 - 2\sqrt{D_+\tau}\tilde{\theta}_2, \quad (65)$$

and since for all  $\theta_0 \in \partial\Omega$

$$\begin{aligned} & \mathbb{1}_{\partial\Omega}(\theta_0) - \mathbb{1}_{\partial\Omega}(\theta_0) \mathbb{1}_{\partial\Omega + 2\sqrt{D_+(t - \tau)}\tilde{\theta}_1}(\theta_0) \cdot \\ & \cdot \mathbb{1}_{\partial\Omega + 2\sqrt{D_+(t - \tau)}\tilde{\theta}_1 + 2\sqrt{D_+\tau}\tilde{\theta}_2}(\theta_0) = 0, \end{aligned}$$

we obtain the separation of variables on  $\tilde{\theta}_2$  from  $(\theta_0, s_1, \tilde{\theta}_1)$ :

$$G_{+++}^{\theta_0} \# \mathcal{R}_{\theta_1} G_{+++}^{\theta_0} = \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]^3} ds ds_1 ds_2 \phi(s, s_1, s_2, t, \tau, \theta_0) \\ \cdot \prod_{i=1}^{n-1} \left[ \int_{\mathbb{R}} d\tilde{\theta}_1^i \frac{e^{-(\tilde{\theta}_1^i)^2}}{\sqrt{\pi}} \int_{\mathbb{R}} d\tilde{\theta}_2^i \frac{e^{-(\tilde{\theta}_2^i)^2}}{\sqrt{\pi}} \left( C_1^i (2(\tilde{\theta}_2^i)^2 - 1) - C_2^i \tilde{\theta}_2^i \right) \right],$$

where  $C_1^i$  and  $C_2^i$  are the functions of  $s_1, \theta_0, \tilde{\theta}_1$ , but not of  $\tilde{\theta}_2$ , and consequently

$$G_{+++}^{\theta_0} \# \mathcal{R}_{\theta} G_{+++}^{\theta_0} = 0.$$

By the same reason we have Eq. (61). Changing variables  $\theta_i$  to  $\tilde{\theta}_i$  from (64)–(65), we also obtain the last part of (62).  $\square$  Let us now prove Theorem 4.

**Proof.** To find Eq. (48), we study Eq. (58) using proved relations (60)–(62). For instance, we have

$$G_{+++}^{\theta_0} \# (\mathcal{R}_s^{\theta_0} + \mathcal{R}_{\theta}^{\theta_0}) G_{+++}^{\theta_0} = G_{+++}^{\theta_0} \# \mathcal{R}_s^{\theta_0} G_{+++}^{\theta_0} \\ = \int_0^t d\tau \int_{\Omega_\epsilon} ds d\theta_0 |J(s, \theta_0)| \int_{\Omega_\epsilon} G_{+++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t - \tau) \cdot \\ \cdot \int_{\Omega_\epsilon} \mathcal{R}_{s_1}^{\theta_0}(s_1, \theta_1) G_{+++}^{\theta_0}(s_1, \theta_1, s_2, \theta_2, \tau) d\theta_2 ds_2 d\theta_1 ds_1.$$

As  $G_{+++}^{\theta_0}(s, \theta_0, s_1, \theta_1, t) = (h_+^{\theta_0}(s, s_1, t) - f_+^{\theta_0}(s, s_1, t))K(\theta_0, \theta_1, D_+(\theta_0)t)$ , we have

$$G_{+++}^{\theta_0} \# \mathcal{R}_s^{\theta_0} G_{+++}^{\theta_0} = (h_+^{\theta_0} - f_+^{\theta_0})K \# \mathcal{R}_s^{\theta_0} (h_+^{\theta_0} - f_+^{\theta_0})K.$$

Now we perform the change of variables (64)–(65). Since locally  $k_i(\theta) \in C^1$ , then for all  $\theta_0 \in \partial\Omega$ , for  $t \rightarrow +0$  we can develop

$$k_i(\theta_0 - 2\sqrt{D_+(t-\tau)}\tilde{\theta}_1) = k_i(\theta_0) - \nabla k_i(\theta_0)2\sqrt{D_+(t-\tau)}\tilde{\theta}_1 + O(t-\tau).$$

Consequently, by definition of  $\mathcal{R}_{s_1}(s_1, \theta_1)$  in (43), which is a composition of the operator of the first derivative by  $s_1$  and of a multiplication by a function of the class  $C^1$  on  $\theta_1$  (locally, in the sense of local variables), we also have for  $t \rightarrow +0$

$$\mathcal{R}_{s_1}(s_1, \theta_1) = \mathcal{R}_{s_1}(s_1, \theta)[1 + O(t-\tau)] - \nabla_{\theta} \mathcal{R}_{s_1}(s_1, \theta)2\sqrt{D_+(t-\tau)}\tilde{\theta}_1.$$

As  $2\sqrt{D_+(t-\tau)} \int_{\mathbb{R}} d\tilde{\theta}_i^1 e^{-(\tilde{\theta}_i^1)^2} \tilde{\theta}_i^1 = 0$ , we obtain

$$G_{+++}^{\theta_0} \# \mathcal{R}_s^{\theta_0} G_{+++}^{\theta_0} = \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \int_{[0,\epsilon]} ds_1 \int_{[0,\epsilon]} ds_2 \\ \cdot (h_+^{\theta_0} - f_+^{\theta_0})(s, s_1, t - \tau) \mathcal{R}_{s_1}(s_1, \theta_0) [1 + O(t - \tau)] (h_+^{\theta_0} - f_+^{\theta_0})(s_1, s_2, \tau) \\ \cdot \prod_{i=1}^{n-1} \left[ \int_{\mathbb{R}} d\tilde{\theta}_i^1 \frac{e^{-(\tilde{\theta}_i^1)^2}}{\sqrt{\pi}} \int_{\mathbb{R}} d\tilde{\theta}_i^2 \frac{e^{-(\tilde{\theta}_i^2)^2}}{\sqrt{\pi}} \right] = \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \int_{[0,\epsilon]} ds_1 \\ \cdot \int_{[0,\epsilon]} ds_2 (h_+^{\theta_0} - f_+^{\theta_0})(s, s_1, t - \tau) \mathcal{R}_{s_1}(s_1, \theta_0) [1 + O(t - \tau)] (h_+^{\theta_0} - f_+^{\theta_0})(s_1, s_2, \tau),$$

from which it follows

$$\begin{aligned}
P(t) &= \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds \int_{[0,\epsilon]} ds_1 (h_+^{\theta_0}(s, s_1, t) - f_+^{\theta_0}(s, s_1, t)) |J(s, \theta_0)| \\
&+ [1 + O(t)] \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \int_{[0,\epsilon]} ds_1 (h_+^{\theta_0} - f_+^{\theta_0})(s, s_1, t - \tau) \\
&\cdot \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) (h_+^{\theta_0} - f_+^{\theta_0})(s_1, s_2, \tau) \\
&+ [1 + O(t)]^2 \int_{\partial\Omega} d\theta_0 (h_+^{\theta_0} - f_+^{\theta_0}) \# \mathcal{R}_s(s, \theta_0) (h_+^{\theta_0} - f_+^{\theta_0}) \# \mathcal{R}_s(s, \theta_0) u_\epsilon^+. \tag{66}
\end{aligned}$$

We notice that the solution  $\hat{u}(s, \theta_0, t)$  of the one-dimensional system (45)–(47) is given by

$$\begin{aligned}
\hat{u}(s, \theta_0, t) &= \int_{[0,\epsilon]} ds_1 (h_+^{\theta_0}(s, s_1, t) - f_+^{\theta_0}(s, s_1, t)) \\
&+ \int_0^t d\tau \int_{[0,\epsilon]} ds_1 (h_+^{\theta_0} - f_+^{\theta_0})(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) (h_+^{\theta_0} - f_+^{\theta_0})(s_1, s_2, \tau) \\
&+ (h_+^{\theta_0} - f_+^{\theta_0}) \# \mathcal{R}_s(s, \theta_0) (h_+^{\theta_0} - f_+^{\theta_0}) \# \mathcal{R}_s(s, \theta_0) \hat{u}.
\end{aligned}$$

To obtain (48) of Theorem 4 from formula (66), we estimate

$$\begin{aligned}
NN^2(t) &= O(t) \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \\
&\cdot \int_{[0,\epsilon]} ds_1 (h_+^{\theta_0} - f_+^{\theta_0})(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) (h_+^{\theta_0} - f_+^{\theta_0})(s_1, s_2, \tau). \tag{67}
\end{aligned}$$

In fact, from (52), proven in what follows, it holds (see (68) for the definition of  $NN^1(t)$ )

$$NN^2(t) = O(t)NN^1(t) = O(t) \begin{cases} O(t^{\frac{3}{2}}), & 0 < \lambda < \infty \\ O(t), & \lambda = \infty \end{cases} = \begin{cases} O(t^{\frac{5}{2}}), & 0 < \lambda < \infty \\ O(t^2), & \lambda = \infty \end{cases}.$$

To conclude, we note that if all principal curvatures  $k_j(\theta)$  on  $\Omega_\epsilon$  are constant, then for all  $\theta \in \partial\Omega$

$$\mathcal{R}_s(s, \theta) \equiv \mathcal{R}_s(s, \theta_0),$$

and thus

$$N(t) = \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds (1 - \hat{u}(s, \theta_0, t)) |J(s, \theta_0)| + O(e^{-\frac{1}{t^\alpha}}).$$

To show (52), we need to estimate

$$NN^j(t) = \sum_{l=1}^j \Gamma_{++}^{\theta_0} \# \mathcal{R}_s \Gamma_{++}^{\theta_0} \# \dots \# \mathcal{R}_s \Gamma_{++}^{\theta_0} \quad (l\text{-fold}), \tag{68}$$

where

$$\Gamma_{++}^{\theta_0} = (h_+^{\theta_0} - f_+^{\theta_0}).$$

More precisely we want to prove that for all  $j \geq 1$

$$|NN^j(t)| \leq C \begin{cases} t^{\frac{1+j}{2}} \mu(\partial\Omega, \sqrt{4D_+t}), & 0 < \lambda < \infty \\ t^{\frac{j}{2}} \mu(\partial\Omega, \sqrt{4D_+t}), & \lambda = \infty \end{cases}. \quad (69)$$

Due to Lemma 2, we start with (see (66))

$$\begin{aligned} NN^1(t) &= \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \\ &\cdot \int_{[0,\epsilon]} ds_1 (h_+^{\theta_0} - f_+^{\theta_0})(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) (h_+^{\theta_0} - f_+^{\theta_0})(s_1, s_2, \tau). \end{aligned}$$

Therefore, we have to estimate four terms:

$$NN^1(t) = \sum_{j=1}^4 MM_j(t),$$

where

$$\begin{aligned} MM_1(t) &= \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \\ &\cdot \int_{[0,\epsilon]} ds_1 h_+^{\theta_0}(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) h_+^{\theta_0}(s_1, s_2, \tau), \\ MM_2(t) &= - \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \\ &\cdot \int_{[0,\epsilon]} ds_1 f_+^{\theta_0}(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) h_+^{\theta_0}(s_1, s_2, \tau), \\ MM_3(t) &= - \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \\ &\cdot \int_{[0,\epsilon]} ds_1 h_+^{\theta_0}(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) f_+^{\theta_0}(s_1, s_2, \tau), \\ MM_4(t) &= \int_0^t d\tau \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds |J(s, \theta_0)| \\ &\cdot \int_{[0,\epsilon]} ds_1 f_+^{\theta_0}(s, s_1, t - \tau) \int_{[0,\epsilon]} ds_2 \mathcal{R}_{s_1}(s_1, \theta_0) f_+^{\theta_0}(s_1, s_2, \tau). \end{aligned}$$

We aim to approximate  $\mathcal{R}_{s_1}(s_1, \theta_0) = R(s_1, \theta_0) \partial_{s_1}$  from Eq. (43) near the point  $(s, \theta_0)$ . For  $t \rightarrow +0$  and  $0 < s_1 < \epsilon = O(\sqrt{t})$ , we find that

$$\frac{1}{1 - s_1 k_i(\theta_0)} = 1 + s_1 k_i(\theta_0) + O(s_1^2),$$

which gives

$$R(s_1, \theta_0) = -D_+ \left( \sum_{i=1}^{n-1} k_i(\theta_0) + s_1 \sum_{i=1}^{n-1} k_i^2(\theta_0) + O(s_1^2) \right).$$

Introducing the notations

$$C_{\pm} = s_1 \pm s_2, \quad I_{s_1 \pm s_2}(\tau) = \exp\left(-\frac{(s_1 \pm s_2)^2}{4D_+ \tau}\right), \quad c = \frac{1}{8\pi D_+^2(0, \theta_0) \sqrt{(t-\tau)} \tau^{\frac{3}{2}}},$$

we find

$$\begin{aligned} h_+^{\theta_0}(s, s_1, t-\tau) \mathcal{R}_{s_1}(s_1, \theta_0) h_+^{\theta_0}(s_1, s_2, \tau) &= -cR(s_1, \theta_0) (C_- [I_{s-s_1}(t-\tau) I_{s_1-s_2}(\tau) \\ &+ a(\lambda, 0, \theta_0) I_{s+s_1}(t-\tau) I_{s_1-s_2}(\tau)] \\ &+ a(\lambda, 0, \theta_0) C_+ [I_{s-s_1}(t-\tau) I_{s_1+s_2}(\tau) + a(\lambda, 0, \theta_0) I_{s+s_1}(t-\tau) I_{s_1+s_2}(\tau)]). \end{aligned}$$

We now change  $s_1$  to  $z_1$  and  $s_2$  to  $z_2$  by the following change of variables:

- for  $P_{s_1 \mp s_2}(\tau)$ :  $z_2 = \frac{s_1 \mp s_2}{2\sqrt{D_+ \tau}}$  and  $s_2 = \pm s_1 \mp 2\sqrt{D_+ \tau} z_2$ ,
- for  $P_{s \mp s_1}(t-\tau)$ :  $z_1 = \frac{s \mp s_1}{2\sqrt{D_+(t-\tau)}}$  and  $s_1 = \pm s \mp 2\sqrt{D_+(t-\tau)} z_1$ .

Let us notice that  $t$  is a constant parameter and, as  $\tau$  takes its values between 0 and  $t$ , hence,  $z_1$  and  $z_2$  are in  $\mathbb{R}^+$  or  $\mathbb{R}$ . But at the same time  $2\sqrt{D_+(t-\tau)} z_1 = s \pm s_1$  and  $2\sqrt{D_+ \tau} z_2 = s_1 \pm s_2$  are bounded to the interval  $[-\epsilon, 2\epsilon]$  and hence are of the order of  $O(\sqrt{t})$ . In what follows, we suppose that  $\tau$  and  $t-\tau$  have the same order of smallness as  $t$ :

$$O(t) = O(\tau) = O(t-\tau).$$

Therefore, for  $0 < s_1 = \pm s \mp 2\sqrt{D_+(t-\tau)} z_1 < \epsilon$  we have

$$\mathcal{R}_{s_1}(s_1, \theta_0) = [\phi(\theta_0) \mp \psi(s, z_1, \theta_0)] \frac{\partial}{\partial s_1},$$

where

$$\begin{aligned} \phi(\theta_0) &= -D_+ \sum_{i=1}^{n-1} k_i(\theta_0), \\ \psi(s, z_1, \theta_0) &= (s - 2\sqrt{D_+(t-\tau)} z_1) D_+ \sum_{i=1}^{n-1} k_i^2(\theta_0) + O(t). \end{aligned}$$

If we develop  $R$  in the neighborhood of  $(0, \theta_0)$ , we find

$$R(s_1, \theta_0) = R(0, \theta_0) + O(\sqrt{t}) = \phi(\theta_0) + O(\sqrt{t}). \quad (70)$$

For  $\lambda = \infty$  on  $\partial\Omega$ , we simply have

$$MM_2(t) = MM_3(t) = MM_4(t) = 0,$$

and

$$\begin{aligned} |NN^1(t)| &= |MM_1(t)| = |h_+^{\theta_0} \# \mathcal{R}_s h_+^{\theta_0}| \\ &\leq C \left| \int_0^t d\tau \frac{1}{\sqrt{\tau}} \int_{\partial\Omega} d\theta_0 \int_0^\epsilon ds |J(s, \theta_0)| \right| \leq C \sqrt{t} \mu(\partial\Omega, \sqrt{4D_+ t}). \end{aligned}$$

By iteration of the proof, we show that  $|NN^j(t)| \leq Ct^{\frac{j}{2}}\mu(\partial\Omega, \sqrt{4D_+t})$  for  $j \geq 1$ .

Now, for  $\lambda < \infty$ ,

$$\begin{aligned} MM_1(t) &= h_+^{\theta_0} \# \mathcal{R}_s h_+^{\theta_0} = - \int_0^t d\tau \frac{1}{\sqrt{\tau}} \int_{\partial\Omega} d\theta_0 \frac{\pi}{\sqrt{D_+}} \int_{\mathbb{R}} ds |J(s, \theta_0)| \int_{\mathbb{R}} dz_1 e^{-z_1^2} \\ &\cdot \left[ \int_{\mathbb{R}} dz_2 \phi(\theta_0) z_2 e^{-z_2^2} \right] \sum_{i=1}^4 \chi_i(s, z_1, z_2) - \int_0^t d\tau \frac{1}{\sqrt{\tau}} \int_{\partial\Omega} d\theta_0 \frac{\pi}{\sqrt{D_+}} \int_{\mathbb{R}} ds |J(s, \theta_0)| \\ &\cdot \int_{\mathbb{R}} dz_1 e^{-z_1^2} \cdot \left[ \int_{\mathbb{R}} dz_2 \psi(s, z_1, \theta_0) z_2 e^{-z_2^2} \right] \cdot \left( \sum_{i=1}^2 \chi_i(s, z_1, z_2) - \sum_{i=3}^4 \chi_i(s, z_1, z_2) \right). \end{aligned}$$

Here for  $v_+(z_1) = 2\sqrt{D_+(t-\tau)}z_1\mathbb{1}_{\mathbb{R}^+}(z_1)$ ,  $v(z_1) = 2\sqrt{D_+(t-\tau)}z_1$ ,  $w^+(z_2) = 2\sqrt{D_+\tau}z_2\mathbb{1}_{\mathbb{R}^+}(z_2)$  and  $w(z_2) = 2\sqrt{D_+\tau}z_2$

$$\begin{aligned} \chi_1(s, z_1, z_2) &= \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(s-v(z_1)) \mathbb{1}_{[0,\epsilon]}(s-v(z_1)-w(z_2)), \\ \chi_2(s, z_1, z_2) &= \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(s-v(z_1)) \mathbb{1}_{[0,\epsilon]}(-s+v(z_1)+w^+(z_2)), \\ \chi_3(s, z_1, z_2) &= \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s+v_+(z_1)) \mathbb{1}_{[0,\epsilon]}(-s+v_+(z_1)-w(z_2)), \\ \chi_4(s, z_1, z_2) &= \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s+v_+(z_1)) \mathbb{1}_{[0,\epsilon]}(s-v_+(z_1)+w^+(z_2)). \end{aligned}$$

Considering two formulas:

$$\eta = \mathbb{1}_{[0,\epsilon]}(s) - \mathbb{1}_{[0,\epsilon]}(s-v) \mathbb{1}_{[0,\epsilon]}(s), \quad \text{and } \zeta = \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s+v_+),$$

we find that

$$\eta \neq 0 \iff \begin{cases} 0 < v < \epsilon & 0 < s < v \\ -\epsilon < v < 0 & \epsilon + v < s < \epsilon \end{cases} \quad (71)$$

$$\zeta \neq 0 \iff \begin{cases} 0 < v < \epsilon & 0 < s < v \\ \epsilon < v < 2\epsilon & v - \epsilon < s < \epsilon \end{cases} \quad (72)$$

It means that for  $0 < v = v_+ < \epsilon$ , it holds

$$\eta(s) = \zeta(s) = \mathbb{1}_{[0,v_+]}(s)$$

and for  $-\epsilon < v < 0$  and  $\epsilon < v_+ = v + 2\epsilon < 2\epsilon$  it holds

$$\eta(s) = \zeta(s) = \mathbb{1}_{[v_+-\epsilon,\epsilon]}(s) = \mathbb{1}_{[\epsilon+v,\epsilon]}(s).$$

Consequently, we found the formula

$$\mathbb{1}_{[0,\epsilon]}(s) - \mathbb{1}_{[0,\epsilon]}(s-v) \mathbb{1}_{[0,\epsilon]}(s) = \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s+v_+), \quad (73)$$

from which it follows

$$\chi_1 = \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(s-v) - \chi_2, \quad \chi_4 = \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s+v_+) - \chi_3.$$

Therefore, we have

$$\sum_{i=1}^4 \chi_i(s, z_1, z_2) = \mathbb{1}_{[0,\epsilon]}(s) \quad \text{and } \chi_1 + \chi_2 - \chi_3 - \chi_4 = \mathbb{1}_{[0,\epsilon]}(s) - 2 \cdot \mathbb{1}_{[0,v_+(z_1)]}(s),$$

which are independent of  $z_2$ . Since

$$\int_{\mathbb{R}} z_2 e^{-z_2^2} dz_2 = 0 \quad \text{and} \quad \int_{\mathbb{R}} (2z_2^2 - 1) e^{-z_2^2} dz_2 = 0,$$

we obtain exactly

$$|MM_1(t)| = |h\sharp\mathcal{R}_s h| = 0.$$

For  $MM_2$  we find in completely analogous way

$$\begin{aligned} MM_2(t) &= \frac{2}{\sqrt{\pi}} \int_0^t d\tau \frac{\sqrt{t-\tau}}{\sqrt{\tau}} \int_{\partial\Omega} d\theta_0 \frac{\lambda(\theta_0)}{D_+} \int_{\mathbb{R}} ds |J(s, \theta_0)| \\ &\cdot \int_{\mathbb{R}^+} dz_1 e^{2\lambda(\theta_0)\alpha z_1 \sqrt{t-\tau} + \lambda(\theta_0)^2 \alpha^2 (t-\tau)} \operatorname{Erfc}(z_1 + \lambda(\theta_0)\alpha\sqrt{t-\tau}) \\ &\cdot \int_{\mathbb{R}} dz_2 \left[ \phi(\theta_0) z_2 e^{-z_2^2} \right] (\chi_3(s, z_1, z_2) + \chi_4(s, z_1, z_2)) \\ &+ \frac{2}{\sqrt{\pi}} \int_0^t d\tau \frac{\sqrt{t-\tau}}{\sqrt{\tau}} \int_{\partial\Omega} d\theta_0 \frac{\lambda(\theta_0)}{D_+} \int_{\mathbb{R}} ds |J(s, \theta_0)| \int_{\mathbb{R}^+} dz_1 \\ &\cdot e^{2\lambda(\theta_0)\alpha z_1 \sqrt{t-\tau} + \lambda(\theta_0)^2 \alpha^2 (t-\tau)} \operatorname{Erfc}(z_1 + \lambda(\theta_0)\alpha\sqrt{t-\tau}) \\ &\cdot \int_{\mathbb{R}} dz_2 \left[ \psi(s, z_1, \theta_0) z_2 e^{-z_2^2} \right] (\chi_3(s, z_1, z_2) - \chi_4(s, z_1, z_2)). \end{aligned}$$

Since

$$\begin{aligned} \chi_3(s, z_1, z_2) + \chi_4(s, z_1, z_2) &= \mathbf{1}_{[0, \epsilon]}(s) \mathbf{1}_{[0, \epsilon]}(-s + v_+(z_1)) = \mathbf{1}_{[0, v_+(z_1)]}(s), \\ \chi_3(s, z_1, z_2) - \chi_4(s, z_1, z_2) &= 2\chi_3(s, z_1, z_2) - \mathbf{1}_{[0, \epsilon]}(s) \mathbf{1}_{[0, \epsilon]}(-s + v_+(z_1)), \end{aligned}$$

the parts of  $MM_2$ , which contain the integration over  $s$  on  $[0, 2\sqrt{D_+(t-\tau)}z_1]$ , are equal to zero. In addition, for  $\ell = 0, 1$

$$\left| \int_{\mathbb{R}^+} dz_1 z_1^\ell e^{2\lambda(\theta_0)\alpha z_1 \sqrt{t-\tau} + \lambda(\theta_0)^2 \alpha^2 (t-\tau)} \operatorname{Erfc}(z_1 + \lambda(\theta_0)\alpha\sqrt{t-\tau}) \right| \leq C.$$

As  $\psi$  is of the order  $O(\sqrt{t})$  and linear on  $z_1$ , and  $\epsilon = O(\sqrt{t})$ , we directly obtain

$$\begin{aligned} |MM_2(t)| &= \left| \frac{2}{\sqrt{\pi}} \int_0^t d\tau \frac{\sqrt{t-\tau}}{\sqrt{\tau}} \int_{\partial\Omega} d\theta_0 \frac{\lambda(\theta_0)}{D_+} \int_{\mathbb{R}} ds |J(s, \theta_0)| \int_{\mathbb{R}^+} dz_1 \right. \\ &\cdot e^{2\lambda(\theta_0)\alpha z_1 \sqrt{t-\tau} + \lambda(\theta_0)^2 \alpha^2 (t-\tau)} \operatorname{Erfc}(z_1 + \lambda(\theta_0)\alpha\sqrt{t-\tau}) \\ &\cdot \left. \int_{\mathbb{R}} dz_2 \psi(s, z_1, \theta_0) z_2 e^{-z_2^2} 2\chi_3(s, z_1, z_2) \right| \\ &\leq C \int_0^t d\tau \sqrt{\tau} \int_{\partial\Omega} d\theta_0 \int_0^\epsilon ds |J(s, \theta_0)| \leq Ct^{\frac{3}{2}} \mu(\partial\Omega, \sqrt{4D_+t}). \end{aligned}$$

Since  $\mu(\partial\Omega, \sqrt{4D_+t}) = C\sqrt{t}$  for a regular boundary, then  $|MM_2(t)| \leq Ct^2$ .



To estimate  $MM_3$  we find

$$\partial_{s_1} f_+^{\theta_0}(s_1, s_2, \tau) = \frac{\lambda(\theta_0)\alpha}{\sqrt{D_+}} f_+^{\theta_0}(s_1, s_2, \tau) - \frac{\lambda(\theta_0)}{D_+} \frac{1}{\sqrt{\pi D_+ \tau}} \exp\left(-\frac{(s_1 + s_2)^2}{4D_+ \tau}\right).$$

In our notations, using (70), we have

$$\begin{aligned} h_+^{\theta_0}(s, s_1, t - \tau) \mathcal{R}_{s_1}(s_1, \theta_0) f_+^{\theta_0}(s_1, s_2, \tau) &= \frac{P_{s-s_1} + P_{s+s_1}}{\sqrt{4\pi D_+(t-\tau)}} \cdot \\ &\cdot (\phi(\theta_0) + O(\sqrt{t})) \left\{ \frac{\lambda(\theta_0)\alpha}{\sqrt{D_+}} f_+^{\theta_0}(s_1, s_2, \tau) - \frac{\lambda(\theta_0)}{D_+} \frac{1}{\sqrt{\pi D_+ \tau}} P_{s_1+s_2} \right\}. \end{aligned}$$

Changing variables  $s_1$  to  $z_1$  and  $s_2$  to  $z_2$ , we obtain  $\chi_2 \pm \chi_4$  for the area of  $s$ , which gives intervals (linearly) depending on the values of  $z_1$  and  $z_2$ . Thus, we majorate  $s$  by  $\epsilon$  and estimate  $MM_3$ :

$$\begin{aligned} |MM_3(t)| &\leq C \left| \int_0^t d\tau \sqrt{\tau} \int_{\partial\Omega} d\theta_0 \int_0^\epsilon ds |J(s, \theta_0)| \right. \\ &\cdot \left. \int_{\mathbb{R}} dz_1 e^{-z_1^2} \int_{\mathbb{R}^+} dz_2 (\phi(\theta_0) + O(\sqrt{t})) f(z_2, \tau) \right|, \end{aligned}$$

where

$$\begin{aligned} f(z_2, \tau) &= \frac{\lambda(\theta_0)\alpha}{\sqrt{D_+}} \exp(2\lambda(\theta_0)\alpha z_2 \sqrt{\tau} + \lambda(\theta_0)^2 \alpha^2 \tau) \cdot \\ &\text{Erfc}(z_2 + \lambda(\theta_0)\alpha \sqrt{\tau}) - \frac{1}{\sqrt{\pi D_+ \tau}} e^{-z_2^2}. \end{aligned}$$

We see that

$$\sqrt{\tau} \left| \int_{\mathbb{R}^+} dz_2 (\phi(\theta_0) + O(\sqrt{t})) f(z_2, \tau) \right| \leq C.$$

Therefore, we have

$$|MM_3(t)| \leq Ct \mu(\partial\Omega, \sqrt{4D_+ t}).$$

In the same way, since  $\chi_4$  depends on  $z_1$  and  $z_2$  at the same time, we have

$$\begin{aligned} |MM_4(t)| &\leq C \left| \int_0^t d\tau \sqrt{\tau(t-\tau)} \int_{\partial\Omega} d\theta_0 \int_0^\epsilon ds |J(s, \theta_0)| \right. \\ &\cdot \int_0^{+\infty} dz_1 e^{2\lambda(\theta_0)\alpha z_1 \sqrt{t-\tau} + \lambda(\theta_0)^2 \alpha^2 (t-\tau)} \text{Erfc}(z_1 + \lambda(\theta_0)\alpha \sqrt{t-\tau}) \\ &\cdot \left. \int_0^{+\infty} dz_2 (\phi(\theta_0) + O(\sqrt{t})) f(z_2, \tau) \right| \leq Ct^{\frac{3}{2}} \mu(\partial\Omega, \sqrt{4D_+ t}). \end{aligned}$$

By iteration of the proof, we show for  $j \geq 1$  that

$$|NN^j(t)| \leq Ct^{\frac{1+j}{2}} \mu(\partial\Omega, \sqrt{4D_+ t}).$$

□

## 6 Relation of the heat content expansion with the interior Minkowski sausage

Let us start with a heat problem with just a discontinuous initial condition.

### 6.1 Particular case $D_+ = D_- = \text{const}$

**Lemma 3** *Let  $\Omega \subset \mathbb{R}^n$  be a compact connected bounded domain with a connected boundary  $\partial\Omega$  of the Hausdorff dimension  $d$  and  $u$  is the solution of the following problem:*

$$\partial_t u - D\Delta u = 0 \quad x \in \mathbb{R}^n, \quad t > 0, \quad (74)$$

$$u|_{t=0} = \mathbf{1}_\Omega, \quad (75)$$

Then for  $t \rightarrow +0$  we have

$$N(t) = \int_0^2 \frac{e^{-z^2}}{\sqrt{\pi}} \mu(\partial\Omega, 2\sqrt{Dt}z) dz + o\left(t^{\frac{n-d}{2}}\right). \quad (76)$$

Moreover, it can be approximated by

$$N(t) = \beta_{n-d} \mu(\partial\Omega, 2\sqrt{Dt}) + o\left(t^{\frac{n-d}{2}}\right), \quad (77)$$

with the prefactor

$$\beta_x \equiv \int_0^2 \frac{z^x e^{-z^2}}{\sqrt{\pi}} dz = \frac{1}{2\sqrt{\pi}} \gamma\left(\frac{x+1}{2}, 4\right) \quad (78)$$

is expressed through the incomplete Gamma function.

**Proof.** Let us prove formula (76). By definition

$$N(t) = \int_{\mathbb{R}^n \setminus \bar{\Omega}} \int_{\mathbb{R}^n} G(x, y, t) \mathbf{1}_\Omega dx dy,$$

where this time  $G$  is the heat kernel in  $\mathbb{R}^n$

$$G(x, y, t) = (4D\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4Dt}\right).$$

Therefore, we have

$$\begin{aligned} N(t) &= \text{Vol}(\Omega) - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(4\pi Dt)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4Dt}} \mathbf{1}_\Omega(x) \mathbf{1}_\Omega(y) dx dy \\ &= \text{Vol}(\Omega) - \int_{\mathbb{R}^n} \frac{1}{\pi^{\frac{n}{2}}} e^{-|v|^2} \left( \int_{\mathbb{R}^n} \mathbf{1}_\Omega(x) \mathbf{1}_\Omega(x + 2\sqrt{Dt}v) dx \right) dv \\ &= \int_{\mathbb{R}^n} \frac{1}{\pi^{\frac{n}{2}}} e^{-|v|^2} \left[ \int_{\Omega} (\mathbf{1}_\Omega(x) - \mathbf{1}_{\Omega - 2\sqrt{Dt}v}(x)) dx \right] dv, \end{aligned}$$

where  $\mathbf{1}_{\Omega - 2\sqrt{Dt}v}(x) = \mathbf{1}_\Omega(x + 2\sqrt{Dt}v)$  and the notation  $\Omega - 2\sqrt{Dt}v$  means that  $\Omega$  is shifted by the vector  $-2\sqrt{Dt}v \in \mathbb{R}^n$ .

Let us firstly suppose that  $\partial\Omega$  is regular, i.e of the class  $C^3$ . We see that for all points  $x \in \Omega$  for which  $d(x, \partial\Omega) \geq 2\sqrt{Dt}\|v\|$ , it holds  $(x + 2\sqrt{Dt}v) \in \Omega$ . Thus, it follows that for  $\epsilon = 2\sqrt{Dt}\|v\|$ ,

$$\mathbb{1}_\Omega(x) (\mathbb{1}_\Omega(x) - \mathbb{1}_{\Omega - 2\sqrt{Dt}v}(x)) = 0 \text{ for all } x \in \Omega \setminus \Omega_\epsilon.$$

Therefore, only  $x$  belonging to  $\Omega_\epsilon$  with  $\|v\| < \frac{\epsilon}{2\sqrt{Dt}}$  contribute to  $N(t)$  and we can write:

$$N(t) = \int_{\mathbb{R}^n} \frac{1}{\pi^{\frac{n}{2}}} e^{-|v|^2} \left[ \int_{\Omega_\epsilon} (\mathbb{1}_{\Omega_\epsilon}(x) - \mathbb{1}_{\Omega_\epsilon - 2\sqrt{Dt}v}(x)) dx \right] dv + O\left(e^{-\frac{1}{t^\delta}}\right),$$

where the exponentially small error with a  $\delta > 0$  is defined by the integral

$$\int_{\|v\| > \frac{\epsilon}{2\sqrt{Dt}}} \frac{1}{\pi^{\frac{n}{2}}} e^{-|v|^2} dv.$$

Since  $\partial\Omega$  is regular, we introduce (see Section 4) the local coordinates  $x = (\theta, s)$  and thus have  $\hat{x}(\theta) \in \partial\Omega$  and  $x \in \Omega_\epsilon$  iff  $0 < s < \epsilon$ . In this case,  $\chi_{2\sqrt{Dt},v}(x) = \mathbb{1}_{\Omega_\epsilon}(x) - \mathbb{1}_{\Omega_\epsilon - 2\sqrt{Dt}v}(x) \neq 0$  iff  $x \in \Omega_\epsilon$  and  $\hat{x}(\theta) - sn(\theta) + 2\sqrt{Dt}v \notin \Omega$ . Moreover, with the notation  $(v, n)$  for the Euclidean inner product of two vectors in  $\mathbb{R}^n$ ,

$$(\hat{x}(\theta) - sn(\theta) + 2\sqrt{Dt}v) \cdot n(\theta) = -s + 2\sqrt{Dt}(v, n).$$

We deduce that

$$\chi_{2\sqrt{Dt},v}(x) \neq 0 \text{ iff } s - 2\sqrt{Dt}(v, n) < 0.$$

Consequently, if  $(v, n) < 0$ , as  $s > 0$ , it is not possible to have  $s - 2\sqrt{Dt}(v, n) < 0$ . In turn, if  $0 < (v, n)$  then  $s \in ]0, 2\sqrt{Dt}(v, n)[$ . Considering only  $(v, n) > 0$ , we can define  $\epsilon = 2\sqrt{Dt}(v, n)$  and, since  $v = \frac{x-y}{\sqrt{4Dt}}$  and  $x, y \in \Omega_{2\sqrt{Dt}(v, n)}$ , we have  $0 < (v, n) < 2$ . Thus, the vector  $v$  can be locally decomposed in two parts:  $v = ((v, n), (v, \hat{x})) = (v_n, v_{\hat{x}})$ . Thus, returning to  $N(t)$ , we obtain with the error  $O(t)$  which comes from the Jacobian approximation (see  $|J(s, \theta)|$  in Section 4)

$$\begin{aligned} N(t) &= \int_{\mathbb{R}^{n-1}} \frac{1}{\pi^{\frac{n-1}{2}}} e^{-|v_{\hat{x}}|^2} dv_{\hat{x}} \int_0^2 \frac{1}{\sqrt{\pi}} e^{-|v_n|^2} \left( \int_{\Omega_\epsilon} \chi_{2\sqrt{Dt},v_n}(x) dx \right) dv_n + O(t) \\ &= \int_0^2 \frac{e^{-z^2}}{\sqrt{\pi}} \mu(\partial\Omega, 2\sqrt{Dt}z) dz + o(t^{\frac{n-d}{2}}). \end{aligned}$$

If  $\partial\Omega$  is regular, then  $d = n - 1$  and  $o(t^{\frac{n-d}{2}}) = o(\sqrt{t})$ , which, as it was mentioned, is actually  $O(t)$ . The last formula that depends only on a volume of the interior Minkowski sausage, holds for all types of connected boundaries described in Subsection 2.2.

The formula (77) follows from Eq. (76) and the relation

$$\mu(\partial\Omega, \epsilon z) = z^{n-d} \mu(\partial\Omega, \epsilon) + O(\epsilon^{2(n-d)}), \quad (79)$$

which, for a fixed  $z$  and  $\epsilon \rightarrow +0$ , is evident for the regular case and can be proved by approximating the fractal volume by a converging sequence of the volumes for smooth boundaries. For  $d = n - 1$  in Eq. (77), one has  $\beta_1 = \frac{1-e^{-4}}{2\sqrt{\pi}} \approx 0.2769$ .  $\square$

A comparison between the asymptotic formula (77) and a numerical solution of the problem (74)–(75) is illustrated in Fig. 4 (for a square and a prefractal domain).

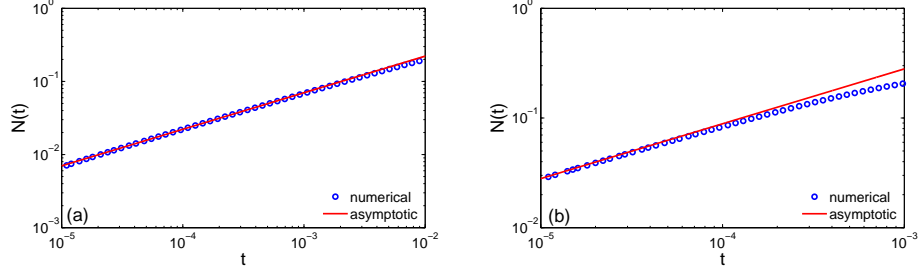


Figure 4: Comparison between the asymptotic formula (77) (solid line) and a FreeFem++ numerical solution of the problem (74)–(75) (circles) for two domains: (a) the unit square (with  $\text{Vol}(\partial\Omega) = 4$ ) and (b) the second generation of the Minkowski fractal, with  $\text{Vol}(\partial\Omega) = 2^2 \cdot 4$ . We set  $D_+ = D_- = D = 1$ .

## 6.2 General case

Let us come back to the problem (1)–(4).

According to Theorem 4 (Eq. (52)), the heat content can be found up to the terms either  $t^{\frac{3}{2}}$ , or  $t$  (depending on values of  $\lambda$ ), by integrating over all boundary points  $\theta$  of the solution  $\hat{u}^{hom}$  of the homogeneous problem (49)–(51) with constant coefficients taken at a boundary point  $(0, \theta)$ . Obviously, Eq. (52) is valid only for regular boundaries. Let us reformulate it to allow an explicit calculation of the heat content for all types of boundaries mentioned in Section 2.

For this purpose, given  $\epsilon = O(\sqrt{t})$ ,  $\epsilon > \sqrt{4D_+t}$ , we divide  $\partial\Omega$  (which is still supposed to be regular) into  $J$  disjoint parts  $B_j$  ( $j = 1, \dots, J$ ) of the size  $\delta^{n-1}$  with  $0 < \delta \leq \epsilon$  such that  $\partial\Omega = \sqcup_{j=1}^J B_j$ .

For  $t \rightarrow +0$ ,  $\delta \rightarrow 0$  and thus, due to regularity of  $\partial\Omega$  on each  $B_j \times ]0, \epsilon[$  the local change of variables from Section 4 is a  $C^1$ -diffeomorphism. In addition, since  $u$  continuously depends on  $\lambda$  (see Theorem 2),  $\hat{u}^{hom}$ , considered as a function of  $\theta$ , by the continuity of  $\lambda$ , is continuous on  $\theta$ . Therefore, by the mean value theorem and due to the positivity of  $|J(s, \theta)|$ , we deduce that for all  $j = 1, \dots, J$  there exists  $\theta_0^j \in \overline{B_j}$  such that

$$\int_{B_j} d\theta \int_{[0, \epsilon]} ds (1 - \hat{u}^{hom}(s, \theta, t)) |J(s, \theta)| = \int_{[0, \epsilon]} ds (1 - \hat{u}^{hom}(s, \theta_0^j, t)) \int_{B_j} d\theta |J(s, \theta)|.$$

From Eq. (69), Eq. (52) becomes

$$\begin{aligned} N(t) &= \sum_{j=1}^J \int_{[0, \epsilon]} ds (1 - \hat{u}^{hom}(s, \theta_0^j, t)) \int_{B_j} d\theta |J(s, \theta)| \\ &= \begin{cases} O(t \mu(\partial\Omega, \sqrt{t})), & 0 < \lambda < \infty \\ O(\sqrt{t} \mu(\partial\Omega, \sqrt{t})), & \lambda = \infty \end{cases}. \end{aligned}$$

Hence we prove

**Theorem 5** *The heat content for the solution of the problem (1)–(4) can be explicitly found for all types of boundaries  $\partial\Omega$  (a connected boundary of a compact domain described in Subsection 2.2) using the following expressions:*

1. for  $\lambda < \infty$  on  $\partial\Omega$  :

$$\begin{aligned} N(t) &= \frac{2\sqrt{t}}{\sqrt{D_+} \text{Vol}(\partial\Omega)} \left[ \mu(\partial\Omega, \sqrt{4D_+t}) \int_{\partial\Omega} d\sigma \lambda(\sigma) \int_1^2 dz f(\sigma, z, t) \right. \\ &\quad - \int_1^2 dz \mu(\partial\Omega, \sqrt{4D_+t}(z-1)) \int_{\partial\Omega} d\sigma \lambda(\sigma) f(\sigma, z, t) \\ &\quad \left. - \int_0^1 dz \mu(\partial\Omega, \sqrt{4D_+t}z) \int_{\partial\Omega} d\sigma \lambda(\sigma) f(\sigma, z, t) \right] + O(t\mu(\partial\Omega, \sqrt{t})), \end{aligned} \quad (80)$$

where  $\alpha = \frac{1}{\sqrt{D_-}} + \frac{1}{\sqrt{D_+}}$  and

$$f(\sigma, z, t) = \exp\left(2\lambda(\sigma)\alpha\sqrt{t}z + \lambda(\sigma)^2\alpha^2t\right) \text{Erfc}(z + \lambda(\sigma)\alpha\sqrt{t}). \quad (81)$$

2. for  $\lambda = \infty$  on  $\partial\Omega$  :

$$N(t) = \frac{2\sqrt{D_-}}{\sqrt{D_-} + \sqrt{D_+}} \int_0^2 \frac{e^{-z^2}}{\sqrt{\pi}} \mu(\partial\Omega, \sqrt{4D_+t}z) dz + O(\sqrt{t} \mu(\partial\Omega, \sqrt{t})). \quad (82)$$

Formulas (80) and (82) can be approximated by

1. for  $\lambda < \infty$  on  $\partial\Omega$  :

$$\begin{aligned} N(t) &= \frac{2\sqrt{t} \mu(\partial\Omega, \sqrt{4D_+t})}{\sqrt{D_+} \text{Vol}(\partial\Omega)} \left[ \int_{\partial\Omega} d\sigma \lambda(\sigma) \int_1^2 dz f(\sigma, z, t) \right. \\ &\quad - \int_1^2 dz (z-1)^{n-d} \int_{\partial\Omega} d\sigma \lambda(\sigma) f(\sigma, z, t) \\ &\quad \left. - \int_0^1 dz z^{n-d} \int_{\partial\Omega} d\sigma \lambda(\sigma) f(\sigma, z, t) \right] + O(\sqrt{t} \mu(\partial\Omega, \sqrt{t})^2), \end{aligned} \quad (83)$$

2. for  $\lambda = \infty$  on  $\partial\Omega$  :

$$N(t) = \frac{2\sqrt{D_-} \beta_{n-d}}{\sqrt{D_-} + \sqrt{D_+}} \mu(\partial\Omega, \sqrt{4D_+t}) + O(\mu(\partial\Omega, \sqrt{t})^2), \quad (84)$$

where  $\beta_x$  was defined in Eq. (78).

**Proof.** Using Eqs. (56)–(60),  $N(t)$  becomes

$$\begin{aligned} N(t) &= \mu(\partial\Omega, \epsilon) + \sum_{j=1}^J \int_{[0, \epsilon]^2} ds_1 ds \left( h_+^{\theta_j}(s, s_1, t) - f_+^{\theta_j}(s, s_1, t) \right) \int_{B_j} d\theta |J(s, \theta)| \\ &= \begin{cases} O(t \mu(\partial\Omega, \sqrt{4D_+t})), & 0 < \lambda < \infty \\ O(\sqrt{t} \mu(\partial\Omega, \sqrt{4D_+t})), & \lambda = \infty \end{cases}. \end{aligned}$$

Let us calculate it explicitly. We start with the part

$$Nh_j(t) = \int_{[0,\epsilon]^2} ds_1 ds h_+^{\theta_0^j}(s, s_1, t) \int_{B_j} d\theta |J(s, \theta)|.$$

Changing variables as in the proof of Theorem 4,  $Nh_j(t)$  becomes

$$\begin{aligned} Nh_j(t) &= \int_{B_j} d\theta \int_{\mathbb{R}} \frac{e^{-z^2}}{\sqrt{\pi}} \left( \int_{\mathbb{R}} \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(s - \sqrt{4D_+tz}) |J(s, \theta)| ds \right) dz \\ &+ a(\lambda, 0, \theta_0^j) \int_{B_j} d\theta \int_{\mathbb{R}} \frac{e^{-z^2}}{\sqrt{\pi}} \left( \int_{\mathbb{R}} \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s + \sqrt{4D_+tz}) |J(s, \theta)| ds \right) dz. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} Nh_j(t) &= \int_{B_j} d\theta \left[ \int_{[0,\epsilon]} |J(s, \theta)| ds \right. \\ &- \int_{\mathbb{R}} \frac{e^{-z^2}}{\sqrt{\pi}} \left( \int_{[0,\epsilon]} (\mathbb{1}_{[0,\epsilon]}(s) - \mathbb{1}_{[0,\epsilon]+\sqrt{4D_+tz}}(s)) |J(s, \theta)| ds \right) dz \\ &\left. + a(\lambda, 0, \theta_0^j) \int_{\mathbb{R}} \frac{e^{-z^2}}{\sqrt{\pi}} \left( \int_{\mathbb{R}} \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[-\epsilon,0]+\sqrt{4D_+tz}}(s) |J(s, \theta)| ds \right) dz \right]. \end{aligned}$$

Applying formula (73) with  $v = \sqrt{4D_+t} z$  (see also Subsection 6.1), we find

$$Nh_j(t) = \mu(B_j, \epsilon) - (1 - a(\lambda, 0, \theta_0^j)) \int_0^2 \frac{e^{-z^2}}{\sqrt{\pi}} \mu(B_j, \sqrt{4D_+tz}) dz.$$

Thus, for  $\lambda < \infty$   $Nh_j(t) = \mu(B_j, \epsilon)$  since  $a = 1$ .

We treat the second part in the same way,

$$Nf_j(t) = - \int_{[0,\epsilon]^2} ds_1 ds f_+^{\theta_0^j}(s, s_1, t) \int_{B_j} d\theta |J(s, \theta)|,$$

which is equal to zero for  $\lambda = \infty$ . For  $f(\theta_0^j, z, t)$  from Eq. (81), we find that

$$\begin{aligned} Nf_j(t) &= - \frac{2\lambda(\theta_0^j)\sqrt{t}}{\sqrt{D_+}} \int_{\mathbb{R}^2} ds dz \mathbb{1}_{[0,\epsilon]}(s) \mathbb{1}_{[0,\epsilon]}(-s + 2\sqrt{D_+tz}) f(z, t) \int_{B_j} |J(s, \theta)| d\theta \\ &= - \frac{2\lambda(\theta_0^j)\sqrt{t}}{\sqrt{D_+}} \left[ \int_1^2 dz f(\theta_0^j, z, t) \int_{B_j} \int_{(z-1)\sqrt{4D_+t}}^{\sqrt{4D_+t}} |J(s, \theta)| ds d\theta \right. \\ &+ \left. \int_0^1 dz f(\theta_0^j, z, t) \int_{B_j} \int_0^{\sqrt{4D_+tz}} |J(s, \theta)| ds d\theta \right] \\ &= - \frac{2\lambda(\theta_0^j)\sqrt{t}}{\sqrt{D_+}} \left[ \mu(B_j, \sqrt{4D_+t}) \int_1^2 f(\theta_0^j, z, t) dz \right. \\ &- \left. \int_1^2 f(\theta_0^j, z, t) \mu(B_j, \sqrt{4D_+t}(z-1)) dz + \int_0^1 f(\theta_0^j, z, t) \mu(B_j, \sqrt{4D_+tz}) dz \right]. \end{aligned}$$

Putting two results together, we obtain the following approximations for  $N(t)$ :

1. for  $\lambda < \infty$  on  $\partial\Omega$ :

$$\begin{aligned}
N(t) &= \sum_{j=1}^J \mu(B_j, \sqrt{4D_+t}) \frac{2\lambda(\theta_0^j)\sqrt{t}}{\sqrt{D_+}} \int_1^2 f(\theta_0^j, z, t) dz \\
&\quad - \sum_{j=1}^J \frac{2\lambda(\theta_0^j)\sqrt{t}}{\sqrt{D_+}} \left[ \int_1^2 f(\theta_0^j, z, t) \mu(B_j, \sqrt{4D_+t}(z-1)) dz \right. \\
&\quad \left. - \int_0^1 f(\theta_0^j, z, t) \mu(B_j, \sqrt{4D_+tz}) dz \right] + O(t \mu(\partial\Omega, \sqrt{t})), \tag{85}
\end{aligned}$$

2. for  $\lambda = \infty$  on  $\partial\Omega$ :

$$\begin{aligned}
N(t) &= \frac{2\sqrt{D_-}}{\sqrt{D_-} + \sqrt{D_+}} \sum_{j=1}^J \int_0^2 \frac{e^{-z^2}}{\sqrt{\pi}} \mu(B_j, \sqrt{4D_+t} z) dz \\
&\quad + O(\sqrt{t} \mu(\partial\Omega, \sqrt{t})). \tag{86}
\end{aligned}$$

It means that if the formulas for  $\mu(B_j, \delta)$  are known, we get the approximation of  $N(t)$  up to terms of the order of  $t^{\frac{n-d+2}{2}}$  for  $\lambda < \infty$ , and of the order of  $t^{\frac{1+n-d}{2}}$  for  $\lambda = \infty$ . Moreover, this approximation, depending only on the volume of  $\partial\Omega$ , holds for all types of boundaries, even fractals (see Subsection 2.2 and p. 378 of Ref. [11] for a similar conclusion).

Let us now change the sum over  $j$  with the integral over  $z$  and make  $J \rightarrow +\infty$ :

$$\lim_{J \rightarrow +\infty} \sum_{j=1}^J C(z, t, \theta_0^j) \mu(B_j, \sqrt{4D_+tz}) = \int_{\partial\Omega} C(z, t, \sigma) \text{dist}(\sigma, \sqrt{4D_+tz}) d\sigma,$$

where  $d\sigma$  is understood in the sense of the Hausdorff measure ( $d$ -measure) defined on  $\partial\Omega$ . Thus, again with the help of the mean value theorem, we have

$$\int_{\partial\Omega} C(z, t, \sigma) \text{dist}(\sigma, \sqrt{4D_+tz}) d\sigma = \frac{\mu(\partial\Omega, \sqrt{4D_+t})}{\text{Vol}(\partial\Omega)} \int_{\partial\Omega} C(z, t, \sigma) d\sigma,$$

from which Eqs. (80) and (82) follow. We use Eq. (79) to obtain formulas (83) and (84).

□

## 7 Regular case

In the case of a regular boundary we provide the asymptotic expansion of the heat content up to the third-order term.

In this case, we can approximate the solution of the system (29)–(33) by the solution

$v$  of the following problem (instead of (35)–(38), as previously)

$$\frac{\partial}{\partial t}u_+ - D_+ \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_+ + D_+ \sum_{i=1}^{n-1} k_i(\theta_0) \frac{\partial u_+}{\partial s} = 0, \quad 0 < s < \epsilon \quad (87)$$

$$\frac{\partial}{\partial t}u_- - D_- \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_- + D_- \sum_{i=1}^{n-1} k_i(\theta_0) \frac{\partial u_-}{\partial s} = 0, \quad -\epsilon < s < 0 \quad (88)$$

$$u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \quad (89)$$

$$D_- \frac{\partial u_-}{\partial s} \Big|_{s=-0} = \lambda(\theta_0)(u_- - u_+) \Big|_{s=0},$$

$$D_+ \frac{\partial u_+}{\partial s} \Big|_{s=+0} = D_- \frac{\partial u_-}{\partial s} \Big|_{s=-0}. \quad (90)$$

In this approximation the remainder terms of the system (29)–(33) contain only the coefficients of the order  $\sqrt{t}$  (to compare with (70)):

$$R(s_1, \theta_0) = s_1 D_+ \sum_{i=1}^{n-1} k_i^2(\theta_0) + O(s_1^2),$$

that gives

$$R(s_1, \theta_0) = (s \mp 2\sqrt{D_+(t-\tau)z_1}) D_+ \sum_{i=1}^{n-1} k_i^2(\theta_0) + O(t) = O(\sqrt{t}). \quad (91)$$

The basis of the parametrix is the Green function given by (see Section B.1)

$$\begin{aligned} h_+^{\theta_0}(s_1, s_2, t) &= \frac{1}{\sqrt{4\pi D_+ t}} \left( \exp \left( -\frac{(s_1 - s_2 - tD_+\gamma(\theta_0))^2}{4D_+ t} \right) \right. \\ &\quad \left. + a(\lambda, 0, \theta_0) \exp \left( -\frac{(s_1 + s_2 - tD_+\gamma(\theta_0))^2}{4D_+ t} \right) \right), \end{aligned} \quad (92)$$

$$\begin{aligned} f_+^{\theta_0}(s_1, s_2, t) &= b(\lambda, 0, \theta_0) \frac{\lambda(\theta_0)}{D_+} \cdot \\ &\quad \cdot \exp \left( \frac{\lambda(\theta_0)\alpha}{\sqrt{D_+}} (s_1 + s_2 - tD_+\gamma(\theta_0)) + \lambda(\theta_0)^2 \alpha^2 t \right) \cdot \\ &\quad \cdot \operatorname{Erfc} \left( \frac{s_1 + s_2 - tD_+\gamma(\theta_0)}{2\sqrt{D_+ t}} + \lambda(\theta_0)\alpha\sqrt{t} \right), \end{aligned} \quad (93)$$

where  $\gamma(\theta_0) = \sum_{i=1}^{n-1} k_i(\theta_0)$ . The estimate (69) becomes

$$N(t) - \int_{\partial\Omega} d\theta_0 \int_{[0,\epsilon]} ds (1 - \hat{u}_\epsilon^{\text{hom}}(s, \theta_0, t)) |J(s, \theta_0)| = \begin{cases} O(t^2), & 0 < \lambda < \infty \\ O(t^{\frac{3}{2}}), & \lambda = \infty \end{cases} \quad (94)$$

Consequently, for the regular case we have



**Theorem 6** Let  $\Omega$  be a compact domain of  $\mathbb{R}^n$  with a connected boundary  $\partial\Omega \in C^\infty(\mathbb{R}^n)$ . Then for  $\lambda = \infty$  we have

$$N(t) = 2 \frac{1 - e^{-4}}{\sqrt{\pi}} \frac{\sqrt{D_+ D_-}}{\sqrt{D_+} + \sqrt{D_-}} \text{Vol}(\partial\Omega) \sqrt{t} + O(t^{\frac{3}{2}}). \quad (95)$$

In the case of  $0 < \lambda < \infty$ , we have

$$\begin{aligned} N(t) = & 4C_0 t \int_{\partial\Omega} \lambda(\sigma) d\sigma - \frac{2}{3} C_1 t^{\frac{3}{2}} \left[ 2 \left( \frac{1}{\sqrt{D_+}} + \frac{1}{\sqrt{D_-}} \right) \int_{\partial\Omega} \lambda^2(\sigma) d\sigma \right. \\ & \left. - \sqrt{D_+} (n-1) \int_{\partial\Omega} \lambda(\sigma) H(\sigma) d\sigma \right] + O(t^2), \end{aligned} \quad (96)$$

where  $H$  is the mean curvature, and

$$C_0 = 1 + \frac{3}{2} \text{erf}(1) - \frac{9}{4} \text{erf}(2) + \frac{1}{\sqrt{\pi}} \left( \frac{1}{e} - \frac{1}{e^4} \right) \approx 0.2218, \quad (97)$$

$$C_1 = \frac{1}{\sqrt{\pi}} - 6 + \frac{5e^{-4} - 4e^{-1}}{\sqrt{\pi}} - 5 \text{erf}(1) + 11 \text{erf}(2) \approx 0.5207. \quad (98)$$

**Proof.** Let us consider the case  $\lambda = \infty$ . Using the Green function given in Eqs. (92) and (93), we obtain

$$N(t) = \sum_{j=1}^J \beta \int_0^{2 - \sqrt{tD_+} \frac{\gamma(\theta_0^j)}{2}} \frac{e^{-z^2}}{\sqrt{\pi}} \mu(B_j, \sqrt{4D_+ t z} + tD_+ \gamma(\theta_0^j)) dz + O(t^{\frac{3}{2}}), \quad (99)$$

where  $\beta = \frac{2\sqrt{D_-}}{\sqrt{D_-} + \sqrt{D_+}}$ . In Eq. (99) the remainder term also contains the integrals  $\int_{-\sqrt{tD_+} \frac{\gamma(\theta_0^j)}{2}}^0 dz$ . From Eq. (99) we find

$$\begin{aligned} N(t) = & \sum_{j=1}^J \beta \int_0^{2 - \sqrt{tD_+} \frac{\gamma(x_j)}{2}} \frac{e^{-z^2}}{\sqrt{\pi}} \int_{B_j} d\theta \int_0^{2\sqrt{D_+ t z} + \gamma(x_j) D_+ t} ds (1 - s(n-1)H(\theta)) \\ & + O(t^{\frac{3}{2}}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} N(t) = & \sqrt{t} \left( 2C \sqrt{D_+} \sum_{j=1}^J \text{Vol}(B_j) \right) \\ & - t(n-1) \left( \sum_{j=1}^J \xi \left[ \text{Vol}(B_j) H(x_j) - \int_{B_j} H(\sigma) d\sigma \right] \right) + O(t^{\frac{3}{2}}), \end{aligned} \quad (100)$$

where

$$\begin{aligned} C &= \frac{1 - e^{-4}}{\sqrt{\pi}} \frac{\sqrt{D_-}}{\sqrt{D_+} + \sqrt{D_-}}, \\ \xi &= \left( 4 \frac{e^{-4}}{\sqrt{\pi}} - \text{erf}(2) \right) \frac{D_+ \sqrt{D_-}}{\sqrt{D_+} + \sqrt{D_-}}. \end{aligned}$$

In addition, for all  $\sigma \in B_j$ , the distance between  $x_j$  (which also belongs to  $B_j$ ) and  $\sigma$  goes to 0 as  $J \rightarrow +\infty$ . Thus, since

$$|H(x_j) - H(\sigma)| \leq H'(\sigma)|x_j - \sigma| \leq C \text{Vol}(B_j),$$

we have

$$\lim_{J \rightarrow +\infty} \sum_{j=1}^J \left| \text{Vol}(B_j)H(x_j) - \int_{B_j} H(\sigma)d\sigma \right| = 0.$$

Hence, from Eq. (100) we obtain Eq. (95).

The case  $0 < \lambda < \infty$  can be treated in the similar way using in Eq. (85) the expansion of the  $f(\sigma, t, z)$ :

$$\begin{aligned} f(\sigma, t, z) &= \exp\left(2\lambda(\sigma)\alpha\sqrt{t}z + \lambda^2(\sigma)\alpha^2(\sigma)t\right) \text{Erfc}(z + \lambda(\sigma)\alpha\sqrt{t}) \\ &= \text{Erfc}(z) - 2\lambda(\sigma)\alpha\sqrt{t} \left( \frac{1}{\sqrt{\pi}}e^{-z^2} - z \text{Erfc}(z) \right) + O(t). \end{aligned}$$

□

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## A Definitions of Besov spaces on fractals

Let us define the Besov space  $B_\beta^{2,2}(\partial\Omega)$  on a  $d$ -set  $\partial\Omega$  (see Ref. [21] p.135 and Ref. [22]).

There are many equivalent definitions[23, 31] of Besov spaces. To give one of them, we introduce[21, 23] a net  $\mathcal{N}$  with mesh  $2^{-\nu}$ ,  $\nu \in \mathbb{N}$ , i.e. a division of  $\mathbb{R}^n$  into half-open non-overlapping cubes  $W$  with edges of length  $2^{-\nu}$ , obtained by intersecting  $\mathbb{R}^n$  with hyperplanes orthogonal to the axes. In addition, we denote by  $\mathcal{P}_k(\mathcal{N})$  the set of functions which on each cube  $W$  in the net  $\mathcal{N}$  coincide with a polynomial of degree at most  $k$ .

**Definition 4 (Besov space  $B_\beta^{p,q}(\Gamma)$ ,  $\beta > 0$ , see Ref. [21])** Let  $\Gamma$  be a closed subset of  $\mathbb{R}^n$  which is a  $d$ -set preserving Markov's inequality for  $0 < d \leq n$  and let  $m_d$  be a fixed  $d$ -measure on  $\Gamma$ . We say that  $f \in B_\beta^{p,q}(\Gamma)$ ,  $\beta > 0$ ,  $1 \leq p, q \leq +\infty$ , if  $f \in L^p(m_d)$  and there is a sequence  $B = (B_\nu)_{\nu \in \mathbb{N}} \in \ell^q$  such that for every net  $\mathcal{N}$  with mesh  $2^{-\nu}$ ,  $\nu \in \mathbb{N}$  there exists a function  $s(\mathcal{N}) \in \mathcal{P}_{[\beta]}(\mathcal{N})$  (by  $[\beta]$  is denoted the integer part of  $\beta$ ) satisfying

$$\|f - s(\mathcal{N})\|_{L^p(m_d)} \leq 2^{-\nu\beta} B_\nu.$$

The norm of  $f$  in  $B_\beta^{p,q}(\Gamma)$  is given by the formula

$$\|f\|_{B_\beta^{p,q}(\Gamma)} = \|f\|_{L^p(m_d)} + \inf_B \|B\|_{\ell^q},$$

where the infimum is over all such sequences  $B$ .

The dual Besov space  $(B_{\beta}^{2,2}(\partial\Omega))' = B_{-\beta}^{2,2}(\partial\Omega)$  is introduced in Ref. [23]. To give the definition of the Besov space  $B_{-\beta}^{2,2}(\partial\Omega)$  we need to define the atoms:

**Definition 5 (Atom[23])** Let  $\beta > 0$ ,  $1 \leq p \leq \infty$ , and let  $W$  with  $W \cap \Gamma \neq \emptyset$  be a cube with edge length  $2^{-\nu}$ ,  $\nu \in \mathbb{N}$ . A function  $a = a_W \in L^p(m_d)$  is a  $(-\beta, p)$ -atom associated with  $W$  if

1.  $\text{supp } a \subset 2W$ , where  $2W$  is the cube obtained by expanding  $W$  twice from its center,
2.  $\int x^\gamma a(x) dm_d = 0$  for  $|\gamma| \leq [\beta]$  if  $\nu > 0$ ,
3.  $\|a\|_{L^p(m_d)} \leq 2^{\nu\beta}$ .

Let  $\mathcal{N}_\nu(\Gamma) = \{W \in \mathcal{N}_\nu \mid W \cap \Gamma \neq \emptyset\}$  with the notation  $\mathcal{N}_\nu$  of the net with mesh  $2^{-\nu}$  such that the origin is a corner of some cube in the net. Then we can define the Besov space with a negative parameter  $-\beta$ ,  $B_{-\beta}^{2,2}(\partial\Omega)$ , which is actually[23] the dual Besov space of  $B_{\beta}^{2,2}(\partial\Omega)$ :

**Definition 6 (Besov space  $B_{-\beta}^{p,q}(\Gamma)$ ,  $\beta > 0$ , see Ref. [21])** The space  $B_{-\beta}^{p,q}(\Gamma)$ ,  $\beta > 0$ ,  $1 \leq p, q \leq \infty$  consists of functions  $f \in \mathcal{D}'(\mathbb{R}^n)$  which are given by

$$\forall \phi \in \mathcal{D}(\mathbb{R}^n) \quad \langle f, \phi \rangle = \sum_{\nu \in \mathbb{N}} \sum_{W \in \mathcal{N}_\nu(\Gamma)} s_W \int a_W \phi dm_d,$$

where  $a_W$  are  $(-\beta, p)$ -atoms and  $s_W$  are numbers such that  $S = (S_\nu)_{\nu \in \mathbb{N}} \in \ell^q$  and  $S_\nu$  is defined by

$$S_\nu = \left( \sum_{W \in \mathcal{N}_\nu(\Gamma)} |s_W|^p \right)^{\frac{1}{p}}.$$

The norm of  $f$  is defined by

$$\|f\|_{B_{-\beta}^{p,q}(\Gamma)} = \inf \|S\|_{\ell^q},$$

where the infimum is taken over all possible atomic decompositions of  $f$ :

$$f = \sum_{\nu \in \mathbb{N}} \sum_{W \in \mathcal{N}_\nu(\Gamma)} s_W a_W.$$

## B Explicit computations for half space problem with constant coefficients

## B.1 Case $\lambda = \infty$

The Green function of the one-dimensional problem (49)–(51) with  $\lambda = \infty$  and  $s \in \mathbb{R}$  was treated in Ref. [6, 1] and it is given by

$$\Gamma(s, s_1, t) = \mathbb{1}_{\{s>0, s_1>0\}}\Gamma_{++}(s, s_1, t) + \mathbb{1}_{\{s<0, s_1>0\}}\Gamma_{-+}(s, s_1, t)$$

with

$$\Gamma_{++}(s, s_1, t) = \frac{1}{\sqrt{4\pi D_+ t}} \left( \exp\left(-\frac{(s-s_1)^2}{4D_+ t}\right) + A \exp\left(-\frac{(s+s_1)^2}{4D_+ t}\right) \right), \quad (101)$$

$$\Gamma_{-+}(s, s_1, t) = B \frac{1}{\sqrt{\pi D_+ t}} \exp\left(-\frac{\left(s-s_1\sqrt{\frac{D_-}{D_+}}\right)^2}{4D_- t}\right), \quad (102)$$

where  $A = \frac{\sqrt{D_+} - \sqrt{D_-}}{\sqrt{D_+} + \sqrt{D_-}}$  and  $B = \frac{\sqrt{D_+}}{\sqrt{D_+} + \sqrt{D_-}}$ .

Let us use this result to find the Green function  $\Gamma^{reg}(s, s_1, t)$  of the following one-dimensional problem

$$\frac{\partial}{\partial t} u_+ - D_+ \frac{\partial^2}{\partial s^2} u_+ + D_+ k \frac{\partial}{\partial s} u_+ = 0, \quad s > 0 \quad (103)$$

$$\frac{\partial}{\partial t} u_- - D_- \frac{\partial^2}{\partial s^2} u_- + D_- k \frac{\partial}{\partial s} u_- = 0, \quad s < 0 \quad (104)$$

$$u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \quad (105)$$

$$u_+|_{s=+0} = u_-|_{s=-0}, \quad D_+ \frac{\partial}{\partial s} u_+|_{s=+0} = D_- \frac{\partial}{\partial s} u_-|_{s=-0}, \quad (106)$$

The constant coefficient problem

$$\frac{\partial}{\partial t} u - D \frac{\partial^2}{\partial s^2} u + Dk \frac{\partial}{\partial s} u = 0, \quad s \in \mathbb{R}, \quad (107)$$

$$u|_{t=0} = u_0, \quad (108)$$

has the Green function of the form

$$K(s, s_1, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(s-y-tDk)^2}{4Dt}},$$

that means that the change of variables  $s - tDk = X$  transforms (107) to

$$\frac{\partial}{\partial t} u - D \frac{\partial^2}{\partial X^2} u = 0$$

with the Green function

$$K_0(X, Y, t) = \frac{1}{4\pi Dt} e^{-\frac{(X-Y)^2}{4Dt}}.$$

In addition[6], we know (see (101)–(102)) the Green function for the constant coefficient problem

$$\begin{aligned} \frac{\partial}{\partial t} u_+ - D_+ \frac{\partial^2}{\partial X^2} u_+ &= 0 \quad X > 0, \\ \frac{\partial}{\partial t} u_- - D_- \frac{\partial^2}{\partial X^2} u_- &= 0 \quad X < 0, \\ u_+|_{t=0} &= 1, \quad u_-|_{t=0} = 0, \\ u_+|_{X=+0} &= u_-|_{X=-0}, \quad D_+ \frac{\partial}{\partial X} u_+|_{X=+0} = D_- \frac{\partial}{\partial X} u_-|_{X=-0}. \end{aligned}$$

Consequently, we perform the following change of variables in Eqs. (103)–(106):

$$X = \mathbf{1}_{s>0}(s) (s + tD_+k) + \mathbf{1}_{s<0}(s) \left( \frac{D_-}{D_+} s - tD_-k \right)$$

and obtain for  $z = \frac{D_\pm}{D_-} X$  that

$$\begin{aligned} \frac{\partial}{\partial t} u_+ - D_+ \frac{\partial^2}{\partial X^2} u_+ &= 0 \quad X > tD_+k, \\ \frac{\partial}{\partial t} u_- - D_- \frac{\partial^2}{\partial z^2} u_- &= 0 \quad z < tD_+k, \\ u_+|_{t=0} &= 1, \quad u_-|_{t=0} = 0, \\ u_+|_{X=+tD_+k} &= u_-|_{z=-tD_+k}, \quad D_+ \frac{\partial}{\partial X} u_+|_{X=+tD_+k} = D_- \frac{\partial}{\partial z} u_-|_{z=-tD_+k}. \end{aligned}$$

Thus,

$$\begin{aligned} \Gamma_{++}^{reg}(s, s_1, t) &= \frac{1}{\sqrt{4\pi D_+ t}} \left( \exp \left( -\frac{(s - s_1 - tD_+k)^2}{4D_+ t} \right) \right. \\ &\quad \left. + A \exp \left( -\frac{(s + s_1 - tD_+k)^2}{4D_+ t} \right) \right) \end{aligned}$$

for  $s, s_1 > 0$ , and

$$\Gamma_{-+}^{reg}(s, s_1, t) = \frac{1}{\sqrt{\pi D_+ t}} B \exp \left( -\frac{([D_+/D_-]s - s_1 \sqrt{D_+/D_-} + tD_+k)^2}{4D_- t} \right)$$

for  $s < 0, s_1 > 0$ . Now, to obtain the Green function of the multidimensional problem

$$\begin{aligned} \frac{\partial}{\partial t} u_+ - D_+ \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_+ + D_+ k \frac{\partial}{\partial s} u_+ &= 0, \quad s > 0, \theta_i \in \mathbb{R}, \\ \frac{\partial}{\partial t} u_- - D_- \left( \frac{\partial^2}{\partial s^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \theta_i^2} \right) u_- + D_- k \frac{\partial}{\partial s} u_- &= 0, \quad s < 0, \theta_i \in \mathbb{R} \\ u_+|_{t=0} &= 1, \quad u_-|_{t=0} = 0, \\ u_+|_{s=+0} &= u_-|_{s=-0}, \quad D_+ \frac{\partial}{\partial s} u_+|_{s=+0} = D_- \frac{\partial}{\partial s} u_-|_{s=-0}, \end{aligned}$$

we apply the Fourier transform in  $s_i$  variables and, due to the boundary conditions depending only on  $s$ , we obtain that  $\hat{G}_{\pm\pm}$ , the Fourier transform of the Green function  $G_{\pm\pm}$ , can be found by the formula

$$\hat{G}_{\pm\pm}(s, s_1, \xi, t) = e^{-D_{\pm}|\xi|^2 t} \Gamma_{\pm\pm}(s, s_1, t),$$

where  $\Gamma_{\pm\pm}(s, s_1, t)$  is the Green function of the corresponding one-dimensional problem. This implies

$$\begin{aligned} \Gamma_{++}^{reg}(s, \theta, s_1, \theta_1, t) &= \frac{1}{(4\pi D_+ t)^{\frac{n}{2}}} \left( \exp\left(-\frac{(s - s_1 - tD_+ k)^2}{4D_+ t}\right) \right. \\ &+ A \exp\left(-\frac{(s + s_1 - tD_+ k)^2}{4D_+ t}\right) \left. \right) \exp\left(-\frac{|\theta - \theta_1|^2}{4D_+ t}\right) \quad \text{for } s, s_1 > 0, \theta, \theta_1 \in \mathbb{R}^{n-1}, \end{aligned}$$

$$\begin{aligned} \Gamma_{-+}^{reg}(s, \theta, s_1, \theta_1, t) &= \frac{B}{(4\pi D_- t)^{\frac{n-1}{2}} \sqrt{\pi D_+ t}} \exp\left(-\frac{(\frac{D_+}{D_-} s - s_1 \sqrt{\frac{D_+}{D_-}} + tD_+ k)^2}{4D_- t}\right) \\ &\cdot \exp\left(-\frac{|\theta - \theta_1|^2}{4D_- t}\right) \quad \text{for } s < 0, s_1 > 0, \theta, \theta_1 \in \mathbb{R}^{n-1}. \end{aligned}$$

For  $\Gamma$  and  $\Gamma^{reg}$  we also have Varadhan's bounds[30] for  $s \neq s_1$

$$\begin{aligned} \lim_{t \rightarrow 0^+} t \ln \Gamma_{++}(s, s_1, t) &= \lim_{t \rightarrow 0^+} t \ln \Gamma_{++}^{reg}(s, s_1, t) = -\frac{d(s, s_1)^2}{4D_+}, \\ \lim_{t \rightarrow 0^+} t \ln \Gamma_{-+}(s, s_1, t) &= \lim_{t \rightarrow 0^+} t \ln \Gamma_{-+}^{reg}(s, s_1, t) = -\frac{d(s, s_1 \sqrt{\frac{D_-}{D_+}})^2}{4D_-}, \end{aligned}$$

where  $d(s, s_1)$  is the Riemannian distance between  $s$  and  $s_1$ , which is equal here to the Euclidean distance, since  $D_+$  and  $D_-$  are constant in  $\Omega_+$  and  $\Omega_-$  respectively.

## B.2 Case $0 < \lambda < \infty$

Let us consider the one-dimensional problem (49)–(51) with  $\lambda \equiv \lambda(\theta_0)$  and  $s \in \mathbb{R}$ . The associated problem for the heat kernel is then given by

$$\begin{aligned} (\partial_t - D_{\pm} \partial_s^2) G(s, s_1, t) &= 0, \\ G|_{t=0} &= \delta(s, s_1) \quad \text{for } s > 0, \\ D_- \frac{\partial}{\partial s} G(-0, s_1, t) &= \lambda(G(-0, s_1, t) - G(+0, s_1, t)), \end{aligned} \tag{109}$$

$$D_+ \frac{\partial}{\partial s} G(+0, s_1, t) = D_- \frac{\partial}{\partial s} G(-0, s_1, t). \tag{110}$$

We search the explicit solution of the problem[6] with

$$G(s, s_1, t) = \begin{cases} G_{-+}, & s < 0, s_1 > 0 \\ G_{++}, & s > 0, s_1 > 0 \end{cases}.$$

We seek for  $G_{-+}$  and  $G_{++}$  in terms of free heat kernel  $K(s, s_1, D_{\pm}t)$  (see Eq. (55)) and single layer heat potentials for  $s_1 > 0$ :

$$G_{++}(s, s_1, t) = K(s, s_1, D_+t) + D_+ \int_0^t K(s, 0, D_+(t-\tau))\alpha_+(s_1, \tau)d\tau \quad (s > 0),$$

$$G_{-+}(s, s_1, t) = D_- \int_0^t K(s, 0, D_-(t-\tau))\alpha_-(s_1, \tau)d\tau \quad (s < 0),$$

where  $\alpha_{\pm}(s_1, \tau)$  are unknown densities to be determined. Considering the boundary conditions (109)–(110) and the jumps of the first derivatives of  $G_{\pm+}$ ,

$$\frac{\partial}{\partial s}G_{++}|_{s=+0} = -\frac{1}{2}\alpha_+(s_1, t) + \frac{\partial}{\partial s}K(0, s_1, D_+t),$$

$$\frac{\partial}{\partial s}G_{-+}|_{s=-0} = -\frac{1}{2}\alpha_-(s_1, t),$$

we obtain two relations

$$D_- \alpha_-(s_1, t) = -D_+ \alpha_+(s_1, t) + 2D_+ \frac{\partial}{\partial s}K(0, s_1, D_+t),$$

$$D_- \alpha_-(s_1, t) = 2\lambda K(0, s_1, D_+t) + \lambda \frac{\sqrt{D_+}}{\sqrt{\pi}} \int_0^t \frac{\alpha_+(s_1, \tau)}{\sqrt{t-\tau}} d\tau$$

$$- \lambda \frac{\sqrt{D_-}}{\sqrt{\pi}} \int_0^t \frac{\alpha_-(s_1, \tau)}{\sqrt{t-\tau}} d\tau.$$

Following the method from Ref. [6], we solve the system corresponding to  $\alpha_-(s_1, t)$  and  $\alpha_+(s_1, t)$ :

$$D_- \alpha_-(s_1, t) + D_+ \alpha_+(s_1, t) = 2D_+ \frac{\partial}{\partial s}K(0, s_1, D_+t),$$

$$D_- \alpha_-(s_1, t) + \frac{\lambda\sqrt{D_-}}{\sqrt{\pi}} \left(1 + \sqrt{\frac{D_-}{D_+}}\right) \int_0^t \frac{\alpha_-(s_1, \tau)}{\sqrt{t-\tau}} d\tau = 4\lambda K(0, s_1, D_+t).$$

We obtain therefore the Abel integral equation of the second kind for  $\alpha_-(s_1, t)$

$$\alpha_-(s_1, t) + \gamma \int_0^t \frac{\alpha_-(s_1, \tau)}{\sqrt{t-\tau}} d\tau = \frac{4\lambda}{D_-} K(0, s_1, D_+t),$$

where  $\gamma = \frac{\lambda}{\sqrt{\pi D_-}} \left(1 + \sqrt{\frac{D_-}{D_+}}\right)$ . Consequently,

$$\alpha_-(s_1, t) = \frac{4\lambda}{D_-} K(0, s_1, D_+t) - \gamma \frac{4\lambda}{D_-} \int_0^t \frac{K(0, s_1, D_+\tau)}{\sqrt{t-\tau}} d\tau$$

$$+ \pi\gamma^2 \frac{4\lambda}{D_-} \int_0^t e^{\pi\gamma^2(t-\tau)} \left( K(0, s_1, D_+\tau) - \gamma \int_0^\tau \frac{K(0, s_1, D_+s)}{\sqrt{\tau-s}} ds \right) d\tau.$$

Using the Laplace transform yields, after simplifications:

$$G_{++}(s, s_1, t) = \frac{1}{\sqrt{4\pi D_+ t}} \left( \exp\left(-\frac{(s-s_1)^2}{4D_+ t}\right) + \exp\left(-\frac{(s+s_1)^2}{4D_+ t}\right) \right) - \frac{\lambda}{D_+} \exp\left(\frac{\lambda\alpha}{\sqrt{D_+}}(s+s_1) + \lambda^2\alpha^2 t\right) \operatorname{Erfc}\left(\frac{s+s_1}{2\sqrt{D_+ t}} + \lambda\alpha\sqrt{t}\right),$$

where  $\alpha = \frac{1}{\sqrt{D_-}} + \frac{1}{\sqrt{D_+}}$ . By the same way,

$$G_{-+}(s, s_1, t) = \frac{\lambda}{\sqrt{D_- D_+}} \exp\left(\frac{\lambda\alpha}{\sqrt{D_-}}\left(-s + s_1\sqrt{\frac{D_-}{D_+}}\right) + \lambda^2\alpha^2 t\right) \cdot \operatorname{Erfc}\left(\frac{-s + s_1\sqrt{\frac{D_-}{D_+}}}{2\sqrt{D_- t}} + \lambda\alpha\sqrt{t}\right).$$

We see that the Green function  $G_{++}$  for  $\lambda = 0$  becomes the Green function of the problem with the Neumann boundary conditions and in this case  $N(t) = 0$ , as  $u_- \equiv 0$ . This property,  $N(t) = 0$ , can be also directly found using the Green function.

In  $\mathbb{R}^n$  for  $x = (s, \theta)$  and  $y = (s_1, \theta_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$  we have

$$\begin{aligned} G_{++}(s, \theta, s_1, \theta_1, t)_{\mathbb{R}^n} &= G_{++}(s, s_1, t)_{\mathbb{R}} K(\theta, \theta_1, D_+ t)_{\mathbb{R}^{n-1}}, \\ G_{-+}(s, \theta, s_1, \theta_1, t)_{\mathbb{R}^n} &= G_{-+}(s, s_1, t)_{\mathbb{R}} K(\theta, \theta_1, D_- t)_{\mathbb{R}^{n-1}}. \end{aligned}$$

Therefore in  $\mathbb{R}^n$  for Varadhan's bounds with  $x \neq y$  we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} t \ln G_{++}(x, y, t)_{\mathbb{R}^n} &= -\frac{d(x, y)^2}{4D_+}, \\ \lim_{t \rightarrow 0^+} t \ln G_{-+}(x, y, t)_{\mathbb{R}^n} &= -\frac{d(s, s_1\sqrt{\frac{D_-}{D_+}})^2 + d(\theta, \theta_1)^2}{4D_-}. \end{aligned}$$

**Remark 3** *Applying this framework to the same system but with the transmittal boundary*



condition for  $0 < \lambda < \infty$ , we obtain

$$\begin{aligned}
G_{++}(s, \theta, s_1, \theta_1, t) &= \frac{1}{(4\pi D_+ t)^{\frac{n}{2}}} \left( \exp\left(-\frac{(s - s_1 - tD_+ k)^2}{4D_+ t}\right) \right. \\
&+ \exp\left(-\frac{(s + s_1 - tD_+ k)^2}{4D_+ t}\right) \left. \right) \exp\left(-\frac{d(\theta, \theta_1)^2}{4D_+ t}\right) \\
&- \frac{1}{(4\pi D_+ t)^{\frac{n-1}{2}}} \frac{\lambda}{D_+} \exp\left(\frac{\lambda\alpha}{\sqrt{D_+}}(s + s_1 - tD_+ k) + \lambda^2\alpha^2 t\right) \cdot \\
&\cdot \operatorname{Erfc}\left(\frac{s + s_1 - tD_+ k}{2\sqrt{D_+ t}} + \lambda\alpha\sqrt{t}\right) \exp\left(-\frac{d(\theta, \theta_1)^2}{4D_+ t}\right), \\
G_{-+}(s, \theta, s_1, \theta_1, t) &= \frac{1}{(4\pi D_- t)^{\frac{n-1}{2}}} \frac{\lambda}{\sqrt{D_- D_+}} \\
&\cdot \exp\left(\frac{\lambda\alpha}{\sqrt{D_-}}\left(-\frac{D_+}{D_-}s + s_1\sqrt{\frac{D_+}{D_-}} + tD_+ k\right) + \lambda^2\alpha^2 t\right) \\
&\cdot \operatorname{Erfc}\left(\frac{-\frac{D_+}{D_-}s + s_1\sqrt{\frac{D_+}{D_-}} + tD_+ k}{2\sqrt{D_- t}} + \lambda\alpha\sqrt{t}\right) \exp\left(-\frac{d(\theta, \theta_1)^2}{4D_- t}\right).
\end{aligned}$$

We also notice that for a fixed  $t > 0$  for  $\lambda \rightarrow +\infty$  we obtain

$$G_{++}(s, s_1, t) \rightarrow \Gamma_{++}(s, s_1, t) \quad \text{and} \quad G_{-+}(s, s_1, t) \rightarrow \Gamma_{-+}(s, s_1, t).$$

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