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# Nonsmooth modal analysis of a $N$ degree-of-freedom system undergoing a purely elastic impact law

Mathias Legrand<sup>†</sup>, Stéphane Junca<sup>‡§</sup>, and Sokly Heng<sup>‡</sup>

**Abstract** The dynamics of a  $N$  degree-of-freedom autonomous oscillator undergoing an energy-preserving impact law on one of its mass is investigated in the light of nonlinear modal analysis. The impacted rigid foundation provides a natural Poincaré section of the investigated system from which is formulated a well-defined smooth First Return Map away from the grazing trajectory. Continuous one-parameter families of  $T$ -periodic orbits featuring one impact per period and lying on two-dimensional invariant manifolds in the state-space are shown to exist. The geometry of these piecewise-smooth manifolds is such that a linear “flat” portion (on which contact is not activated) is continuously attached to a purely nonlinear portion (on which contact is activated once per period) exhibiting a velocity discontinuity through a grazing orbit. These features explain the newly introduced terminology “Nonsmooth modal analysis”. The stability of the periodic orbits lying on the invariant manifolds is also explored by calculating the eigenvalues of the linearized First Return Map. Internal resonances and multiple impacts per period are not addressed in this work.

**Key words** nonlinear modal analysis; vibro-impact systems; frequency-energy plots; spectral stability analysis

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**1. Introduction** In the framework of linear vibration theory of conservative autonomous systems, natural modes of vibration, uniquely defined as a countable sequence of natural frequency and normalized modeshape pairs, exhibit interrelated properties appealing to the engineer and mathematician: (1) they span the state-space through the principle of superposition, (2) they are invariant (that is linearly independent), orthogonal, and uncouple the equations of motion thus enabling the construction of reduced-order models, (3) they efficiently predict potential vibratory resonances of periodically forced systems [16].

Nonlinear modes of vibration (NLM) are conceived as an extension of linear modes when a nonlinear term arises in the governing equations. Their existence in the vicinity of fixed points is ensured by the center-stable manifold theorem [23]. For autonomous conservative nonresonant systems,

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they are defined as continuous one-parameter families of periodic orbits realized on two-dimensional curved manifolds in the state-space reducing to the common “flat” eigenspaces in the linear framework. Similarly to the invariance property of linear modes, such manifolds are also invariant under the flow: trajectories stemming from an initial condition in the manifold will remain in the manifold as time unfolds [23, 14]. On such manifolds, the modal dynamics can be regarded as a single degree-of-freedom second-order nonlinear oscillator from which all position and velocity coordinates can be functionally parametrized. For certain classes of systems, equivalence between nonlinear modal analysis and normal form theory is also established [25]. Evidently, the principle of superposition does no longer apply for nonlinear systems but NLM are useful for the analysis of mode bifurcations and forced resonances as well as for the construction of reduced-order models [27]. Nonlinear oscillations are also known to exhibit energy-dependent natural frequencies (and corresponding modeshapes) [19], feature that is conveniently illustrated in frequency–energy plots (FEP) where they form the backbone of the nonlinear forced response curves [26].

The present work explores the free dynamics of  $N$  degree-of-freedom *impact oscillators* within the nonlinear modal analysis framework. Such systems commonly consist of  $N$  linearly coupled one-degree-of-freedom oscillators one of which undergoes a motion limiting constraint induced by the presence of a rigid foundation against which it impacts. The impenetrability condition is incorporated in the formulation through (1) an energy-preserving impact law acting on the impacting mass velocity and characterized by the *restitution coefficient*  $e = 1$  which introduces impulsive contact forces into the dynamics [5] and (2) a one-sided *admissibility* condition to be satisfied by the solution which has to be realized on the “appropriate” side of the rigid foundation. Between successive impacts, the motion is governed by a linear second-order ordinary differential equation in time but the two previous conditions introduce nonlinearity into the dynamics. An initially positive separating clearance between the impacting mass and the rigid foundation is assumed.

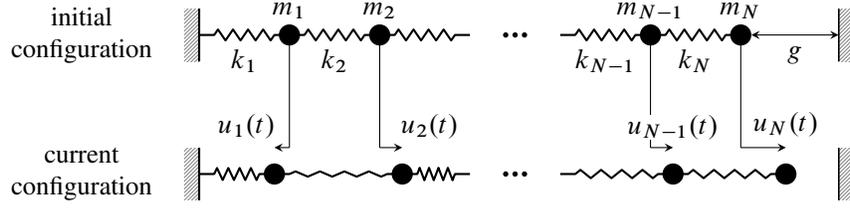
It should be noted that the literature on (mainly low-dimensional) vibro-impact oscillators is vast (among many other references, see [9, 7, 1] for an overview). Nevertheless, it seems much more limited when it comes to the nonlinear modal analysis of such systems. Three distinct formulations are then available: piecewise linear formulation [11, 12, 6], regularized unilateral formulation [18, 15], and purely nonsmooth formulation [21, 20]. Nonsmooth terms used to be replaced by simplified smooth approximations whose mathematical properties are better understood. This approach is acceptable in many situations but it is no longer justified to neglect such nonsmooth effects. In the, by now traditional, research area on nonsmooth systems, focus has been directed towards a differential inclusions, convex analysis, or complementarity formulation that deal with standard questions like existence and uniqueness of solutions.

The paper is organized as follows: as explained in section 2, the proposed procedure involves the construction of a First Return Map where the switching hyperplane defined by the boundary of the rigid foundation impacted by the oscillator serves as a Poincaré section in the state-space. Calculating the targeted families of periodic orbits reduces to determining the fixed points of this map. The corresponding main results are given in section 3. In section 4, (quasi-) closed-form solutions are provided but the first return time has to be approximated numerically. Admissible trajectories defining nonlinear modes of vibration are discussed in section 5. Stability together with an introductory bifurcation analyses are then undertaken by computing the eigenvalues of the linearized First Return Map numerically evaluated at a fixed-point in section 6.

## 2. Assumptions and formulations

**2.1. Governing equations of motion** A generic, initially unstressed, autonomous and conservative  $N$  degree-of-freedom oscillator, similar in essence to the one illustrated in Fig. 2.1, is investigated.

The  $N$  masses  $m_i$  and stiffnesses  $k_i$ ,  $i = 1, \dots, N$  are associated to the  $N$  degrees-of-freedom, or



**Figure 2.1:** Investigated  $N$  degree-of-freedom vibratory system subject to a unilateral contact constraint on  $u_N(t)$ .

displacements,  $u_i(t)$  where  $t$  stands for time. The last degree-of-freedom  $u_N(t)$  is subject to an impenetrability condition due to the presence of a rigid foundation. The corresponding equations governing the dynamics of the system shall be expressed in a compact matrix form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{f}^c = \mathbf{0} \quad (2.1)$$

where  $\mathbf{M} \in \mathbb{R}^{N,N}$  and  $\mathbf{K} \in \mathbb{R}^{N,N}$  are the mass and stiffness matrices of the system, respectively. Displacements are stored in vector  $\mathbf{u} = [u_1, \dots, u_{N-1}, u_N]^\top \in \mathbb{R}^N$  and vector  $\mathbf{f}^c = [0, \dots, 0, f_N^c]^\top \in \mathbb{R}^N$  stands for the contact force emerging from the unilateral condition. At this point, the initial conditions in displacement  $\mathbf{u}$  and velocity  $\dot{\mathbf{u}}$  are not discussed.

Unilateral contact constraints are commonly expressed as complementarity conditions such that the contact force vector  $\mathbf{f}^c$  in Eq. (2.1) becomes

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \lambda\mathbf{B}^\top &= \mathbf{0} \\ \mathbf{B}\mathbf{u} - g &\leq 0 \quad ; \quad \lambda \leq 0 \quad ; \quad \lambda(\mathbf{B}\mathbf{u} - g) = 0 \end{aligned} \quad (2.2)$$

where, in this work, matrix  $\mathbf{B} \in \mathbb{R}^{1,N}$  maps the displacement vector  $\mathbf{u}$  to the  $N$ th displacement  $u_N$ , that is  $\mathbf{B} = \mathbf{e}_N^\top = (0 \ 0 \ \dots \ 0 \ 1)$ . It is well-known that formulation (2.2) is ill-posed [8, 5, 4] and the complementarity conditions should be supplemented with an impact law governing the dynamics during impact, that is

$$u_N(t) = g \quad \Rightarrow \quad \dot{u}_N^+(t) = -\dot{u}_N^-(t) \quad (2.3)$$

where the notations

$$\dot{u}_N^+(t) \stackrel{\text{def}}{=} \lim_{0 < \epsilon \rightarrow 0} \dot{u}_N(t + \epsilon) \quad \text{and} \quad \dot{u}_N^-(t) \stackrel{\text{def}}{=} \lim_{0 < \epsilon \rightarrow 0} \dot{u}_N(t - \epsilon) \quad (2.4)$$

are used. It is understood that  $\dot{u}_N$ , the velocity of the  $N$ th degree-of-freedom, is a function of bounded variations. Usually, the reflection coefficient  $e$  belongs to the interval  $[0; 1]$  with the reflection rule  $\dot{u}_N^+(t) = -e\dot{u}_N^-(t)$ . Since we are targeting periodic motions without impact dissipation,  $e = 1$  in (2.3) by assumption.

Following [1, 2, 3], the formulation of the problem of interest now takes the form: Find the displacement  $\mathbf{u}(t)$  and the contact force  $\lambda(t)$  satisfying

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \lambda\mathbf{B}^\top, \quad (2.5a)$$

$$\lambda \text{ is a non-positive measure,} \quad (2.5b)$$

$$u_N(t) - g \leq 0, \quad \forall t, \quad (2.5c)$$

$$\text{suppr } \lambda \subset \{t; u_N(t) = g\}, \quad (2.5d)$$

$$u_N(t) = g \quad \Rightarrow \quad \dot{u}_N^+(t) = -\dot{u}_N^-(t). \quad (2.5e)$$

The measure  $\lambda$  depends on the solution  $\mathbf{u}$ . It arises as a nonlinear impulsive restoring force in (2.5a).

**2.2. Periodic solutions with a single impact per period** This work targets one-parameter continuous families of periodic solutions. If such families exist, it is expected that they will be organized on two-dimensional invariant manifolds which usually characterize modes of vibration for nonlinear mechanical systems [14, 11, 21].

A major assumption is made: trajectories experience a single impact per period where the period  $T$  is to be found. Since the system of interest is autonomous, the phase is free and formulation (2.5) can be advantageously simplified by considering, without loss of generality, that the impact, if any, occurs at  $t = T$ . Accordingly, the problem shall be reformulated and simplified as follows (see [6] for a similar formulation for piecewise linear systems):

**Definition 2.1 [Extended formulation]** Find  $T$  and  $\mathbf{u}(t)$  such that:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}, \quad \forall t \in ]0; T[ \quad (2.6a)$$

$$\mathbf{u}(T) = \mathbf{u}(0) \quad (2.6b)$$

$$u_N(0) = g \quad (2.6c)$$

$$\dot{\mathbf{u}}^-(T) = \mathbf{S}\dot{\mathbf{u}}^+(0) \quad (2.6d)$$

where  $\mathbf{S}$  is the matrix of symmetry with respect to the hyperplane  $\dot{u}_N = 0$ , or jump matrix:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix} \quad (2.7)$$

System (2.6) says the following:

- Away from an impact, the system is free and its dynamics governed by Eq. (2.6a).
- The displacement is a continuous and periodic function of time.
- The period  $T$  is an unknown of the problem.
- At the beginning of the period  $t = 0$ , the impacting mass is located on the rigid wall.
- As reflected by the matrix  $\mathbf{S}$ , the velocities of masses 1 to  $N - 1$  are continuous and periodic functions of time and the velocity of mass  $N$  is discontinuous at  $t = T$  where it satisfies  $\dot{u}_N^+(t) = -\dot{u}_N^-(t)$ .

In (2.6), the measure  $\lambda$  does not seem to arise explicitly. In fact, it emerges as the zero measure in (2.6a) when  $u_N(t) - g < 0$  and as a Dirac measure at  $t = T$  such that  $u_N(T) = g$  in (2.6d) which defines its magnitude. Also, the less restrictive formulation (2.6) exhibits more  $T$ -periodic solutions than (2.5) because the unilateral condition  $u_N(t) - g < 0$  might be violated on interval  $]0; T[$ : their *admissibility* will have to be systematically verified. Still, this extended formulation can be completely solved through closed-form expressions as detailed later.

**Definition 2.2 [Admissible solution]** A solution to the extended formulation 2.1 is said to be admissible if

$$\forall t, \quad u_N(t) \leq g. \quad (2.8)$$

**2.3. Eligible linear eigenspace structure** The construction of the nonlinear periodic solutions is tightly related to the eigenspace structure of the underlying linear mechanical system for which the unilateral contact constraint is discarded. In Eq. (2.1), matrices  $\mathbf{M}$  and  $\mathbf{K}$  are assumed to be positive

definite thus excluding rigid-body motions. Accordingly, there exists a matrix  $\mathbf{P}$  of  $\mathbf{M}$ -orthogonal eigenmodes which diagonalizes both the mass and stiffness matrices, that is

$$\begin{aligned}\mathbf{P}^\top \mathbf{M} \mathbf{P} &= \mathbf{I}_N \\ \mathbf{P}^\top \mathbf{K} \mathbf{P} &= \boldsymbol{\Omega}^2 = \text{diag}(\omega_i^2)_{i=1, \dots, N}\end{aligned}\tag{2.9}$$

where  $\mathbf{I}_N$  is the identity matrix in  $\mathbb{R}^{N, N}$  and  $\omega_i^2$ ,  $i = 1, \dots, N$  are the  $N$  eigenfrequencies of the underlying linear system. In the remainder, it is assumed that eigenfrequencies are all distinct, *ie*

$$0 < \omega_1 < \dots < \omega_N \quad \text{and} \quad 0 < T_N < \dots < T_1\tag{2.10}$$

where  $\omega_i T_i = 2\pi$ ,  $i = 1, \dots, N$ . In order to efficiently expose the coming developments, a definition based on the modal matrix  $\mathbf{P}$  is needed:

**Definition 2.3 [Interaction coefficient]** For all  $j = 1, \dots, N$ , the quantity:

$$a_j = P_{Nj} P_{jN}^{-1}\tag{2.11}$$

is called the interaction coefficient between the displacement of mass  $N$  along the  $j$ th linear mode and the unilateral contact constraint.

It should be noted that the interaction coefficients satisfy

$$\sum_{j=1}^N a_j = \mathbf{e}_N^\top \mathbf{P} \mathbf{P}^{-1} \mathbf{e}_N = 1.\tag{2.12}$$

**Assumption 2.1** None of the  $N$  interaction coefficients shall vanish, that is  $a_j \neq 0$ ,  $j = 1, \dots, N$ .

Mechanically, this simply means that the last mass always experiences a displacement along a linear modal motion. Further interpretations will be provided in the remainder of the paper.

**Definition 2.4 [Linear grazing orbit]** A linear grazing orbit associated to the  $j$ th linear mode is a periodic trajectory  $\mathbf{u}$  such that

$$\max_{t \in \mathbb{R}} u_N(t) = g.\tag{2.13}$$

Assumption 2.1 implies that  $P_{Nj} \neq 0$ ,  $j = 1, \dots, N$  which also reflects the existence and uniqueness of the  $j$ th linear grazing orbit

$$\mathbf{u}(t) = \frac{g}{P_{Nj}} \cos(\omega_j t) \mathbf{P}_{\cdot j}\tag{2.14}$$

where  $\mathbf{P}_{\cdot j}$  is column  $j$  of matrix  $\mathbf{P}$ . In other words, the last row  $\mathbf{e}_N^\top \mathbf{P} = \mathbf{P}_{N\cdot}$  of the eigenvectors matrix has no vanishing terms. Hence, a linear grazing orbit is automatically admissible since  $u_N(t) \leq g$  for all time  $t$ . Since  $\mathbf{P}^{-1} = \mathbf{P}^\top \mathbf{M}$ , Assumption 2.1 also entails

$$P_{jN}^{-1} = \mathbf{e}_j^\top \mathbf{P}^{-1} \mathbf{e}_N = \mathbf{e}_j^\top \mathbf{P}^\top \mathbf{M} \mathbf{e}_N = (\mathbf{M} \mathbf{e}_N)^\top \mathbf{P} \mathbf{e}_j \neq 0, \quad j = 1, \dots, N\tag{2.15}$$

where  $\mathbf{e}_j^\top = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^N$  is a row vector with a 1 located at its  $j$ th coordinate. In particular, if the mass matrix is the constant diagonal matrix  $\mathbf{M} = m \mathbf{I}_N$  with  $m > 0$  then  $\mathbf{M} \mathbf{e}_N = m \mathbf{e}_N$ ,  $P_{jN}^{-1} = m P_{Nj}$ ,  $a_j = m P_{Nj}^2 > 0$ , and  $a_j \neq 0$  reduces to  $P_{Nj} \neq 0$ . For a diagonal mass matrix, the sign of the interaction coefficients is known through Assumption 2.1

$$\mathbf{M} = m \mathbf{I}_N \quad \Rightarrow \quad a_j = m P_{Nj}^2 > 0, \quad j = 1, \dots, N.\tag{2.16}$$

*Remark 1.* In this work, all numerical examples and corresponding illustrations are provided for the generic oscillator depicted in Fig. 2.1 with  $N = 5$ ,  $m_j = m = 1/N$ , and  $k_j = k = N$ ,  $i = j, \dots, N$ , that is

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix}. \quad (2.17)$$

As such, it falls into the framework explained above. Yet, the proposed construction of nonsmooth modes of vibration is still valid for a much broader class of discrete oscillators.

**2.4. First Return Map** The admissible state-space of the investigated system takes the form

$$\mathcal{D} = \mathcal{D}^\circ \cup H^+ \cup H^- \cup H^0 \quad (2.18)$$

where

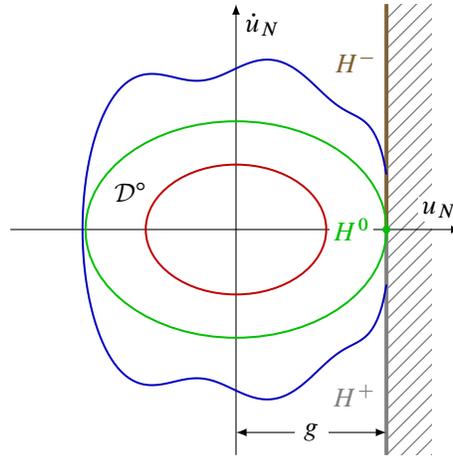
$$\mathcal{D}^\circ = \{(\mathbf{u}, \dot{\mathbf{u}}) \in \mathbb{R}^{2N} \text{ such that } u_N < g\}, \quad (2.19a)$$

$$H^+ = \{(\mathbf{u}, \dot{\mathbf{u}}^+) \in \mathbb{R}^{2N} \text{ such that } u_N = g \text{ and } \dot{u}_N^+ < 0\}, \quad (2.19b)$$

$$H^- = \{(\mathbf{u}, \dot{\mathbf{u}}^-) \in \mathbb{R}^{2N} \text{ such that } u_N = g \text{ and } \dot{u}_N^- > 0\}, \quad (2.19c)$$

$$H^0 = \{(\mathbf{u}, \dot{\mathbf{u}}) \in \mathbb{R}^{2N} \text{ such that } u_N = g \text{ and } \dot{u}_N = 0\}. \quad (2.19d)$$

It is illustrated in Fig. 2.2.



**Figure 2.2:** Cross-section of the admissible state-space. The hypersurface  $H = H^+ \cup H^0 \cup H^-$  partitions the state-space into two regions. The hatched area is the zone of non-admissible displacements and its complementary  $\mathcal{D}^\circ$  in the state-space is populated by admissible orbits of which are one admissible periodic orbit with one impact per period [blue], one admissible periodic orbit with no impact (linear system) [red], and one grazing periodic orbit [green].

Seeking solutions of the extended formulation 2.1 can be equivalently seen as finding fixed-points of a First Return Map, the so called Poincaré map. Among all eligible Poincaré cross-sections, the half hyperplane  $H^+$  plays a crucial role in the impact dynamics and is selected. Loosely speaking, it is expected that the Poincaré map is well defined from  $H^+$  to  $H^+$ . Indeed, we prove later that near an

admissible periodic solution with initial data in  $H^+$ , the First Return Map is locally well-defined and smooth. Since our focus is on periodic solutions with a unique impact per period, trajectories  $(\mathbf{u}, \dot{\mathbf{u}})$  defined as follows

$$H^+ \rightarrow \mathcal{D}^\circ \rightarrow H^- \rightarrow H^+ \quad (2.20)$$

are considered. More challenging situations shall emerge<sup>1</sup> with potential grazing impacts on  $H^0$  between two impacts on  $H^+$  as illustrated in Fig. 5.2.

In order to enjoy an almost closed-form expression of the Poincaré map, Eq. (2.6a) is projected onto the linear modal space through the transformation  $\mathbf{u} = \mathbf{P}\mathbf{q}$  to become

$$\mathbf{I}_N \ddot{\mathbf{q}} + \boldsymbol{\Omega}^2 \mathbf{q} = \mathbf{0} \quad (2.21)$$

by invoking Eq. (2.9). Assume the existence of the first return time  $T$  from  $H^+$  to  $H^-$  which is a nonlinear function of the initial state  $(\mathbf{u}(0), \dot{\mathbf{u}}^+(0))$  of the system. It is then straightforward to map the state of system at  $t = 0^+$  to its state at  $t = T^-$  by solving the linear differential system (2.21) [6, 20] as follows:

$$\mathbf{q}^-(T) = \cos(T\boldsymbol{\Omega})\mathbf{q}^+(0) + \boldsymbol{\Omega}^{-1} \sin(T\boldsymbol{\Omega})\dot{\mathbf{q}}^+(0) \quad (2.22a)$$

$$\dot{\mathbf{q}}^-(T) = -\boldsymbol{\Omega} \sin(T\boldsymbol{\Omega})\mathbf{q}^+(0) + \cos(T\boldsymbol{\Omega})\dot{\mathbf{q}}^+(0) \quad (2.22b)$$

where  $\cos(T\boldsymbol{\Omega}) \equiv \text{diag}(\cos(\omega_i T))_{i=1,\dots,N}$  and  $\sin(T\boldsymbol{\Omega}) \equiv \text{diag}(\sin(\omega_i T))_{i=1,\dots,N}$  are used as notations. In modal coordinates, the continuity of the displacement as well as the jump in velocity occurring at  $t = T$  become

$$\mathbf{q}^+(T) = \mathbf{q}^-(T) \quad (2.23a)$$

$$\dot{\mathbf{q}}^+(T) = \mathcal{S}\dot{\mathbf{q}}^-(T) \quad (2.23b)$$

where the expression of the jump matrix in modal coordinates is

$$\mathcal{S} = \mathbf{P}^{-1}\mathbf{S}\mathbf{P}. \quad (2.24)$$

When  $T$  is known and by inserting Eq. (2.23) into Eq. (2.22), a  $2N \times 2N$  Return Map matrix  $\mathbf{R}(T)$  mapping the system from  $t = 0^+$  to  $t = T^+$  can be built in modal coordinates. It includes the free flight in  $\mathcal{D}^\circ$  as well as one impact on  $H^-$ :

**Definition 2.5 [First Return Map]** *The modal state of the system at time  $t$  is denoted by the quantity  $\mathbf{Q}(t) = (\mathbf{q}(t), \dot{\mathbf{q}}^+(t)) = (\mathbf{P}^{-1}\mathbf{u}(t), \mathbf{P}^{-1}\dot{\mathbf{u}}^+(t))$ . If  $(\mathbf{u}(0), \dot{\mathbf{u}}^+(0)) \in H^+$ ,  $T = T(\mathbf{Q}(0)) > 0$  is the first return time to  $H^-$  where  $\mathbf{Q}(0)$  is the initial modal state of the system. Then, the modal state at time  $T^+$  is given by the First Return Map*

$$\mathbf{F}(\mathbf{Q}(0)) = \mathbf{Q}(T) = \mathbf{R}(T(\mathbf{Q}(0)))\mathbf{Q}(0) \quad (2.25)$$

with

$$\mathbf{R}(T) = \begin{bmatrix} \cos(T\boldsymbol{\Omega}) & \boldsymbol{\Omega}^{-1} \sin(T\boldsymbol{\Omega}) \\ -\mathcal{S}\boldsymbol{\Omega} \sin(T\boldsymbol{\Omega}) & \mathcal{S} \cos(T\boldsymbol{\Omega}) \end{bmatrix}. \quad (2.26)$$

<sup>1</sup> Trajectories with more than one impact per period are investigated in [24] for a two degree-of-freedom system.

To better “visualize” the expression of the First Return Map, Eq. (2.25) is expanded as

$$\begin{pmatrix} \mathbf{q}(T) \\ \dot{\mathbf{q}}^+(T) \end{pmatrix} = \begin{bmatrix} \cos(T\Omega) & \Omega^{-1} \sin(T\Omega) \\ -\mathcal{S}\Omega \sin(T\Omega) & \mathcal{S} \cos(T\Omega) \end{bmatrix} \begin{pmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}^+(0) \end{pmatrix}. \quad (2.27)$$

From the periodicity conditions (2.6b) and (2.6d), the extended formulation (2.1) is transformed into a nonlinear fixed point problem: find  $\mathbf{Q}(0)$  such that  $\mathbf{F}(\mathbf{Q}(0)) = \mathbf{Q}(0)$  also arising in the form of Eq. (2.25) as

$$\mathbf{Q}(0) = \mathbf{R}(T(\mathbf{Q}(0)))\mathbf{Q}(0) \quad \Leftrightarrow \quad (\mathbf{R}(T(\mathbf{Q}(0))) - \mathbf{I}_{2N})\mathbf{Q}(0) = \mathbf{0}_{2N}. \quad (2.28)$$

Determining the fixed-points of the First Return Map thus reduces to solving a generalized eigenvalue problem in  $(T, \mathbf{Q}(0))$ . Since  $\mathbf{Q}(0) \in H^+$ , half hyperplane of dimension  $2N - 1$ , there are  $2N = 1 + 2N - 1$  unknowns for  $2N$  equations. Although the scalar  $T$  is an implicit function of the  $2N - 1$  dimensional variable  $\mathbf{Q}(0)$ , the key point to solve nonlinear system (2.28) is to fix  $T$  such that system (2.28) is then linear in  $\mathbf{Q}(0)$ . The constraint  $\mathbf{Q}(0) \in H^+$  yields a unique solution (the proof is given later). In other words, we obtain a one-to-one relation between the period and the periodic solution with one impact per period<sup>2</sup>.

We might think that the First Return Map is a linear map, but this is a wrong statement. First, the first return time  $T$  from  $H^+$  to  $H^-$  is an unknown of the nonlinear transcendental algebraic Eq. (2.6c). Second, since  $\mathcal{S}$  is a non diagonal matrix, all the linear modes are fully coupled through (2.23b), which is a well known feature of nonlinearity.

Let us emphasize the expression of the impact nonlinearity in terms of the matrix  $\mathcal{S}$  and the interaction coefficients  $a_j$ . It is clear that  $\mathbf{S}$  is a diagonal matrix but  $\mathcal{S}$  is not. In the unrealistic case where  $\mathcal{S} = \mathbf{I}_N$  (which is strictly impossible), the reflection on  $H^-$  is removed and the problem would be simply linear. In other words, the nonlinearity shall be quantified by the difference  $\mathcal{S} - \mathbf{I}_N$  which is directly linked to the interaction coefficients  $a_j$  from definition 2.3. The identity  $\mathbf{S} = \mathbf{I}_N - 2\mathbf{e}_N\mathbf{e}_N^\top$  yields

$$\mathcal{S} = \mathbf{I}_N - 2\mathbf{P}^{-1}\mathbf{e}_N\mathbf{e}_N^\top\mathbf{P} = \mathbf{I}_N - 2\mathbf{P}_{\cdot N}^{-1}\mathbf{P}_{N\cdot}^\top. \quad (2.29)$$

The matrix  $\mathbf{e}_N\mathbf{e}_N^\top$  is the rank one matrix of the orthogonal projection onto  $\mathbf{e}_N$ . The interaction coefficients are the diagonal terms of the rank one matrix  $\mathbf{P}_{\cdot N}^{-1}\mathbf{P}_{N\cdot}^\top$ , uniquely defined by the last row of the eigenvector matrix and the last column of its inverse

$$a_j = (\mathbf{P}^{-1}\mathbf{e}_N\mathbf{e}_N^\top\mathbf{P})_{jj} = (\mathbf{P}_{\cdot N}^{-1}\mathbf{P}_{N\cdot}^\top)_{jj}. \quad (2.30)$$

The condition  $a_j \neq 0$  from assumption 2.1 simply means that the diagonal of the projector on  $\mathbf{e}_N$  expressed in the modal basis  $\mathbf{P}_{\cdot N}^{-1}\mathbf{P}_{N\cdot}^\top$  does not vanish. Accordingly, all the coefficients of this matrix are nonzero since it is a rank one matrix. More precisely, the condition  $a_j = 0$  has a mechanical interpretation:

- if  $P_{jN}^{-1} = 0$ , the entire row  $j$  of  $(\mathbf{P}_{\cdot N}^{-1}\mathbf{P}_{N\cdot}^\top)$  vanishes. Thus, equation  $j$  of (2.23b) becomes  $q_j^+(T) = q_j^-(T)$  which means that the linear mode  $j$  is not coupled to any of the other modes  $k \neq j$  on  $H^-$ .
- if  $P_{Nj} = 0$ , none of the modes  $k \neq j$  are affected by mode  $j$  on  $H^-$  since the  $N$ th mass does not move along linear mode  $j$ .

<sup>2</sup> The solution to Eq. (2.28) is depicted in the form of a Frequency-Energy plot in Fig. 3.2: it is the union of the thin grey and thick black lines. As long as the non-admissible solutions are not discarded, the system is shown to exhibit a continuous spectrum.

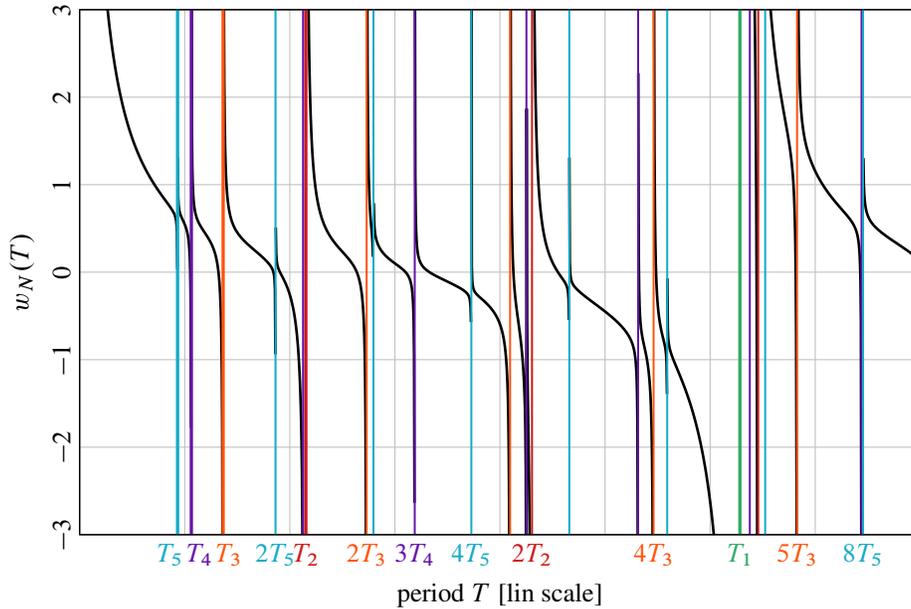
**3. Main results** This paper is mainly concerned with the construction of nonlinear modes of vibration of a conservative and autonomous  $N$  degree-of-freedom impact oscillator defined as families of periodic solutions with one impact per period. These periodic solutions are found to be organized on two-dimensional invariant manifolds in the state-space. The proposed formulation gives access to (almost) closed-form solutions, admissibility criteria, as well as spectral stability (and bifurcation) of these nonlinear modes.

*Remark 2.* In this contribution, derivations are mostly expressed in terms of the period  $T$  but the plots are displayed in terms of  $\omega = 2\pi/T$  in line with conventional Frequency-Energy Plots. As stated in [14], “[a] nonlinear modal motion is represented by a point in Frequency-Energy Plots, which is drawn at a frequency corresponding to the minimal period of the periodic motion and at an energy equal to the conserved total energy during the motion. A branch is a family of modal motions possessing the same qualitative features.”

A majority of the proposed results are related to the key function

$$w_N(T) = \sum_{j=1}^N a_j \phi_j(T), \quad \phi_j(T) = \frac{\sin(\omega_j T)}{\omega_j(1 - \cos(\omega_j T))} = \frac{1}{\omega_j \tan(\omega_j T/2)} \quad (3.1)$$

illustrated in Fig. 3.1. The existence and uniqueness of solutions to the extended formulation 2.1 is



**Figure 3.1:** Function  $w_N$  versus period  $T$  for the system of interest. The linear periods and corresponding subharmonics  $kT_i$ ,  $i = 1, \dots, 5$ ,  $k \in \mathbb{N}^{*+}$  [coloured vertical lines] are the poles of this function. All  $T$  such that  $w_N(T) = 0$  are the zeros of the function

proven as soon as  $0 < |w_N(T)| < +\infty$ .

*Remark 3.* The function  $w_N$  is a barycenter of the functions  $\phi_j$  through (2.12).

*Remark 4.* The poles of  $w_N$ , for which  $|w_N(T)| = \infty$ , are all the linear periods and their subharmonics, that is  $T \in \cup_{j=1}^N T_j \mathbb{N}$  (see Fig. 3.1).

*Remark 5.* The function  $w_N$  admits at most a countable set of zeros.

**Assumption 3.1 [No internal resonance assumption]** *The linear periods of the system of interest satisfy  $\bigcap_{j=1}^N T_j \mathbb{N} = \emptyset$ .*

Through assumption 3.1, difficulties induced by internal resonances are discarded. We then have the following results:

**Theorem 3.1 [Extended formulation: existence and uniqueness of solutions]** *Assume a positive  $T$  that is neither a pole nor a zero of  $w_N$ , then for all such  $T$ :*

- *Extended formulation 2.1 admits one and only one solution.*
- *The set of initial data*

$$\mathcal{V}_0 = \{(\mathbf{u}(0; T), \dot{\mathbf{u}}^+(0; T)), 0 < T, 0 < |w_N(T)| < +\infty\} \quad (3.2)$$

*for the extended formulation is an analytic curve which generates a set of periodic solutions forming a piecewise analytic manifold of dimension 2 defined as*

$$\mathcal{V} = \{(\mathbf{u}(t; T), \dot{\mathbf{u}}^+(t; T)), 0 \leq t \leq T, 0 < T, 0 < |w_N(T)| < \infty\} \quad (3.3)$$

*in the state-space  $\mathbb{R}^{2N}$ .*

- *If both assumptions 2.1 and 3.1 are satisfied, then the Lipschitz extension  $\mathbf{u}(\cdot; T)$  when  $T \rightarrow T_j$  is the linear grazing solution denoted  $\mathbf{u}(\cdot; T_j)$  here.*

Note that the analytic curve is not connected when  $w_N(T) = 0$ . When assumption 3.1 holds and  $w_N(T) = \pm\infty$ , then there is an analytic connection with the initial data associated to a linear grazing orbit.

Solutions to the extended formulation are not necessarily admissible solutions since many of them violate the non-penetration condition (2.8): they are called ‘‘phantom solutions’’. As detailed later, a one-sided condition naturally arises near the linear period  $T_j$  [21].

**Theorem 3.2 [One-sided condition for admissible periodic solutions]** *Assumptions 2.1 and 3.1 hold. Then, for all  $T > 0$  such that  $T \approx T_j$  and  $a_j(T - T_j) < 0$ :*

- *The unique  $T$ -periodic solution  $\mathbf{u}(\cdot; T)$  of theorem 3.1 is admissible.*
- *The invariant manifold of admissible periodic solutions with one impact per period near the linear grazing orbit is a piecewise smooth manifold in the admissible state-space  $\mathcal{D}^\circ$ .*

*Moreover:*

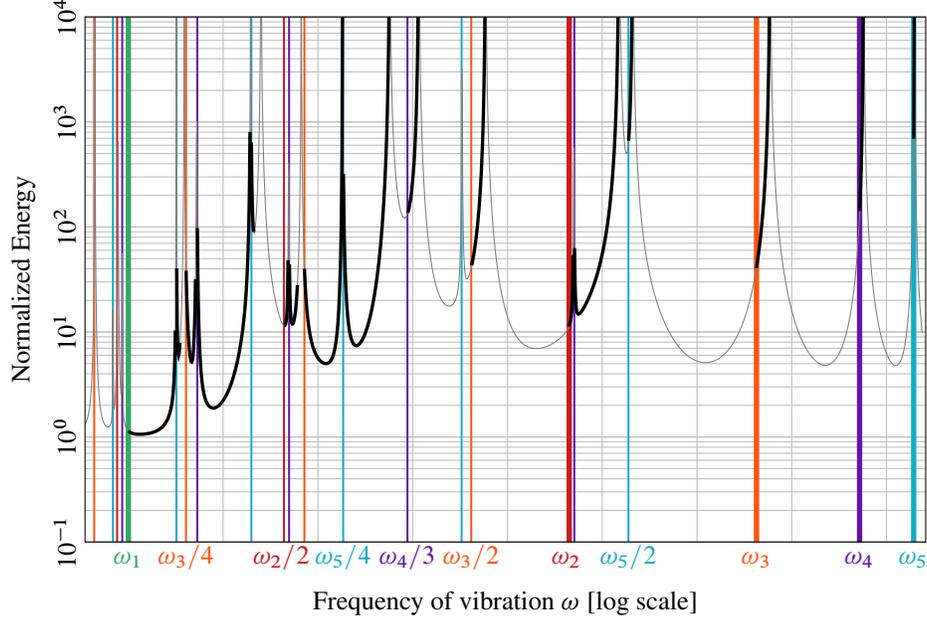
- *For  $a_j(T - T_j) > 0$  and  $T \approx T_j$ , the solution of the extended formulation is not admissible.*
- *If for all  $j$ ,  $a_j > 0$  then the admissible frequencies are bounded.*

The extended formulation 2.1 also exhibits solutions near linear subharmonics. The question of their admissibility is more challenging since in the most general case, a numerical verification is required. Still, a simple criteria for subharmonics of order one-half exists.

**Theorem 3.3 [Admissible solutions near linear subharmonics of order one-half]** *Assumptions 2.1 and 3.1 hold. Then, for  $T \approx 2T_j$  such that  $a_j(T - 2T_j) < 0$  and  $\mathbf{u}(T/2; T) \leq g$ , the solution  $t \mapsto \mathbf{u}(t; T)$  is admissible.*

For subharmonics of lower order ( $1/3, 1/4, \dots$ ), the admissibility is discussed in Section 5.2.

Theorems 3.1, 3.2 and 3.3 are illustrated in Fig. 3.2 for the system of interest. It should be noted that this system features only strictly positive interaction coefficients  $a_j$ ,  $j = 1, \dots, 5$ , stemming from Eq. (2.16): from a mechanical standpoint, only stiffening is observed when contact is initiated. When phantom solutions are excluded, the initially continuous spectrum becomes a set of admissible branches organized by the zeros and poles of  $w_N$ .



**Figure 3.2:** Frequency-Energy Plot: non-admissible spectrum [thin gray line] and admissible spectrum [thick black line]. Linear eigenfrequencies  $\omega_i$ ,  $i = 1, \dots, 5$  [thick vertical coloured lines] as well as the corresponding subharmonics  $\omega_i/k$ ,  $i = 1, \dots, 5$ ,  $k = 1, \dots, 10$  [thin vertical coloured lines] are displayed. The frequency interval  $\omega \in [0; \omega_1]$  shows admissible branches that are not displayed here for the sake of clarity.

The key-tool for the stability analysis is the First Return Map whose fixed-points can be expressed in a closed form up as soon as the first return time  $T$  is (numerically) known. Moreover, in the vicinity of an impacting periodic orbit, the First Return Map is locally well defined and a fully explicit differential of this map is also available. Stability can then be evaluated numerically by computing its eigenvalues. The corresponding results are expressed in modal coordinates for conciseness where  $\mathbf{Q}(t; T) = (\mathbf{q}(t; T), \dot{\mathbf{q}}^+(t; T))$  denotes a  $T$ -periodic solution. We also choose the notations  $\bar{\mathbf{e}}_N^\top = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{2N}$  ( $\bar{\mathbf{e}}_N^\top$  is a row vector with a 1 located at the  $N$ th coordinate) and  $\mathbf{P} = \mathbf{B} \text{diag}(\mathbf{P}, \mathbf{P})$ .

**Theorem 3.4 [First Return Map near an impacting periodic orbit]** *Let  $T_0 \notin \bigcup_{j=1}^N T_j \mathbb{N}$  be a nonlinear period and  $\mathbf{Q}_0 = \mathbf{Q}(0; T_0) \in H^+$  be the initial condition generating an orbit of period  $T_0 = T(\mathbf{Q}_0)$ . Then, for all  $\mathbf{Q} \simeq \mathbf{Q}_0 \in H^+$ , the first return time  $T(\mathbf{Q})$  and the First Return Map  $\mathbf{F}(\mathbf{Q})$  are well defined and smooth. Moreover, the gradient of the first return time and the matrix of the linearized First Return Map at  $\mathbf{Q}_0$  are respectively given by*

$$\nabla T(\mathbf{Q}_0) = \frac{w_N(T_0)}{g} \bar{\mathbf{e}}_N^\top \bar{\mathbf{P}} \mathbf{R}(T_0), \quad (3.4)$$

$$\mathbf{DF}(\mathbf{Q}_0) = \mathbf{R}(T_0) + \dot{\mathbf{R}}(T_0)[\mathbf{Q}_0 \nabla T(\mathbf{Q}_0)]. \quad (3.5)$$

Furthermore, 1 is an eigenvalue of  $\mathbf{DF}(\mathbf{Q}_0)$ .

Numerical examples seem to show that 1 is a simple eigenvalue as opposed to usual (smooth) Hamiltonian systems for which 1 is a double eigenvalue [17]. Another derivation of  $\mathbf{DF}$  in the space variables is proposed in appendix A.

**4. Explicit solutions to the extended formulation** In this section, existence and uniqueness of solution for the extended formulation are proven. Closed-form initial data generating periodic solutions with one impact per period is derived for all nonlinear periods  $T \notin \bigcup_{j=1}^N T_j \mathbb{N}$  and the associated invariant manifolds are investigated. Admissibility is discussed later.

The following quantities are well defined:

$$\dot{\mathbf{L}}(T) = \mathbf{\Omega}^{-1}(\mathbf{I} - \cos(T\mathbf{\Omega}))^{-1} \sin(T\mathbf{\Omega}) \quad (4.1a)$$

$$\mathbf{w}(T) = (\mathbf{P}\dot{\mathbf{L}}(T)\mathbf{P}^{-1})\mathbf{e}_N \quad (4.1b)$$

$$w_N(T) = \mathbf{e}_N^\top \mathbf{w}(T) \quad (4.1c)$$

Notice that  $w_N$  is exactly the function defined in (3.1). A first important result is established:

**Proposition 4.1 [Zero velocities, analytic manifolds, and symmetry]**

1. For all  $T \notin \bigcup_{j=1}^N T_j \mathbb{N}$ , there exists a unique solution  $t \mapsto \mathbf{u}(t; T)$  to the extended formulation with period  $T$  given by its initial data

$$\begin{aligned} \mathbf{u}(0; T) &= \dot{\mathbf{u}}_N^+(0; T)\mathbf{w}(T) \\ \dot{\mathbf{u}}^+(0; T) &= \dot{\mathbf{u}}_N^+(0; T)\mathbf{e}_N \end{aligned} \quad (4.2)$$

where  $\dot{\mathbf{u}}_N^+(0; T) = g/w_N(T)$ .

2. The initial data set of solutions is an analytic manifold as soon as  $0 < |w_N(T)| < +\infty$  and the set of periodic solutions is a piecewise analytic invariant manifold in the state-space.
3. The solution is symmetric with respect to  $T/2$  that is  $\mathbf{u}(T/2 + t; T) = \mathbf{u}(T/2 - t; T)$ ,  $\forall t$  and  $\dot{\mathbf{u}}(T/2; T) = 0$ .

Before proving proposition 4.1, we explicitly express the eigenspace associated to the eigenvalue 1 of  $\mathbf{R}(T)$  from Eq. (2.26).

**Lemma 4.1 [Fixed-points of the Linear Return Map  $\mathbf{R}$ ]** If  $T \notin \bigcup_{j=1}^N T_j \mathbb{N}$  then system (2.28) defines a one-dimensional vector space parametrized by  $c \in \mathbb{R}$ :

$$\begin{pmatrix} \mathbf{u}(0) \\ \dot{\mathbf{u}}^+(0) \end{pmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix} \begin{pmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}^+(0) \end{pmatrix} = c \begin{pmatrix} \mathbf{w}(T) \\ \mathbf{e}_N \end{pmatrix} \quad (4.3)$$

Note that all velocities except velocity  $N$  are zero when the  $N$ th mass hits the wall, as already known for a two degree-of-freedom system [20]. The solution in Eq. (4.3) can be seen as the ‘‘shape’’  $\mathbf{w}(T)$  of the mode (as commonly found for linear modes that are standing waves) to which is added an initial velocity on the  $N$ th degree-of-freedom.

*Proof.* The kernel  $\ker(\mathbf{R}(T) - \mathbf{I}_{2N})$  can be explicitly expressed through a simple block-manipulation of  $\mathbf{R}(T)$  detailed below:

$$\begin{aligned} & \begin{bmatrix} \cos(T\mathbf{\Omega}) - \mathbf{I}_N & \mathbf{\Omega}^{-1} \sin(T\mathbf{\Omega}) \\ -\mathbf{S}\mathbf{\Omega} \sin(T\mathbf{\Omega}) & \mathbf{S} \cos(T\mathbf{\Omega}) - \mathbf{I}_N \end{bmatrix} \sim \begin{bmatrix} \mathbf{I}_N - \cos(T\mathbf{\Omega}) & -\mathbf{\Omega}^{-1} \sin(T\mathbf{\Omega}) \\ -\mathbf{\Omega} \sin(T\mathbf{\Omega}) & \cos(T\mathbf{\Omega}) - \mathbf{S} \end{bmatrix} \\ & \sim \begin{bmatrix} \mathbf{I}_N - \cos(T\mathbf{\Omega}) & -\mathbf{\Omega}^{-1} \sin(T\mathbf{\Omega}) \\ \mathbf{0} & (\mathbf{I}_N - \cos(T\mathbf{\Omega}))(\cos(T\mathbf{\Omega}) - \mathbf{S}) - \sin^2(T\mathbf{\Omega}) \end{bmatrix} \end{aligned} \quad (4.4)$$

where  $\sim$  stands for ‘‘equivalent homogeneous systems’’. The lower-right block can be simplified as follows:

$$\begin{aligned} & (\mathbf{I}_N - \cos(T\mathbf{\Omega}))(\cos(T\mathbf{\Omega}) - \mathbf{S}) - \sin^2(T\mathbf{\Omega}) \\ &= -(\mathbf{I}_N - \cos(T\mathbf{\Omega}))\mathbf{S} + \cos(T\mathbf{\Omega}) - \cos^2(T\mathbf{\Omega}) - \sin^2(T\mathbf{\Omega}) \\ &= -(\mathbf{I}_N - \cos(T\mathbf{\Omega}))\mathbf{S} + (\cos(T\mathbf{\Omega}) - \mathbf{I}_N) \\ &= -(\mathbf{I}_N - \cos(T\mathbf{\Omega}))(\mathbf{S} + \mathbf{I}_N) \\ &= -(\mathbf{I}_N - \cos(T\mathbf{\Omega}))\mathbf{P}^{-1}(\mathbf{S} + \mathbf{I}_N)\mathbf{P}. \end{aligned} \quad (4.5)$$

By assumption, the matrix  $\mathbf{I}_N - \mathbf{cos}(T\boldsymbol{\Omega})$  is invertible and system (4.4) is equivalent to

$$\begin{bmatrix} \mathbf{I}_N - \mathbf{cos}(T\boldsymbol{\Omega}) & -\boldsymbol{\Omega}^{-1} \mathbf{sin}(T\boldsymbol{\Omega}) \\ \mathbf{0} & (\mathbf{S} + \mathbf{I}_N)\mathbf{P} \end{bmatrix}. \quad (4.6)$$

Since  $\dot{\mathbf{u}} = \mathbf{P}\dot{\mathbf{q}}$ , the right lower block in (4.6) yields  $(\mathbf{S} + \mathbf{I}_N)\dot{\mathbf{u}} = \mathbf{0}$ , that is  $\dot{\mathbf{u}} = c\mathbf{e}_N$  with  $c \in \mathbb{R}$ . Similarly, the upper block yields  $\mathbf{q} = c\boldsymbol{\Phi}(T)\mathbf{P}^{-1}\mathbf{e}_N$ .  $\square$

We now prove proposition 4.1 and theorem 3.1.

**Proof.** The identity  $c = \dot{u}_N^+(0)$  is a direct consequence of the last equality of (4.3) which means that the outgoing velocity at the boundary parametrizes every periodic solution. If  $w_N(T) \neq 0$ , condition (2.6c) implies

$$\dot{u}_N^+(0) = \frac{g}{w_N(T)}. \quad (4.7)$$

Due to the velocity jump, only piecewise regularity of  $\mathcal{V}$  can be expected. Nevertheless, the projection of  $\mathcal{V}$  on the displacement

$$\mathcal{V}_u = \{\mathbf{u}(t; T), 0 \leq t \leq T, 0 < T, 0 < |w_N(T)| < \infty\} \quad (4.8)$$

is at least continuous since the free flight response is governed by a linear differential system and is accordingly very smooth. The regularity of the manifold  $\mathcal{V}$  can be completely identified through the restriction of  $\mathcal{V}$  to the Poincaré section

$$\mathcal{V}_0 = \{(\mathbf{u}(0; T), \dot{\mathbf{u}}^+(0; T)), 0 < T, 0 < |w_N(T)| < \infty\}. \quad (4.9)$$

Singularities of  $\mathcal{V}_0$  shall arise when  $T$  approaches the poles  $\bigcup_{j=1}^N T_j\mathbb{N}$  of  $w_N$ . Consider the case  $T \rightarrow T_j$ . The only singular term in  $\mathbf{w}(T)$  stems from the matrix  $(\mathbf{I}_N - \mathbf{cos}(T\boldsymbol{\Omega}))^{-1}$  through assumption 3.1. More precisely, only the diagonal term

$$\phi_j(T) = \frac{\sin(\omega_j T)}{\omega_j(1 - \cos(\omega_j T))} \sim \frac{2}{\omega_j^2(T - T_j)} \quad (4.10)$$

becomes singular when  $T$  approaches  $T_j$  since all linear periods are distinct. From assumption 2.1,  $w_N(T)$  is precisely expanded as

$$w_N(T) = \sum_{k=1}^N a_k \phi_k(T) \sim a_j \phi_j(T) \quad \text{when } T \rightarrow T_j \quad (4.11)$$

and the condition  $u_N(0; T) = g = \dot{u}_N^+(0; T)w_N(T)$  yields

$$\dot{u}_N^+(0; T) \sim \frac{g}{a_j \phi_j(T)} \sim \frac{g}{2} \frac{\omega_j^2}{a_j} (T - T_j). \quad (4.12)$$

Thus  $\dot{u}_N^+(0; T) \rightarrow 0$  when  $T \rightarrow T_j$ : the manifold  $\mathcal{V}_0$  is smooth at  $T = T_j$  and recovers the initial data of the corresponding linear grazing orbit. Let  $\mathbf{e}_k$  be the vector in  $\mathbb{R}^N$  such that all coordinates are 0 except the  $k$ th one equal to 1. The computation is straightforward:

- if  $P_{kj} \neq 0$ :

$$w_k(T) = \mathbf{e}_k^\top \mathbf{w}(T) = \sum_{p=1}^N P_{kp} P_{pN}^{-1} \phi_p(T) \sim P_{kj} P_{jN}^{-1} \phi_j(T) \quad \text{when } T \rightarrow T_j \quad (4.13)$$

and thus

$$u_k(0; T) = \dot{u}_N^+(0; T)w_k(T) = g w_k(T)/w_N(T) \rightarrow g P_{kj}/P_{Nj}, \quad k = 1, \dots, N \quad (4.14)$$

and  $\dot{\mathbf{u}}^+(0; T) \rightarrow \mathbf{0}$  which forms the initial data of the linear grazing mode.

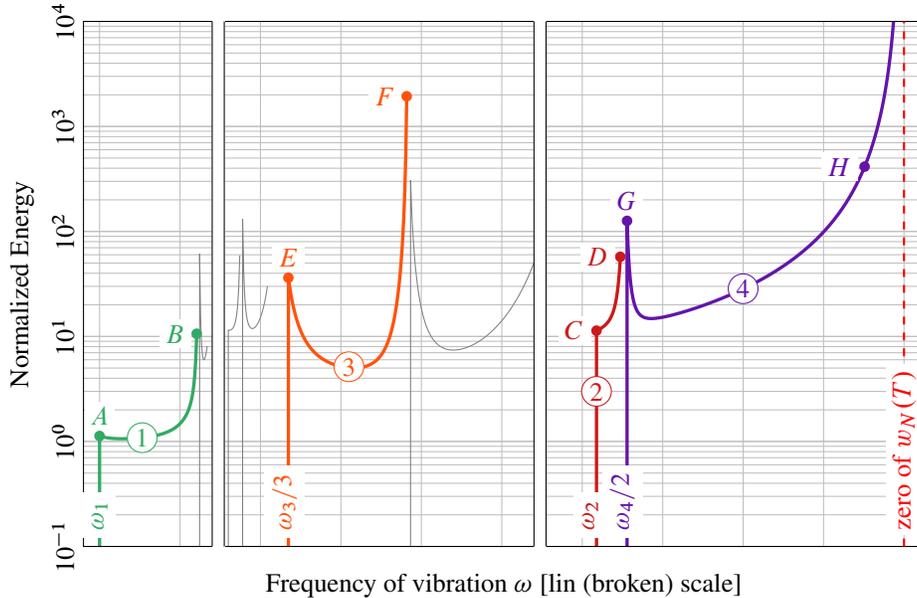
- if  $P_{k_j} = 0$ , then  $w_k(T) = \mathcal{O}(1)$  so  $u_k(0; T) \rightarrow 0$  and the previous statement still holds true. Moreover, the function  $T \mapsto (T - T_j)\phi_j(T)$  is analytic at  $T = T_j$  and then  $\mathcal{V}_0$  is an analytic manifold.

The same investigation is valid for the subharmonics  $T \simeq kT_j, k \in \mathbb{Z}^*$ .

Finally we prove the symmetry of the solutions. The proof only uses the uniqueness initial data theorem for linear differential system (without contact). To this end, let us define  $\mathbf{z}(t) := \mathbf{u}(T/2 + t), -T/2 < t < T/2$ : it is a solution of the linear differential system and can be extended to all time  $t \in \mathbb{R}$ . This vectorial function is very smooth and also well defined at  $t = \pm T/2$  with  $\mathbf{z}(T/2) = \mathbf{z}(-T/2) = \mathbf{u}(0)$  and  $\dot{\mathbf{z}}(T/2) = -\dot{\mathbf{z}}(-T/2) = -\dot{\mathbf{u}}^+(0)$ . As a consequence,  $\mathbf{z}(t)$  and  $\mathbf{z}(-t)$  satisfy the same linear differential system with the same initial data, that is  $\mathbf{z}(t) = \mathbf{z}(-t), \forall t \in \mathbb{R}$ . As a direct consequence,  $\dot{\mathbf{z}}(0) = 0$  that is  $\mathbf{u}(t) = \mathbf{u}(-t), \forall |t| \leq T/2$  and  $\dot{\mathbf{u}}(T/2) = \mathbf{0}$ , which concludes the proof.  $\square$

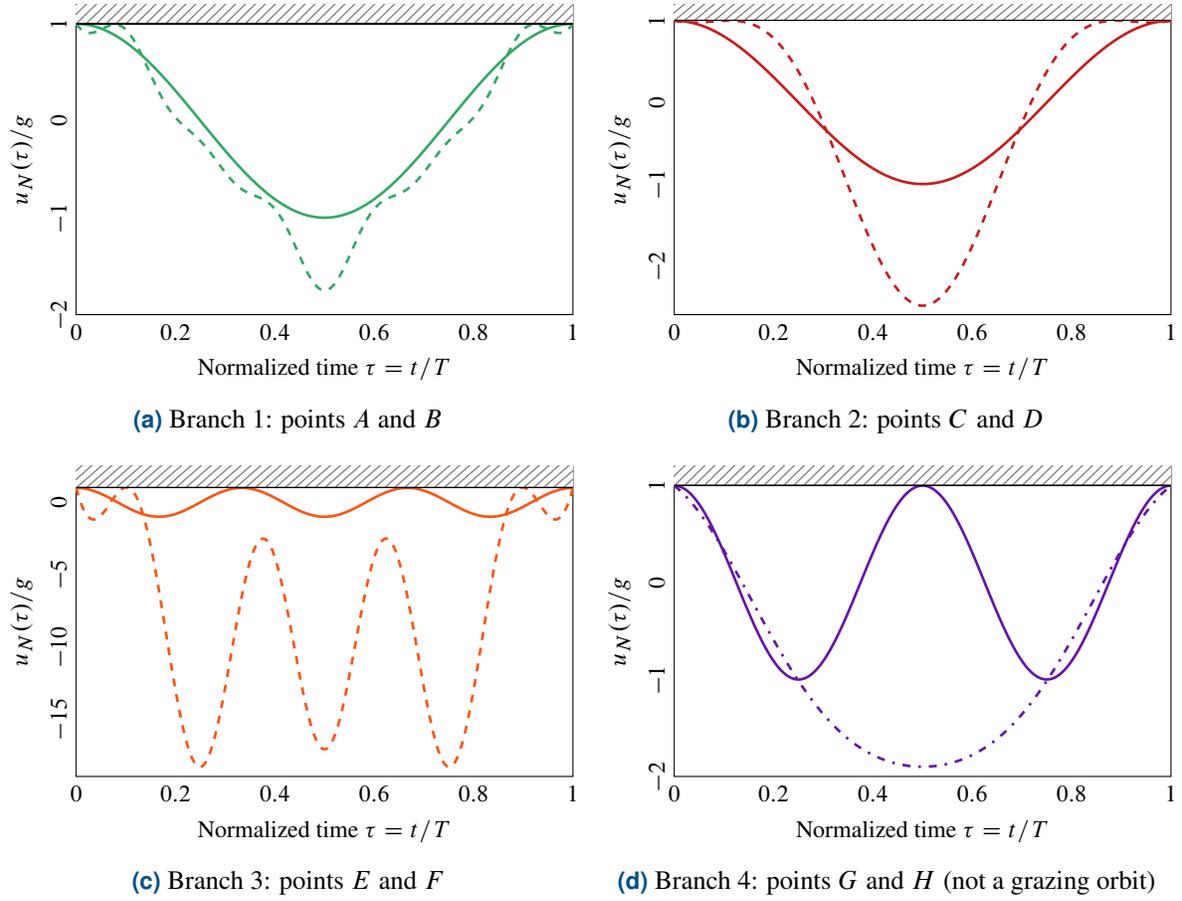
**5. Nonsmooth modes of vibration with one impact per period** The previous section defines the set of all solutions to the extended formulation 2.1. They are not all physically acceptable and the admissible solutions only form the so-called ‘‘Nonsmooth modes of vibration’’.

Among all the admissible branches shown in Fig. 3.2, special attention is paid to four of them illustrated in Fig. 5.1 since they embed the main properties of all branches: they are defined in the vicinity of fundamental and sub-harmonics of the linear modes and their domain of definition is either bounded by a nonlinear grazing orbit (branches 1, 2, and 3) or not bounded (branch 4). These nonlinear



**Figure 5.1:** Zoomed-in Frequency-Energy plot: investigated branches of the admissible spectrum [see Fig. 3.2]. Colored vertical straight segments are the linear portions of the nonlinear modes of vibration [contact not activated] while the colored curved segments are the nonlinear portions [one impact per period] of the nonlinear modes of vibration: the corresponding supporting manifolds are depicted in figures 5.4, 5.8, 5.5, and 5.9 with increasing frequencies. Points A, C, E, and G correspond to the linear grazing orbits in these figures, while points B, D, and F correspond to the nonlinear grazing orbits. One zero of  $w_N(T)$  is highlighted.

grazing orbits are plotted in Fig. 5.2, together with their linear counterparts. It should be understood that these are solutions where the system (probably) bifurcates from one impact per period orbits to multiple impacts per period orbits. The proposed approach is not capable of investigating these new types of response and a sophisticated formulation tackling more than one impact per period is needed [24].



**Figure 5.2:** Linear [solid line] and nonlinear [dashed line] grazing displacements mentioned in Fig. 5.1. Nonlinear grazing orbits feature additional grazing impact instants thus limiting the domain of admissibility of one impact per period orbits. Note that the time axis is rescaled for each orbit for viewing purposes.

All invariant manifolds are plotted in the cross-section  $(u_{N-1}, u_N, \dot{u}_N)$  of the state-space as generically shown in Fig. 5.3. The terminology “Hyperplane view” and “Top view” used later is self-explanatory. The jump in velocity induced by the impact law is highlighted by a thick black line representing the one-dimensional manifold  $\mathcal{V}_0$ .

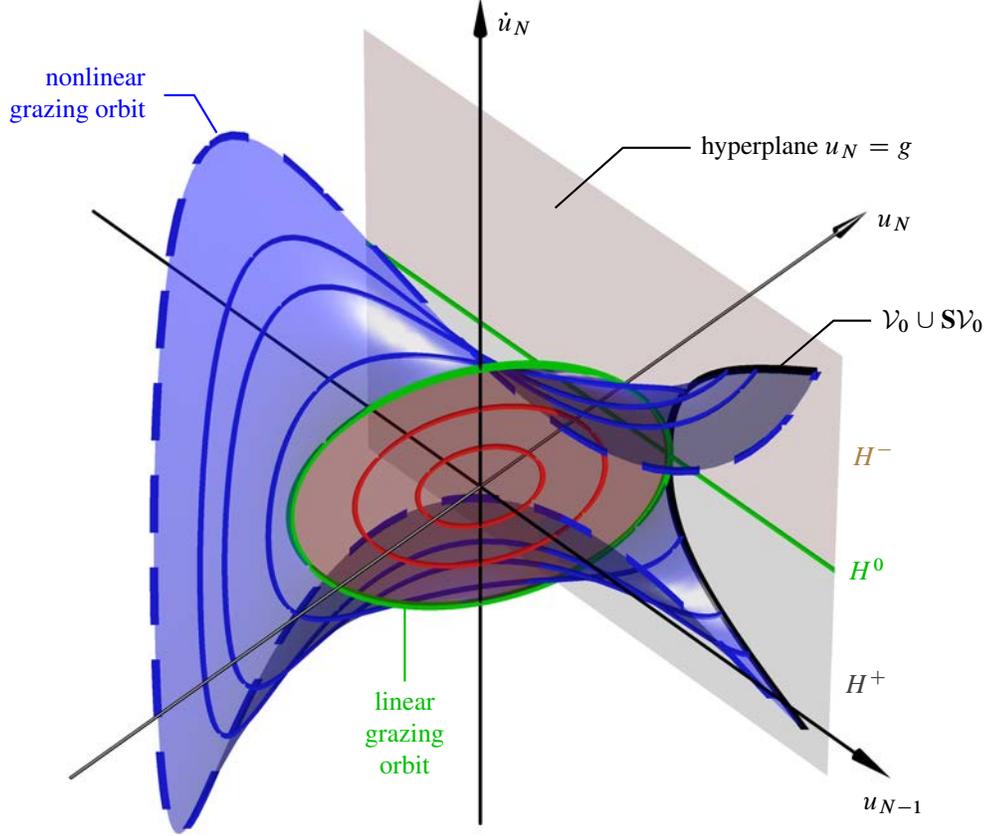
**5.1. In the vicinity of the linear natural frequencies** The solution of the eigenvalue problem (2.28) given in proposition 4.1 has various consequences on the existence and admissibility of a periodic solution satisfying  $u_N(0) = g$  and  $\dot{u}_N^+(0) < 0$ .

**Proposition 5.1 [Admissibility]** *Assumptions 2.3, 3.1, and  $T \notin \bigcup_{j=1}^N T_j \mathbb{N}$  hold. Then a periodic solution  $\mathbf{u}(\cdot; T)$  will be an admissible solution if:*

1. *Necessary admissibility conditions:  $\dot{u}_N^+(0) < 0$  and  $w_N(T) < 0$ .*
2. *Sufficient admissibility condition:  $a_j(T - T_j) < 0$  and  $T \simeq T_j$ .*

This proposition is a reformulation of theorem 3.2 which is proved below.

**Proof.** The first necessary condition  $\dot{u}_N^+(0; T) \leq 0$  should be satisfied otherwise mass  $N$  penetrates the rigid wall. Since  $\dot{u}_N^+(0; T) \neq 0$  for the nonlinear period  $T$ , the condition  $\dot{u}_N^+(0; T) \leq 0$  becomes  $\dot{u}_N^+(0; T) < 0$ .



**Figure 5.3:** Generic admissible manifold for illustration purposes only. It shows (1) the linear portion [red] for which contact is not activated during one period and (2) the nonlinear portion [blue] supporting periodic orbits with one impact per period. The linear grazing orbit joins the two portions defining the manifold. The continuum of periodic orbits “hits” the hyperplane  $u_N = g$  along the one-dimensional manifold  $\mathcal{V}_0 \cup \mathcal{S}\mathcal{V}_0$  [thick black line].

The second necessary condition stems from the fact that  $\mathbf{u}(0; T) = c\mathbf{w}(T)$  with  $c = \dot{u}_N^+(0; T)$  and  $0 < g = u_N(0; T) = c w_N(T)$ .

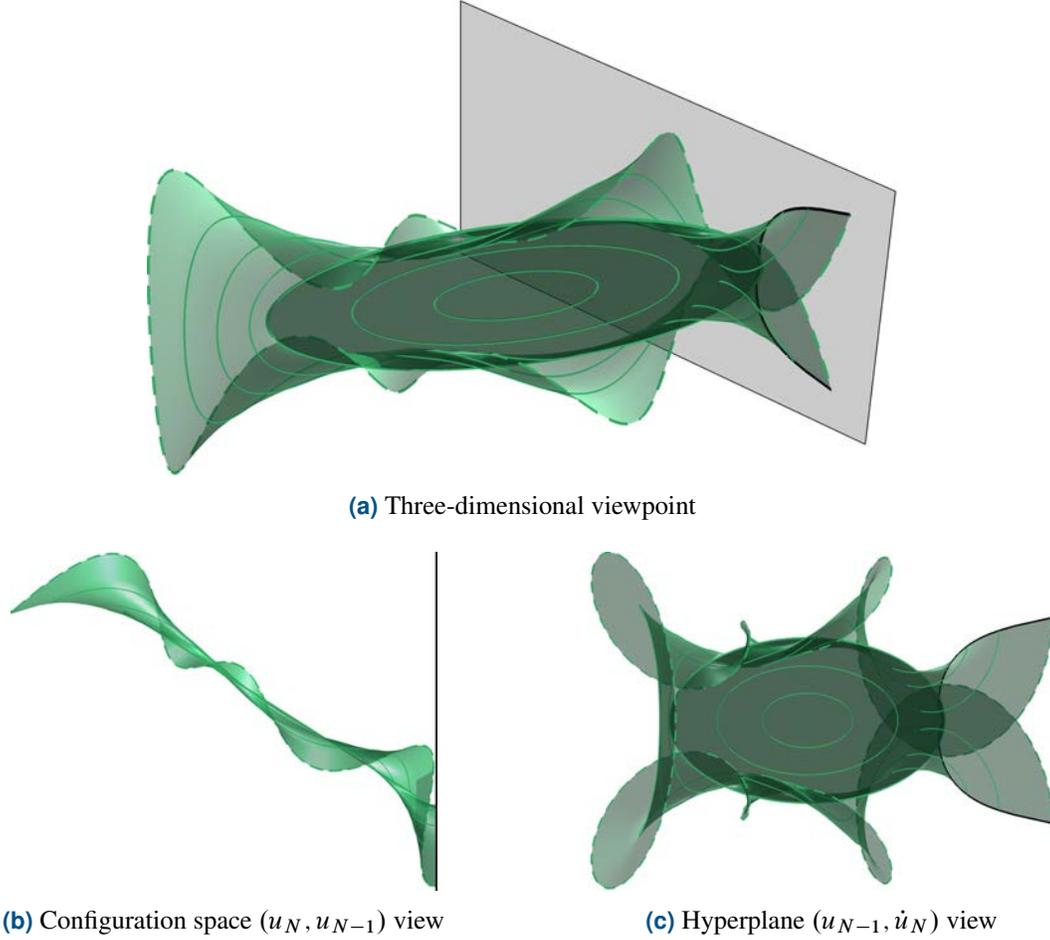
Equivalent (4.12) for  $T \simeq T_j$  implies the one-sided condition  $a_j(T - T_j) < 0$ . Assume a positive interaction coefficient  $a_j$ . The admissibility of a  $T$ -periodic solution  $u_N(t; T)$  with only one impact per period means

$$u_N(t; T) < g, \quad \forall t \in ]0; T[. \quad (5.1)$$

A continuity argument on  $\mathbf{u}(t; T)$  along  $t$  and  $T$  belonging to the compact set  $K_\epsilon = \{(t, T), 0 \leq t \leq T, T_j - \epsilon \leq T \leq T_j\}$  parametrized by  $0 < \epsilon \ll 1$  shows that  $\mathbf{u}(t; T) \approx \mathbf{u}(t; T_j)$  which is the linear grazing solution satisfying  $u_N(t; T_j) < g$  except for  $t = 0$  and  $t = T$ : as a consequence,  $u_N(t; T) < g$  except for  $t \approx 0$  and  $t \approx T$ . Let us investigate inequality (5.1) around  $t \approx 0$  only since the other configuration is handled in an identical fashion. The solutions  $u_N(0; T) = g$  for  $T < T_j$  have negative velocity on the wall  $\dot{u}_N^+(0; T) < 0$  without any uniform estimate since  $\dot{u}_N(0; T) \rightarrow 0$  when  $T \rightarrow T_j$ . We cannot conclude directly on the existence of a time zone  $]0; t_0[$  independent of  $T$  such that  $u_N(t; T) < g$  for all  $(t, T) \in K_\epsilon$ ,  $0 < t < t_0$ . Fortunately, there exist  $t_0 > 0$  and  $\epsilon > 0$  such that

$$\sup_{[0; t_0] \times [T_j - \epsilon; T_j]} \ddot{u}_N(t; T) < 0 \quad (5.2)$$

which is a sufficient condition to ensure admissibility because  $\dot{u}_N(t; T)$  decreases on  $[0; t_0]$  and  $\dot{u}_N(0; T) \leq 0$  so  $\dot{u}_N(t; T) < 0$  on  $]0; t_0[ \times [T_j - \epsilon; T_j]$ , and then  $u_N(t; T) < g$  on  $]0; t_0[ \times [T_j - \epsilon; T_j]$ . Let us



**Figure 5.4:** Nonsmooth mode for  $T \approx T_1$  corresponding to branch 1 in Fig. 5.1.

calculate the second derivatives by starting with

$$\mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \Leftrightarrow \sum_{p=1}^N P_{kp}^{-1} P_{pj} = \delta_{kj}. \quad (5.3)$$

The expansions are now performed in a neighborhood of  $T_j$  through (4.10), (4.11), and (4.14) when  $T \rightarrow T_j$ . For  $0 < t < T$ , the explicit solution is

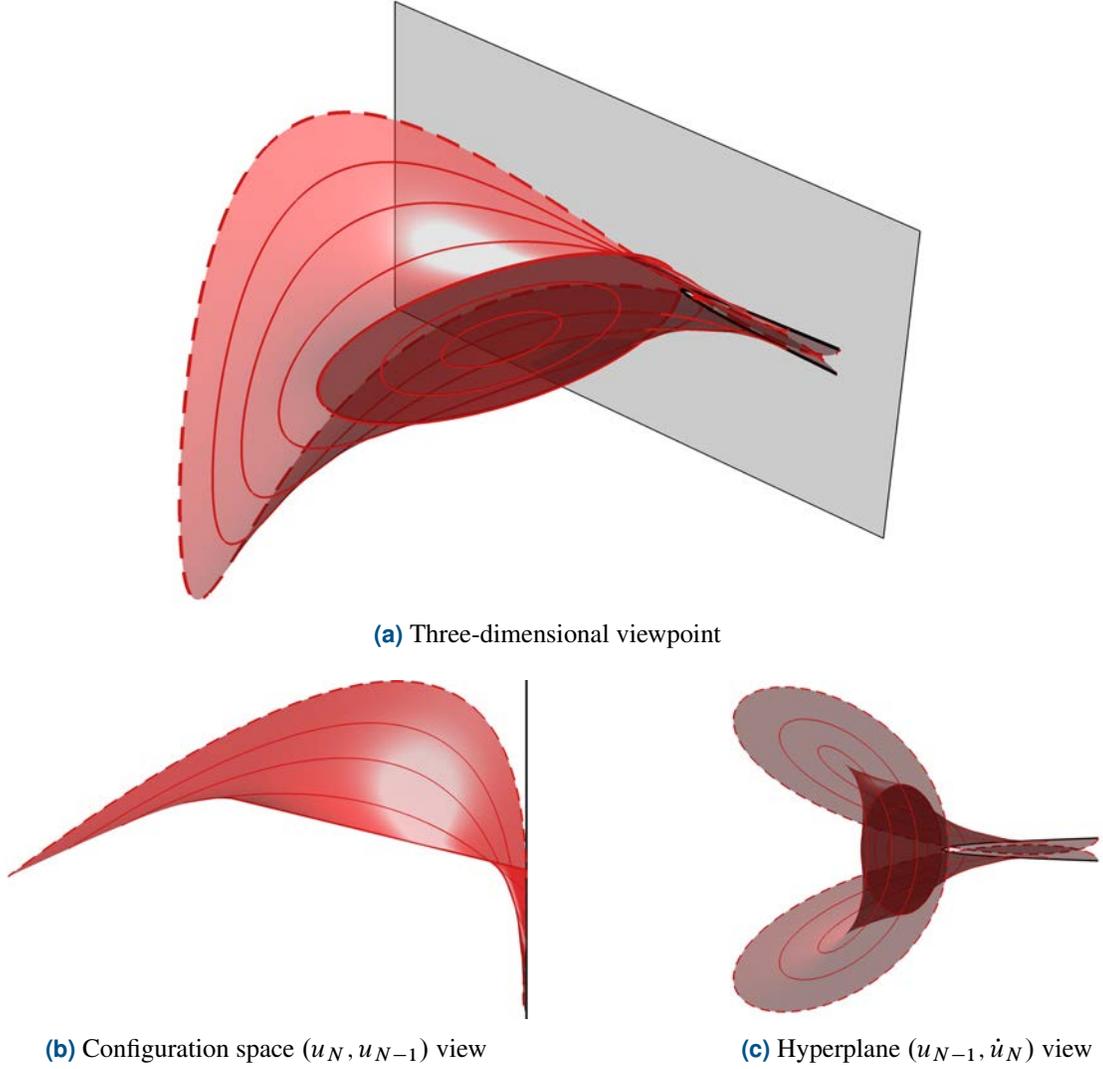
$$\mathbf{u}(t) = \mathbf{P} \cos(T\boldsymbol{\Omega})\mathbf{P}^{-1}\mathbf{u}(0) + \mathbf{P}\boldsymbol{\Omega}^{-1} \sin(T\boldsymbol{\Omega})\mathbf{P}^{-1}\dot{\mathbf{u}}^+(0) \quad (5.4)$$

which leads to the expressions for

- the displacement:

$$u_N(t; T) = \frac{g}{w_N(T)} \left( \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \cos(\omega_\ell t) w_k(T) + \sum_{k=1}^N a_k \frac{\sin(\omega_k t)}{\omega_k} \right) \quad (5.5)$$

$$u_N(0; T) = g, \quad (5.6)$$



**Figure 5.5:** Nonsmooth mode for  $T \approx T_2$  corresponding to branch 2 in Fig. 5.1.

- the velocity:

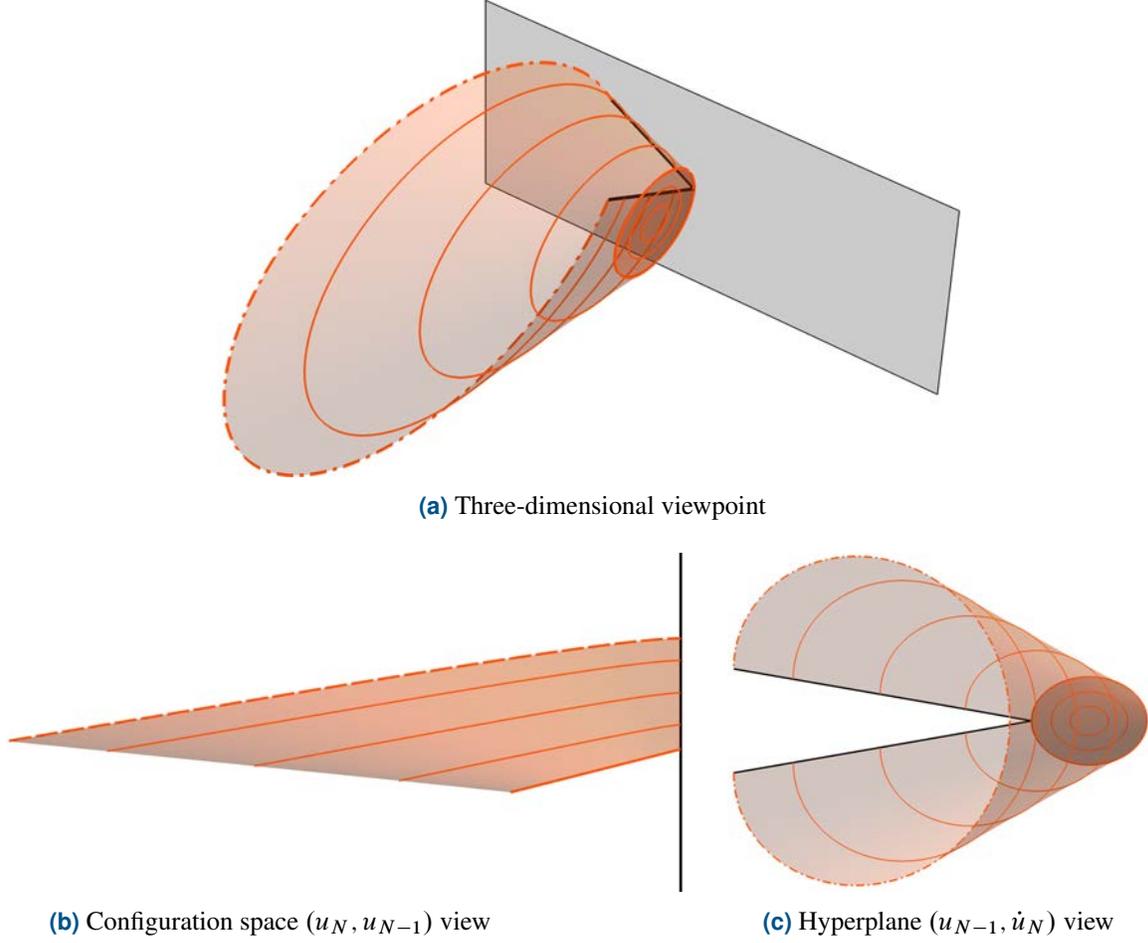
$$\dot{u}_N(t; T) = \frac{g}{w_N(T)} \left( - \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \omega_\ell \sin(\omega_\ell t) w_k(T) + \sum_{k=1}^N a_k \cos(\omega_k t) \right) \quad (5.7)$$

$$\dot{u}_N^+(0; T) = \frac{g}{w_N(T)} \left( \sum_{k=1}^N a_k \right) = \frac{g}{w_N(T)}, \quad (5.8)$$

- the acceleration:

$$\ddot{u}_N(t; T) = - \frac{g}{w_N(T)} \left( \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \omega_\ell^2 \cos(\omega_\ell t) w_k(T) + \sum_{k=1}^N a_k \omega_k \sin(\omega_k t) \right) \quad (5.9)$$

$$\ddot{u}_N^+(0; T) = - \frac{g}{w_N(T)} \left( \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \omega_\ell^2 w_k(T) \right) = -g \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \omega_\ell^2 \frac{w_k(T)}{w_N(T)},$$



**Figure 5.6:** Nonsmooth mode for  $T \approx T_3$ . It is not bounded by a nonlinear grazing orbit. The corresponding branch is not shown in Fig. 5.1. It should be compared to the manifold emerging in the vicinity of  $T \approx 3T_3$  reported in Fig. 5.8. Both share the same *linear* part but bifurcate differently at the linear grazing orbit.

which becomes, by passing to the limit when  $T \rightarrow T_j$

$$\lim_{T \rightarrow T_j} \ddot{u}_N^+(0; T) = -g \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \omega_\ell^2 \frac{P_{kj}}{P_{Nj}} = -g \sum_{\ell=1}^N \omega_\ell^2 \frac{P_{N\ell}}{P_{Nj}} \left( \sum_{k=1}^N P_{\ell k}^{-1} P_{kj} \right) \quad (5.10)$$

$$= -g \sum_{\ell=1}^N \omega_\ell^2 \frac{P_{N\ell}}{P_{Nj}} \delta_{\ell j} = -g \omega_j^2 < 0. \quad (5.11)$$

To summarize, we can state that  $u_N(0; T) = g$ ,  $\dot{u}_N^+(0; T) \leq 0$  and  $\ddot{u}_N^+(0; T) < 0$  for  $T \approx T_j$  and  $a_j(T - T_j) < 0$ . Thus  $u_N(t; T) < g$  for  $0 < t < \epsilon$ ,  $a_j(T - T_j) < 0$ ,  $|T - T_j| < \epsilon$  and  $\epsilon > 0$ , which is sufficient to prove the admissibility of the solution.  $\square$

Two manifolds emerging in the neighborhood of the linear natural frequencies  $\omega_1$  and  $\omega_2$  are displayed in figures 5.4 and 5.5, respectively. They both exhibit a nonlinear grazing orbit limiting their domain of definition. Accordingly, these two manifolds have boundaries. A third one, unbounded, defined around  $\omega_3$  is shown in Fig. 5.6. It should be compared to its counterpart arising about  $\omega_3/3$ : they both share the same linear portion but then bifurcate differently at the linear grazing orbit. A

similar behavior is expected for all admissible subharmonics of all (linear) modes. As such, this is a typical example of a mechanical system for which the number of nonlinear modes of vibration exceeds the number of degrees-of-freedom.

**5.2. In the vicinity of the linear subharmonics** In this section, we prove theorem 3.3 and discuss the more challenging case of linear subharmonics.

*Proof.* The solution  $\mathbf{u}(t; 2T_j)$  is simply the  $j$ th linear grazing solution with two loops and two (grazing) contacts per period. By continuity, it is clear that the solution  $\mathbf{u}(t; T) \simeq \mathbf{u}(t; 2T_j)$  when  $T \approx 2T_j$  (see Fig. 5.7). To test admissibility, attention is paid to the scalar function  $t \mapsto u_N(t; T)$ : from the closed-form expression (5.4), it clearly lies near  $u_N(t; 2T_j) = g \cos(\omega_j t/2)$  in all  $C^k$  topologies (outside the impact). The solution  $\mathbf{u}(t; 2T_j)$  and all its derivatives with respect to time  $t$  have simple roots only, and this holds true for  $T \approx 2T_j$  as well. Accordingly,  $\mathbf{u}(t; T)$  has only two local maxima on  $[0; T[$  where the condition  $\mathbf{u}(t; T) \leq g$  has to be tested. If the one-sided condition  $a_j(T - 2T_j) < 0$  is satisfied for all  $T \approx 2T_j$ , the first maximum is located at  $t = 0$ : this is a straightforward consequence of the previous proof. Due to the symmetry of the solution, the second maximum is at  $t = T/2$ . We also have  $\ddot{u}_N(T/2; T) \sim \ddot{u}_N(T_j; 2T_j) = \ddot{u}_N(0; T_j) < 0$  and the inequality  $u_N(T/2; T) \leq g$  guarantees admissibility.  $\square$

As detailed in Proposition 4.1, interesting features of the modal dynamics can also be observed by examining responses for various initial energy levels in the phase portraits. Fig. 5.10 depicts phase plane diagrams for masses 4 and 5: the loops correspond to periodic responses whose amplitude depends on the initial energy level. They are symmetric along the velocity axis because of the conservative nature of the system and the symmetry of the impulsive impact force in velocity. It also show how mass 4 is affected in its motion by the nonlinear restitution force.

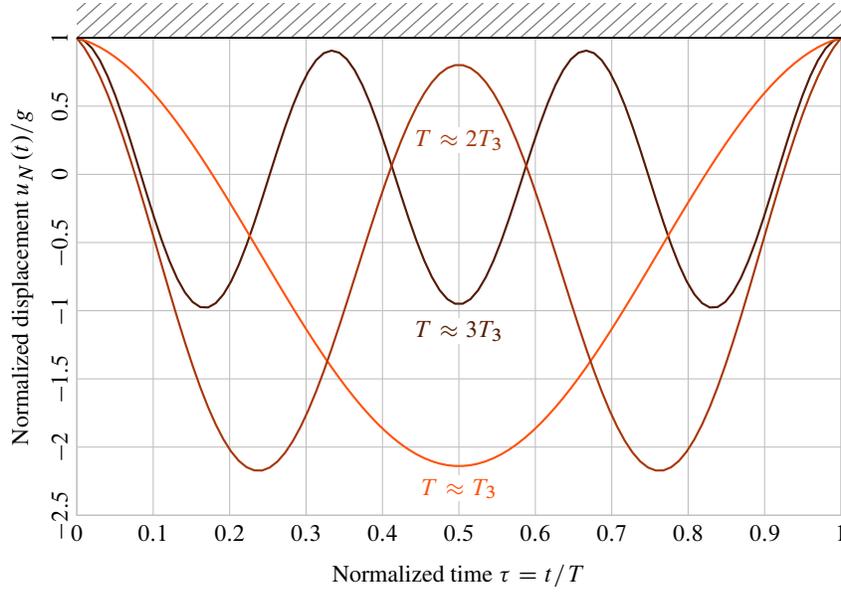
*Remark 6.* An important feature that distinguishes the investigated vibro-impact oscillator from polynomial nonlinear oscillators is the existence of physically observable backbone curves near the linear subharmonics of vibration. In Fig. 5.1, the vibratory motions corresponding to the two linear segments located at  $\omega_3/3$  and  $\omega_4/2$  are not readily observable as they turn to be identical to the ones corresponding to the two linear segments located at  $\omega_3$  and  $\omega_4$  (that is linear modes 3 and 4), respectively. Nevertheless, the nonlinear branches emanating from the natural frequencies  $\omega_i$  and from their subharmonics  $\omega_i/j$ ,  $(i, j) \in \mathbb{N}^{*+}$ , correspond to distinct co-existing manifolds sharing the same linear portion, as illustrated in Figs. 5.6 and 5.8. This is in contrast with polynomial nonlinear systems where manifolds are defined in the neighborhood of linear modes of vibration only.

**Admissibility for  $T \simeq kT_j$ ,  $k > 2$**  For lower-order subharmonics ( $1/3, 1/4, \dots$ ), numerical approximations are required to test admissibility conditions:

1. Check  $a_j(T - kT_j) < 0$ .
2. With a numerical solver, find  $t_\ell \approx \ell T/k$  such that  $\dot{u}_N(t_\ell; T) = 0$  for all integers  $\ell$  such that  $1 \leq \ell \leq k/2$ .
3. Check the conditions  $u_N(t_\ell; T) \leq g$  for  $1 \leq \ell \leq k/2$ .

As shown in Fig. 5.7, the displacement  $u_N(\cdot; T)$  on  $[0; T]$  has a graph very similar to  $u_N(\cdot; kT_j)$  on  $[0; kT_j]$  with the same number of extrema nearly at the same locations. By symmetry, only half a period should be scrutinized. Moreover, if  $k$  is even,  $t_{k/2} = T/2$  as in theorem 3.3.

**5.3. High Frequencies** In [21, p. 1008], it is shown that very high frequency (that is  $T \rightarrow 0$  here) admissible solutions exist. This is not true in our investigation. The last statement of theorem 3.2 is a consequence of proposition 5.2.



**Figure 5.7:** Admissible nonsmooth periodic solutions in the vicinity of the third linear mode of period  $T_3$ , and in the vicinity of two of its subharmonics  $2T_3$  and  $3T_3$ . For readability, the time scale is normalized with respect to each  $T$ .

**Proposition 5.2 [Nonexistence of high-frequency nonlinear modes]** *If the condition*

$$\alpha := \sum_{j=1}^N a_j \omega_j^{-2} > 0 \quad (5.12)$$

*is satisfied, then the frequencies of the admissible nonlinear modes are bounded.*

As previously highlighted, our illustrative example (2.17) is such the interaction coefficients are all positive and the admissible frequencies are bounded. To have the opportunity to observe high-frequency admissible nonlinear modes, the condition  $\alpha < 0$  is required which is a bit challenging to fulfil as  $\sum_j a_j = 1$  from (2.12).

**Proof.** Within the current mathematical framework, it is possible to derive an asymptotic behaviour when  $T \rightarrow 0$ . The limit  $\lim_{T \rightarrow 0} \mathbf{u}(0; T)$  can be explicitly exploited and the only non vanishing velocity  $\dot{u}_N^+(0; T) = g/w_N(T) = \mathcal{O}(T)$  converges to 0. Recall that  $u_k(0; T) = g w_k(T)/w_N(T)$ . The asymptotic behavior of  $w_k(T)$  for  $T \approx 0$  is

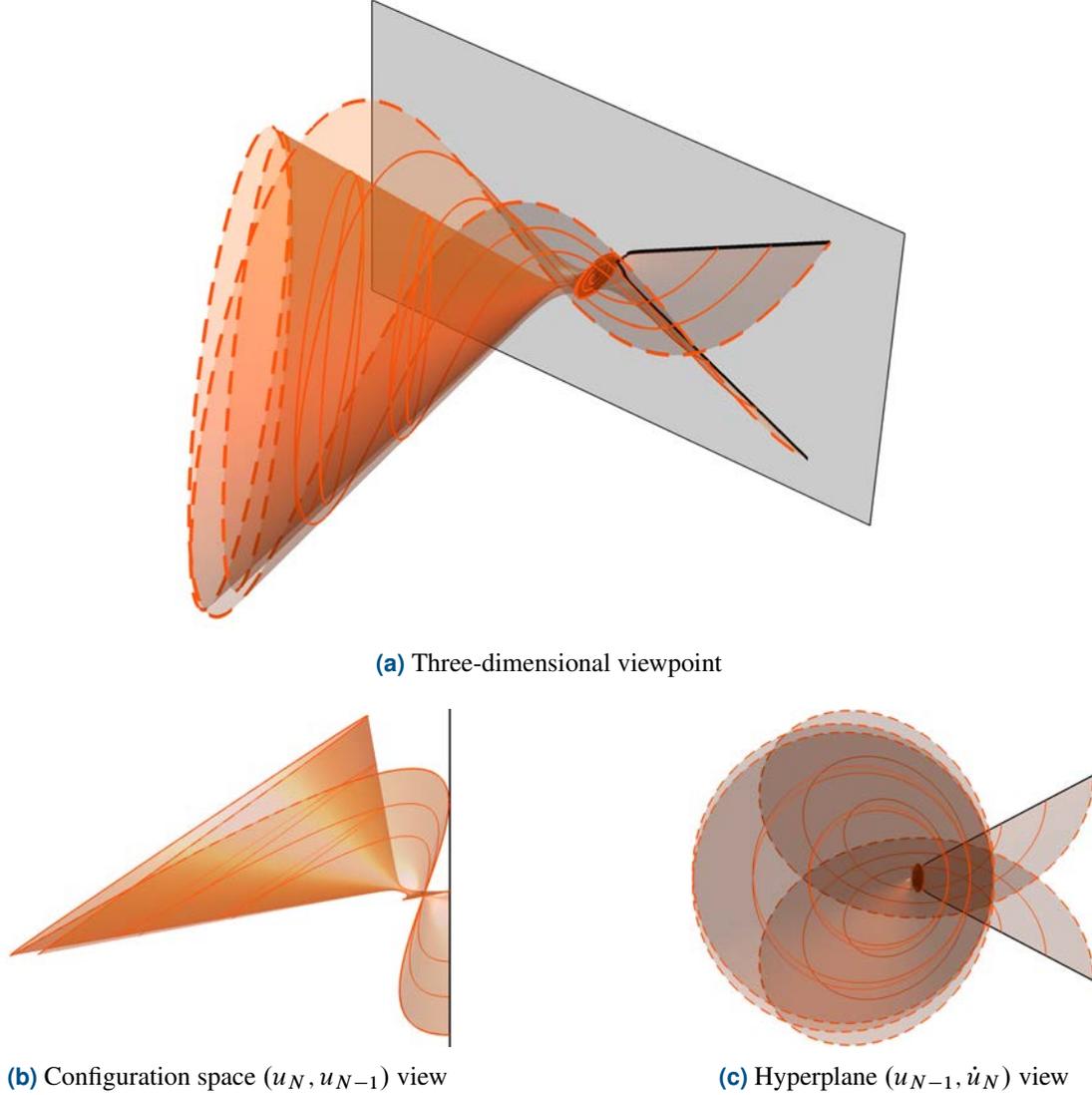
$$w_k(T) = \sum_{p=1}^n P_{kp} P_{pN}^{-1} \phi_p(T) = \frac{2}{T} \sum_{p=1}^N P_{kp} P_{pN}^{-1} \omega_p^{-2} + \mathcal{O}(1) \text{ when } T \rightarrow 0 \quad (5.13)$$

In particular, for  $k = N$ , we simply have:

$$w_N(T) \sim \frac{2\alpha}{T} \text{ when } T \rightarrow 0. \quad (5.14)$$

Consequently, the limiting initial data are:

$$\begin{aligned} \lim_{T \rightarrow 0} u_k(0; T) &= \frac{g}{\alpha} \sum_{p=1}^N P_{kp} P_{pN}^{-1} \omega_p^{-2}, \quad k = 1, \dots, N \\ \lim_{T \rightarrow 0} \dot{u}_k(0; T) &= 0, \quad k = 1, \dots, N \end{aligned} \quad (5.15)$$



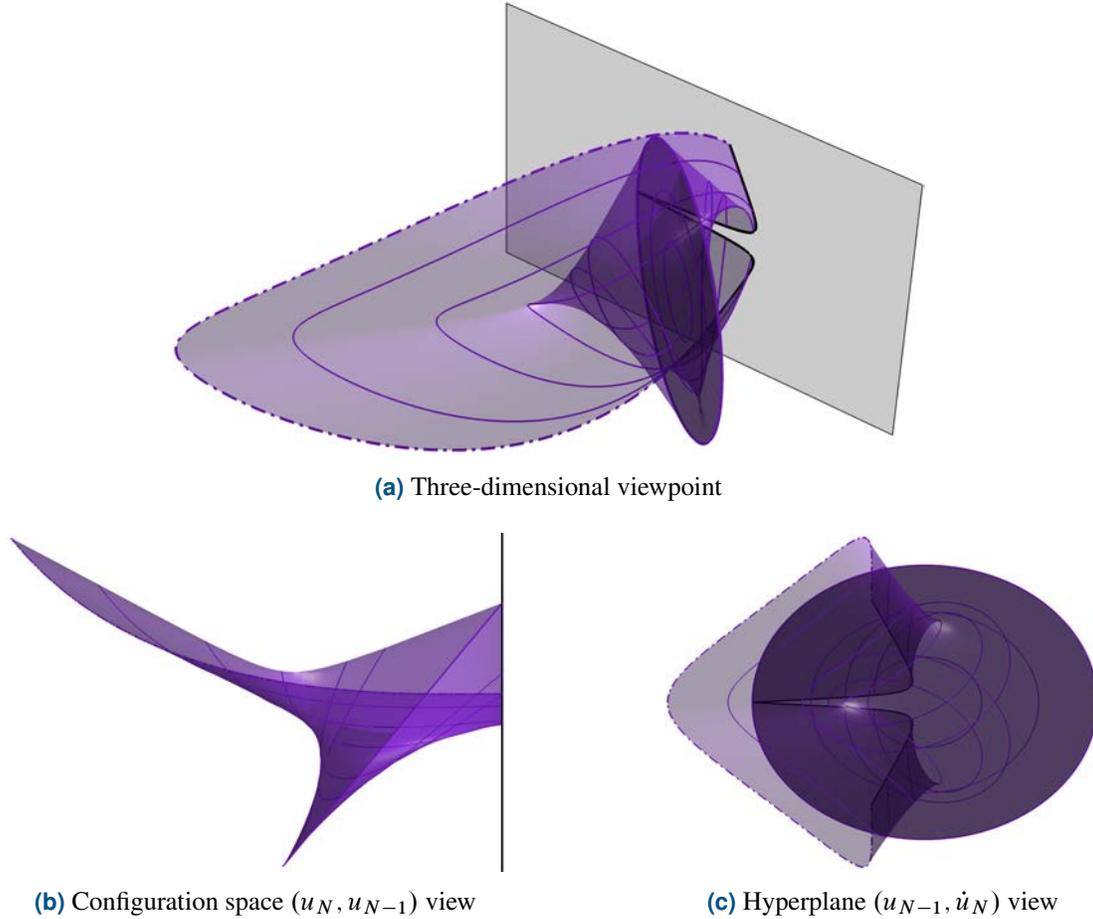
**Figure 5.8:** Nonsmooth mode for  $T \approx 3T_3$  corresponding to branch 3 in Fig. 5.1.

In particular,  $u_N(0; 0) = g$  and a grazing contact trajectory is retrieved. The admissibility criterion on the velocity requires  $\dot{u}_N(0; T) < 0$  but for  $T \simeq 0$  and  $T > 0$ ,  $\dot{u}_N^+(0; T) = g/w_N(T) \sim Tg/(2\alpha) > 0$ . Thus the solution of the extended formulation becomes inadmissible. Moreover, through Eqs. (5.10) and (5.3), we can derive an approximation of the second right derivative for small  $T$ :

$$\begin{aligned} \ddot{u}_N^+(0; T) &= -g \sum_{\ell=1}^N \sum_{k=1}^N P_{N\ell} P_{\ell k}^{-1} \omega_\ell^2 \frac{w_k(T)}{w_N(T)} \approx -\frac{g}{\alpha} \sum_{\ell=1}^N \sum_{k=1}^N \sum_{p=1}^N \frac{\omega_\ell^2}{\omega_p^2} P_{N\ell} P_{\ell k}^{-1} P_{kp} P_{pN}^{-1} \\ &\approx -\frac{g}{\alpha} \sum_{\ell=1}^N \sum_{p=1}^N \frac{\omega_\ell^2}{\omega_p^2} P_{N\ell} P_{pN}^{-1} \delta_{\ell p} = -\frac{g}{\alpha} \sum_{\ell=1}^N P_{N\ell} P_{\ell N}^{-1} \sim -\frac{g}{\alpha} < 0. \end{aligned} \quad (5.16)$$

Accordingly, the trajectory leaves the rigid foundation on the “wrong” non-admissible side  $u_N > g$  with decreasing velocity.  $\square$

**6. Spectral stability analysis** Nonlinear systems shall exhibit a wide range of exotic behaviours, including instabilities, bifurcations, and chaos. Of high interest is the systematic investigation of the



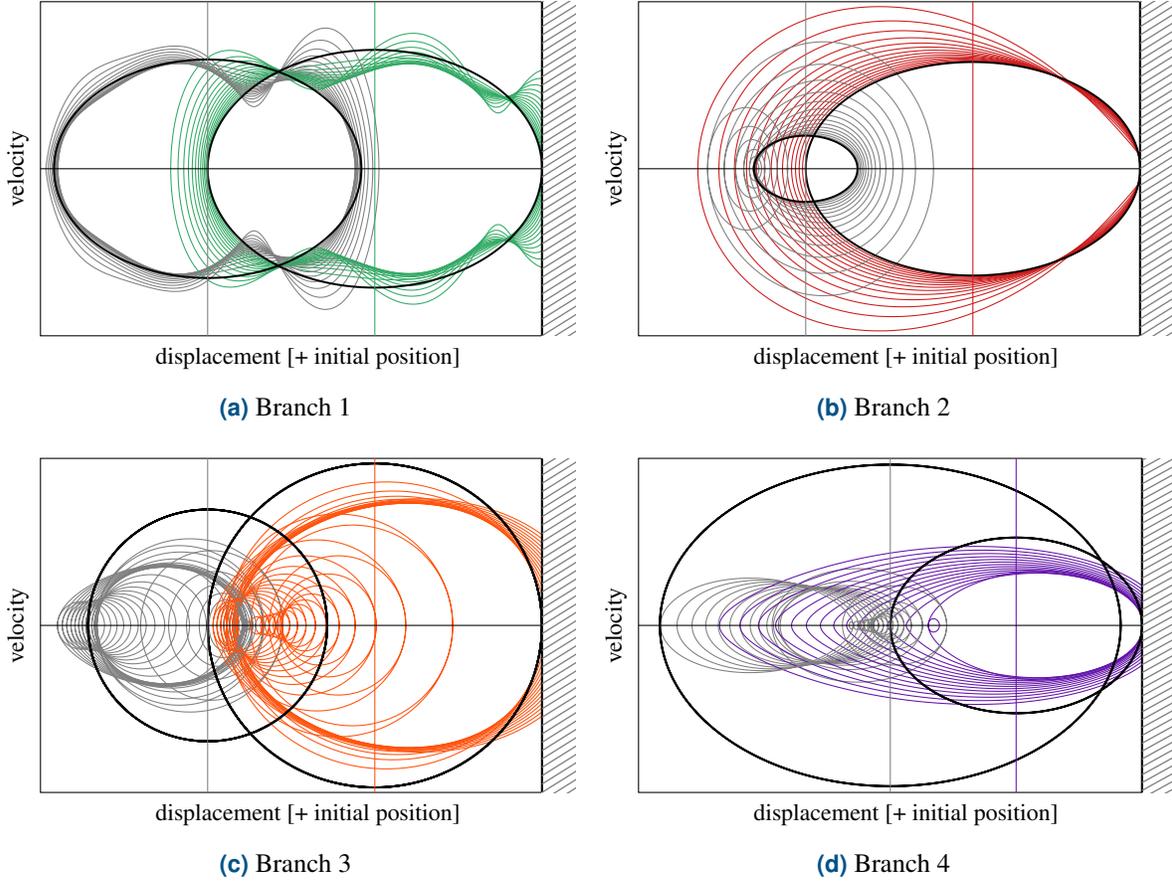
**Figure 5.9:** Nonsmooth mode for  $T \approx 2T_4$  corresponding to branch 4 in Fig. 5.1.

stability of the previously found periodic orbits as their frequency—or energy—increases. Stability of a periodic solution manifests itself in the way neighboring trajectories behave. It is known for autonomous conservative *linear* systems that the periodic orbits are *neutrally stable*. In general, this property is destroyed for nonlinear systems and as a result, some of the previously found orbits shall be unstable. In order to clarify the situation, a dedicated analysis should be undertaken: it commonly prescribes a perturbation of a calculated periodic trajectory in order to explore its dynamical behavior with time. This can be conveniently achieved by either calculating the eigenvalues of the linearized Poincaré map or, equivalently, by exploiting the characteristic multipliers determined from the monodromy matrix [22].

In this work, since the First Return Map is found to be locally well defined and smooth for non-grazing solutions<sup>3</sup>, Lyapunov’s theorems can be used and the stability analysis reduces to the classical *spectral* stability analysis of the Poincaré map fixed points [10]. The sufficient condition for (spectral) stability is that all the eigenvalues of the linearized Poincaré map lie inside the unit circle in the complex plane; the necessary condition is that they lie either inside or on the circle. In other words, (nonlinear) instability is achieved when the modulus of one eigenvalue of the linearized First Return Map is larger than unity.

Some additional features of the response are also readily accessible from this analysis: depending

<sup>3</sup> The stability analysis of the grazing orbits involves quite complicated arguments [4, 13] that are not discussed in this paper.



**Figure 5.10:** Phase portraits of the periodic motion on the invariant manifolds: mass  $N$  [color] and mass  $N - 1$  [gray]. Linear grazing orbit [thick black]. Branches are defined in Fig. 5.1

on where the eigenvalue or pair of complex conjugate eigenvalues crosses the unit circle defined in the complex plane, different types of bifurcation occur. Evidently, the more challenging bifurcation mechanisms induced by grazing orbits, that are specific to nonlinear systems involving switching manifolds in the state-space, are not addressed with this approach.

**6.1. Linearized First Return Map** In order to prove theorem 3.4, let us first notice that the First Return Map  $\mathbf{F}$  is fully defined by the first return time  $T$  when it exists. Also, we need the following ingredient: the mechanical energy  $E(\dot{\mathbf{u}}, \mathbf{u}) = \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \mathbf{u}^T \mathbf{K} \mathbf{u}$  is conserved after an impact when the last mass is concentrated.

**Proposition 6.1 [Conserved energy]** *The impact law conserves the mechanical energy if and only if the mass matrix of the investigated system satisfies  $M_{Nk} = 0 = M_{kN}$  for all  $k = 1, \dots, N - 1$ .*

*Proof.* In space coordinates, the energy is clearly conserved during the free flight. The impact rule reads  $\dot{\mathbf{u}}^+ = \mathbf{S} \dot{\mathbf{u}}^-$  and the energy is conserved only when  $\mathbf{S} \mathbf{M} \mathbf{S} = \mathbf{M}$  which concludes the proof.  $\square$

We now prove theorem 3.4.

*Proof.* To avoid the impact-induced jump in the orbit, the closed-form solution is considered on interval  $]0; T[$  as a function of initial data  $\mathbf{Q}$  and time  $t$

$$\mathbf{q}(t) = \mathbf{L}(t)\mathbf{Q} = \cos(t\Omega)\mathbf{q}(0) + \Omega^{-1} \sin(t\Omega)\dot{\mathbf{q}}^+(0) \quad (6.1)$$

where the  $N \times 2N$  matrix  $\mathbf{L}(t)$  stores the first  $N$  rows of  $\mathbf{R}(t)$ . Linear relation (6.1) is valid only on  $[0; T]$ . Away from impact,  $\mathbf{q}(t)$  can be differentiated with respect to time such that  $\dot{\mathbf{q}}(t) = \dot{\mathbf{L}}(t)\mathbf{Q}$  while at the impact,  $\dot{\mathbf{q}}^-(T) = \dot{\mathbf{L}}(T)\mathbf{Q}$ .

The proof involves several distinct steps:

1. Existence and smoothness of  $T$  in the neighborhood of  $\mathbf{Q}_0$  — The scalar function  $T(\mathbf{Q})$  is implicitly defined by  $u_N(T(\mathbf{Q})) = g$  in physical coordinates which becomes, in modal coordinates

$$f(T, \mathbf{Q}) := \mathbf{e}_N^\top \mathbf{P}\mathbf{L}(T)\mathbf{Q} = g. \quad (6.2)$$

Note also that  $f(T, \mathbf{Q}) = \bar{\mathbf{e}}_N^\top \bar{\mathbf{P}}\mathbf{R}(T)\mathbf{Q}$ . In order to use the implicit function theorem at  $(T_0, \mathbf{Q}_0)$ , it is sufficient to prove that  $\partial_T f(T_0, \mathbf{Q}_0) \neq 0$ . This partial derivative has a simple interpretation:  $\partial_T f(T_0, \mathbf{Q}_0) = \dot{u}_N^-(T_0; T_0)$  since  $u_N(t; T_0) = f(t, \mathbf{Q}_0)$  for all  $t \in ]0; T_0[$ . As a consequence, the non-grazing condition  $\dot{u}_N^+(0; T_0) \neq 0$ , the periodicity, and the reflection law  $\dot{u}_N^-(T_0; T_0) = -\dot{u}_N^+(0; T_0)$  show that

$$\partial_T f(T_0, \mathbf{Q}_0) = \mathbf{e}_N^\top \mathbf{P}\dot{\mathbf{L}}(T_0)\mathbf{Q}_0 = -\dot{u}_N^+(0; T_0) \neq 0. \quad (6.3)$$

2. Explicit formula for  $\mathbf{DF}$  — Differentiating equality (6.2) with respect to  $\mathbf{Q}$  at  $\mathbf{Q} = \mathbf{Q}_0$  leads to

$$\begin{aligned} \nabla T &= -\frac{1}{\partial_T f} \nabla_{\mathbf{Q}} f = -\frac{1}{\mathbf{e}_N^\top \mathbf{P}\dot{\mathbf{L}}(T(\mathbf{Q}_0))\mathbf{Q}_0} \mathbf{e}_N^\top \mathbf{P}\mathbf{L}(T(\mathbf{Q}_0)) = \frac{1}{\dot{u}_N^+(0; T_0)} \mathbf{e}_N^\top \mathbf{P}\mathbf{L}(T_0) \\ &= \frac{w_N(T_0)}{g} \mathbf{e}_N^\top \mathbf{P}\mathbf{L}(T_0) = \frac{w_N(T_0)}{g} \bar{\mathbf{e}}_N^\top \bar{\mathbf{P}}\mathbf{R}(T_0). \end{aligned} \quad (6.4)$$

Finally, it is sufficient to differentiate  $\mathbf{F}$  through the standard chain rule formula.

3. 1 is an eigenvalue — There are two ways to see that 1 is an eigenvalue of  $\mathbf{DF}$ . First, we have a smooth family of initial data  $\mathbf{Q}(0; T)$  for a non-grazing nonsmooth mode, so  $\mathbf{F}(\mathbf{Q}(0; T)) = \mathbf{Q}(0; T)$  and differentiating with respect to the time yields

$$\mathbf{DF}(\mathbf{Q}(0; T))\partial_T \mathbf{Q}(0; T) = \partial_T \mathbf{Q}(0; T). \quad (6.5)$$

The explicit formula for  $\mathbf{Q}(0; T)$  shows that  $\partial_T \mathbf{Q}(0; T) \neq \mathbf{0}$  at least for almost all  $T$ . In this case  $\partial_T \mathbf{Q}(0; T)$  is a left eigenvector of  $\mathbf{DF}$  associated with the eigenvalue 1<sup>4</sup>.

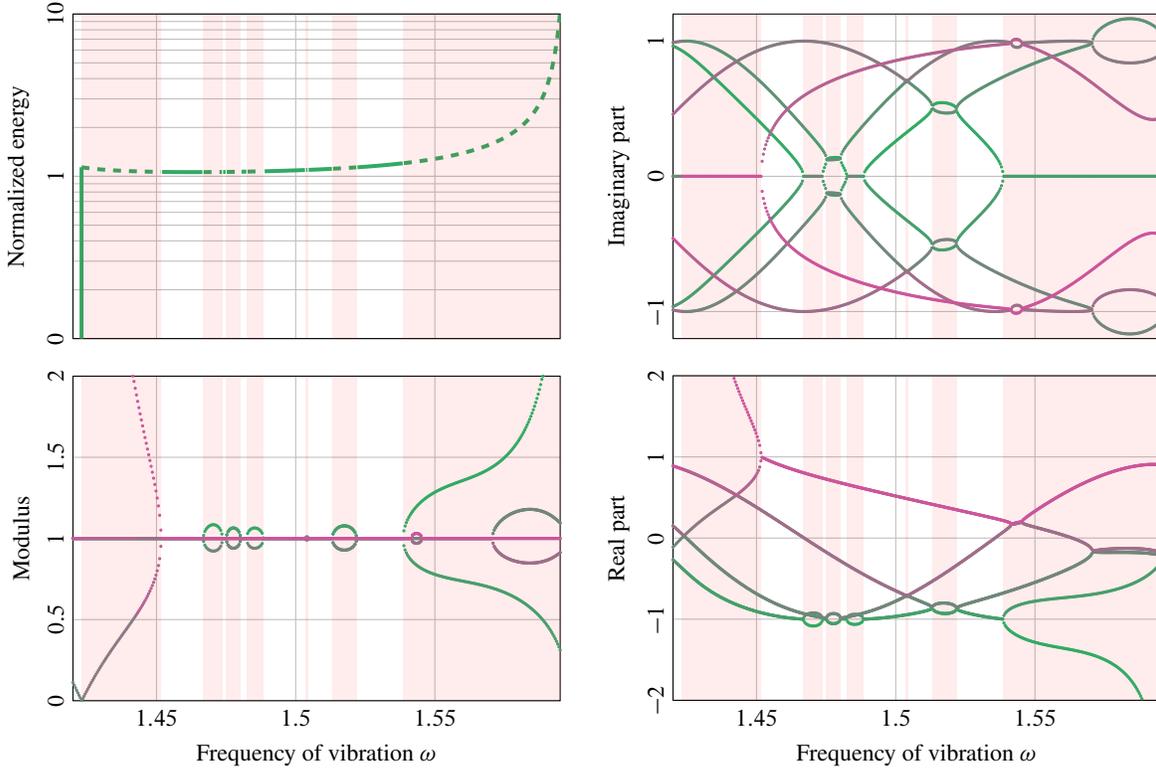
Second, if the mass matrix satisfies the assumptions of proposition 6.1, differentiating the conservation of energy  $E(\mathbf{F}(\mathbf{Q})) = E(\mathbf{Q})$  yields  $\nabla E(\mathbf{F}(\mathbf{Q}))\mathbf{DF}(\mathbf{F}(\mathbf{Q})) = \nabla E(\mathbf{Q})$ . Note that the energy is not degenerate, that is  $\nabla E(\mathbf{Q}) \neq 0$  except for  $\mathbf{Q} = 0$  which is not an initial condition for periodic solutions. Furthermore, if  $\mathbf{F}(\mathbf{Q}) = \mathbf{Q}$  then  $\nabla E(\mathbf{Q})$  is a left eigenvector of  $\mathbf{DF}(\mathbf{Q})$  with eigenvalue 1. We cannot deduce that 1 is an eigenvalue with multiplicity 2 as it is for smooth Hamiltonian systems.

This ends the proof.  $\square$

The First Return Map is conventionally defined on the Poincaré section only, that is the hyperplane  $H^+$  in this work. In the derivations above, the First Return Map and its differential in the neighborhood of an admissible nonlinear trajectory are expressed in the *entire* state-space, which adds one dimension, and thus one eigenvalue to the Jacobian. An obvious question then arises: “What is the magnitude and meaning of the additional eigenvalue?” Consider the orbit of the nonlinear mode which leaves  $H^+$  at time  $t = 0^+$ :  $\mathbf{F}(\mathbf{Q}(t; T_0)) = \mathbf{Q}(0; T_0) = \mathbf{Q}_0$  such that  $\mathbf{DF}(\mathbf{Q}_0)$  satisfies  $\mathbf{DF}(\mathbf{Q}_0)\mathbf{v} = \mathbf{0}$  where  $\mathbf{v} = \dot{\mathbf{Q}}(0; T_0) \neq \mathbf{0}$  does not belong to  $H$ . Consequently, in this formulation, where the unnecessary new dimension makes it easier to handle, 0 is always an extraneous eigenvalue of the matrix  $\mathbf{DF}(\mathbf{Q}_0)$ , that should be formally removed in the analysis.

<sup>4</sup>The derivative  $\partial_T \mathbf{Q}(0; T)$  can eventually be  $\mathbf{0}$  for an isolated time only. Thus, for almost all time, this vector is non-zero and 1 is an eigenvalue. Since the eigenvalue depends continuously on the matrix coefficients that are smooth with respect to  $T$ , 1 is an eigenvalue for all time.

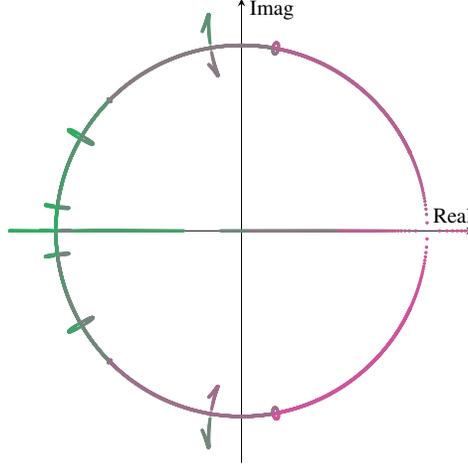
**6.2. Numerical example: first nonlinear mode** Computational work can be directly conducted in physical coordinates where the  $N$ th row of  $\mathbf{DF}$  vanishes since  $\mathbf{e}_N^\top \mathbf{DF} = \mathbf{0}$  which is a direct consequence of  $\mathbf{F}(H) \subset H$ . The varied control parameter is the frequency of the autonomous periodic response. The corresponding stability/instability pattern is shown in Fig. 6.1. As depicted in Fig. 6.2,



**Figure 6.1:** Spectral stability analysis of the first nonlinear mode in the vicinity of  $\omega_1$  [Branch 1 in Fig. 5.1]. Instability is achieved when the modulus of one eigenvalue of the linearized First Return Map  $\mathbf{DF}$  is larger than unity. In every subfigure, zones of instability are highlighted in light red rectangles. The color scheme based on the magnitude of the real parts allows for the convenient identification of the various branches. Eigenvalue 1 is removed for readability purposes.

instability arises through the three well-known bifurcation scenarios usually identified for autonomous and conservative systems: a pair of complex conjugate eigenvalues leaves the unit circle, defined in the complex plane, either (i) at  $(1, 0)$ , or (ii) at  $(-1, 0)$ , or (iii) anywhere else. For each of the above scenarios, eigenvalues initially travel on the unit circle and escape it, for critical values of the control parameter, yielding unstable behavior. Some of the additional features of the bifurcated solutions (fold, flip or Neimark-Sacker bifurcations, for instance) can be graphically displayed by numerically integrating the equations of motion initiated on the nonlinear mode. The Poincaré map is then employed to explore and visualize the dynamics in the unstable region. This is not further discussed in this work.

**7. Conclusion** Nonlinear normal modes are one-parameter continuous families of periodic motions of multi-degrees-of-freedom systems. They have been investigated since the 1950's for vibratory systems involving smooth and differentiable nonlinearities. In this work, an extension to a vibro-impact system exhibiting non-differentiable displacement and discontinuous velocity is proposed. Based on the construction of a convenient Poincaré map and with the assumption of a single impact per period, (quasi-) closed-form solutions are obtained and nonlinear modes of vibration are shown to exist. They



**Figure 6.2:** Spectral stability analysis of the first nonlinear mode in the vicinity of  $\omega_1$  [Branch 1 in Fig. 5.1]. Eigenvalues of the linearized First Return Map  $\mathbf{DF}$  in the complex plane.

can be visualized as piecewise analytic two-dimensional manifolds in the state-space, formed by the union of one “flat” submanifold (contact is not activated and the normal modes are invariant elliptic disks) and one “curved” submanifold (contact occurs once per period) sharing a common grazing orbit. From the expressions for the normal modes, the frequency-energy relationships on each mode are captured and are (obviously) of piecewise type.

Interestingly, it is shown that the manifolds emanating from the linear modes of period  $T_i$  and from the subharmonics of the linear modes of period  $nT_i$ ,  $n \in \mathbb{N}^+$  and  $i \in \{1, \dots, N\}$  are distinct. Altogether, such manifolds “shape” the state-space of the system and graphically illustrate the dynamical complexities induced by the presence of a perfectly elastic impact law into the (otherwise linear) governing equations.

A succinct spectral stability analysis is conducted for the first nonlinear mode. It is numerically demonstrated that an intricate pattern of stable and instable solutions arises as the frequency of the periodic solution is increased. The three commonly reported bifurcation mechanisms are found: they depend on where the eigenvalues of the linearized First Return Map leave the unit circle in the complex plane. Other more advanced bifurcations involving grazing orbits transitioning from a single impact per period to multiple impacts per period are conjectured.

**Appendix A. Stability analysis through perturbed orbits** The stability analysis of a periodic orbit lying on the previously constructed manifolds as explained in theorem 3.4 is revisited in a more pragmatic and intuitive manner. Let’s first define the notation  $\mathbf{U}_t := (\mathbf{u}_t, \dot{\mathbf{u}}_t) := (\mathbf{u}(t), \dot{\mathbf{u}}^+(t))$ . The initial condition on the Poincaré section of this periodic orbit is denoted by  $\mathbf{U}_0$  and its period,  $T_0$ . By definition of the Poincaré section,  $\bar{\mathbf{e}}_N^\top \mathbf{U}_0 = u_N(0) = g$ . We want to know how slightly disturbed initial conditions  $\mathbf{U}_0 + \delta \mathbf{U}_0$  defined on the Poincaré section, *i.e.* satisfying  $\bar{\mathbf{e}}_N^\top \delta \mathbf{U}_0 = 0$ , are mapped back onto the Poincaré section  $\mathbf{U}_T = \mathbf{U}_0 + \delta \mathbf{U}_T$  after a time  $T = T_0 + \delta T$  with the condition  $\bar{\mathbf{e}}_N^\top \delta \mathbf{U}_T = 0$ . If their amplitude is magnified, then the periodic orbit is unstable. The Poincaré map (2.27) in physical coordinates reads

$$\begin{pmatrix} \mathbf{u}(T) \\ \dot{\mathbf{u}}^+(T) \end{pmatrix} = \begin{bmatrix} \mathbf{P} \cos(T\Omega)\mathbf{P}^{-1} & \mathbf{P}\Omega^{-1} \sin(T\Omega)\mathbf{P}^{-1} \\ -\mathbf{S}\mathbf{P}\Omega \sin(T\Omega)\mathbf{P}^{-1} & \mathbf{S}\mathbf{P} \cos(T\Omega)\mathbf{P}^{-1} \end{bmatrix} \begin{pmatrix} \mathbf{u}(0) \\ \dot{\mathbf{u}}^+(0) \end{pmatrix} = \begin{bmatrix} \Gamma_1(T) & \Gamma_2(T) \\ \Gamma_3(T) & \Gamma_4(T) \end{bmatrix} \begin{pmatrix} \mathbf{u}(0) \\ \dot{\mathbf{u}}^+(0) \end{pmatrix} \quad (\text{A.1})$$

that is  $\Gamma(T) = \bar{\mathbf{P}}\mathbf{R}(T)\bar{\mathbf{P}}^{-1}$ . An admissible perturbed orbit starting on the Poincaré section at  $t = 0$  in

a neighborhood of the actual periodic orbit and coming back to the Poincaré section at  $T = T_0 + \delta T$  is thus governed by the system

$$\begin{pmatrix} \mathbf{u}_0 + \delta \mathbf{u}_T \\ \dot{\mathbf{u}}_0 + \delta \dot{\mathbf{u}}_T \end{pmatrix} = \begin{bmatrix} \mathbf{\Gamma}_1(T_0 + \delta T) & \mathbf{\Gamma}_2(T_0 + \delta T) \\ \mathbf{\Gamma}_3(T_0 + \delta T) & \mathbf{\Gamma}_4(T_0 + \delta T) \end{bmatrix} \begin{pmatrix} \mathbf{u}_0 + \delta \mathbf{u}_0 \\ \dot{\mathbf{u}}_0 + \delta \dot{\mathbf{u}}_0 \end{pmatrix} \quad (\text{A.2})$$

with the property

$$\mathbf{e}_N^\top \delta \mathbf{u}_T = \mathbf{e}_N^\top \delta \mathbf{u}_0 = 0. \quad (\text{A.3})$$

Neglecting higher order terms, system (A.2) simplifies to

$$\begin{pmatrix} \mathbf{u}_0 + \delta \mathbf{u}_T \\ \dot{\mathbf{u}}_0 + \delta \dot{\mathbf{u}}_T \end{pmatrix} = \begin{bmatrix} \mathbf{\Gamma}_1(T_0) + \dot{\mathbf{\Gamma}}_1(T_0)\delta T & \mathbf{\Gamma}_2(T_0) + \dot{\mathbf{\Gamma}}_2(T_0)\delta T \\ \mathbf{\Gamma}_3(T_0) + \dot{\mathbf{\Gamma}}_3(T_0)\delta T & \mathbf{\Gamma}_4(T_0) + \dot{\mathbf{\Gamma}}_4(T_0)\delta T \end{bmatrix} \begin{pmatrix} \mathbf{u}_0 + \delta \mathbf{u}_0 \\ \dot{\mathbf{u}}_0 + \delta \dot{\mathbf{u}}_0 \end{pmatrix}. \quad (\text{A.4})$$

Since Eq. (A.1) is satisfied by the periodic orbit, Eq. (A.4) becomes

$$\begin{pmatrix} \delta \mathbf{u}_T \\ \delta \dot{\mathbf{u}}_T \end{pmatrix} = \begin{bmatrix} \mathbf{\Gamma}_1(T_0) & \mathbf{\Gamma}_2(T_0) \\ \mathbf{\Gamma}_3(T_0) & \mathbf{\Gamma}_4(T_0) \end{bmatrix} \begin{pmatrix} \delta \mathbf{u}_0 \\ \delta \dot{\mathbf{u}}_0 \end{pmatrix} + \delta T \begin{bmatrix} \dot{\mathbf{\Gamma}}_1(T_0) & \dot{\mathbf{\Gamma}}_2(T_0) \\ \dot{\mathbf{\Gamma}}_3(T_0) & \dot{\mathbf{\Gamma}}_4(T_0) \end{bmatrix} \begin{pmatrix} \mathbf{u}_0 \\ \dot{\mathbf{u}}_0 \end{pmatrix} \quad (\text{A.5})$$

From Eq. (A.3), we can express  $\delta T$  in terms of  $(\delta \mathbf{u}_0, \delta \dot{\mathbf{u}}_0)$  since

$$\begin{pmatrix} \mathbf{e}_N \\ \mathbf{0} \end{pmatrix}^\top \begin{bmatrix} \mathbf{\Gamma}_1(T_0) & \mathbf{\Gamma}_2(T_0) \\ \mathbf{\Gamma}_3(T_0) & \mathbf{\Gamma}_4(T_0) \end{bmatrix} \begin{pmatrix} \delta \mathbf{u}_0 \\ \delta \dot{\mathbf{u}}_0 \end{pmatrix} + \delta T \begin{pmatrix} \mathbf{e}_N \\ \mathbf{0} \end{pmatrix}^\top \begin{bmatrix} \dot{\mathbf{\Gamma}}_1(T_0) & \dot{\mathbf{\Gamma}}_2(T_0) \\ \dot{\mathbf{\Gamma}}_3(T_0) & \dot{\mathbf{\Gamma}}_4(T_0) \end{bmatrix} \begin{pmatrix} \mathbf{u}_0 \\ \dot{\mathbf{u}}_0 \end{pmatrix} = 0 \quad (\text{A.6})$$

which leads to

$$\delta T = -\frac{\bar{\mathbf{e}}_N^\top \mathbf{\Gamma}(T_0) \delta \mathbf{U}_0}{\bar{\mathbf{e}}_N^\top \dot{\mathbf{\Gamma}}(T_0) \mathbf{U}_0}. \quad (\text{A.7})$$

System (A.5) thus becomes

$$\delta \mathbf{U}_T = \left( \mathbf{\Gamma} - \frac{\dot{\mathbf{\Gamma}}(T_0) \mathbf{U}_0 \bar{\mathbf{e}}_N^\top \mathbf{\Gamma}(T_0)}{\bar{\mathbf{e}}_N^\top \dot{\mathbf{\Gamma}}(T_0) \mathbf{U}_0} \right) \delta \mathbf{U}_0 \quad (\text{A.8})$$

which takes the compact form

$$\delta \mathbf{U}_T = \delta \mathbf{\Gamma} \delta \mathbf{U}_0 \quad (\text{A.9})$$

where  $\delta \mathbf{\Gamma}$  is a notation. The matrices  $\delta \mathbf{\Gamma}$  in Eq. (A.9) and  $\mathbf{DF}$  in Eq. (3.5) are two expressions of the same linearized First Return Map, in physical and modal coordinates, respectively (see transformation (4.3) between the two systems). They both exhibit the additional extraneous eigenvalue 0 which should be discarded. To this end, let's consider the *rectangular* mapping  $\mathbf{Q}_N^\top$  which removes row  $N$  of a vector it multiplies;  $\mathbf{Q}_N$  is a rectangular matrix of size  $2N \times 2N - 1$  storing "1" on its "fake diagonal". Let's also use two new reduced quantities:  $\mathbf{U}_{0,N} = \mathbf{Q}_N \mathbf{U}_0$  is  $\mathbf{U}_0$  with its  $N$ th coordinate removed; same for  $\mathbf{U}_{T,N}$  with  $\mathbf{U}_T$ , as well as their corresponding perturbations. System (A.9) simplifies to

$$\delta \mathbf{U}_{T,N} = \mathbf{Q}_N^\top \delta \mathbf{\Gamma} \mathbf{Q}_N \delta \mathbf{U}_{0,N} = \delta \mathbf{\Gamma}_N \delta \mathbf{U}_{0,N}. \quad (\text{A.10})$$

A similar reduction could be proposed for  $\mathbf{DF}$  in the modal coordinates system. The eigenvalues of the reduced maps dictate the instability of the targeted periodic orbit. For a given periodic response, a

sufficient condition for instability is that at least one of these eigenvalues in modulus lies outside the unit circle in the complex plane.

**Appendix B. Eigenvalues of the Return Map Matrix** The spectrum of the Return Map Matrix  $\mathbf{R}(T)$  lies on the unit circle. Unfortunately, this is not sufficient to conclude on the stability of the nonlinear modes but it could be useful for further works on this topic. It is determined by finding  $\lambda$  such that

$$\det \left( \begin{bmatrix} \cos(T\boldsymbol{\Omega}) - \lambda \mathbf{I} & \boldsymbol{\Omega}^{-1} \sin(T\boldsymbol{\Omega}) \\ -\mathcal{S}\boldsymbol{\Omega} \sin(T\boldsymbol{\Omega}) & \mathcal{S} \cos(T\boldsymbol{\Omega}) - \lambda \mathbf{I} \end{bmatrix} \right) = 0. \quad (\text{B.1})$$

The first row of blocks stores diagonal matrices and the determinant can be calculated blockwise

$$\det((\cos(T\boldsymbol{\Omega}) - \lambda \mathbf{I})(\mathcal{S} \cos(T\boldsymbol{\Omega}) - \lambda \mathbf{I}) + \mathcal{S} \sin(T\boldsymbol{\Omega})^2) = 0 \quad (\text{B.2})$$

or equivalently

$$\det(\mathcal{S} - \lambda \cos(T\boldsymbol{\Omega})(\mathcal{S} + \mathbf{I}) + \lambda^2 \mathbf{I}) = 0. \quad (\text{B.3})$$

We can respectively premultiply and postmultiply by the non-singular matrices  $\mathbf{P}$  and  $\mathbf{P}^{-1}$  such that

$$\det \mathbf{P} \det(\mathcal{S} - \lambda \cos(T\boldsymbol{\Omega})(\mathcal{S} + \mathbf{I}) + \lambda^2 \mathbf{I}) \det \mathbf{P}^{-1} = 0 \quad (\text{B.4})$$

which is equivalent to

$$\det(\mathbf{P}(\mathcal{S} - \lambda \cos(T\boldsymbol{\Omega})(\mathcal{S} + \mathbf{I}) + \lambda^2 \mathbf{I})\mathbf{P}^{-1}) = 0 \quad (\text{B.5})$$

which itself simplifies to

$$\det(\mathcal{S} - \lambda \cos(T\boldsymbol{\Omega})(\mathcal{S} + \mathbf{I}) + \lambda^2 \mathbf{I}) = 0. \quad (\text{B.6})$$

This becomes the determinant of a diagonal system with the following discussion:

1. entries  $i = 1, \dots, N - 1$ :

$$1 - 2\lambda \cos(\omega_i T) + \lambda^2 = 0 \quad (\text{B.7})$$

2. entry  $N$ :

$$-1 + \lambda^2 = 0 \quad (\text{B.8})$$

All eigenvalues are of modulus 1.

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