

Nonsmooth modal analysis: investigation of spring-mass system subject to an elastic impact law

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McGill



IDETC 2015, Boston

State space

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Equivalence

Mechanical state



Point in the phase space

Linear mode 1

Linear mode 2

Linear modal analysis

What about nonlinear systems?

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- **Linear modes:**

- rely on **superposition principle**
- solutions live in a vector space whose basis are the linear modes
- linear algebra (\sim end of 19th)
- physical importance of eigenvalues: resonances and internal resonances

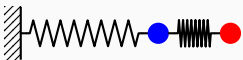
- **Nonlinear modes:**

- superposition principle no longer applies
- *but* maths gives a way to calculate particular surfaces of the state space
- centre manifold theorem (\sim 1970)
- applied to dynamical systems (Rosenberg, Shaw, Pierre, Vakakis, etc.)

Nonlinear modes

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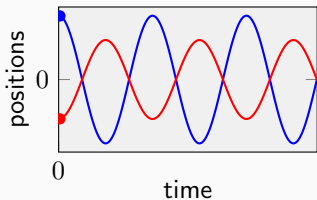
Nonlinear system



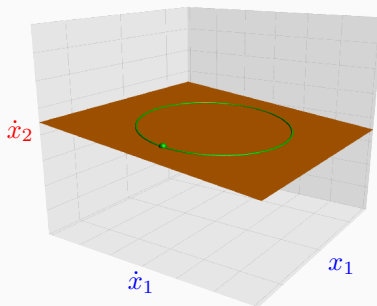
$$M\ddot{\mathbf{X}} + \mathbf{K}(\mathbf{X} + \alpha\mathbf{X}^3) = \mathbf{0}$$



Time-evolution



State space



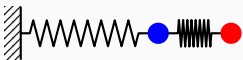
Definition of nonlinear mode

Invariant surface of the state space,
tangent to a linear mode.

Nonlinear modes

5/18

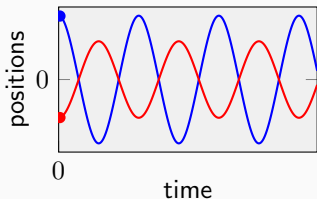
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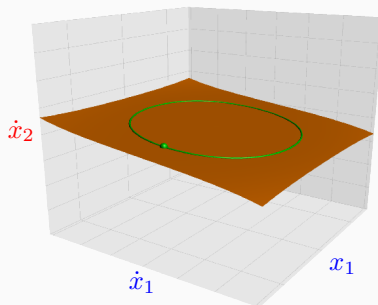
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Time-evolution



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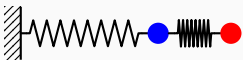
Definition of nonlinear mode

Invariant surface of the state space, tangent to a linear mode.

Nonlinear modes

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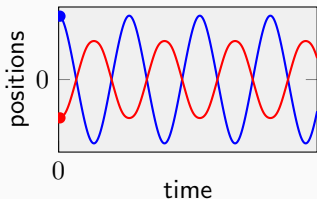
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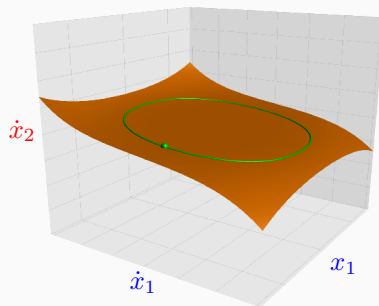
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Time-evolution



State space



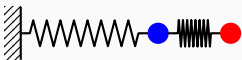
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Nonlinear modes

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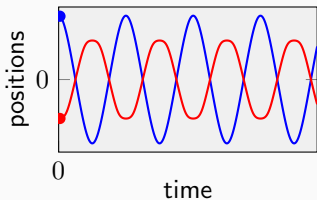
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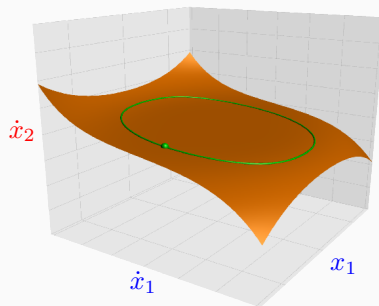
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Time-evolution



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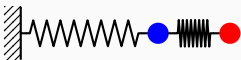
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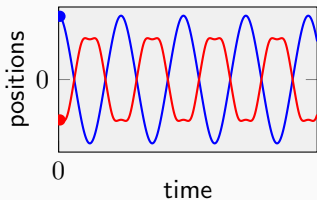
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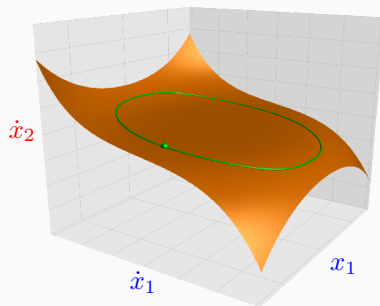
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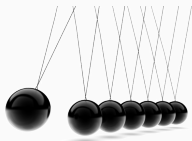
Nonlinear modes

Limitation

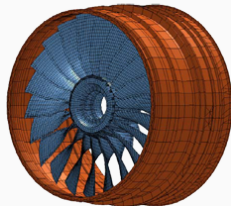
Calculation of nonlinear modes relies on *smoothness*.

→ **What about *nonsmooth* systems?**

- Nonsmooth system: system undergoing impacts or/and friction.



Newton's cradle [Inria]



Aircraft engine [Mech. Eng., McGill]

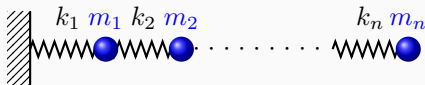
- **Long-term objective:** extend nonlinear modal analysis to nonsmooth systems

Outline

- 1 Context
- 2 Description of the non-smooth system
- 3 1 impact per period
- 4 2 impacts per period
- 5 k impacts per period

Spring-mass oscillator: impact condition

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- Free flight

Dynamics $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$

First-order form $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with $\mathbf{X} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}$ and $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$

Solutions $\mathbf{X}(t) = e^{t\mathbf{A}}\mathbf{X}(0)$

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Impact law $\dot{x}_n(t_{\text{imp}}^+) = -e\dot{x}_n(t_{\text{imp}}^-)$ with $e = 1$

Matrix form $\mathbf{X}(t_{\text{imp}}^+) = \mathbf{diag}(1, \dots, 1, -1)\mathbf{X}(t_{\text{imp}}^-)$

Spring-mass oscillator: impact condition

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- Free flight

Dynamics $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$ ← unforced

First-order form $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with $\mathbf{X} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}$ and $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$

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How to build nonsmooth modes for this system?

Linear algebra + Geometric interpretation

Change of basis

8/18

After an appropriate change of basis:

- Free flight

$$\mathbf{X}(t) = e^{t\mathbf{A}}\mathbf{X}(0) \text{ becomes } \mathbf{q}(t) = \mathbf{S}(t)\mathbf{q}(0)$$

where diagonal blocks of $\mathbf{S}(t)$ are $\begin{bmatrix} \cos(\omega_i t) & -\sin(\omega_i t) \\ \sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix}$

$\implies \mathbf{S}(t)$ is a direct isometry with n orthogonal stable planes

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- Impact condition

$$\mathbf{X}(t_{\text{imp}}^+) = \mathbf{diag}(1, \dots, 1, -1)\mathbf{X}(t_{\text{imp}}^-) \text{ becomes } \mathbf{q}(t_{\text{imp}}^+) = \mathbf{N}\mathbf{q}(t_{\text{imp}}^-)$$

$\implies \mathbf{N}$ is an indirect isometry (reflexion w.r.t. hyperplane)

Geometric interpretation

- Free flight can be seen as the image of \mathbf{q} by n plane rotations.
- Impact = image of $\mathbf{q}(t)$ by reflexion with respect to a hyperplane.

1 i.p.p.: governing equations

9/18

Summary

- free flight from t_{imp} to t : $\mathbf{q}(t) = \mathbf{S}(t - t_{\text{imp}})\mathbf{q}(t_{\text{imp}})$ n plane rotations
- impact at t : $\mathbf{q}(t_{\text{imp}}^+) = \mathbf{N}\mathbf{q}(t_{\text{imp}}^-)$ 1 reflexion

1 i.p.p.: governing equations

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t_1 : impact instant ($t_1 = T$)

initial state

$\mathbf{q}(0) \mid \mathbf{q}(0^+)$

1 i.p.p.: governing equations

9/18

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initial state
free flight

$$\mathbf{q}(0) \quad \Bigg| \quad \mathbf{q}(0^+)$$

$$\mathbf{S}(t_1)\mathbf{q}(0) \quad \Bigg| \quad \mathbf{q}(t_1^-)$$

1 i.p.p.: governing equations

9/18

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initial state

free flight

impact

$$\begin{array}{l} \mathbf{q}(0) \\ \mathbf{S}(t_1)\mathbf{q}(0) \\ \mathbf{NS}(t_1)\mathbf{q}(0) \end{array} \left| \begin{array}{l} \mathbf{q}(0^+) \\ \mathbf{q}(t_1^-) \\ \mathbf{q}(t_1^+) \end{array} \right.$$

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9/18

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- free flight from t_{imp} to t : $\mathbf{q}(t) = \mathbf{S}(t - t_{\text{imp}})\mathbf{q}(t_{\text{imp}})$ *n plane rotations*
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Periodicity conditions

Find $\mathbf{q}(0)$ and t_1 such that $\mathbf{q}(0) = \mathbf{q}(t_1)$

Mathematical problem

Find $\mathbf{q}(0)$ and t_1 such that $\mathbf{q}(0) \in \ker(\mathbf{NS}(t_1) - \mathbf{Id})$

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initial state

free flight

impact

$$\begin{array}{l|l} \mathbf{q}(0) & \mathbf{q}(0^+) \\ \mathbf{S}(t_1)\mathbf{q}(0) & \mathbf{q}(t_1^-) \\ \mathbf{NS}(t_1)\mathbf{q}(0) & \mathbf{q}(t_1^+) \end{array}$$

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1 i.p.p.: Illustration

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Mathematical problem

Find $\mathbf{q}(0)$ and t_1 such that $\mathbf{q}(0) \in \ker(\mathbf{NS}(t_1) - \mathbf{Id})$

- Solve $\det(\mathbf{NS}(t_1) - \mathbf{Id}) = 0$
- Choose $\mathbf{q}(0) \in \ker(\mathbf{NS}(t_1) - \mathbf{Id})$ such that $x_n(0) = d$

1 i.p.p.: Building nonsmooth mode

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- Start from a grazing linear mode
 - Vary continuously t_1 (recall that t_1 is the period)
- ⇒ continuum of nonsmooth orbits

2 i.p.p.: governing equations

12/18

Summary

- free flight from t_{imp} to t : $\mathbf{q}(t) = \mathbf{S}(t - t_{\text{imp}})\mathbf{q}(t_{\text{imp}})$ n plane rotations
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t_1, t_2 : impact instant ($t_2 = T$)

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12/18

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initial state
free flight 1

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initial state
free flight 1
impact 1

$$\begin{array}{l} \mathbf{q}(0) \\ \mathbf{S}(t_1)\mathbf{q}(0) \\ \mathbf{NS}(t_1)\mathbf{q}(0) \end{array} \left| \begin{array}{l} \mathbf{q}(0^+) \\ \mathbf{q}(t_1^-) \\ \mathbf{q}(t_1^+) \end{array} \right.$$

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12/18

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impact 1	$\mathbf{N}\mathbf{S}(t_1)\mathbf{q}(0)$	$\mathbf{q}(t_1^+)$
free flight 2	$\mathbf{S}(t_2 - t_1)\mathbf{N}\mathbf{S}(t_1)\mathbf{q}(0)$	$\mathbf{q}(t_2^-)$

2 i.p.p.: governing equations

12/18

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impact 2	$\mathbf{NS}(t_2 - t_1)\mathbf{NS}(t_1)\mathbf{q}(0)$	$\mathbf{q}(t_2^+)$

Periodicity conditions

Find $\mathbf{q}(0)$ and t_1, t_2 such that $\mathbf{q}(0) = \mathbf{q}(t_2)$

Mathematical problem

Find $\mathbf{q}(0)$ and t_1 such that $\mathbf{q}(0) \in \ker(\mathbf{NS}(t_2 - t_1)\mathbf{NS}(t_1) - \mathbf{Id})$

2 i.p.p.: governing equations

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Direct approach: fails

13/18

Mathematical problem

Find $\mathbf{q}(0)$ and t_1, \dots, t_k s.t. $\mathbf{q}(0) \in \ker \left(\underbrace{\mathbf{NS}(t_2 - t_1)\mathbf{NS}(t_1)}_{:=\mathbf{A}(t_1, t_2)} - \mathbf{Id} \right)$

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Idea:

- 1 Solve $\det(\mathbf{A}(t_1, t_2)) = 0$ for unknowns (t_1, t_2)
- 2 calculate $\ker(\mathbf{A}(t_1, t_2))$

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Issues:

- 1 $\det(\mathbf{A}(t_1, t_2)) = 0$ is a cumbersome condition
- 2 dimension of \ker has to be greater than rank of the system of conditions
 $x_n(t_1) = x_n(t_2) = d$.

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Not possible to solve in practice for $n \geq 3$.

\implies More subtle reasoning is required

2 i.p.p.: Illustration

See last part for details on how this was obtained.

2 i.p.p.: Building nonsmooth mode

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- Start from a grazing linear mode
- Vary continuously t_1, t_2 (recall that t_2 is the period)
- Difficulty: (t_1, t_2) must always verify $\dim(\ker \mathbf{A}(t_1, t_2)) = 2$
 \implies continuum of nonsmooth orbits

Mathematical results

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Main mathematical results

- **Existence of** t_1, \dots, t_k

There exist a known isomorphism φ and a known matrix $\mathbf{\Pi}$ such that:

$$\mathbf{q}(0) \in \ker(\mathbf{A} - \mathbf{Id}) \xrightarrow{\sim} \ker(\mathbf{\Pi}), \quad \mathbf{\Pi} \in \text{Skew}_k$$

$$\text{So } \boxed{\mathbf{A}(t_1, \dots, t_k)\mathbf{q}(0) = \mathbf{q}(0) \iff \mathbf{\Pi}(t_1, \dots, t_k)\varphi(\mathbf{q}(0)) = \mathbf{0}}$$

- **Conditions** $x_n(t_1) = d, \dots, x_n(t_k) = d$

There exists a known matrix $\mathbf{\Sigma}$ such that these conditions write

$$\boxed{\mathbf{\Sigma}(t_1, \dots, t_k)\varphi(\mathbf{q}(0)) = d[1, \dots, 1]^\top}$$

Procedure (with $\boldsymbol{\lambda} = \varphi(\mathbf{q}(0))$)

Given k , find $\boldsymbol{\lambda}$ and t_1, \dots, t_k such that

- $\mathbf{M}(t_1, \dots, t_k)\boldsymbol{\lambda} = \mathbf{0} \longrightarrow \frac{k(k-1)}{2}$ equations

- $\mathbf{\Sigma}(t_1, \dots, t_k)\boldsymbol{\lambda} = d[1, \dots, 1]^\top \longrightarrow k$ equations

for $k + k$ unknowns (t_1, \dots, t_k and $\boldsymbol{\lambda}$)

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for $k + k$ unknowns $(t_1, \dots, t_k$ and $\boldsymbol{\lambda}$); **independent of** n

Mathematical results: application

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Procedure (with $\lambda = \varphi(\mathbf{q}(0))$)

Given k , find λ and t_1, \dots, t_k such that

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for $k + k$ unknowns (t_1, \dots, t_k and λ)

- $k = 1$: almost always a unique solution

$$\mathbf{\Pi} = [0] \tag{1}$$

Mathematical results: application

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Procedure (with $\lambda = \varphi(\mathbf{q}(0))$)

Given k , find λ and t_1, \dots, t_k such that

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- $\mathbf{\Sigma}(t_1, \dots, t_k)\lambda = d[1, \dots, 1]^T \quad \longrightarrow \quad k$ equations

for $k + k$ unknowns (t_1, \dots, t_k and λ)

- $k = 1$: almost always a unique solution
- $k = 2$: when $\Pi_{1,2}(t_1, t_2) = 0$, almost always a unique solution

$$\mathbf{\Pi} = \begin{bmatrix} 0 & \Pi_{1,2}(t_1, t_2) \\ -\Pi_{1,2}(t_1, t_2) & 0 \end{bmatrix} \quad (1)$$

Mathematical results: application

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Mathematical results: application

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- $k = 3$: difficulty is to solve $\Pi_{1,2} = \Pi_{1,3} = \Pi_{2,3} = 0$
- $k \geq 4$: too many $\Pi_{1,2} = \dots = \Pi_{k-1,k}$ but solutions based on symmetries

Conclusions & Prospects

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Conclusions

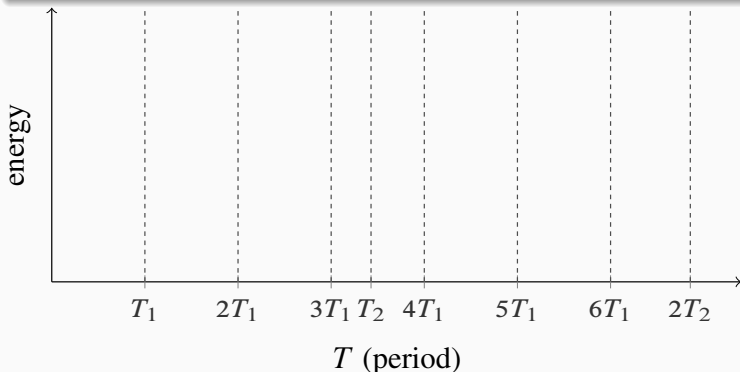
- Existence of invariant manifolds, tangent to linear modes: **nonsmooth modes**
- Classification by number of impact(s) per period
- Highlights importance of mathematics
- **Extremely rich behaviour induced by contact:**

Conclusions & Prospects

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Conclusions

- Existence of invariant manifolds, tangent to linear modes: **nonsmooth modes**
- Classification by number of impact(s) per period
- Highlights importance of mathematics
- **Extremely rich behaviour induced by contact:** linear modes

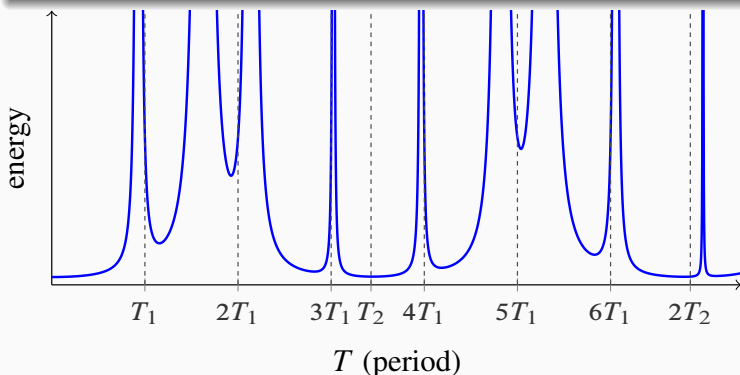


Conclusions & Prospects

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Conclusions

- Existence of invariant manifolds, tangent to linear modes: **nonsmooth modes**
- Classification by number of impact(s) per period
- Highlights importance of mathematics
- **Extremely rich behaviour induced by contact: 1 impact per period**

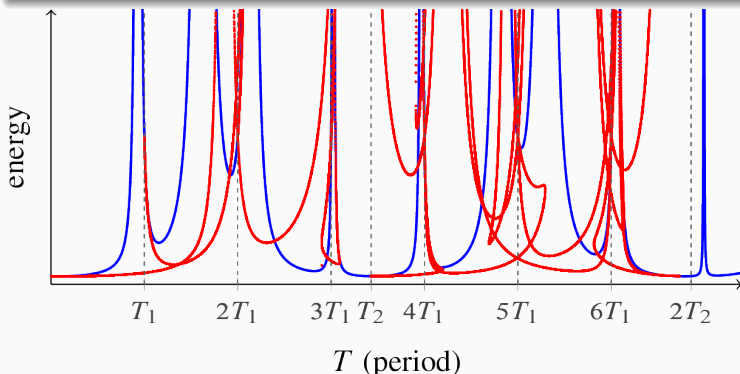


Conclusions & Prospects

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Conclusions

- Existence of invariant manifolds, tangent to linear modes: **nonsmooth modes**
- Classification by number of impact(s) per period
- Highlights importance of mathematics
- **Extremely rich behaviour induced by contact: 2 impacts per period**



Conclusions & Prospects

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Conclusions

- Existence of invariant manifolds, tangent to linear modes: **nonsmooth modes**
- Classification by number of impact(s) per period
- Highlights importance of mathematics
- **Extremely rich behaviour induced by contact:**

Future work: is it usable in practice?

- Study of stabilities (Lyapunov, orbital, etc.)
- Behaviour of *forced* system
- Generalisation to more complex geometries